Università degli Studi di Roma "Tor Vergata"
Macroarea di Scienze Matematiche, Fisiche e Naturali
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Long time behaviour of Optimal Control problems and the Turnpike Property

Candidato:
Dario Pighin

Relatore:
prof. Alessio Porretta

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Introduction

This dissertation is devoted to the study of the so called "Turnpike Property" in Optimal Control Theory. In abstract terms, the problem can be formulated as follows. For appropriate Banach Spaces U and X and for every time horizon $T \in (0, +\infty)$, we consider a NonStationary Control System taking place in $[0, T]$ described by an operator Λ. For every Admissible Control Function $u \in \mathcal{U}$ the corresponding State of the System is $x \in AC([0, T]; X)$ solution of the functional equation

$$\frac{d}{dt}x + \Lambda(x, u) = 0 \qquad \text{in } (0, T).$$

The aim of the optimization is to minimize a certain functional, which is defined as an integral of an appropriate Lagrangian function

$$\mathcal{L} : X \times U \longmapsto \mathbb{R}$$

i.e.:

$$\begin{aligned} J^T : \mathcal{U} &\longmapsto \mathbb{R} \\ u &\longrightarrow \int_0^T \mathcal{L}(x(t), u(t))dt. \end{aligned}$$

Therefore, $\forall T \in (0, +\infty)$, the Optimal Control Problem $(OCP)^T$ consists in finding

$$\inf_{\mathcal{U}} J^T$$

On the other hand, we formulate the stationary version $(OCP)^s$. First of all, we define

$$M \stackrel{\text{def}}{=} \{(x, u) \in X \times U \mid \Lambda(x, u) = 0\}.$$

Then, $(OCP)^s$ is the problem of minimizing $\mathcal{L} \restriction_M$, namely:

$$\inf_M \mathcal{L}.$$

At this stage, it is natural to find a certain link between $(OCP)^T$ and $(OCP)^s$, at least if we suppose that both $(OCP)^T$ and $(OCP)^s$ are well posed and admit a unique minimizer. The idea of the Turnpike is that for time horizon T large enough, the optimal triple (x^T, p^T, u^T) of $(OCP)^T$, far away from the initial point and the terminal point, is close to the optimal triple $(\overline{x}, \overline{p}, \overline{u})$ of $(OCP)^s$. More precisely the idea is that, for a positive T-independent time $\tau \in [0, T)$:

1. in the interval $[0, \tau]$ the optimal triple (x^T, p^T, u^T) for $(OCP)^T$ moves roughly from $(x^T(0), p^T(0), u^T(0))$ to $(\overline{x}, \overline{p}, \overline{u})$;

2. for a long time arc $[\tau, T - \tau]$ (x^T, p^T, u^T) stays near $(\overline{x}, \overline{p}, \overline{u})$;

3. for a final arc $[T - \tau, T]$ (x^T, p^T, u^T) moves roughly from $(\overline{x}, \overline{p}, \overline{u})$ to $(x^T(T), p^T(T), u^T(T))$.

For general initial-terminal conditions, it is not possible to expect a proximity of (x^T, p^T, u^T) to $(\overline{x}, \overline{p}, \overline{u})$ $\forall t \in [0, T]$. For example, if $x(0) = x_0$ and x_0 is away from $\overline{x}$, in an initial time interval $[0, \tau]$, x will be far away from $\overline{x}$. Moreover, the transversality condition $p^T(T) = 0$ entails that, if the norm of $\overline{p}$ is big, p^T is away from $\overline{p}$ in a final arc $[T - \tau, T]$.

Historically, this question arises in many fields of application. For example, in Econometry, where the stationary optimal state $\overline{x} \in X$ is named "Von Neumann point". One of the pioneers was the econometrician Paul Samuelson, awarded the Nobel Prize in Economic Sciences in 1970. Samuelson's reference [8] gave this property the name of "Turnpike": *...if we are planning long-run growth, no matter where we start and where we desire to end up, it will pay in the intermediate stages to get into a growth phase of this kind. It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.* Later on, Paul Samuelson investigated the fulfillment of this property in [17] and in [13] and this subject has been widely studied all throughout the second half of the 20^{th} Century both in Mathematics and in fields of application like Econometry. More recently, Alexander J. Zaslavski wrote a book [23] on this topic, where different definitions of the Turnpike Property are presented and corresponding Optimization Problems, where they are fulfilled, are studied. Furthermore, some results in Mean Field Games Theory (see [5] and [6]) motivated a new source of investigation of the Turnpike Property in Optimal Control Theory, specifically in an Infinite Dimensional Setting, where the nearness of (x^T, p^T, u^T) to $(\overline{x}, \overline{p}, \overline{u})$ on $[\tau, T - \tau]$ is required to be exponential.

This led to the paper "Long time versus steady state optimal control" by Alessio Porretta and Enrique Zuazua [16], which is the origin of the present thesis. Furthermore, in "The turnpike property in finite-dimensional nonlinear optimal control" by Emmanuel Trélat and Enrique Zuazua [20], it is proved a Local Turnpike Property with exponential nearness for a wide class of NonLinear Optimal Control Problems. Moreover, [20] has been employed by some fields of application like biology (e.g. [11]). Finally, Turnpike Property has surprising consequences in Numerical Analysis. In fact, when it is fulfilled, $(x^T(\frac{T}{2}), p^T(\frac{T}{2}), u^T(\frac{T}{2}))$ is close to $(\overline{x}, \overline{p}, \overline{u})$. This provides a good initialization for Indirect Shooting Methods employed to solve numerically $(OCP)^T$. This idea is described by Emmanuel Trélat and Enrique Zuazua in [20].

In the present dissertation, we analyse the Turnpike Property for certain classes of Optimal Control Problems. We have divided this thesis into 4 chapters. The first one introduces the reader to some concepts that will be useful in the understanding of what follows. In the following chapters, for several classes of problems, we will prove a Turnpike Property, where the nonstationary optimal triple (x^T, p^T, u^T) is exponentially near to the stationary one $(\overline{x}, \overline{p}, \overline{u})$ on a long time interval $[\tau, T-\tau]$.

2$^{\text{nd}}$ Chapter

The 2$^{\text{nd}}$ Chapter deals with Finite Dimensional Linear Quadratic Problems. We will work with a triple of matrices $(A, B, C) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R}) \times \mathcal{M}(N, N; \mathbb{R})$, where (A, B) is controllable and (A, C) is observable. Moreover, we will take into account an arbitrary initial data $x_0 \in \mathbb{R}^N$ and an arbitrary target $z \in \mathbb{R}^N$. For any $u \in L^2((0,T); \mathbb{R}^M)$, we will consider the unique x solution of:

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & \text{a.e. } t \in (0,T) \\ x(0) = x_0. \end{cases} \tag{1}$$

The nonstationary optimization $(OCP)^T$ consists in minimizing the following functional:

$$J^T : L^2((0,T); \mathbb{R}^M) \longmapsto \mathbb{R}$$

$$u \longrightarrow \frac{1}{2} \int_0^T \left[\|u\|^2 + \|Cx - z\|^2 \right] dt.$$

On the other hand, in order to define the stationary functional we need to define the vector subspace:

$$M \stackrel{\text{def}}{=} \left\{ (x, u) \in \mathbb{R}^N \times \mathbb{R}^M \mid Ax = Bu \right\}. \tag{2}$$

The stationary functional to be minimized is:

$$J^s : M \longmapsto \mathbb{R}$$

$$(x, u) \longrightarrow \frac{1}{2}\left[\|u\|^2 + \|Cx - z\|^2\right].$$

At the beginning we will show that the Kalman-Controllability of (A, B) is essential. As a matter of fact, we will show a counterexample where the pair (A, B) is not Kalman-Controllable and the Turnpike Property does not hold. The conclusion of this 2^{nd} chapter is the following Theorem.

Theorem 0.1 (Global Turnpike Property)**.** *We assume (A, B) controllable and (A, C) observable. Hence, there exists $(C, \mu) \in (0, +\infty)^2$ such that for every $T \in (0, +\infty)$, $\forall t \in [0, T]$:*

$$\|x^T(t)-\overline{x}\|+\|p^T(t)-\overline{p}\|+\|u^T(t)-\overline{u}\| \leq C\left[\|x_0 - \overline{x}\|e^{-\mu t} + \|\overline{p}\|e^{-\mu(T-t)}\right]. \quad (3)$$

The proof of this Theorem is based on the dynamic approach of [16].

3^{rd} Chapter

The purpose of the 3^{rd} Chapter is to show a Global Turnpike Property for the NonLinear Convex Case. In the first section, we will prove again the Turnpike Property for the Finite Dimensional Linear Quadratic Case, with some different hypotheses and by an algebraic approach. This approach essentially deals with a decomposition of the optimality system into a contracting part and an expanding one. It was firstly presented in [2] and [21]. More recently it has been applied to the Turnpike Theory in [20]. In the second section, we begin working with the NonLinear Convex case. We will take (A, B) Kalman-Controllable and 2 functions $(F, L) \in C^2(\mathbb{R}^N, \mathbb{R}) \times C^2(\mathbb{R}^M, \mathbb{R})$ characterized by the existence of $(\alpha, \beta) \in (0, +\infty)^2$ such that $\alpha I_N \leq F_{xx}$ and $\alpha I_M \leq L_{uu} \leq \beta I_M$. For every control function $u \in L^2((0, T); \mathbb{R}^M)$ we take into account x the unique solution of (1). The functional to be minimized is:

$$J^T : L^2((0, T); \mathbb{R}^M) \longmapsto \mathbb{R}$$

$$u \longrightarrow \int_0^T [L(u) + F(x)]\, dt$$

Furthermore, we define M as in (2). When we address the stationary problem, we seek for a minimizer of:

$$J^s : M \longmapsto \mathbb{R}$$

$$(x, u) \longrightarrow [F(x) + L(u)].$$

The first object of the first section is the well posedeness of the problem. Then, we will prove that the time-averages of the optimal triple (x^T, p^T, u^T) converge to $(\overline{x}, \overline{p}, \overline{u})$. In the third section we show a Local Turnpike Property for the NonLinear Convex Case, by studying the Linearised Optimality System employing the algebraic approach already mentioned. In the last section, the Convergence of Averages and the Local Turnpike Property allow us to prove a Global Turnpike Property for the NonLinear Convex Case, i.e.:

Theorem 0.2 (Global Turnpike Property). *We suppose* (A, B) *Kalman-Controllable and* $(F, L) \in C^2(\mathbb{R}^N, \mathbb{R}) \times C^2(\mathbb{R}^M, \mathbb{R})$ *such that* $(\alpha, \beta) \in (0, +\infty)^2$ *such that* $\alpha I_N \leq F_{xx}$ *and* $\alpha I_M \leq L_{uu} \leq \beta I_M$*. Hence, there exists* $(C, \mu) \in (0, +\infty)^2$ *such that the optimal triple* (x^T, p^T, u^T) *satisfies:*

$$\|x^T(t) - \overline{x}\| + \|p^T(t) - \overline{p}\| + \|u^T(t) - \overline{u}\| \leq C \left[e^{-\mu t} + e^{-\mu(T-t)}\right] \qquad \forall t \in [0, T]. \tag{4}$$

4th Chapter

In the last Chapter we work in an Infinite Dimensional Context. We will construct a framework which corresponds to Parabolic Distributed Control Case. This Chapter is the generalization of the 2nd Chapter to the Infinite Dimensional Case. At the end of the chapter, we will make an example with a Distributed Control of a Parabolic Divergential System. We will follow the dynamic approach of [16].

Notation

- For every A set, $\#A$ is the cardinality of A;

- $\mathbb{N}$ are the natural numbers including 0;

- $\{e_1, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^N$;

- $\mathcal{M}(M, N; \mathbb{R})$ is the set of $M \times N$ matrices with real coefficients;

- $GL(N; \mathbb{R}) = \{A \in \mathcal{M}(N, N; \mathbb{R}) \mid A \text{ is invertible}\}$;

- $AC([0, T], \mathbb{R}^N) = \big\{f : [0, T] \longmapsto \mathbb{R}^N \mid f \text{ is absolutely continuous}\big\}$;

- For any $I \subset \mathbb{R}$ interval

$$AC_{loc}(I, \mathbb{R}^N) = \big\{f : I \longmapsto \mathbb{R}^N \mid f \text{ is absolutely continuous in each compact subinterval of } I\big\};$$

- For any given Ω open subset of $\mathbb{R}^M$ and $k \in \mathbb{N} \cup \{+\infty\}$

  $C^k(\Omega, \mathbb{R}^N) = \big\{f : \Omega \longmapsto \mathbb{R}^N \mid f \text{ is k times continuously differentiable on } \Omega\big\}$

- for every $\Omega \subset \mathbb{R}^N$ open and for all

$$F : \Omega \longmapsto \mathbb{R}^p \in C^1(\Omega, \mathbb{R}^p)$$

$\forall x \in \Omega$ we define the Jacobian of F at x as follows:

$$F_x(x) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial}{\partial x_1} F_1(x) & \dots & \frac{\partial}{\partial x_N} F_1(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_1} F_p(x) & \dots & \frac{\partial}{\partial x_N} F_p(x) \end{pmatrix} \in \mathcal{M}(p, N; \mathbb{R}).$$

- For any given F closed subset of $\mathbb{R}^M$ and $k \in \mathbb{N} \cup \{+\infty\}$

$$C^k(F, \mathbb{R}^N) =$$

$$\Big\{ f : F \longmapsto \mathbb{R}^N \mid f \text{ has an extention on a open neighborhood of F}$$
$$\text{such that is k times continuously differentiable} \Big\}$$

- $$(.\,,.)_{\mathbb{R}^N} : \mathbb{R}^N \times \mathbb{R}^N \longmapsto \mathbb{R}$$
$$(a, b) \longrightarrow \sum_{i=1}^{N} a_i b_i$$

  denotes the Euclidean scalar product.

- in $\mathbb{R}^N$, $\| \cdot \|$ indicates the Euclidean norm;

- in $\mathcal{M}(M, N; \mathbb{R})$, $\| \cdot \|$ denotes the matricial norm induced by the Euclidean norm;

- in $\mathbb{R}^N$, $d_{\mathbb{R}^N}$ the Euclidean distance;

- $Sym(N, \mathbb{R})$ is the set of N-dimensional symmetric real matrices.

- for any $(X, \| \ \|_X)$ and $(Y, \| \ \|_Y)$ normed spaces $B(X, Y)$ indicates the set of bounded linear operators from X to Y.

- let $(X, \| \ \|_X)$ and $(Y, \| \ \|_Y)$ be 2 normed spaces, $K(X, Y)$ denotes the set of compact linear operators from X to Y.

- Let (X, d) be a metric space, $(x_0, r) \in X \times \mathbb{R}^+$ we denote by

$$B^X(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$$

and by

$$S^X(x_0, r) = \{x \in X \mid d(x, x_0) = r\};$$

- We take into account (X, d_x) and (Y, d_y) 2 metric spaces, we name

$$C_b(X, Y) = \big\{f : X \longmapsto Y \mid f \in C^0(X, Y) \wedge f \text{ is bounded}\big\};$$

- Let (X, d_x) and (Y, d_y) be 2 metric spaces, we define henceforth

$$d_\infty : C_b(X, Y) \times C_b(X, Y) \longmapsto \mathbb{R}^+$$

$$(f, g) \longrightarrow \sup\{d_y(f(x), g(x)) \mid x \in X\};$$

- Let V_1 and V_2 be 2 vector spaces over $\mathbb{R}$, $T : V_1 \longmapsto V_2$ a linear map, then we define

$$Ker(T) = \{v \in V_1 \mid T(v) = 0\} \quad \text{and}$$

$$R(T) = \{T(v) \mid v \in V_1\}$$

- μ_{leb} is the Lebesgue Measure.
- Given $(X, \|\|_X)$ a Banach space, we name $\sigma(X, X')$ the weak topology of X.
- $$x_k \underset{k \to +\infty}{\rightharpoonup} x$$

  means the sequence $\{x_k\}_{k \in \mathbb{N}}$ converge weakly to x.
- If Ω is an open subset of $\mathbb{R}^N$, $C_c^{+\infty}(\Omega, \mathbb{R}) = \{f \in C^\infty(\Omega, \mathbb{R}) \mid supp(f) \text{ is compact}\}$;
- $sp(N, \mathbb{R})$ is the Lie Algebra associated to the Lie Group $Sp(N; \mathbb{R})$ of Symplectic Matrices.

Chapter 1

Preliminaries

Let us introduce some notions that are useful for the understanding of the present work. First of all, the reader interested in an introduction on Finite Dimensional Optimal Control Theory should refer to [19]. Furthermore, a classical book on Optimal Control of Partial Differential Equations is [14]. Another book about Control of PDEs is [3], whose approach is based on Semigroup Theory. First of all we give some basic notions on Ordinary Differential Equations. We follow the approach of [18] page 474.

Definition 1.1. Let $I \subset (0, +\infty)$ be an interval and let $\Omega \subset \mathbb{R}^N$ be an open set and

$$f : I \times \Omega \longmapsto \mathbb{R}^N$$

be such that

1. $\forall x \in \Omega$

   $$f(\cdot, x) : I \longmapsto \mathbb{R}^N$$

   is measurable;

2. $\forall t \in I$

   $$f(t, \cdot) : \Omega \longmapsto \mathbb{R}^N \in C^0(\Omega, \mathbb{R}^N)$$

   is continuous.

$\forall (y_0, t_0) \in \Omega \times I$, if $J \subset I$ is a subinterval of I containing t_0, then a function

$$y(\cdot; y_0) : J \longmapsto \mathbb{R}^N \in AC_{loc}(J, \mathbb{R}^N)$$

is a solution of the Cauchy Problem

$$\begin{cases} \frac{d}{dt}y(t) = f(t, y(t)) & \text{a.e. } t \in I \\ y(0) = y_0 \end{cases} \tag{1.1}$$

if

1.
$$f(\cdot, y(\cdot; y_0)) \in L^1_{loc}(J; \mathbb{R}^N);$$

2. $\forall t \in J$
$$y(t; y_0) = y_0 + \int_{t_0}^{t} f(s, y(s; y_0))ds.$$

Furthermore, $y(\cdot; y_0) \in AC_{loc}(J; \mathbb{R}^N)$ is called a maximal solution to the Cauchy Problem (1.1) if for all $J' \subset J$ such that there exists

$$y'(\cdot; y_0) : J' \longmapsto \mathbb{R}^N \in AC_{loc}(J'; \mathbb{R}^N)$$

solution of the Cauchy Problem (1.1) above,

$$J' \subseteq J$$

and
$$y(t; y_0) = y'(t; y_0) \qquad \forall t \in J'.$$

Then, we name $J(y_0) \stackrel{\text{def}}{=} J$ the maximal interval of definition of the solution.

Whenever $f \in C^0(I \times \Omega, \mathbb{R}^N)$, for every solution $y(\cdot; y_0) \in AC(J, \mathbb{R}^N)$ of (1.1), by the Fundamental Theorem of Calculus, $y(\cdot; y_0) \in C^1(J, \mathbb{R}^N)$, (1.1) holds for every $t \in J$ and $y(\cdot; y_0)$ is named a classical solution of (1.1). An essential tool in (ODE) Theory is the Gronwall's Lemma.

Lemma 1.1 (Gronwall)**.** *For every*

$$v : [a, b] \longmapsto \mathbb{R}^+ \in C^0([a, b]; \mathbb{R}^+),$$
$$w : [a, b] \longmapsto \mathbb{R}^+ \in C^0([a, b]; \mathbb{R}^+)$$

and $C_0 \in (0, +\infty)$ *such that*

$$v(t) \leq C_0 + \int_a^t w(s)v(s)ds \qquad \forall t \in [a, b]$$

then,
$$v(t) \leq C_0 e^{\int_a^t w(s)ds} \qquad \forall t \in [a, b].$$

By using the transformation

$$T : [a, b] \longmapsto [a, b]$$
$$t \longrightarrow (b + a) - t$$

it is possible to prove the following alternative version of Gronwall's Lemma.

Lemma 1.2 (Gronwall)**.** *For every*

$$v : [a, b] \longmapsto \mathbb{R}^+ \in C^0([a, b]; \mathbb{R}^+),$$

$$w : [a, b] \longmapsto \mathbb{R}^+ \in C^0([a, b]; \mathbb{R}^+)$$

and $C_0 \in (0, +\infty)$ *such that*

$$v(t) \le C_0 + \int_t^b w(s)v(s)ds \qquad \forall t \in [a, b]$$

then,

$$v(t) \le C_0 e^{\int_t^b w(s)ds} \qquad \forall t \in [a, b].$$

Sometimes, it is necessary to employ a more general version of Gronwall's Lemma. First of all, we provide a "forward" version.

Lemma 1.3 (Gronwall)**.** *Let*

$$\beta : [a, b] \longmapsto \mathbb{R}^+ \in C^0([a, b], \mathbb{R}^+)$$

$$\gamma : [a, b] \longmapsto (0, +\infty) \in C^0([a, b], \mathbb{R}^+)$$

$$w : [a, b] \longmapsto \mathbb{R}^+ \in C^0([a, b], \mathbb{R}^+).$$

Then,

$$\forall g : [a, b] \longmapsto \mathbb{R}^+ \in C^0([a, b], \mathbb{R}^+)$$

such that:

$$g(t) \le \beta(t) + \int_a^t \gamma(t)w(s)g(s)ds \qquad \forall t \in [a, b],$$

it holds:

$$g(t) \le \gamma(t)e^{\int_a^t \gamma(\xi)w(\xi)d\xi} \int_a^t e^{-\int_a^s \gamma(\xi)w(\xi)d\xi}\beta(s)w(s)ds + \beta(t) \qquad t \in [a, b].$$

Employing the transformation:

$$T : [a, b] \longmapsto [a, b]$$

$$t \longrightarrow a + b - t$$

one can deduce a "backward" version of the generalised Gronwall's Lemma.

Lemma 1.4 (Gronwall). *For any*

$$\beta : [a,b] \longmapsto \mathbb{R}^+ \in C^0([a,b],\mathbb{R}^+)$$

$$\gamma : [a,b] \longmapsto (0,+\infty) \in C^0([a,b],\mathbb{R}^+)$$

$$w : [a,b] \longmapsto \mathbb{R}^+ \in C^0([a,b],\mathbb{R}^+),$$

whenever we take into account

$$g : [a,b] \longmapsto \mathbb{R}^+ \in C^0([a,b],\mathbb{R}^+)$$

such that:

$$g(t) \leq \beta(t) + \int_t^b \gamma(t)w(s)g(s)ds \qquad \forall t \in [a,b],$$

it follows that:

$$g(t) \leq \gamma(t)e^{\int_t^b \gamma(\xi)w(\xi)d\xi} \int_t^b e^{-\int_t^b \gamma(\xi)w(\xi)d\xi}\beta(s)w(s)ds + \beta(t) \qquad t \in [a,b].$$

After Gronwall's Lemmas, another key tool to prove the local existence and uniqueness for Cauchy Problems is the Contraction Mapping Theorem.

Theorem 1.1. *Let (X,d) be a complete metric space. For every map*

$$f : (X,d) \longmapsto (X,d)$$

which is a contraction, i.e. there exists $\alpha \in (0,1)$ such that:

$$d(f(x_1), f(x_2)) \leq \alpha\ d(x_1,x_2) \qquad \forall (x_1,x_2) \in X^2$$

there exists a unique $\hat{x} \in X$ fixed point for f, namely:

$$f(\hat{x}) = \hat{x}.$$

At this point, we state a local existence and global uniqueness result.

Theorem 1.2. $\forall (a,b) \in (0,+\infty)^2$ *and* $\forall \Omega \subset \mathbb{R}^N$ *open set, it is given*

$$f : I \times \Omega \longmapsto \mathbb{R}^N$$

enjoying the following properties:

1. $\forall x \in \Omega$

$$f(\cdot, x) : I \longmapsto \mathbb{R}^N \tag{1.2}$$

is measurable;

2. $\forall t \in I$

$$f(t,\cdot) : \Omega \longmapsto \mathbb{R}^N \in C^0(\Omega, \mathbb{R}^N) \tag{1.3}$$

is continuous;

3. f *is locally Lipschitz on* x*, i.e. for all* $x_0 \in \Omega$ *there exists* $\rho \in (0, +\infty)$ *and* $\alpha \in L^1_{loc}(I, \mathbb{R})$ *such that* $B(x_0, \rho) \subset \Omega$ *and*

$$\|f(t,y) - f(t,x)\| \leq \alpha(t)\|y - x\| \qquad \forall (t,x,y) \in I \times B(x_0,\rho) \times B(x_0,\rho);$$

4. f *is locally integrable on* t*, namely* $\forall x_0 \in \Omega$ *there exists* $\beta \in L^1_{loc}(I; \mathbb{R})$ *such that:*

$$\|f(t, x_0)\| \leq \beta(t) \qquad \forall t \in I.$$

Then, there exists a unique

$$y(\cdot; y_0) : J(y_0) \longmapsto \mathbb{R}^N \in AC_{loc}(J(y_0); \mathbb{R}^N)$$

maximal solution of (1.1)

$$\begin{cases} \frac{d}{dt} y(t) = f(t, y(t)) \qquad \forall t \in J(y_0) \\ y(0) = y_0, \end{cases} \tag{1.4}$$

where $J(y_0) \subset I$ *is a subinterval of* I *open in the relative topology of* I*.*

We give now a global existence and uniqueness result.

Theorem 1.3. $\forall I \subset (0, +\infty)$ *interval and* $\forall \Omega \subset \mathbb{R}^N$ *open set. We take into account*

$$f : I \times \Omega \longmapsto \mathbb{R}^N$$

fulfilling (1.2) *and* (1.2)*,*

- f *is globally Lipschitz on* x*, namely there exists* $\alpha \in L^1((a,b), \mathbb{R})$ *such that:*

$$\|f(t,y) - f(t,x)\| \leq \alpha(t)\|y - x\| \qquad \forall (t,x,y) \in I \times \mathbb{R}^N \times \mathbb{R}^N;$$

- f *is globally integrable on* t*, i.e.* $\forall x_0 \in \Omega$ *there exists* $\beta \in L^1((a,b); \mathbb{R})$ *such that*

$$\|f(t, x_0)\| \leq \beta(t) \qquad \forall t \in (a,b).$$

Then, there exists a unique global solution of (1.1)

$$y(\cdot; y_0) : [a,b] \longmapsto \mathbb{R}^N \in AC([a,b]; \mathbb{R}^N).$$

For the stability theory, we work with classical maximal solutions.

Definition 1.2. We consider a dynamics

$$f : \Omega \longmapsto \mathbb{R}^N$$

locally Lipschitz, i.e. for all $x_0 \in \Omega$ there exists $\rho \in (0, +\infty)$ and $\alpha \in \mathbb{R}^+$ such that $B(x_0, \rho) \subset \Omega$ and

$$\|f(y) - f(x)\| \leq \alpha \|y - x\| \qquad \forall (x, y) \in B(x_0, \rho) \times B(x_0, \rho);.$$

A point $y_0 \in \Omega$ is called an equilibrium point if

$$f(y_0) = 0.$$

We define now the different notions of stability of an equilibrium point.

Definition 1.3. We take into account a map

$$f : \Omega \longmapsto \mathbb{R}^N$$

locally Lipschitz. Moreover, let us consider $y_0 \in \Omega$ an equilibrium point.

- y_0 is said to be Locally Stable if:

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \mid \forall x \in B(y_0, \delta_\varepsilon)$$

  1. $$[0, +\infty) \subset J(y_0);$$
  2. $$y(t; y_0) \in B(y_0, \varepsilon) \qquad \forall t \in [0, +\infty);$$

- y_0 is named Locally Asymptotically Stable if:
  1. y_0 is Locally Stable;
  2. $$\exists \delta > 0 \mid y(t; y_0) \underset{t \to +\infty}{\longrightarrow} 0 \qquad \forall y_0 \in B(0, \delta);$$

- y_0 is named Globally Asymptotically Stable if:
  1. y_0 is Locally Stable;
  2. $$y(t; y_0) \underset{t \to +\infty}{\longrightarrow} 0 \qquad \forall y_0 \in \mathbb{R}^N;$$

- y_0 is called Exponentially Globally Asymptotically Stable if:
    1. y_0 is Locally Stable;
    2. $$\exists (C,\omega) \in (0,+\infty)^2 \mid \|y(t;y_0)\| \leq Ce^{-\omega t}\|y_0\| \qquad \forall (y_0,t) \in \mathbb{R}^N \times [0,+\infty).$$

A useful tool in Stability Theory is the Lie Derivative.

Definition 1.4. We consider a dynamics

$$f : \Omega \longmapsto \mathbb{R}^N$$

locally Lipschitz and a function

$$V : \Omega \longmapsto \mathbb{R} \in C^1(\Omega, \mathbb{R}).$$

The Lie Derivative of V is

$$L_V : \Omega \longmapsto \mathbb{R}$$
$$x \longrightarrow (\nabla V(x), f(x))_{\mathbb{R}^N}\,.$$

All throughout the thesis will be essential the following definition of Kalman-Observability and Kalman-Controllability.

Definition 1.5. An arbitrary pair of matrices $\forall (A,C) \in \mathcal{M}(N,N;\mathbb{R}) \times \mathcal{M}(M,N;\mathbb{R})$ and $\forall T \in (0,+\infty)$ is said to be Kalman-Observable at time $T \in (0,+\infty)$ if the observability map

$$\eta : \mathbb{R}^N \longmapsto C^0([0,T];\mathbb{R}^M)$$
$$x_0 \longrightarrow Ce^{At}x_0$$

is one to one.

We are going to define now the Kalman-Controllability.

Definition 1.6. $\forall (A,B,T) \in \mathcal{M}(N,N;\mathbb{R}) \times \mathcal{M}(N,M;\mathbb{R}) \times (0,+\infty)$, the pair of matrices (A,B) is Kalman-Controllable at time $T \in (0,+\infty)$ if $\forall (z_0,z_1) \in \mathbb{R}^N \times \mathbb{R}^N$ there exists a control function $u \in C^0([0,T];\mathbb{R}^M)$ such that the unique solution $z \in C^1([0,T];\mathbb{R}^N)$ of

$$\begin{cases} \frac{d}{dt}z(t) = Az(t) + Bu(t) & \forall t \in (0,T) \\ z(0) = z_0 \end{cases} \tag{1.5}$$

satisfies the final condition

$$z(T) = z_1.$$

These notions are related together by the following Theorem.

Theorem 1.4. *$\forall T \in (0,+\infty)$ the pair $\forall (A,B) \in \mathcal{M}(N,N;\mathbb{R}) \times \mathcal{M}(N,M;\mathbb{R})$ is Kalman-Controllable at time T if and only if (A^*,B^*) is Kalman-Observable at time T.*

In what follows we will introduce 2 algebraic conditions which are equivalent to the notion of Kalman-Observability and Kalman-Controllability respectively.

Theorem 1.5 (Kalman)**.** *$\forall (A,C) \in \mathcal{M}(N,N;\mathbb{R}) \times \mathcal{M}(M,N;\mathbb{R})$ is Kalman-Observable at time T if an only if*

$$rk\left[C, CA, \dots, CA^{N-1}\right] = N.$$

Theorem 1.6 (Kalman)**.** *$\forall (A,B) \in \mathcal{M}(N,N;\mathbb{R}) \times \mathcal{M}(N,M;\mathbb{R})$ is Kalman-Controllable at time T if an only if*

$$rk\left[B, AB, \dots, A^{N-1}B\right] = N.$$

From this theorems, we deduce that the Kalman-Observability and Kalman-Controllability in finite dimension are independent of $T \in (0,+\infty)$. There is another important notion related to Kalman-Observability and Kalman-Controllability. It is the object of the next definition.

Definition 1.7. $\forall (A,B) \in \mathcal{M}(N,N;\mathbb{R}) \times \mathcal{M}(M,N;\mathbb{R})$, the pair (A,B) is said to be stabilizable if there exists $F \in B(\mathbb{R}^N,\mathbb{R}^M)$ such that $\forall z_0 \in \mathbb{R}^N$ the unique solution $z \in C^1([0,+\infty),\mathbb{R}^N)$ of the linear Cauchy Problem

$$\begin{cases} \frac{d}{dt}z(t) = (A+BF)z(t) & \forall t \in [0,+\infty) \\ z(0) = z_0 \end{cases} \tag{1.6}$$

satisfies

$$z(t) \underset{t\to+\infty}{\longrightarrow} 0.$$

Then, F is called stabilizing feedback function and Fz is called feedback control function.

Kalman-Controllability implies exponential Stabilizability.

Theorem 1.7. *Let (A,B) be a Kalman-Controllable pair. Then, (A,B) is a Stabilizable. Moreover, $\forall \omega \in (0,+\infty)$ there exists $(F,M) \in B(\mathbb{R}^N,\mathbb{R}^M) \times$*

$(0,+\infty)$ *such that* $\forall z_0 \in \mathbb{R}^N$ *the unique solution* $z \in C^1([0,+\infty),\mathbb{R}^N)$ *of the linear system*

$$\begin{cases} \frac{d}{dt}z(t) = (A+BF)z(t) & \forall t \in [0,+\infty) \\ z(0) = z_0 \end{cases} \tag{1.7}$$

fulfills

$$\|z(t)\| \le Me^{-\omega t}\|z_0\| \qquad \forall t \in [0,+\infty).$$

A powerful tool to prove the existence of the minimum of a lower semi-continuous function is the Weierstrass Theorem.

Theorem 1.8. *For every* (X,d) *metric space. If*

$$f : (X,d) \longmapsto \mathbb{R} \cup \{+\infty\}$$

such that

- $$\exists x_0 \in X \mid f(x_0) < +\infty;$$
- f *is lower semi-continuous;*
- $$\forall k \in \mathbb{R} \qquad \{x \in X \mid f(x) \le k\}$$

  is a compact set.

Then,

$$\exists \min_X f.$$

Of course, an analogous result is true for the maximum.

Theorem 1.9. *For every* (X,d) *metric space. If*

$$f : (X,d) \longmapsto \mathbb{R} \cup \{-\infty\}$$

such that

- $$\exists x_0 \in X \mid f(x_0) > -\infty;$$
- f *is upper semi-continuous;*
- $$\forall k \in \mathbb{R} \qquad \{x \in X \mid f(x) \ge k\}$$

  is a compact set.

Then,

$$\exists \max_X f.$$

The following Proposition is very useful proving the convergence of a sequence into a metric space.

Proposition 1.1. *Let (X, d) be a metric space. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be such that there exists $\overline{x} \in X$ fulfilling:*

$$\forall \{x_{n_k}\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$$

$$\exists \left\{x_{n_{kh}}\right\}_{h\in\mathbb{N}} \subset \{x_{n_k}\}_{k\in\mathbb{N}}$$

such that

$$x_{n_{k_h}} \underset{h\to+\infty}{\longrightarrow} \overline{x} \qquad \text{in } (X, d).$$

Then,

$$x_n \underset{n\to+\infty}{\longrightarrow} \overline{x} \qquad \text{in } (X, d).$$

At this stage we give a useful consequence of the Dominated Convergence Theorem.

Theorem 1.10. *Let $I \subset \mathbb{R}$ an interval, (X, Σ, μ) a measure space and*

$$f : I \times X \longmapsto \mathbb{R}$$

such that:

1. *$\forall t \in I$, $f(t, \cdot)$ is μ-measurable;*
2. *$\exists t_0 \in I$ such that $f(t_0, \cdot) \in L^1((X, \Sigma, \mu); \mathbb{R})$;*
3. *$\exists A \in \Sigma$ such that $\mu(X \setminus A) = 0$ and $\forall x \in A$ $f(\cdot, x) \in C^1(I, \mathbb{R})$;*
4. *$\exists g \in L^1((X, \Sigma, \mu); \mathbb{R})$ such that*

$$\|\frac{\partial}{\partial t} f(t, x)\| \leq g(x) \qquad \forall (t, x) \in I \times A.$$

Then,

$$\begin{aligned} F : I &\longmapsto \mathbb{R} \\ t &\longrightarrow \int_X f(t, x) d\mu(x) \end{aligned}$$

is well defined and $F \in C^1(I, \mathbb{R})$. Furthermore, it is possible to compute its derivative as follows:

$$\frac{d}{dt} F(t) = \int_X \frac{\partial}{\partial t} f(t, x) d\mu(x).$$

At this point, we define the concept of the derivative in Vector Spaces and Banach Spaces. First of all, we give the following definition.

Definition 1.8. Let V be a Vector Space on $\mathbb{R}$, $v \in V$, $D \subset V$, $x_0 \in V$ such that $x_0 + \mathbb{R}v \subset D$ and

$$f : D \longmapsto \mathbb{R}$$

if there exists

$$\lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h} \in \mathbb{R},$$

f is said to be differentiable along v at x_0 and

$$\frac{\partial}{\partial v} f(x_0) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}.$$

We introduce now the notion of Gateaux differentiability and Frechet differentiability.

Definition 1.9. For all $(X, \|\|_X)$, $(Y, \|\|_Y)$ Banach Spaces, for each $D \subset X$ open set, for any $x_0 \in D$

-
  $$f : D \subset X \longmapsto Y$$

  is said to be Gateaux differentiable at $x_0 \in D$ if there exists a bounded linear operator $A \in B(X, Y)$ such that:

  $$\forall v \in X \qquad \exists \lim_{h \to 0}^{\|\|_Y} \left[\frac{f(x_0 + hv) - f(x_0)}{h} \right] = Av;$$

-
  $$f : D \subset X \longmapsto Y$$

  is named Frechet differentiable in $x_0 \in D$ if there exists a bounded linear operator $A \in B(X, Y)$ such that:

  $$\exists \lim_{h \to 0} \frac{\|f(x_0 + hv) - f(x_0) - A(x - x_0)\|_Y}{\|x - x_0\|_X} = 0;$$

  Then, we call

  $$d^{Frechet} f(x_0) \stackrel{\text{def}}{=} A.$$

We will discuss now some properties of convex functions. First of all, we define the concept of convex set.

Definition 1.10. Let V be a vector space on $\mathbb{R}$. $C \subseteq V$ is named convex set if

$$\forall (t, x_1, x_2) \in [0,1] \times C \times C \qquad tx_1 + (1-t)x_2 \in C.$$

At this moment, we are in position to define the notion of convex function.

Definition 1.11. For every V vector space on $\mathbb{R}$, we take into account a convex subset $C \subseteq V$. Then,

- $$f : C \longmapsto \mathbb{R} \cup \{+\infty\}$$
  is said to be convex if
  $$\forall (t, x_1, x_2) \in [0,1] \times C \times C \qquad f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2);$$

- $$f : C \longmapsto \mathbb{R}$$
  is named strictly convex if $\forall (t, x_1, x_2) \in (0,1) \times C \times C$ such that $x_1 \neq x_2$
  $$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$$

Henceforth, for every f convex function, we will name

$$D(f) \stackrel{\text{def}}{=} \{x \in C \mid f(x) < +\infty\}.$$

One can observe that, by the convexity of f, $D(f)$ is a convex set. Strictly convex functions have a pleasant structure for minimization. In fact, the following Proposition holds.

Proposition 1.2. *Let V be a Vector Space on $\mathbb{R}$, $C \subset V$ a convex subset and*

$$f : C \longmapsto \mathbb{R} \cup \{+\infty\}$$

a strictly convex function such that there exists $\tilde{x} \in C$ such that $f(\tilde{x}) < +\infty$. Then:

1. *if $\exists x_0 \in C$ a global minimizer, then it is unique;*
2. *$\forall x_0 \in C$, we define*
   $$\mathcal{D}_{x_0} \stackrel{\text{def}}{=} \{v \in V \mid \forall h \in \mathbb{R} \ x_0 + hv \in D(f)\}.$$
   Assuming
   $$D(f) - \{x_0\} \stackrel{\text{def}}{=} \{x - x_0 \mid x \in D(f)\} \subset \{v \in V \mid \forall h \in \mathbb{R} \ x_0 + hv \in D(f)\}$$

and $\forall (h_0, v) \in \mathbb{R} \times D(f) - x_0$, f *is differentiable along* v *at* $x_0 + h_0 v$, *then* x_0 *is a global minimizer for* f *if and only if*

$$\forall v \in D(f) - x_0 \qquad \frac{\partial}{\partial v} f(x_0) = 0.$$

The following definition will be useful in this dissertation.

Definition 1.12.

$$f : \mathbb{R}^N \longmapsto \mathbb{R} \in C^2(\mathbb{R}^N, \mathbb{R})$$

is said to be strongly convex of parameter $\alpha \in (0, +\infty)$ if and only if

$$\nabla^2 f(x) \geq \alpha I_N \qquad \forall x \in \mathbb{R}^N.$$

A strongly convex function is strictly convex. Let us now introduce a classical representation of solutions to Nonhomogeneous Differential Equations. We begin with the finite dimensional versions.

Proposition 1.3 (Duhamel's representation formula)**.** *Let* $A \in \mathcal{M}(N, N; \mathbb{R})$ *and* $f \in C^0([0,T], \mathbb{R}^N)$. $\forall y_0 \in \mathbb{R}^N$ *we consider* $y \in C^1([0,T], \mathbb{R}^N)$ *the unique solution of the Cauchy Problem:*

$$\begin{cases} \frac{d}{dt} y(t) + A y(t) = f(t) & \forall t \in (0,T) \\ y(0) = y_0. \end{cases} \tag{1.8}$$

Then the following representation formula holds:

$$y(t) = e^{-At} y_0 + \int_0^t e^{-A(t-s)} f(s) ds \qquad \forall t \in [0,T].$$

Another version reads as follows.

Proposition 1.4 (Duhamel's representation formula)**.** $\forall A \in \mathcal{M}(N, N; \mathbb{R})$, $\forall f \in C^0([0,T], \mathbb{R}^N)$, *for any initial data* $y_0 \in \mathbb{R}^N$, *we take into account* $y \in C^1([0,T], \mathbb{R}^N)$ *the unique solution of the Cauchy Problem:*

$$\begin{cases} -\frac{d}{dt} y(t) + A y(t) = f(t) & \forall t \in (0,T) \\ y(T) = y_0. \end{cases} \tag{1.9}$$

Then, y *admits the following representation:*

$$y(t) = e^{-A(T-t)} y_0 + \int_t^T e^{-A(\xi - t)} f(\xi) d\xi \qquad \forall t \in [0,T].$$

An infinite dimensional version is presented in Chapter 4. We provide now a useful Proposition from the Linear Alegbra.

Proposition 1.5. *$\forall A \in Sym(N;\mathbb{R})$ positive semidefinite there exists a unique $C \in Sym(N;\mathbb{R})$ such that $C^2 = A$.*

Henceforth, we will call C the symmetric square root of A. The following Theorem will be a key tool in Chapter 3 proving the Global Turnpike Property.

Theorem 1.11 (Mean Value Theorem for Definite Integrals)**.** *$\forall (a,b) \in \mathbb{R}^2$ such that $a < b$, for every*

$$f : [a,b] \longmapsto \mathbb{R} \in C^0([a,b],\mathbb{R})$$

there exists $t_0 \in [a,b]$ such that

$$f(t_0) = \frac{1}{b-a}\int_a^b f(t)dt.$$

Let us introduce now some notions on functional analysis. For an overview on Functional Analysis the reader should refer to [7]. First of all, we state a classical Theorem establishing the isomorphism between an Hilbert space and its dual.

Theorem 1.12 (Riesz Theorem)**.** *For every H Hilbert space on $\mathbb{R}$, $\forall \varphi \in H'$ continuous linear functional, then:*

1. $$\exists!\ x_\varphi \in H \mid \varphi(x) = (x_\varphi, x)_H \quad \forall x \in H;$$

2. $$\|x_\varphi\|_H = \|\varphi\|_{H'};$$

3. $$\Phi_H : H \longmapsto H'$$ $$x \longrightarrow (x,\cdot)_H$$

   is a surjective isometry named Riesz isomorphism.

Riesz Theorem implies that each H Hilbert space is reflexive. Moreover, the lower semicontinuity of the norm with respect to the weak convergence, which is a consequence of Banach-Steinhaus Theorem, will prove to be essential showing the existence of minimizers of certain functionals.

Proposition 1.6. *Let* $(X, \|\|_X)$ *a Normed Space.* $\forall \{x_n\}_{n\in\mathbb{N}} \subset X$, $\forall x \in X$ *such that*

$$x_n \underset{n\to+\infty}{\rightharpoonup} x$$

weakly in $(X, \|\|_X)$. *Then:*

1. $$\sup_{n\in\mathbb{N}} \|x_n\|_X < +\infty;$$

2. $$\|x\|_X \le \liminf_{n\to+\infty} \|x_n\|_X.$$

In Reflexive Banach Spaces Banach-Alaoglu Theorem generalizes Bolzano-Weierstrass Theorem.

Theorem 1.13 (Banach-Alaoglu)**.** *For every* $(X, \|\|_X)$ *Reflexive Banach Space, the ball*

$$\overline{B^X(0,1)}^{\|\|_X}$$

is compact with respect to the weak topology $\sigma(X, X')$.

Since all Hilbert Spaces H are reflexive, the above Theorem holds true for all H Hilbert Spaces. There exists an analogous of Proposition 1.1 for weak convergence.

Proposition 1.7. *For every* $(X, \|\|_X)$ *Banach Space, for any sequence* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *such that there exists* $\overline{x} \in X$ *fulfilling:*

$$\forall \{x_{n_k}\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$$

$$\exists \left\{x_{n_{k_h}}\right\}_{h\in\mathbb{N}} \subseteq \{x_{n_k}\}_{k\in\mathbb{N}}$$

such that:

$$x_{n_{k_h}} \underset{h\to+\infty}{\rightharpoonup} \overline{x} \qquad \text{weakly in} X$$

then

$$x_n \underset{n\to+\infty}{\rightharpoonup} \overline{x} \qquad \text{weakly in} X.$$

The proof is accomplished applying $\forall \varphi \in X'$ Proposition 1.1 to the sequence $\{\varphi(x_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}$. At this moment, we explain how is it possible to define the adjoint of a continuous linear operator. The classical construction is the following. We consider H_1 and H_2 2 arbitrary Hilbert Spaces on $\mathbb{R}$. $\forall T \in B(H_1, H_2)$ we take into account a generic $y \in H_2$. Then, by Riesz Theorem 1.12, there exists a unique $x' \in H$ such that:

$$(y, Tx)_{H_2} = (x', x)_{H_1} \qquad \forall x \in H.$$

Definition 1.13. For every H_1 and H_2 Hilbert Spaces on $\mathbb{R}$ and for all

$$T : H_1 \longmapsto H_2 \in B(H_1, H_2),$$

$$T^* : H_2 \longmapsto H_1$$
$$x \longrightarrow x'$$

is called the adjoint operator of T.

The following Proposition affirms that T^* is still a bounded linear operator and $\|T\|_{B(H_1,H_2)} = \|T^*\|_{B(H_2,H_1)}$.

Proposition 1.8. *Let H_1 and H_2 be generic Hilbert Spaces on $\mathbb{R}$. Then*

$$\forall T : H_1 \longmapsto H_2 \in B(H_1, H_2),$$

$$T^* : H_2 \longmapsto H_1$$

is such that

- $$T^* \in B(H_2, H_1);$$
- $$\|T\|_{B(H_1,H_2)} = \|T^*\|_{B(H_2,H_1)}.$$

This concept can be generalised for Banach Spaces with the notion of the transpose of a continuous linear operator.

Definition 1.14. For all $(X_1, \|\|_{X_1})$ and $(X_2, \|\|_{X_2})$ Banach Spaces on $\mathbb{R}$, for every

$$T : X_1 \longmapsto X_2 \in B(X_1, X_2),$$

we define

$$T' : X_2' \longmapsto X_1'$$
$$\psi \longrightarrow \psi \circ T.$$

The transpose of a bounded linear operator is itself a bounded linear operator with the same operatorial norm.

Proposition 1.9. *Let $(X_1, \|\|_{X_1})$ and $(X_2, \|\|_{X_2})$ be generic Banach Spaces on $\mathbb{R}$, for each*

$$T : X_1 \longmapsto X_2 \in B(X_1, X_2),$$
$$T' : X_2' \longmapsto X_1'$$
$$\psi \longrightarrow \psi \circ T$$

inherits some some features from T, i.e.:

- $$T' \in B(X_2', X_1');$$

- $$\|T'\|_{B(X_2',X_1')} = \|T\|_{B(X_1,X_2)}$$

If we work in Hilbert Spaces, the notion of adjoint and transpose are equivalent, up to Riesz Isomorphism. In fact the following Proposition holds true.

Proposition 1.10. *For all H_1 and H_2 Hilbert Spaces on $\mathbb{R}$, for every*

$$T : H_1 \longmapsto H_2 \in B(H_1, H_2),$$

then

$$T^* = \Phi_{H_1}^{-1} \circ T' \circ \Phi_{H_2}.$$

For this reason, working in Hilbert Spaces, $\forall T \in B(H_1, H_2)$, we will call $T' \in B(H_2', H_1')$ the adjoint of T. We provide now the Closed Range Theorem.

Theorem 1.14 (Closed Range Theorem)**.** *Let $(X, \|\|_X)$ and $(Y, \|\|_Y)$ Banach Spaces. For every*

$$T : X \longmapsto Y \in B(X, Y)$$

then, the following statements are equivalent:

1. *$R(T)$ is closed in $(Y, \|\|_Y)$;*
2. *$R(T')$ is closed in $(X', \|\|_{X'})$;*
3. *$R(T) = Ker(T')^\perp = \{y \in Y \mid < y, \psi >_{(Y,Y')}= 0 \ \forall \psi \in Ker(T')\}$;*
4. *$R(T') = Ker(T)^\perp = \{\varphi \in X' \mid < \varphi, x >_{(X',X)}= 0 \ \forall x \in Ker(T)\}$.*

A proof of this Theorem is in [22] at page 205. We define self-adjoint operators.

Definition 1.15. Let H be an Hilbert Space on $\mathbb{R}$. $T \in B(H, H)$ is said to be self-adjoint if and only if $T^* = T$.

There exists a useful formula to compute the norm of a self-adjoint operator.

Proposition 1.11. *For each H Hilbert Space, for every*

$$T : H \longmapsto H \in B(H,H) \text{ self-adjoint,}$$

it holds:

$$\|T\|_{B(H,H)} = \sup\left\{ |\,(Tx,x)_H\,| \mid x \in S^H(0,1) \right\}.$$

At this point, we provide some additional tools in the context of integration and derivation in Banach Spaces. A good reference for integration in Banach Spaces is [9]. We give the definition of Sobolev Space of Banach valued functions. The derivative involved in this definition is intended to be distributional.

Definition 1.16. We take into account $(E, \|\|_E)$, $(F, \|\|_F)$ and $(G, \|\|_G)$ 3 generic Banach Spaces such that there exist:

$$i_1 : E \hookrightarrow G \in B(E,G)$$

one to one and

$$i_2 : F \hookrightarrow G \in B(F,G)$$

one to one. Then, $\forall (p,q) \in [1,+\infty] \times [1,+\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we define

$$W^{p,q}((0,T);(E,F)) \stackrel{\text{def}}{=} \left\{ y \in L^p((0,T);E) \mid \frac{d}{dt} y \in L^q((0,T);F). \right\}$$

Furthermore, on this space we define the norm

$$\|\|_{W^{p,q}((0,T);(E,F))} : W^{p,q}((0,T);(E,F)) \longmapsto \mathbb{R}^+$$

$$y \longrightarrow \|y\|_{L^p((0,T);E)} + \|\frac{d}{dt} y\|_{L^q((0,T);F)}.$$

Properties of Sobolev Spaces for Banach Valued functions are investigated in [9]. In this dissertation we will take into account 2 arbitrary Hilbert Spaces $(X, (,)_X)$ and $(H, (,)_H)$, such that there exists a dense inclusion, namely:

$$i : X \hookrightarrow H \in B(X,H)$$

one to one with $\overline{i(X)}^H = H$. Furthermore, we identify H with its topological dual H'. Indeed, by Theorem 1.12, there exists an isomorphism of Hilbert Spaces Φ_H such that:

$$\Phi_H : H \longmapsto H'$$

$$h \longrightarrow (h, \cdot)_H$$

Moreover, we consider the transpose of the inclusion i

$$i^* : H' \longmapsto X'$$

$$\phi \longrightarrow \phi \circ i$$

By Proposition 1.9, $i^* \in B(H', X')$. Furthermore, $\overline{i^*(H')}^{X'} = X'$. In this background, we name:

$$\Lambda : X \longmapsto H'$$

$$x \longrightarrow \Phi_H \circ i(x) = (i(x), \cdot)_H$$

This map is well posed and $\Lambda \in B(X, H')$. On the other hand, the action of an arbitrary element h of H on H is usually defined as $\Phi_H(h)$. Finally, we call:

- $$(E, \|\|_E) \stackrel{\text{def}}{=} (X, \|\|_X);$$

- $$(F, \|\|_F) \stackrel{\text{def}}{=} (X', \|\|_{X'});$$

- $$(G, \|\|_G) \stackrel{\text{def}}{=} (X', \|\|_{X'}).$$

This leads us to take into account the Sobolev Space:

$$W^{1,2}((0,T);(X,X')) \stackrel{\text{def}}{=} \left\{ y \in L^2((0,T);X) \mid \frac{d}{dt}y \in L^2((0,T);X'). \right\}.$$

Each function $y \in W^{1,2}((0,T);(X,X'))$ admits a continuous representative. This is the object of the following Proposition.

Theorem 1.15. *There exists*

$$i : W^{1,2}((0,T);(X,X')) \hookrightarrow C^0([0,T];H) \in B(W^{1,2}((0,T);(X,X')), C^0([0,T];H))$$

$$u \longrightarrow \tilde{u}\,[\tilde{u} \in u]$$

one to one.

At this stage, we formulate the Integration by Parts.

Proposition 1.12 (Integration by Parts)**.** $\forall (f,g) \in W^{1,2}((0,T);(X,X'))$ $\forall (s,t) \in [0,T]^2$

$$\int_s^t < \frac{d}{dt}g(\xi), f(\xi) >_{(X',X)} d\xi =$$

$$= (g(t), f(t))_H - (g(s), f(s))_H - \int_s^t < g(\xi), \frac{d}{dt}f(\xi) >_{(X,X')} d\xi.$$

The following compactness Theorem in Parabolic Spaces, will be useful in the last Chapter.

Theorem 1.16 (Simon's Theorem)**.** *Let $(V, \|\|_V)$, $(E, \|\|_E)$ and $(F, \|\|_F)$ be 3 Banach Spaces such that:*

- $$\exists i : V \hookrightarrow E \in K(V, E)$$

  one to one;

- $$\exists j : E \hookrightarrow F \in B(E, F)$$

  one to one.

Moreover, we consider an arbitrary pair of exponents $\forall (p, q) \in (1, +\infty] \times [1, +\infty]$. Whenever $D \subset W^{1,p}((0, T); (F, F))$ is bounded and $D \subset L^q((0, T); V)$ is bounded, then

1. *$D \subset C^0([0, T]; F)$ is relatively compact;*
2. *$D \subset L^q((0, T); E)$ is relatively compact.*

The reader can find a proof of this Theorem at page 51 of [9]. In this last chapter, we are going to make an example on the internal control of a divergential parabolic equation. It will be useful the following version of the Maximum Principle.

Theorem 1.17. *Let $\Omega \subset \mathbb{R}^N$ bounded open set with $\partial\Omega \in C^1$, $A \in L^\infty(\Omega, \mathcal{M}(N, N; \mathbb{R}))$ coercive and $\beta \in L^N(\Omega; \mathbb{R})$. Then, $\forall u \in H^1(\Omega)$ such that*

- *$u_+ \in H^1_0(\Omega)$;*
- *$\forall \varphi \in C^\infty_c(\Omega, \mathbb{R})$, $\varphi \geq 0$:*

$$\int_\Omega (A\nabla u, \nabla\varphi)_{\mathbb{R}^N} \, dx \leq \int_\Omega \beta \|\nabla u\| \varphi dx.$$

Then, it holds:

$$u \leq 0 \qquad a.e. \text{ on } \Omega.$$

An interesting reference on this subject is [15].

Chapter 2

Finite Dimensional Linear Quadratic Case

In this chapter we will work in a Finite Dimensional Linear Quadratic context, under some controllability and observability hypotheses. This analysis will provide useful insights both for the nonlinear generalization and the infinite dimensional generalization. It is chosen the "dynamic" approach used in [16].

We consider a triple of matrices $(A, B, C) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R}) \times \mathcal{M}(N, N; \mathbb{R})$, an initial data $x_0 \in \mathbb{R}^N$ and a target $z \in \mathbb{R}^N$. For every control function $u \in L^2((0, T); \mathbb{R}^M)$ there exists a unique absolutely continuous dynamics

$$x : [0, T] \longmapsto \mathbb{R}^N$$

satisfying the linear system

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & \text{a.e. } t \in (0, T) \\ x(0) = x_0 \end{cases} \tag{2.1}$$

In this setting, the non-stationary problem $(OCP)^T$ consists in minimizing the functional

$$J^T : L^2((0, T); \mathbb{R}^M) \longmapsto \mathbb{R}$$

$$u \longrightarrow \frac{1}{2} \int_0^T \left(\|u(t)\|^2 + \|Cx(t) - z\|^2 \right) dt$$

One can prove that the minimizer $u^T \in L^2((0, T); \mathbb{R}^M)$ exists and it is unique. Moreover, it satisfies the state-adjoint state Pontryagin system:

$$
\begin{cases}
\frac{d}{dt}x^T(t) + Ax^T(t) = Bu^T(t) & \text{a.e. } t \in (0,T) \\
-\frac{d}{dt}p^T(t) + A^*p^T(t) = C^*(Cx^T(t) - z) & \text{a.e. } t \in (0,T) \\
u^T(t) = -B^*p^T(t) & \text{a.e. } t \in (0,T) \\
x^T(0) = x_0 \\
p^T(T) = 0
\end{cases}
\tag{2.2}
$$

The triple (x^T, p^T, u^T) is called the optimal triple for the optimal control problem $(OCP)^T$. As a consequence of the fulfillment of the Pontryagin first order conditions, it is possible to prove more regularity for the components of the optimal triple. In fact $x^T \in AC([0,T], \mathbb{R}^N)$, then $p^T \in C^1([0,T], \mathbb{R}^N)$. Furthermore, from the relation $u^T = -B^*p^T$ one gets $u \in C^1([0,T], \mathbb{R}^M)$. Now, using the first equation, it is possible to prove that $x \in C^2([0,T], \mathbb{R}^N)$. Iterating this procedure, one can conclude that the optimal triple $(x^T, p^T, u^T) \in C^\infty([0,T], \mathbb{R}^N) \times C^\infty([0,T], \mathbb{R}^N) \times C^\infty([0,T], \mathbb{R}^M)$.

On the other hand, the stationary problem is defined as follows. First of all, we introduce a vector subspace of $\mathbb{R}^N \times \mathbb{R}^M$, where the stationary functional will be defined.

Definition 2.1.

$$
M = \left\{ (x,u) \in \mathbb{R}^N \times \mathbb{R}^M \mid Ax = Bu \right\}
$$

The problem is minimizing the map

$$
\begin{aligned}
J^s : M &\longmapsto \mathbb{R} \\
(x,u) &\longrightarrow \frac{1}{2}\left[\|u\|^2 + \|Cx - z\|^2\right]
\end{aligned}
$$

The existence and the uniqueness of the minimizer are considered later. Whenever it exists and it is unique, the minimizer $(\overline{x}, \overline{u})$ is called the optimal pair for $(OCP)^s$.

In the following we introduce the setting in which the turnpike property is proved.

Assumptions for the Linear Quadratic Case

1. $(A, B, C) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R}) \times \mathcal{M}(N, N; \mathbb{R})$;

2. (A, B) is Kalman-controllable, i.e. using Kalman Controllability Theorem:

$$
\operatorname{rank}[B, AB, A^2B, \ldots, A^{N-1}B] = N \tag{2.3}
$$

3. the pair (A, C) is Kalman-observable, i.e. using Kalman Observability Theorem:
$$\operatorname{rank}[C, CA, CA^2, \dots, CA^{N-1}] = N \tag{2.4}$$
4. the initial data $x_0 \in \mathbb{R}^N$;
5. the target $z \in \mathbb{R}^N$;
6. the control functions belongs to $L^2((0,T); \mathbb{R}^M)$.

By Theorem 1.4, (A^*, B^*) is Kalman-Observable and that (A^*, C^*) is Kalman-Controllable.

Remark 2.1. The hypothesis of Kalman-controllability of the pair (A, B) is essential. Without this hypothesis the turnpike property doesn't hold in general.

Proof. Indeed, one can define a triple of matrices as follows

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

a target $z = 0$ and an initial data $x_0 = e_1$ the first element of the canonical basis of $\mathbb{R}^2$. Then, for every control function $u \in L^2((0,T); \mathbb{R}^M)$, the corresponding state satisfies the linear differential system:

$$\begin{cases} \frac{d}{dt}x_1 = x_1 \\ \frac{d}{dt}x_2 = u_2 \\ x_1(0) = 1 \\ x_2(0) = 0 \end{cases} \tag{2.5}$$

Therefore, for every control function $u \in L^2((0,T); \mathbb{R}^M)$, the first component x_1 of the state satisfies the 1-dimensional Cauchy Problem

$$\begin{cases} \frac{d}{dt}x_1 = x_1 \\ x_1(0) = 1 \end{cases} \tag{2.6}$$

Which implies

$$x_1 : [0, T] \longmapsto \mathbb{R}$$

$$t \longrightarrow e^t$$

Moreover, the stationary functional becomes:

$$J^s : M \longmapsto \mathbb{R}$$

$$(x,u) \longrightarrow \frac{1}{2}\left[\|u\|^2 + \|x\|^2\right]$$

This functional has a unique global minimizer given by $(\overline{x}, \overline{u}) = (0,0)$. At this stage, if one considers the unique optimal triple (x^T, p^T, u^T) for $(OCP)^T$ is such that for every $t \in [0,T]$

$$\|x^T(t) - \overline{x}\| \geq \|x_1^T(t)\| = e^t$$

Hence, even

$$\frac{1}{T}\int_0^T x_1^T dt = \frac{e^T - 1}{T} \underset{T\to+\infty}{\longrightarrow} +\infty.$$

Therefore, the turnpike property fails to be realised, even in an averaged version.

□

The following Lemma will be useful later.

Lemma 2.1. *For every $(A,C) \in \mathcal{M}(N,N;\mathbb{R})^2$ observable, there exists a T-independent constant $C \in (0,+\infty)$ such that $\forall\ T \in [1,+\infty)$ and for any $f \in L^2((0,T),\mathbb{R}^N)$, for every $y \in AC([0,T],\mathbb{R}^N)$ solution of the equation*

$$\frac{d}{dt}y(t) + Ay(t) = f(t) \qquad a.e.\ t \in (0,T),$$

we have:

$$\|y(T)\|^2 \leq C\left[\int_{T-1}^T \|Cy(t)\|^2 dt + \int_{T-1}^T \|f(t)\|^2 dt\right].$$

Proof. First of all, it is necessary to prove the following intermediate result. For each pair (A,C) observable there exists a T-independent constant C such that for every z solution of the homogenous equation

$$z_t(t) + Az(t) = 0 \qquad \forall t \in (0,T)$$

it holds:

$$\|z(T)\|^2 \leq C\int_{T-1}^T \|Cz(s)\|^2 dt$$

This proof relies on the definition of observability 1.5 and on norms equivalence in $\mathbb{R}^N$. One defines the following

$$\|.\|_{new} : \mathbb{R}^N \longmapsto \mathbb{R}^+$$

$$\xi \longrightarrow \left(\int_{T-1}^{T} \|Cz(s;(T,\xi))\|^2 ds \right)^{\frac{1}{2}}$$

where $z(\cdot;(T,\xi))$ is the unique solution of the linear system:

$$\begin{cases} z_t(t) + Az(t) = 0 & \forall t \in (0,T) \\ z(T) = \xi \end{cases} \tag{2.7}$$

The map already defined is a seminorm. Moreover, for every $\xi \in \mathbb{R}^N$ such that $\|\xi\|_{new} = 0$, by measure theory, $Cz(\cdot;(T,\xi)) = 0$ a.e. on $[T-1,T]$. Since $z \in C^0([0,T],\mathbb{R}^N)$, $Cz(t;(T\xi)) = 0 \ \forall t \in [T-1,T]$. By the observability assumption, the linear map

$$\eta : \mathbb{R}^N \longmapsto C^0([T-1,T],\mathbb{R}^N)$$

$$\xi \longrightarrow Cz(\cdot;(T,\xi))$$

is one to one. Therefore $\xi = 0$. Hence, $\|.\|_{new}$ is actually a norm. Now, by the equivalence of norms in finite dimension, one gets that there exists a positive constant C such that for each $\xi \in \mathbb{R}^N$

$$\|\xi\|^2 \leq C \int_{T-1}^{T} \|Cz(t;(T,\xi))\|^2 dt.$$

Furthermore for every $w \in AC([T-1,T],\mathbb{R}^N)$ such that:

$$\begin{cases} w_t(t) + Aw(t) = f(t) & \text{a.e. } t \in (T-1,T) \\ w(T-1) = 0 \end{cases} \tag{2.8}$$

by the Duhamel's formula (Proposition 1.3), $\forall t \in \ [T-1,T]$

$$w(t) = 0 + \int_{T-1}^{t} e^{-A(t-s)} f(s) ds.$$

Therefore, there exists a positive number C independent of T, such that:

$$\|w(t)\|^2 \leq C \int_{T-1}^{T} \|f(s)\|^2 ds \qquad \forall t \in \ [T-1,T]$$

Hence, for every $y \in AC([0,T],\mathbb{R}^N)$ solution of the equation

$$y_t(t) + Ay(t) = f(t) \qquad \text{a.e. } t \in (0,T)$$

one defines z the unique solution of

$$\begin{cases} z_t(t) + Az(t) = 0 & \forall t \in (0,T) \\ z(T) = y(T) \end{cases} \tag{2.9}$$

and w the unique solution of the non homogeneous system

$$\begin{cases} w_t(t) + Aw(t) = f(t) & \text{a.e. } t \in (T-1,T) \\ w(T-1) = 0 \end{cases} \tag{2.10}$$

By uniqueness of the solution for the Cauchy Problem

$$\begin{cases} v_t + Av = f & \text{a.e. } t \in (T-1,T) \\ v(T) = y(T) \end{cases} \tag{2.11}$$

it follows that $y = z + w$. Then,

$$\|y(T)\|^2 \leq 2\left(\|z(T)\|^2 + \|w(T)\|^2\right) \leq C\left[\int_{T-1}^{T} \|C(y-w)(s)\|^2 ds + \|w(T)\|^2\right] \leq$$

$$\leq C\left[\int_{T-1}^{T} \|Cy(s)\|^2 ds + \int_{T-1}^{T} \|Cw(s)\|^2 ds + \|w(T)\|^2\right] \leq$$

$$\leq C\left[\int_{T-1}^{T} \|Cy(s)\|^2 ds + \int_{T-1}^{T} \|f(s)\|^2 ds\right]$$

which concludes the proof. □

After having proved the Lemma 2.1, in order to be prepared for the next stages, it is necessary to prove the following stationary version.

Lemma 2.2. *For each observable pair (A,C) there exists a constant $C \in (0,+\infty)$ such that*

$$\|z\|^2 \leq C[\|Az\|^2 + \|Cz\|^2] \qquad \forall z \in \mathbb{R}^N$$

Proof. For every $z \in \mathbb{R}^N$, defining $f \equiv Az$, $y \equiv z$ is a solution of the equation

$$y_t(t) + Ay(t) = f(t) \qquad \text{a.e. } t \in (0,T)$$

Then, using Lemma 2.1, one gets

$$\|z\|^2 = \|y(T)\|^2 \leq C \left[\int_{T-1}^{T} \|Cz\|^2 dt + \int_{T-1}^{T} \|Az\|^2 dt \right] \leq$$

$$\leq C \left[\|Az\|^2 + \|Cz\|^2 \right]$$

Which yields the desired result. □

At this stage, it is useful to prove the third Lemma required for the proof of the next Theorem.

Lemma 2.3. *For every triple of matrices $(A, B, C) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R}) \times \mathcal{M}(N, N; \mathbb{R})$ such that (A, C) is Kalman-observable, for each target $z \in \mathbb{R}^N$ there exists a unique $(\overline{x}, \overline{u})$ optimal pair for $(OCP)^s$ satisfying the first order condition*

$$\begin{cases} A\overline{x} = B\overline{u} \\ (\overline{u}, v)_{\mathbb{R}^M} + (C\overline{x} - z, C\varphi)_{\mathbb{R}^N} = 0 \qquad \forall (\varphi, v) \in \mathbb{R}^N \times \mathbb{R}^M \ : \ A\varphi = Bv \end{cases} \tag{2.12}$$

Furthermore, there exists some $\overline{p} \in \mathbb{R}^N$ such that

$$A^* \overline{p} = C^* (C\overline{x} - z) \tag{2.13}$$

and therefore

$$(\overline{u}, v)_{\mathbb{R}^M} + (\overline{p}, Bv)_{\mathbb{R}^N} = 0 \qquad \forall v \in \mathbb{R}^M \text{ such that } \exists \varphi \in \mathbb{R}^N \ : \ A\varphi = Bv \tag{2.14}$$

Proof. For the proof of the existence of the minimizer, one can use Weierstrass Theorem 1.8. In fact, $J^s \in C^0(M, \mathbb{R})$ and for every $k \in \mathbb{R}$ the set

$$F_k = \{(x, u) \in M \mid J^s(x, u) \leq k\}$$

is compact, by Lemma 2.2 . Then, by Weierstrass Theorem, there exists $(\overline{x}, \overline{u}) \in M$ minimizer for J^s. In order to prove the uniqueness of such a minimizer, we show the strict convexity of the functional J^s. To this extent it is essential to prove that $\forall ((x_1, u_1), (x_2, u_2)) \in M^2$ such that

$$(x_1, u_1) \neq (x_2, u_2),$$

then:

$$u_1 \neq u_2 \quad \vee \quad Cx_1 \neq Cx_2.$$

Indeed, by contradiction, if there would exist $((x_1, u_1), (x_2, u_2)) \in M^2$ such that $(x_1, u_1) \neq (x_2, u_2)$ and $(Cx_1, u_1) = (Cx_2, u_2)$, by Lemma 2.2,

$$\|x_2 - x_1\|^2 \leq C \left[\|A(x_2 - x_1)\|^2 + \|C(x_2 - x_1)\|^2\right] =$$

$$= C \left[\|B(u_2 - u_1)\|^2 + \|C(x_2 - x_1)\|^2\right] = 0.$$

Therefore,

$$(x_1, u_1) = (x_2, u_2)$$

which contradicts the hypothesis. Hence, $\forall((x_1, u_1), (x_2, u_2)) \in M^2$ such that $(x_1, u_1) \neq (x_2, u_2)$ and $\forall t \in (0, 1)$

$$J^s(t(x_1, u_1)+(1-t)(x_2, u_2)) = \frac{1}{2} \left[\|tu_1 + (1-t)u_2\|^2 + \|t(Cx_1 - z) + (1-t)(Cx_2 - z)\|^2\right] <$$

by the above achievement and the strict convexity of $\| \cdot \|^2$,

$$< t\frac{1}{2} \left[\|u_1\|^2 + \|Cx_1 - z\|^2\right] + (1-t)\frac{1}{2} \left[\|u_2\|^2 + \|Cx_2 - z\|^2\right] =$$

$$= tJ^s(x_1, u_1) + (1-t)J^s(x_2, u_2).$$

Then, J^s is strictly convex. Therefore, by Proposition 1.2, the minimizer is unique. As regards the first order condition, for each direction $(v, \varphi) \in M$, one defines the following map:

$$g : \mathbb{R} \longmapsto \mathbb{R}$$

$$h \longrightarrow J^s((\overline{x}, \overline{u}) + h(v, \varphi))$$

Now, $(\overline{x}, \overline{u}) \in (\overline{x}, \overline{u}) + \mathbb{R}(v, \varphi) \subset M$. Hence, g has a global minimum at 0. By Fermat Theorem, $\frac{d}{dt}g(0) = 0$. Besides,

$$\frac{d}{dt}g(0) = (\overline{u}, v)_{\mathbb{R}^M} + (C\overline{x} - z, C\varphi)_{\mathbb{R}^N}.$$

This implies

$$(\overline{u}, v)_{\mathbb{R}^M} + (C\overline{x} - z, C\varphi)_{\mathbb{R}^N} = 0$$

Moreover, $Ker(A) \times \{0\} \subset M$. Therefore, for every $\varphi \in Ker(A)$

$$(C\overline{x} - z, C\varphi)_{\mathbb{R}^N} = 0$$

By the linear algebra theory, $Range(A^*) = Ker(A)^\perp$. Then $C^*(C\overline{x} - z) \in Ker(A)^\perp = Range(A^*)$, which is equivalent to

$$\exists \overline{p} \in \mathbb{R}^N \quad : \quad A^*\overline{p} = C^*(C\overline{x} - z).$$

Then

$$(\overline{u}, v)_{\mathbb{R}^M} + (A^*\overline{p}, \varphi)_{\mathbb{R}^N} = 0 \qquad \forall (v, \varphi) \in \mathbb{R}^N \times \mathbb{R}^M \ : \ A\varphi = Bv$$

Therefore, knowing $(A^*\overline{p}, \varphi)_{\mathbb{R}^N} = (\overline{p}, A\varphi)_{\mathbb{R}^N} = (\overline{p}, Bv)_{\mathbb{R}^N}$, it is possible to rewrite the first order condition as follows

$$(\overline{u}, v)_{\mathbb{R}^M} + (\overline{p}, Bv)_{\mathbb{R}^N} = 0 \qquad \forall v \in \mathbb{R}^M \text{ such that } \exists \varphi \in \mathbb{R}^N \ : \ A\varphi = Bv.$$

□

A similar Lemma will be proved in the Infinite Dimensional Linear Quadratic Case. We will use similar techniques adapted to the infinite dimensional context. Both in the finite dimensional and in the infinite dimensional case, $\overline{p}$ may not be unique. Indeed, it is defined up to elements of $Ker(A^*)$.

At this stage, we are ready to prove the first Theorem concerning directly the turnpike property. It regards the convergence of averages of $\min_{u\in L^2((0,T);\mathbb{R}^M)} J^T$, x^T, p^T and u^T to the corresponding stationary ones.

Theorem 2.1 (Convergence of averages)**.** *For every triple of matrices $(A, B, C) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R}) \times \mathcal{M}(N, N; \mathbb{R})$ such that (A, B) is Kalman-controllable, (A, C) is Kalman-observable, we have:*

$$\frac{1}{T} \min_{u\in L^2((0,T);\mathbb{R}^M)} J^T \underset{T\to+\infty}{\longrightarrow} \min_{(x,u)\in M} J^s \tag{2.15}$$

$$\frac{1}{T}\int_0^T \left(\|u^T(t) - \overline{u}\|^2 + \|C\left(x^T(t) - \overline{x}\right)\|^2\right) dt = \underset{T\to+\infty}{O}\left(\frac{1}{T}\right) \underset{T\to+\infty}{\longrightarrow} 0 \tag{2.16}$$

Consequently, for every $(a, b) \in [0, 1]^2$ such that $a \neq b$

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} x^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{x} \tag{2.17}$$

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} u^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{u} \tag{2.18}$$

Moreover, there exists a unique $\overline{p} \in \mathbb{R}^N$ satisfying $A^*\overline{p} = C^*(C\overline{x} - z)$ such that $\overline{u} = -B^*\overline{p}$ and

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} p^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{p} \tag{2.19}$$

Proof. Subtracting the first order conditions for the non stationary problem and the stationary one, one gets

$$\begin{cases} \frac{d}{dt}(x^T - \overline{x})(t) + A(x^T(t) - \overline{x}) = B(u^T(t) - \overline{u}) & \forall t \in (0, T) \\ -\frac{d}{dt}(p^T - \overline{p})(t) + A^*(p^T(t) - \overline{p}) = C^*C(x^T(t) - \overline{x}) & \forall t \in (0, T) \end{cases} \tag{2.20}$$

Moreover, we have $u^T = -B^*p^T$. Our aim is to prove that there exists a T-independent constant $C \in (0, +\infty)$ such that for every $T \in (0, +\infty)$

$$\int_0^T \left(\|u^T - \overline{u}\|^2 + \|C\left(x^T - \overline{x}\right)\|^2\right) dt \leq C \tag{2.21}$$

From this inequality, the thesis follows.

Therefore, we find an alternative expression of (2.21) integrating the first order conditions, i.e.

$$\int_0^T \|C(x^T - \overline{x})\|^2 dt = \int_0^T \left(C^*C(x^T - \overline{x}), x^T - \overline{x}\right)_{\mathbb{R}^N} dt =$$

$$= \int_0^T \left(-(p^T - \overline{p})_t + A^*(p^T - \overline{p}), x^T - \overline{x}\right)_{\mathbb{R}^N} dt =$$

$$= \int_0^T \left(-(p^T - \overline{p})_t, x^T - \overline{x}\right)_{\mathbb{R}^N} dt + \int_0^T \left((p^T - \overline{p}), A(x^T - \overline{x})\right)_{\mathbb{R}^N} dt =$$

integrating by parts

$$= \left((p^T(0) - \overline{p}), (x^T(0) - \overline{x})\right)_{\mathbb{R}^N} - \left((p^T(T) - \overline{p}), x^T(T) - \overline{x}\right)_{\mathbb{R}^N} +$$

$$+ \int_0^T \left(p^T - \overline{p}, (x^T - \overline{x})_t\right)_{\mathbb{R}^N} dt + \int_0^T \left((p^T - \overline{p}), A(x^T - \overline{x})\right)_{\mathbb{R}^N} dt =$$

$$= \left(x_0 - \overline{x}, p^T(0) - \overline{p}\right)_{\mathbb{R}^N} + (x(T) - \overline{x}, \overline{p})_{\mathbb{R}^N} + \int_0^T \left(p^T - \overline{p}, B(u^T - \overline{u})\right)_{\mathbb{R}^N} dt$$

In order to get the desired expression, we compute $\forall t \in [0, T]$

$$\left(p^T(t) - \overline{p}, B(u^T(t) - \overline{u})\right)_{\mathbb{R}^N} + \|u^T(t) - \overline{u}\|^2$$

using $u^T = -B^*p^T$

$$\left((p^T(t) - \overline{p}), B(u^T(t) - \overline{u})\right)_{\mathbb{R}^N} + \|u^T(t) - \overline{u}\|^2 = -\left((u^T(t) + B^*\overline{p}), (u^T(t) - \overline{u})\right)_{\mathbb{R}^N} + \|u^T(t) - \overline{u}\|^2 =$$

$$= -\|u^T(t) - \overline{u}\|^2 + \|u^T(t) - \overline{u}\|^2 - \left(\overline{u} + B^*\overline{p}, u^T(t)\right)_{\mathbb{R}^N} = -\left((\overline{u} + B^*\overline{p}), u^T(t)\right)_{\mathbb{R}^N}$$

Hence,

$$\int_0^T \left(\|u^T-\overline{u}\|^2+\|C\left(x^T-\overline{x}\right)\|^2\right)dt= \tag{2.22}$$

$$=\left(x_0-\overline{x},p^T(0)-\overline{p}\right)_{\mathbb{R}^N}+\left(x^T(T)-\overline{x},\overline{p}\right)_{\mathbb{R}^N}-\int_0^T\left((\overline{u}+B^*\overline{p}),u^T\right)_{\mathbb{R}^N}dt$$

It is time to use Lemma 2.1. We are going to apply it twice. First of all, being (A,C) observable,

$$\|x(T)-\overline{x}\|\leq C\left[\int_0^T\|u^T-\overline{u}\|^2dt+\int_0^T\|C(x^T-\overline{x})\|^2dt\right]^{\frac{1}{2}} \tag{2.23}$$

Secondly, employing the observability of the pair (A^*,B^*),

$$\|p^T(0)-\overline{p}\|\leq C\left[\int_0^T\|C(x^T-\overline{x})\|^2dt+\int_0^T\|B^*(p^T-\overline{p})\|^2dt\right]^{\frac{1}{2}} \tag{2.24}$$

$$\|p^T(0)-\overline{p}\|\leq C\left[\int_0^T\|C(x^T-\overline{x})\|^2dt+\int_0^T\|u^T-\overline{u}\|^2dt+\int_0^T\|\overline{u}+B^*\overline{p}\|^2dt\right]^{\frac{1}{2}} \tag{2.25}$$

We are ready to prove a first estimate for (2.21). Indeed, from (2.22), one obtains

$$\int_0^T \left(\|u^T-\overline{u}\|^2+\|C\left(x^T-\overline{x}\right)\|^2\right)dt\leq$$

$$\leq C\left[\|p^T(0)-\overline{p}\|+\|x^T(T)-\overline{x}\|\right]+\left[\int_0^T\|u^T\|^2dt\right]^{\frac{1}{2}}\left[\int_0^T\|\overline{u}+B^*\overline{p}\|^2dt\right]^{\frac{1}{2}}\leq$$

$$\leq C(T^{\frac{1}{2}}+1)\left[\left(\int_0^T\|C(x^T-\overline{x})\|^2dt+\int_0^T\|u^T-\overline{u}\|^2dt\right)^{\frac{1}{2}}+CT^{\frac{1}{2}}\right]$$

Which implies, for T sufficiently large,

$$\int_0^T \left(\|u^T-\overline{u}\|^2+\|C\left(x^T-\overline{x}\right)\|^2\right)dt\leq CT \tag{2.26}$$

Then, the generalised sequences

$$\left\{\frac{1}{T}\int_0^T u^T dt\right\}_{T\in(0,+\infty)}\subset\mathbb{R}^M$$

$$\left\{\frac{1}{T}\int_0^T Cx^T dt\right\}_{T\in(0,+\infty)}\subset\mathbb{R}^N$$

are bounded. We want now to prove that $\left\{\frac{1}{T}\int_0^T x^T dt\right\}_{T\in(0,+\infty)}$ itself is bounded. To this extent, we take the average of the state equation, getting:

$$A\left(\frac{1}{T}\int_0^T x^T dt\right) = \frac{1}{T}\int_0^T Bu^T dt - \frac{x^T(T)-x_0}{T} \tag{2.27}$$

At this stage, thanks to (2.23) and (2.26), the following inequality holds:

$$\frac{\|x^T(T)-x_0\|}{T} \leq \frac{\sqrt{C}\sqrt{T}}{T}.$$

Hence, the last term in the averaged equation vanishes as $T \to +\infty$. This proves that $\left\{\frac{1}{T}\int_0^T Ax(t)dt\right\}_{T\in(0,+\infty)}$ is actually bounded. Another application of Lemma 2.2, allows us to conclude that the generalised sequence

$$\left\{\frac{1}{T}\int_0^T x^T dt\right\}_{T\in(0,+\infty)} \subset \mathbb{R}^N$$

is bounded. At the moment, thanks to Bolzano-Weierstrass theorem, we know that, up to subsequences, the averages converge, but we know nothing about the limits. We are going to investigate it in the next steps of the proof. First of all, from the averaged equation (2.27), one obtains that for every $(\varphi, v) \in \mathbb{R}^N \times \mathbb{R}^M$ such that there exists $\left\{\frac{1}{T_k}\int_0^{T_k} u^{T_k} dt\right\}_{k\in\mathbb{N}} \subset \left\{\frac{1}{T}\int_0^T u^T dt\right\}_{T\in(0,+\infty)}$ and $\left\{\frac{1}{T_k}\int_0^{T_k} Cx^{T_k} dt\right\}_{k\in\mathbb{N}} \subset \left\{\frac{1}{T}\int_0^T Cx^T dt\right\}_{T\in(0,+\infty)}$ such that

$$\frac{1}{T_k}\int_0^{T_k} x^{T_k} dt \underset{k\to+\infty}{\longrightarrow} \varphi$$

$$\frac{1}{T_k}\int_0^{T_k} u^{T_k} dt \underset{k\to+\infty}{\longrightarrow} v$$

it happens that

$$A\varphi = Bv$$

This result enables us to affirm

$$\frac{1}{T_k}\int_0^{T_k} \left(u^{T_k}, \overline{u} + B^*\overline{p}\right)_{\mathbb{R}^M} dt = \left(\overline{u} + B^*\overline{p}, \frac{1}{T_k}\int_0^{T_k} u^{T_k} dt\right)_{\mathbb{R}^M} \underset{k\to+\infty}{\longrightarrow} (v, \overline{u} + B^*\overline{p})_{\mathbb{R}^M} = 0$$

Therefore, the sequence $\left\{\frac{1}{T}\int_0^T \left(u^T, \overline{u} + B^*\overline{p}\right)_{\mathbb{R}^M} dt\right\}_{T\in(0,+\infty)}$ is such that each subsequence admits a further subsequence converging to 0. This, thanks to Proposition 1.1, allows us to prove the stronger result

$$\frac{1}{T}\int_0^T \left(u^T, \overline{u} + B^*\overline{p}\right)_{\mathbb{R}^M} dt \underset{T\to+\infty}{\longrightarrow} 0$$

The above convergence is true for each $\overline{p}$ satisfying $A^*\overline{p} = C^*\left(C\overline{x} - z\right)$. We are now ready to prove that

$$\frac{1}{T}\int_0^T \left(\|u^T - \overline{u}\|^2 + \|C\left(x^T - \overline{x}\right)\|^2\right) dt \underset{T\to+\infty}{\longrightarrow} 0 \tag{2.28}$$

In fact, using (2.22), (2.23), (2.25), one gets

$$\int_0^T \left(\|u^T - \overline{u}\|^2 + \|C\left(x^T - \overline{x}\right)\|^2\right) dt \leq$$

$$\leq C\left\{\left[\int_0^T \left(\|u^T - \overline{u}\|^2 + \|C\left(x^T - \overline{x}\right)\|^2\right) dt\right]^{\frac{1}{2}} + \left[\int_0^T \|\overline{u} + B^*\overline{p}\|^2 dt\right]^{\frac{1}{2}} - \int_0^T \left(u^T, \overline{u} + B^*\overline{p}\right)_{\mathbb{R}^M} dt\right.$$

Now, using the inequality (2.26),

$$\frac{1}{T}\int_0^T \left(\|u^T - \overline{u}\|^2 + \|C(x^T - \overline{x})\|^2\right) dt \leq \frac{C}{\sqrt{T}} - \frac{1}{T}\int_0^T \left(u^T, \overline{u} + B^*\overline{p}\right)_{\mathbb{R}^M} dt$$

Which implies

$$\frac{1}{T}\int_0^T \left(\|u^T - \overline{u}\|^2 + \|C\left(x^T - \overline{x}\right)\|^2\right) dt \underset{T\to+\infty}{\longrightarrow} 0$$

This directly implies

$$\frac{1}{T}\int_0^T Cx^T dt \underset{T\to+\infty}{\longrightarrow} C\overline{x}$$

$$\frac{1}{T}\int_0^T u^T dt \underset{T\to+\infty}{\longrightarrow} \overline{u}$$

Moreover, from the averaged equation (2.27), one obtains

$$\frac{1}{T}\int_0^T Ax^T dt \underset{T\to+\infty}{\longrightarrow} A\overline{x}$$

Finally, using the observability of (A, C), by Lemma 2.2

$$\frac{1}{T}\int_0^T x^T dt \underset{T\to+\infty}{\longrightarrow} \overline{x}$$

We want now to prove that, for a special $\overline{p} \in \mathbb{R}^N$,

$$\frac{1}{T}\int_0^T p^T dt \underset{T\to+\infty}{\longrightarrow} \overline{p}$$

To achieve this result, we take the average of the dual equation satisfied by p^T and we get:

$$-\frac{p^T(T) - p^T(0)}{T} + \frac{1}{T}\int_0^T A^* p^T dt = \frac{1}{T}\int_0^T C^*(Cx^T - z)dt \tag{2.29}$$

This yields

$$\left\| \frac{1}{T}\int_0^T A^* p^T dt \right\| \leq \frac{1}{T}\|p^T(T) - p^T(0)\| + \frac{1}{T}\int_0^T \|C^*(Cx^T - z)\|^2 dt \leq$$

$$\leq \frac{C + C\sqrt{T}}{T} + C \leq C.$$

Now, being the pair (A^*, B^*) observable, one can use Lemma 2.2 to obtain that the generalised sequence

$$\left\{ \frac{1}{T}\int_0^T p^T dt \right\}_{T\in(0,+\infty)} \subset \mathbb{R}^N$$

is actually bounded. Again, thanks Bolzano-Weierstrass theorem, for every subsequences of $\left\{ \frac{1}{T}\int_0^T p^T dt \right\}_{T\in(0,+\infty)}$ there exists a subsubsequence and one $\overline{p} \in \mathbb{R}^N$ such that the subsubsequence converge to this $\overline{p}$. To prove the convergence of the whole sequence we aim at proving the uniqueness of the limit $\overline{p}$. To this extent, we consider again the averaged equation of p^T

$$\frac{p^T(0)}{T} + \frac{1}{T}\int_0^T A^* p^T dt = \frac{1}{T}\int_0^T C^*(Cx^T - z)dt$$

and we deduce that $\overline{p}$ satisfies the equation $A^*\overline{p} = C^*(C\overline{x} - z)$. Furthermore, we can take the average of the equation $u^T = -B^* p^T$, and get a further property for the limit $\overline{p}$

$$\overline{u} = -B^*\overline{p}$$

This achievement is remarkable, because it proves the existence of a $\overline{p}$ such that

$$\begin{cases} A^*\overline{p} = C^*(C\overline{x} - z) \\ \overline{u} = -B^*\overline{p} \end{cases} \tag{2.30}$$

Moreover, we can prove the uniqueness of such a $\overline{p}$ using Lemma 2.2. Then, by Proposition 1.1, we obtain:

$$\frac{1}{T}\int_0^T p^T dt \underset{T\to+\infty}{\longrightarrow} \overline{p}.$$

Furthermore, we can use the special $\overline{p}$ satisfying (2.30) all throughout the proof. Then, using (2.22), (2.23) and (2.25), we can get

$$\int_0^T \|(Cx^T-\overline{x})\|^2+\|u^T-\overline{u}\|^2 dt \leq C\left[\int_0^T \|C(x^T-\overline{x})\|^2+\|u^T-\overline{u}\|^2 dt\right]^{\frac{1}{2}}$$

Finally, this implies the existence of a constant $C\in(0,+\infty)$ such that:

$$\int_0^T \|C(x^T-\overline{x})\|^2+\|u^T-\overline{u}\|^2 dt \leq C \tag{2.31}$$

From this inequality, the thesis follows.

□

There is an interesting byproduct of this Theorem. This is the object of the following Remark.

Remark 2.2. Assuming (A,C) Kalman-Observable and (A,B) Kalman-Controllable, there exists $M\in\mathbb{R}^+$ such that:

$$\left\{x^T(t) \mid T\in(0,+\infty), t\in[0,T]\right\}\subset B^{\mathbb{R}^N}(0,M).$$

Proof. In order to achieve this result, one can use Lemma 2.1 applied to $x^T-\overline{x}$ together with estimate (2.31). □

Now, we are going to investigate how does (x^T,u^T) converge to $(\overline{x},\overline{u})$. To this extent, we need to prove the following Lemma.

Lemma 2.4. *For each $L\in\mathcal{M}(N,N;\mathbb{R})$ such that for the differential linear system*

$$\begin{cases}\frac{d}{dt}z+Lz=0 & \forall t\in\mathbb{R}\\ z(0)=z_0\end{cases} \tag{2.32}$$

0 is Locally Asymptotically Stable, then 0 is Exponentially Globally Asymptotically Stable for the system (2.32).

Proof. 1st Step

First of all, we show the Global Asymptotical Stability of 0. Indeed, by the hypothesis, there exists a $\delta\in(0,+\infty)$ such that $\forall z_0\in B(0,\delta)$

$$\exists \lim_{t\to+\infty}\|e^{-tL}z_0\|=0$$

Hence, $\forall z_0\in\mathbb{R}^N\setminus\{0\}$, we define $\gamma=\frac{\delta}{2\|z_0\|}$, getting

$$\exists \lim_{t\to+\infty}\|e^{-tL}\gamma z_0\|=0.$$

Which yields

$$\exists \lim_{t\to+\infty} \|e^{-tL}z_0\| = 0.$$

Therefore, 0 is a Globally Asymptotically Stable point for the system (2.32).

2$^{\text{nd}}$ Step

We prove that

$$\sup_{x_0\in\overline{B(0,1)}} \|e^{-Lt}\| \underset{t\to+\infty}{\longrightarrow} 0.$$

$\overline{B(0,1)} \subset \mathbb{R}^N$ is compact. This implies that it is totally bounded, i.e.

$$\forall\varepsilon > 0 \; \exists \{x_1,\dots,x_{N_\varepsilon}\} \subset \overline{B(0,1)} \quad \text{such that} \quad \overline{B(0,1)} \subset \bigcup_{i\in\{1,\dots,N\varepsilon\}} B(x_i,\varepsilon).$$

Moreover, for every $x_0 \in \mathbb{R}^N$, $\left\{e^{-tL}x_0\right\}_{t\in(0,+\infty)} \subset \mathbb{R}^N$ is bounded. Then, by the Banach-Steinhaus Theorem, $\left\{e^{-tL}\right\}_{t\in(0,+\infty)} \subset B(\mathbb{R}^N,\mathbb{R}^N)$ is bounded too, namely:

$$\forall t \in \mathbb{R}^+ \qquad \|e^{-tL}\| \leq M$$

Now, in the previous steps, we have proved that for each $x_0 \in \mathbb{R}^N$

$$e^{-tL}x_0 \underset{t\to+\infty}{\longrightarrow} 0$$

This implies that

$$\forall\varepsilon > 0 \; \exists t_\varepsilon \in (0,+\infty) \quad | \quad \|e^{-tL}x_i\| < \varepsilon \; \forall t > t_\varepsilon \; \forall i \in \{1,\dots,N_\varepsilon\}$$

Hence, $\forall\varepsilon > 0$, $t > t_\varepsilon$ and $\forall x_0 \in \mathbb{R}$

$$\|e^{-tL}x_0\| \leq \left[\|e^{-tL}x_i\| + \|e^{-tL}(x_i - x_0)\|\right] \leq \varepsilon + M\|x_i - x_0\| \leq [\varepsilon + M\varepsilon]$$

Therefore

$$\sup_{x_0\in\overline{B(0,1)}} \|e^{-Lt}\| \underset{t\to+\infty}{\longrightarrow} 0.$$

3$^{\text{rd}}$ Step

We want to show that there exist 2 constants $(C,\mu) \in (0,+\infty)^2$ such that

$$\|e^{-tL}\| \leq Ce^{-\mu t} \qquad \forall t \in (0,+\infty)$$

At this stage, one can use the result of the previous step, i.e.

$$\sup_{x_0\in\overline{B(0,1)}} \|e^{-Lt}\| \underset{t\to+\infty}{\longrightarrow} 0$$

which yields the existence of $t_1 \in (0, +\infty)$ such that:

$$\|e^{-Lt}\| \leq \frac{1}{2} \qquad \forall t \geq t_1$$

This entails, by induction on $n \in \mathbb{N}$,

$$\|e^{-Lnt_1}\| \leq \frac{1}{2^n}$$

Indeed,
P(1) $(n = 1)$ by definition of t_1.
Moreover, if P$(n-1)$ holds true, does it imply P(n)? In order to prove this assertion, one can consider an arbitrary $x_0 \in \overline{B(0,1)}$. Then,

$$\|e^{-nt_1L}x_0\| \leq \|e^{-(n-1)t_1L}\|\|e^{-t_1}Lx_0\| \leq \frac{1}{2^{n-1}}\frac{1}{2} = \frac{1}{2^n}$$

Therefore, by the Principle of Induction, one can conclude that

$$\|e^{-Lnt_1}\| \leq \frac{1}{2^n} \qquad \forall n \in \mathbb{N}$$

To finish the proof, defining $C_1 = \sup_{t\in[0,t_1]} \|e^{-tL}\| \in \mathbb{R}^+$, for each $t \in (0, +\infty)$ $n(t) = \lfloor \frac{t}{t_1} \rfloor$ Then,

$$\|e^{-tL}\| = \|e^{-n(t)t_1 - (\frac{t}{t_1} - n(t))t_1 L}\| \leq C_1\|e^{-n(t)t_1}\| \leq C_1\frac{1}{2^{n(t)}}$$

$$\leq C_1 e^{-n(t)\ln(2)} \leq C_1 e^{-(\frac{t}{t_1}-1)\ln(2)} \leq C_1 e^{\ln(2)} e^{-\frac{t}{t_1}\ln(2)}$$

Hence, defining $C = \sup_{t\in[0,t_1]} \|e^{-tL}\|e^{\ln(2)} \in \mathbb{R}^+$ and $\mu = \frac{\ln(2)}{t_1}$, one gets

$$\|e^{-tL}\| \leq Ce^{-\mu t} \qquad \forall t \in [0, +\infty)$$

□

The next Proposition will prove to be extremely useful in the arguments used in the next proofs.

Proposition 2.1. *Let $L \in \mathcal{M}(N, N; \mathbb{R})$ such that there exists a Lyapunov function V, whose square root defines a norm on $\mathbb{R}^N$. Moreover, we suppose that for every $z : \mathbb{R} \longmapsto \mathbb{R}^N$ solution of*

$$\frac{d}{dt}z + Lz = 0 \qquad \forall t \in \mathbb{R}$$

1. there exists $t_0 \in \mathbb{R}$ such that

$$\exists M \in \mathbb{R}^+ \quad : \quad V(z(t)) \leq M \quad \forall t \in [t_0, +\infty)$$

2. if $\exists (t_1, t_2) \in \mathbb{R}^2, \quad t_1 < t_2$ such that $V(z(t_1)) = V(z(t_2))$, then $z \equiv 0$.

Then, the linear differential system

$$\begin{cases} \frac{d}{dt} z + Lz = 0 & \forall t \in \mathbb{R} \\ z(0) = z_0 \end{cases} \tag{2.33}$$

is Exponentially Globally Asymptotically Stable.

Proof. First of all, 0 is an equilibrium point for the system. We divide the proof in 3 Steps.

1$^{\text{st}}$ Step

We are going to prove that for each z solution of the equation, there exists $M_1 \in \mathbb{R}^+$ such that

$$\forall t \in [t_0, +\infty) \quad \|z(t)\| \leq M_1.$$

Indeed, thanks to the norms equivalence in $\mathbb{R}^N$, for every $t \in [t_0, +\infty)$

$$\|z(t)\| \leq C\sqrt{V(z(t))} \leq C\sqrt{M} \stackrel{\text{def}}{=} M_1$$

2$^{\text{nd}}$ Step

In this step, we aim at showing that if there exists $\{t_n\}_{n\in\mathbb{N}} \subset [\max\{t_0, 0\}, +\infty)$ and $\overline{z} \in \mathbb{R}^N$ such that $t_n \underset{n\to+\infty}{\longrightarrow} +\infty$ and

$$z(t_n) \underset{n\to+\infty}{\longrightarrow} \overline{z},$$

then,

$$\overline{z} = 0.$$

To this extent, $\forall\, n \in \mathbb{N}$ we define

$$z_n : \mathbb{R} \longmapsto \mathbb{R}^N$$

$$t \longrightarrow z(t_n + t)$$

This is a solution of the equation too and for each $t \in [t_0, +\infty)$

$$\|z_n(t)\| = \|z(t_n + t)\| \leq M_1$$

Since z_n satisfies the equation, we have also for every $t \in [t_0, +\infty)$

$$\left\| \frac{d}{dt} z_n(t) \right\| \leq \|L\| \, \|z_n(t)\| \leq \|L\| \, M_1$$

Using again the equation, one has

$$\frac{d^2}{dt^2} z_n(t) = -L \frac{d}{dt} z_n \qquad \forall t \in \mathbb{R}$$

Then, for each $t \in [t_0, +\infty)$

$$\left\| \frac{d^2}{dt^2} z_n(t) \right\| \leq \|L\|^2 \, \|z_n(t)\| \leq \|L\|^2 \, M_1$$

At this stage, thanks to Ascoli-Arzelà's Theorem applied to $\{z_n\}_{n \in \mathbb{N}}$ and to $\left\{ \frac{d}{dt} z_n \right\}_{n \in \mathbb{N}}$ in each compact subset of $[t_0, +\infty)$, one obtains the existence of a subsequence $\{z_{n_k}\}_{k \in \mathbb{N}} \subset \{z_n\}_{n \in \mathbb{N}}$ and $y \in C^1([t_0, +\infty), \mathbb{R}^N)$ such that uniformly on compacts subsets

$$z_{n_k} \underset{k \to +\infty}{\longrightarrow} y$$

and

$$\frac{d}{dt} z_{n_k} \underset{k \to +\infty}{\longrightarrow} \frac{d}{dt} y$$

First of all, this implies that y is a solution of the equation, and can be extended to a solution defined on the whole $\mathbb{R}$. To conclude, one recollect that the Lie derivative (see Definition 1.4) of V is less than or qual to 0, by the definition of Lyapunov function. This implies

$$V \circ z : \mathbb{R} \longmapsto \mathbb{R}^+$$

$$t \longrightarrow V(z(t))$$

is not increasing. Hence, there exists $l \in \mathbb{R}^+$ such that

$$V(z(t)) \underset{t \to +\infty}{\longrightarrow} l$$

On the other hand, for each $s \in [t_0, +\infty)$ fixed

$$V(z(t_{n_k} + s)) \underset{k \to +\infty}{\longrightarrow} V(y(s))$$

By the uniqueness of the limit, on can deduce that for every $s \in [t_0, +\infty)$

$$V(y(s)) = l$$

By the second hypothesis, $y(s) = 0 \ \forall s \in [t_0, +\infty)$. Then, $\overline{z} = y(0) = 0$.
3^{rd} Step

We consider the sequence $\{z(t)\}_{t\in[t_0,+\infty)}$. It is bounded. Therefore, thanks to Bolzano Weiestrass Theorem, for every subsequence there exists a convergent subsubsequence. The 2^{nd} Step allows us to affirm that the limit is actually 0. Hence, by Proposition 1.1,

$$z(t) \underset{t\to+\infty}{\longrightarrow} 0$$

Therefore, the linear differential system is globally asymptotically stable.
4^{th} Step

Employing Lemma 2.4, one gets

$$\|e^{-tL}\| \leq Ce^{-\mu t} \qquad t \in (0, +\infty)$$

as required.

□

At this stage, we focus on the case $z = 0$. In this context the non-stationary problem functional becomes

$$J_0^T : L^2((0,T);\mathbb{R}^N) \longmapsto \mathbb{R}$$

$$u \longrightarrow \int_0^T \left(\|u(t)\|^2 + \|Cx(t)\|^2\right) dt$$

and the stationary one

$$J_0^s : M \longmapsto \mathbb{R}$$

$$(x,u) \longrightarrow \left[\|u\|^2 + \|Cx\|^2\right]$$

We will call the corresponding non stationary minimization problem $(OCP)_0^T$ and the stationary one $(OCP)_0^s$. In this framework, the unique minimizer for the stationary problem is $(0,0)$. In the next Lemma we will show that, when $z = 0$, one can define the optimal control for the finite time-horizon problem by a linear feedback law involving the solution of the Riccati Equation. Moreover, it will be proved, under our assumptions, the existence and uniqueness of a positive definite solution of the Algebraic Riccati Equation. This will enable us to construct a stabilizing linear feedback function, which will define the optimal control for the infinite-time horizon problem.

Lemma 2.5. *Let* $(A, B, C) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R}) \times \mathcal{M}(N, N; \mathbb{R})$ *such that* (A, B) *is Kalman-controllable and* (A, C) *is Kalman-observable. Then,*

1.
$$\exists! \quad \mathcal{E} : \mathbb{R}^+ \longmapsto Sym(N,\mathbb{R}) \in C^1(\mathbb{R}^+, \mathcal{M}(N,N;\mathbb{R}))$$

solution of the nonlinear Cauchy Problem

$$\begin{cases} \mathcal{E}_t = C^*C - (\mathcal{E}A + A^*\mathcal{E}) - \mathcal{E}BB^*\mathcal{E} & \forall t \in (0,+\infty) \\ \mathcal{E}(0) = 0 \end{cases} \tag{2.34}$$

2. *for every* $T \in (0,+\infty)$

$$f^T : [0,T] \times \mathbb{R}^N \longmapsto \mathbb{R}^M$$

$$(t,x) \longrightarrow -B^*\mathcal{E}(T-t)x$$

is an optimal feedback law for $(OCP)_0^T$*, i.e. if* x^T *is the optimal state, then* $(x^T, \mathcal{E}(T-\cdot)x^T, -B^*\mathcal{E}(T-\cdot)x^T)$ *is the optimal triple for* $(OCP)_0^T$*.*

3. *there exists* $R \in Sym(N,\mathbb{R})$ *a positive definite matrix such that for each* $T \in \mathbb{R}^+$
$$0 < \mathcal{E}(T) \leq R$$

4. $\forall (t_1,t_2) \in \mathbb{R}^{+2}$
$$\mathcal{E}(t_1) \leq \mathcal{E}(t_2)$$

5.
$$\exists \lim_{T\to+\infty} \mathcal{E}(T) \stackrel{def}{=} \widehat{E}$$

Furthermore, $\widehat{E}$ *is positive definite and it is the unique positive definite solution of the Algebraic Riccati Equation*

$$(\widehat{E}A + A^*\widehat{E}) + \widehat{E}BB^*\widehat{E} = C^*C \qquad (ARE) \tag{2.35}$$

6. *the linear system*

$$\begin{cases} x_t + (A + BB^*\widehat{E})x = 0 & \forall t \in \mathbb{R} \\ x(0) = x_0 \end{cases} \tag{2.36}$$

is exponentially globally asymptotically stable, namely 0 *is an exponentially globally asymptotically stable equilibrium point.*

Proof. First of all, we face the problem of the global existence in positive time for the non linear Cauchy Problem (2.34). Picard's Theorem implies the existence of a maximal solution

$$\mathcal{E} : I \longmapsto \mathcal{M}(N, N; \mathbb{R})$$

for the Cauchy Problem. We define $J = I \bigcap (0, +\infty)$. Furthermore, one can verify that $\mathcal{E}^*$ satisfies the same Cauchy Problem. By the uniqueness Theorem, $\mathcal{E} = \mathcal{E}^*$. Therefore, for every $t \in J$, $\mathcal{E}(t) \in Sym(N, \mathbb{R})$. To prove the global existence in positive times, we want to prove that there exists a compact $K \subset \mathcal{M}(N, N; \mathbb{R})$ such that $\mathcal{E}(t) \in K$ for each $t \in J$. To this extent, $\forall T \in J$ and for every initial data $x_0 \in \mathbb{R}^N$ we consider the Cauchy Problem

$$\begin{cases} \frac{d}{dt} w(t) + (A + BB^* \mathcal{E}(T - t))w = 0 & \forall t \in (0, T) \\ w(0) = x_0 \end{cases} \tag{2.37}$$

The function defining the dynamics

$$g : (0, T) \times \mathbb{R}^N \longmapsto \mathbb{R}^N$$

$$(t, x) \longmapsto (A + BB^* \mathcal{E}(T - t))x$$

is sublinear. Hence, there exists a unique global solution y^T. Got this far, we can define $(y^T, \pi^T, v^T) = (y^T, \mathcal{E}(T - \cdot)y^T, -B^* \mathcal{E}(T - \cdot)y^T)$, which we want to prove to be the optimal triple for $(OCP)_0^T$. To this extent, we show that it satisfies the Pontryagin system. By definition, y^T is the unique solution of

$$\begin{cases} \frac{d}{dt} x^T(t) + Ax^T = Bv^T & \forall t \in (0, T) \\ x^T(0) = x_0 \end{cases} \tag{2.38}$$

Besides, one can compute the derivative of π^T and get for every $t \in [0, T]$

$$\frac{d}{dt} \pi^T(t) =$$

$$= -C^* C y^T(t) + \mathcal{E}(T - t) A y^T(t) + A^* \mathcal{E}(T - t) y^T(t) +$$

$$+ \mathcal{E}(T - t) BB^* \mathcal{E}(T - t) y^T(t) - \mathcal{E}(T - t) A y^T(t) + \mathcal{E}(T - t) B v^T(t) =$$

$$= A^* \mathcal{E}(T-t) y^T(t) - C^* C y^T(t) + \mathcal{E}(T-t) BB^* \mathcal{E}(T-t) y^T(t) - \mathcal{E}(T-t) BB^* \mathcal{E}(T-t) y^T(t) =$$

$$= A^* \pi^T(t) - C^* C y^T(t)$$

Therefore, the triple (y^T, π^T, v^T) satisfies

$$\begin{cases} \frac{d}{dt} y^T + Ay^T = Bv^T & \forall t \in (0,T) \\ -\frac{d}{dt}\pi^T + A^*\pi^T = C^*Cy^T & \forall t \in (0,T) \\ v^T = -B^*\pi^T \\ x(0) = x_0 \\ \pi^T(T) = 0. \end{cases} \tag{2.39}$$

Then, by the existence and uniqueness of the optimal triple for $(OCP)_0^T$ and the convexity of the functional J_0^T, one obtains (y^T, π^T, v^T) is the unique optimal triple for $(OCP)_0^T$. At this moment, we can prove that for each $T \in J$ and $x_0 \in \mathbb{R}^N$

$$x_0^* \mathcal{E}(T) x_0 = \inf_{u \in L^2((0,T);\mathbb{R}^M)} J_0^T$$

Indeed,

$$\frac{d}{dt}\left[\left(y^T, \pi^T\right)_{\mathbb{R}^N}\right] =$$

$$= \left(-Ay^T - BB^*\pi, \pi\right)_{\mathbb{R}^N} + \left(y^T, A^*\pi^T - C^*Cy^T\right)_{\mathbb{R}^N} =$$

$$= -\|B^*\pi\|^2 - \|Cy^T\|^2.$$

Hence, using the Fundamental Calculus Theorem

$$x_0^* \mathcal{E}(T) x_0 = \int_0^T -\frac{d}{dt}\left[\left(\mathcal{E}(T-t)y^T(t), y^T(t)\right)_{\mathbb{R}^N}\right] dt =$$

$$= \int_0^T \|B^*\pi^T\|^2 + \|Cy^T\|^2 dt = \int_0^T \|v^T\|^2 + \|Cy^T\|^2 dt = \inf_{u \in L^2((0,T);\mathbb{R}^M)} J_0^T.$$

We have already determined, when the target $z = 0$, the value function as a quadratic form by the solution of the Riccati Equation. This equality implies that for each $T \in J$

$$x_0^* \mathcal{E}(T) x_0 = \int_0^T \|v^T\|^2 + \|Cy^T\|^2 dt \geq 0$$

Furthermore, whenever $x_0^* \mathcal{E}(T) x_0 = 0$, we have:

$$\int_0^T \|v^T\|^2 + \|Cy^T\|^2 dt = 0$$

which implies

$$\begin{cases} v^T(t) = 0 & \forall t \in [0,T] \\ Cy^T(t) = 0 & \forall t \in [0,T] \end{cases} \tag{2.40}$$

Now, one use the observability of the pair (A, C), obtaining $x_0 = 0$. Therefore, $\mathcal{E}(T)$ is positive definite $\forall T \in J$. We consider again an arbitrary initial data $x_0 \in \mathbb{R}^N$ to conclude the proof of global existence in positive times. Now, one use the controllability of the pair (A, B). This, by Theorem 1.7, implies the existence of a matrix $F \in \mathcal{M}(N, M; \mathbb{R})$ and 2 positive constants $(C, \omega) \in (0, +\infty)^2$, such that for each $w_0 \in \mathbb{R}^N$ the solution of the system

$$\begin{cases} w_t + Aw = BFw & \forall t \in \mathbb{R} \\ w(0) = w_0 \end{cases} \tag{2.41}$$

$$\|w(t)\| \leq Ce^{-\omega t}\|w_0\|$$

Then, choosing $w_0 = x_0$, one gets a pair state-control $(y, v) = (z, Fz)$ (in general, not optimal) such that

$$\int_0^{+\infty} \|v\|^2 + \|Cz\|^2 dt < +\infty$$

Defining

$$R_0 \stackrel{\text{def}}{=} \int_0^{+\infty} \|v\|^2 + \|Cz\|^2 dt$$

one has for every $T \in (0, +\infty)$

$$x_0^* \mathcal{E}(T) x_0 = J_0^T(v^T) \leq J_0^T(v) = \int_0^{+\infty} \|v\|^2 + \|Cz\|^2 dt = R_0$$

Hence, choosing $x_0 \in S^{\mathbb{R}^N}(0, 1)$, we define the positive definite matrix

$$R \stackrel{\text{def}}{=} R_0 I_N$$

and we deduce that $\forall T \in J$

$$0 < \mathcal{E}(T) \leq R$$

This implies that the desired compact set exists and can be chosen as

$$K = \overline{B(0, R_0)} \subset \mathcal{M}(N, N; \mathbb{R})$$

This implies that actually $J = (0, +\infty)$. We have concluded the proof of 1. Moreover, we have proved 2. and 3.. To prove 4. we use again the characterization of the value function as a quadratic form. For every $(t_1, t_2) \in \mathbb{R}^{+2}$ $t_1 \leq t_2$ if we define $(y^{t_1}, \pi^{t_1}, v^{t_1})$ the optimal triple for $(OCP)_0^{t_1}$ and $(y^{t_2}, \pi^{t_2}, v^{t_2})$ the optimal triple for $(OCP)_0^{t_2}$

$$x_0^* \mathcal{E}(t_2) x_0 = \int_0^{t_2} \|v^{t_2}\|^2 + \|Cy^{t_2}\|^2 dt \geq$$

$$\geq \int_0^{t_1} \|v^{t_2}\|^2 + \|Cy^{t_2}\|^2 dt \geq \int_0^{t_1} \|v^{t_1}\|^2 + \|Cy^{t_1}\|^2 dt = x_0^* \mathcal{E}(t_1) x_0$$

By which we conclude

$$\mathcal{E}(t_1) \leq \mathcal{E}(t_2).$$

We use now 3. and 4. to deduce 5. In fact by the bounded monotonicity of $\{\mathcal{E}(t)\}_{t\in(0,+\infty)}$, one gets

$$\exists \lim_{t\to+\infty} \mathcal{E}(t) \stackrel{\text{def}}{=} \widehat{E}$$

$\widehat{E}$ is symmetric and positive definite. Furthermore, by the equation satisfied by $\mathcal{E}$, the sequence $\left\{\frac{d}{dt}\mathcal{E}(t)\right\}_{t\in(0,+\infty)}$ converge too. Using again the fact that for every $T \in (0,+\infty)$ $0 < \mathcal{E}(t) \leq R$ it follows that

$$\frac{d}{dt}\mathcal{E}(T) \underset{T\to+\infty}{\longrightarrow} 0$$

This implies that $\widehat{E}$ is a solution of the Algebraic Riccati Equation (ARE). Before uniqueness, we prove that for all $\widehat{E} \in Sym(N;\mathbb{R})$ positive definite solution of the Algebraic Riccati Equation (ARE), the Cauchy Problem

$$\begin{cases} x_t + (A + BB^*\widehat{E})x = 0 & \forall t \in \mathbb{R} \\ x(0) = x_0 \end{cases} \tag{2.42}$$

is Exponentially Globally Asymptotically Stable. To this purpose, one defines the function

$$V : \mathbb{R}^N \longmapsto \mathbb{R}$$
$$x \longrightarrow x^*\widehat{E}x$$

This function is strictly positive on $\mathbb{R}^N \setminus \{0\}$ and vanishes in 0. Moreover, we compute the Lie derivative (Definition 1.4) as follows:

$$L_V(x) =$$

$$= (-Ax - BB^*\widehat{E}x)^*\widehat{E}x + x^*\widehat{E}(-Ax - BB^*\widehat{E}x) =$$
$$= x^*(-A^*\widehat{E} - \widehat{E}A - \widehat{E}BB^*\widehat{E} - \widehat{E}BB^*\widehat{E})x =$$
$$= x^*(-C^*C - \widehat{E}BB^*\widehat{E})x =$$
$$= -\|Cx\|^2 - \|B^*\widehat{E}x\|^2 \leq 0$$

Therefore, V is a Lyapunov function. Furthermore, since $\widehat{E}$ is positive definite, $\sqrt{V}$ defines a norm. Moreover, by the computation of the Lie derivative, for every $(t_1, t_2) \in (0, +\infty)$, $t_1 \leq t_2$ and for every $x_0 \in \mathbb{R}^N$

$$V(x(t_1)) - V(x(t_2)) = \int_{t_1}^{t_2} \|Cx\|^2 + \|B^*\widehat{E}x\|^2 dt$$

Therefore, if $V(x(t_1)) = V(x(t_2))$, $Cx(t) = 0 \quad t \in [t_1, t_2]$. By the observability of (A, C), $x(t) = 0 \quad t \in [t_1, t_2]$. Hence $x(t) = 0 \quad \forall t \in \mathbb{R}$. Then, using Proposition 2.1, one deduces 6.

For the uniqueness we reason as follows. For every $\widehat{E} \in Sym(N; \mathbb{R})$ positive definite solution of the (ARE), for every $x_0 \in \mathbb{R}^N$ we consider x the unique solution of the Cauchy Problem

$$\begin{cases} x_t + (A + BB^*\widehat{E})x = 0 & \forall t \in \mathbb{R} \\ x(0) = x_0 \end{cases} \tag{2.43}$$

Then, the triple $(x, \widehat{E}x, -B^*\widehat{E}x)$ satisfies the System

$$\begin{cases} \frac{d}{dt}y + Ay = Bv & \forall t \in (0, +\infty) \\ -\frac{d}{dt}\pi + A^*\pi = C^*Cy & \forall t \in (0, +\infty) \\ v = -B^*\pi \\ x(0) = x_0 \\ \pi(t) \underset{t \to +\infty}{\to} 0 \end{cases} \tag{2.44}$$

and

$$\int_0^{+\infty} \left[\| - B^*\widehat{E}\hat{x}(t)\|^2 + \|C\hat{x}(t)\|^2 \right] dt < +\infty.$$

Which entails that the infinite horizon problem $(OCP)_0^{+\infty}$ is well posed. Furthermore, System (2.44) is the first order condition for the minimization of $J_0^{+\infty}$. Then, since J_0^{∞} is strictly convex, by Proposition 1.2, $(x, \widehat{E}x, -B^*\widehat{E}x)$ is the optimal triple for $(OCP)_0^{\infty}$. We proceed with the following computation

$$\begin{aligned} \frac{d}{dt}\left[x^*\widehat{E}x\right] &= L_V(x) = \\ &= -\|Cx\|^2 - \|B^*\widehat{E}x\|^2. \end{aligned}$$

Now, using the Fundamental Calculus Theorem, one obtains

$$x_0^*\widehat{E}x_0 = \int_0^{+\infty} -\frac{d}{dt}\left[x^*\widehat{E}x\right] dt =$$

$$= \int_0^{+\infty} \|Cx\|^2 + \|B^* \widehat{E} x\|^2 dt = \inf_{u \in L^2((0,+\infty),\mathbb{R}^M)} J_0^{+\infty}.$$

Then, for every $(\widehat{E}_1, \widehat{E}_2) \in Sym(N, \mathbb{R})$ of positive definite solution of the Algebraic Riccati Equation, one can use the above computation to get

$$x_0^* (\widehat{E}_2 - \widehat{E}_1) x_0 = \inf_{u \in L^2((0,+\infty),\mathbb{R}^M)} J_0^{+\infty} - \inf_{u \in L^2((0,+\infty),\mathbb{R}^M)} J_0^{+\infty} = 0$$

This yields $\widehat{E}_1 = \widehat{E}_2$ as required.

□

In order to deduce the consequences of the previous Lemma, we need the following Proposition.

Proposition 2.2 (Schauder's Fixed Point Theorem generalization)**.** *Let* $(X, \| \, \|)$ *be a Banach space,* $F \subseteq X$ *nonempty closed convex*

$$A : F \longmapsto F$$

a (possibly nonlinear) continuous map such that $A(F)$ *is relatively compact. Then there exists a fixed point* $\overline{x} \in F$ *for* A*.*

Proof. By hypothesis, $A(F)$ is relatively compact. Then, one can define the convex hull of $A(F)$:

$$H = \left\{ \sum_{i=1}^{N} \lambda_i x_i \;\middle|\; N \in \mathbb{N},\ \{x_1, \dots, x_N\} \subset \overline{A(F)}, \right.$$

$$\left. \{\lambda_i\}_{i \in 1,\dots,N} \subset [0,1] \text{ such that } \sum_{i=1}^{N} \lambda_i = 1. \right\}$$

Since $\overline{A(F)} \subset X$ is compact, H is compact. We define:

$$A \restriction_H : H \longmapsto H$$

Now, thanks to the Shauder Fixed Point Theorem (page 538 of [10]), we get

$$\exists \overline{x} \in X \qquad \text{such that} \qquad A(\overline{x}) = \overline{x}$$

□

We are going now to describe more accurately the convergence of $\mathcal{E}(t)$ to $\widehat{E}$.

Corollary 2.1. *Assuming (A, C) observable and (A, B) controllable, there exists 2 constants $(C, \mu) \in (0, +\infty)$ such that:*

$$\|\mathcal{E}(t) - \widehat{E}\| \leq Ce^{-2\mu t} \qquad \forall t \in \mathbb{R}^+ \tag{2.45}$$

Remark 2.3. As we will see from the proof, the rate μ in (2.45) is any exponential rate of $A + BB^*\widehat{E}$.

Proof. Let us define $L = A + BB^*\widehat{E}$. We look for the equation satisfied by $\mathcal{E} - \widehat{E}$. To this purpose, we compute the derivative of $(\mathcal{E} - \widehat{E})$:

$$\left(\mathcal{E} - \widehat{E}\right)_t = C^*C - (\mathcal{E}A + A^*\mathcal{E}) - \mathcal{E}BB^*\mathcal{E} - C^*C + \widehat{E}A + A^*\widehat{E} + \widehat{E}BB^*\widehat{E} =$$

$$= -(\mathcal{E} - \widehat{E})A - A^*(\mathcal{E} - \widehat{E}) - (\mathcal{E} - \widehat{E})(BB^*\widehat{E}) + (\mathcal{E} - \widehat{E})BB^*\widehat{E} +$$
$$-\mathcal{E}BB^*\mathcal{E} - \widehat{E}BB^*(\mathcal{E} - \widehat{E}) + \widehat{E}BB^*\widehat{E} + \widehat{E}BB^*(\mathcal{E} - \widehat{E}) =$$
$$= -(\mathcal{E}-\widehat{E})L - L^*(\mathcal{E}-\widehat{E}) + \mathcal{E}BB^*\widehat{E} - \widehat{E}BB^*\widehat{E} + \widehat{E}BB^*\mathcal{E} - \widehat{E}BB^*\widehat{E} - \mathcal{E}BB^*\mathcal{E} + \widehat{E}BB^*\widehat{E} =$$
$$= -(\mathcal{E} - \widehat{E})L - L^*(\mathcal{E} - \widehat{E}) - (\mathcal{E} - \widehat{E})BB^*(\mathcal{E} - \widehat{E})$$

Hence,

$$(\mathcal{E} - \widehat{E})_t = -\left((\mathcal{E} - \widehat{E})L + L^*(\mathcal{E} - \widehat{E})\right) - (\mathcal{E} - \widehat{E})BB^*(\mathcal{E} - \widehat{E}) \tag{2.46}$$

By Proposition 2.1, there exist 2 positive constants $(C, \mu) \in (0, +\infty)^2$ such that for every $t \in \mathbb{R}^+$, $\|e^{-tL}\| \leq Ce^{-\mu t}$. By our computations, $\mathcal{E}(t_0 + .) - \widehat{E}$ satisfies $\forall\ t_0 \in (0, +\infty)$ the nonlinear Cauchy Problem

$$\begin{cases} Z_t = -(ZL + L^*Z) - ZBB^*Z & \forall t \in (0, +\infty) \\ Z(t_0) = \mathcal{E}(t_0) - \widehat{E} \end{cases} \tag{2.47}$$

We are going to study Cauchy Problems of this kind, in order to get the desired estimate for the difference $\mathcal{E} - \widehat{E}$. To this extent, we will use a fixed point argument. For this reason, we define $\forall \delta \in (0, +\infty)$:

$$\chi = \left\{W \in C^0([0, +\infty), \mathcal{M}(N, N; \mathbb{R})) \mid \|W(t)\| \leq \delta e^{-2\mu t} \quad \forall t \in \mathbb{R}^+\right\}$$

By the ODEs' theory, for every $\Lambda \in \mathcal{M}(N, N; \mathbb{R})$, for each $W \in \chi$ there exists a unique $Z \in C^0([0, +\infty), \mathcal{M}(N, N; \mathbb{R})) \cap C^1((0, +\infty), \mathcal{M}(N, N; \mathbb{R}))$ solution of the Cauchy Problem

$$\begin{cases} Z_t = -(ZL + L^*Z) - WBB^*W & \forall t \in (0, +\infty) \\ Z(0) = \Lambda \end{cases} \tag{2.48}$$

In the proof below, $\delta \in (0, +\infty)$ and $\Lambda \in \mathcal{M}(N, N; \mathbb{R})$ are degrees of freedom we will use next. We employ now Duhamel's formula to represent the solution:

$$Z(t) = e^{-tL^*}\Lambda e^{-tL} - \int_0^t e^{-(t-s)L^*} WBB^*W e^{-(t-s)L} ds \qquad \forall t \in [0, +\infty)$$

To get an estimate for $\|Z(t)\|$ we estimate the integral as follows for each $t \in [0, +\infty)$

$$\left\| \int_0^t e^{-(t-s)L^*} WBB^*W e^{-(t-s)L} ds \right\| \leq \int_0^t \left\| e^{-(t-s)L^*} WBB^*W e^{-(t-s)L} \right\| ds \leq$$

$$\leq \int_0^t Ce^{-(t-s)\mu}\delta^2 e^{-2\mu t} e^{-(t-s)\mu} ds =$$

Since

$$\int_0^t e^{-(t-s)2\mu} ds = \frac{1 - e^{-t2\mu}}{2\mu} \leq \frac{1}{2\mu}$$

then,

$$\left\| \int_0^t e^{-(t-s)L^*} WBB^*W e^{-(t-s)L} ds \right\| \leq C\delta^2 e^{-2\mu t}$$

This enables us to estimate $\|Z(t)\|$

$$\|Z(t)\| \leq \|\Lambda\| e^{-2t\mu} + C\delta^2 e^{-2\mu t} \leq \left(\|\Lambda\| + C\delta^2\right) e^{-2\mu t}.$$

Now, one can choose $\|\Lambda\| \leq \frac{\delta}{2}$ and δ sufficiently small so that $\frac{\delta}{2} + C\delta^2 \leq \delta$. Therefore, the map

$$T : \chi \longmapsto \chi$$
$$W \longrightarrow Z$$

is well posed. Moreover we can assign to χ the uniform topology. In this context T is a continuous mapping. Furthermore, $T(\chi)$ is relatively compact into $(C_b^0([0, +\infty), \mathcal{M}(N, N; \mathbb{R})), \| \|_\infty)$, as it can be verified using the equation and Ascoli Arzelà Theorem. This enables us to use the fixed point Theorem 2.2. It yields the existence of a fixed point $Z \in \chi$ for the map T. Now, we use the above result to prove the desired estimate for $\mathcal{E} - \widehat{E}$. By Lemma 2.5 (5.),

$$\mathcal{E}(t) \underset{t \to +\infty}{\longrightarrow} \widehat{E}.$$

So, there exists $t_0 \in [0, +\infty)$ such that $\|\mathcal{E}(t_0) - \widehat{E}\| \leq \frac{\delta}{2}$. Then, thanks to the Fixed Point Theorem, there exists a solution Z of the Cauchy Problem

$$\begin{cases} Z_t = -(ZL + L^*Z) - ZBB^*Z & \forall t \in (0, +\infty) \\ Z(t_0) = \mathcal{E}(t_0) - \widehat{E} \end{cases} \tag{2.49}$$

such that

$$\|Z(t)\| \leq \delta e^{-2\mu t} \qquad \forall t \in [0, +\infty)$$

Hence, by the Uniqueness Theorem for ODEs, one obtains:

$$\mathcal{E}(t + t_0) - \widehat{E} = Z(t) \qquad \forall t \in [0, +\infty)$$

which implies

$$\|\mathcal{E}(t) - \widehat{E}\| \leq \delta e^{-2\mu(t - t_0)} \qquad \forall t \in [t_0, +\infty).$$

Then, there exists $C \in (0, +\infty)$ such that

$$\|\mathcal{E}(t) - \widehat{E}\| \leq C e^{-2\mu t}$$

This completes the proof of the Corollary.

□

We are ready to describe more precisely the convergence of the optimal triple (x^T, p^T, u^T) for $(OCP)^T$ by the next Theorem.

Theorem 2.2 (Global Turnpike Property)**.** *Assuming (A, C) Kalman-observable and (A, B) Kalman-controllable, there exists $(C, \mu) \in (0, +\infty)^2$ such that the turnpike property holds, i.e. for every $T \in (0, +\infty)$, $\forall t \in [0, T]$:*

$$\|x^T(t) - \overline{x}\| + \|p^T(t) - \overline{p}\| + \|u^T(t) - \overline{u}\| \leq C \left[\|x_0 - \overline{x}\| e^{-\mu t} + \|\overline{p}\| e^{-\mu(T-t)}\right] \tag{2.50}$$

Proof. Thanks to Theorem 2.1, there exists a unique $\overline{p} \in \mathbb{R}^N$ such that

$$\begin{cases} A^* \overline{p} = C^*(C\overline{x} - z) \\ \overline{u} = -B^* \overline{p} \end{cases} \tag{2.51}$$

At this step, we define

$$h^T : [0, T] \longmapsto \mathbb{R}^N$$
$$t \longrightarrow p^T(t) - \overline{p} - \mathcal{E}(T - t)(x^T(t) - \overline{x})$$

Now, one can use the Pontryagin system and the equation (2.34) to find out the equation satisfied by h^T. In fact, $\forall t \in [0, T]$

$$h_t^T(t) = (p^T - \overline{p})_t(t) + \mathcal{E}_t(T - t)(x^T(t) - \overline{p}) - \mathcal{E}(T - t)(x^T - \overline{x})_t(t) =$$

$$= A^*(p^T(t) - \overline{p}) - C^*C(x^T(t) - \overline{x}) + C^*C(x^T(t) - \overline{x}) +$$
$$-\mathcal{E}(T - t)A(x^T(t) - \overline{x}) - A^*\mathcal{E}(T - t)(x^T(t) - \overline{x}) +$$

$$-\mathcal{E}(T-T)BB^*\mathcal{E}(T-t)(x^T(t)-\overline{x})+\mathcal{E}(T-t)A(x^T(t)-\overline{x})-\mathcal{E}(T-t)B(u^T(t)-\overline{u}) =$$
$$= A^*(p^T(t)-\overline{p})-A^*\mathcal{E}(T-t)(x^T(t)-\overline{x})-(\mathcal{E}(T-t)BB^*\mathcal{E}(T-t))(x^T(t)-\overline{x})+\mathcal{E}(T-t)BB^*(p^T(t)-\overline{p}).$$

Therefore, h^T is a solution of the Cauchy Problem

$$\begin{cases} h_t^T(t) = (A^* + \mathcal{E}(T-t)BB^*)h^T(t) & \forall t \in (0,T) \\ h^T(T) = -\overline{p}. \end{cases} \tag{2.52}$$

This differential system has a sublinear dynamic. Hence, it admits a global solution. As we did before, we define $L = A + BB^*\widehat{E}$. In this notation, h^T satisfies

$$\begin{cases} h_t^T(t) = L^*h^T(t) + (\mathcal{E}(T-t) - \widehat{E})BB^*h^T(t) & \forall t \in (0,T) \\ h^T(T) = -\overline{p}. \end{cases} \tag{2.53}$$

We use the Duhamel's formula to solve the Cauchy Problem, obtaining for every $t \in [0,T]$

$$h^T(t) = -e^{-(T-t)L^*}\overline{p} - \int_t^T e^{-(s-t)L^*}\left((\mathcal{E}(T-s) - \widehat{E})BB^*h^T(s)\right)ds$$

At this stage, we are going to estimate $\|h^T(t)\|$ using Corollary 2.1. For each $t \in [0,T]$

$$\|h^T(t)\| \le \left\|-e^{-(T-t)L^*}\overline{p}\right\| + \left\|\int_t^T e^{-(s-t)L^*}\left((\mathcal{E}(T-s) - \widehat{E})BB^*h^T(s)\right)ds\right\| \le$$
$$\le \|\overline{p}\|e^{-\mu(T-t)} + C\int_t^T e^{-\mu(s-t)}e^{-2\mu(T-s)}\|h^T(s)\|ds \le$$
$$\le Ce^{-\mu(T-t)}\left[\|\overline{p}\| + \int_t^T e^{-\mu(T-s)}\|h^T(s)\|ds\right]$$

Since for every $t \in [0,T]$

$$\int_t^T Ce^{-2\mu(T-\xi)}d\xi \le \frac{C}{2\mu},$$

the Gronwall's Lemma 1.4 yields for every $t \in [0,T]$

$$\|h^T(t)\| \le C\|\overline{p}\|e^{-\mu(T-t)} \qquad \forall t \in [0,T] \tag{2.54}$$

where the constant C is independent of $T \in (0,+\infty)$. Now, by the definition of h^T, $(x^T - \overline{x})$ satisfies for each $t \in (0,T)$

$$(x^T-\overline{x})_t(t)+(A+BB^*\widehat{E})(x^T(t)-\overline{x}) = BB^*(\widehat{E}-\mathcal{E}(T-t))(x^T(t)-\overline{x})-BB^*h^T(t)$$

Consequently, using the Duhamel's formula, for every $t \in [0,T]$,

$$x^T(t)-\overline{x} = e^{-tL}(x_0-\overline{x})+\int_0^t \left[e^{-(t-s)L}\left(BB^*(\widehat{E}-\mathcal{E}(T-s))(x^T(s)-\overline{x})-BB^*h^T(s)\right)\right] ds.$$

In order to estimate $x^T - \overline{x}$, we use Corollary 2.1 and (2.54). We get for every $t \in [0,T]$

$$\|x^T(t)-\overline{x}\| \leq C\|x_0-\overline{x}\|e^{-\mu t}+C\int_0^t e^{-\mu(t-s)}e^{-2\mu(T-s)}\|x^T(s)-\overline{x}\|ds+C\|\overline{p}\|\int_0^t e^{-\mu(t-s)}e^{-\mu(T-s)}ds \leq$$

$$\leq C\left(\|x_0-\overline{x}\|e^{-\mu t}+\|\overline{p}\|e^{-\mu(T-t)}\right)+Ce^{-\mu(T-t)}\int_0^t e^{-2\mu(t-s)}e^{-\mu(T-s)}\|x^T(s)-\overline{x}\|ds.$$

Now, one can compute

$$\int_0^t Ce^{-2\mu(T-s)}ds \leq \frac{C}{2\mu}$$

and

$$e^{-\mu t}\int_0^t e^{-\int_0^s Ce^{-2\mu(T-\xi)}d\xi}C\left(\|x_0-\overline{x}\|e^{-\mu s}+\|\overline{p}\|e^{-\mu(T-s)}\right)e^{-\mu(T-2s)}ds \leq$$

$$\leq C\left(\|x_0-\overline{x}\|e^{-\mu t}+\|\overline{p}\|e^{-\mu(T-t)}\right)$$

Therefore, one can use the Gronwall's Lemma 1.3 to obtain

$$\|x^T(t)-\overline{x}\| \leq C\left[\|x_0-\overline{x}\|e^{-\mu t}+\|\overline{p}\|e^{-\mu(T-t)}\right] \qquad \forall t \in [0,T].$$

By the definition of h^T,

$$p^T(t)-\overline{p} = \mathcal{E}(T-t)(x^T(t)-\overline{x})+h^T(t) \qquad \forall t \in [0,T]$$

This implies for every $t \in [0,T]$

$$\|p^T(t)-\overline{p}\| \leq R_0\|x^T(t)-\overline{x}\|+\|h^T(t)\| \leq C\left[\|x_0-\overline{x}\|e^{-\mu t}+\|\overline{p}\|e^{-\mu(T-t)}\right]$$

Finally, this yields

$$\|u^T(t)-\overline{u}\| = \|B^*p^T(t)-B^*\overline{p}\| \leq C\left[\|x_0-\overline{x}\|e^{-\mu t}+\|\overline{p}\|e^{-\mu(T-t)}\right] \qquad t \in [0,T].$$

The proof is finished.

□

Chapter 3

NonLinear Case

In the present chapter we discuss the fulfillment of the Turnpike Property for some NonLinear Problems. Our analysis will be divided into 4 parts:

1. Global Turnpike Property for Linear Quadratic Case with the approach of [20];
2. Well Posedeness and Convergence of Averages for NonLinear Convex Case;
3. presentation of a Local Turnpike Property for NonLinear Convex Case;
4. proof of Global Turnpike Property for NonLinear Convex Case.

3.1 Global Turnpike Property for the Linear Quadratic Case

This first part follows the approach of [20].

Shooting Method

First of all, we briefly introduce the so called Shooting Method, which will play a key role in this chapter. We take into account $(p,q) \in \mathbb{N}^2$, an open set $\Omega \subset \mathbb{R}^p$, a regular dynamics

$$F : \Omega \longmapsto \mathbb{R}^p \in C^1(\Omega; \mathbb{R}^p),$$

$$R : \Omega^2 \longmapsto \mathbb{R}^q \in C^1(\Omega^2, \mathbb{R}^q).$$

We want to investigate if there exists and it is unique the solution $Z \in C^1([0,T]; \mathbb{R}^p)$ of the following Problem:

$$\begin{cases} \frac{d}{dt}Z(t) = F(Z(t)) & \forall t \in (0,T) \\ R(Z(0), Z(T)) = 0. \end{cases} \tag{3.1}$$

If $q = p$, and there exists $Z_0 \in \Omega$ such that

$$R : \Omega^2 \longmapsto \mathbb{R}^q$$

$$(x, y) \longrightarrow x - Z_0$$

by Picard's Theorem, it holds at least local existence and uniqueness for (3.1). But, whenever R has a different shape, the problem of existence and uniqueness for system (3.1) is a hard one. We present here a powerful tool, even from the numerical point of view, to address these sort of problems. Given a dynamics $F : \Omega \longmapsto \mathbb{R}^p$ and a time horizon $T \in (0, +\infty)$, assume that there exists an open set $\Omega_0 \subset \Omega \subset \mathbb{R}^p$ such that for all $Z_0 \in \Omega_0$ there exists a unique global solution $Z(\cdot; Z_0) \in C^1([0, T], \mathbb{R}^p)$ to the Cauchy Problem:

$$\begin{cases} \frac{d}{dt} Z(t) = F(Z(t)) & \forall t \in (0, T) \\ Z(0) = Z_0. \end{cases} \tag{3.2}$$

The idea of the Shooting Method is to look for a set of initial data such that the solution of the Cauchy Problem (3.2) satisfies the original terminal conditions given by R. Namely, we want to determine the set

$$GS(R) = \{Z_0 \in \Omega_0 \mid R(Z_0; Z(T; Z_0)) = 0\}.$$

If it is not empty, $\forall Z_0 \in GS(R)$ $Z(\cdot; Z_0) \in C^1([0, T]; \mathbb{R}^p)$ is a solution of the differential system (3.1). The Shooting Method is said to be well posed if and only if

$$GS(R)$$

is a singleton. We define the nonlinear function

$$G : \Omega_0 \longmapsto \mathbb{R}^q$$

$$Z_0 \longrightarrow R(Z_0, Z(T; Z_0)).$$

Hence, in order to determine $GS(R)$ we have to solve the nonlinear equation

$$G(Z_0) = 0 \qquad Z_0 \in \Omega_0.$$

To accomplish this task there are many tools available. For example, it is possible to use numerical methods to solve it. Among them the bisection method and the Newton method are quite popular. It is important to underline that the Newton method requires a good initialization. If the reader is interested in Shooting Method, we refer to [19] for an overview on the topic and some links to the Optimal Control Theory.

The idea of the Shooting Method underlies on the proofs for this first part. Moreover, we will use some algebraic techniques from the Algebraic Riccati's Theory. In order to introduce the reader to this approach, we study again the Linear Quadratic Finite Dimensional Case. We will prove a Global Tunrpike result for the Linear Quadratic Finite Dimensional Case. After that, we will show how our techniques can be generalised in order to obtain a Local Turnpike Property for the NonLinear Case.

There will be some different hypotheses with respect to the previous chapter, motivated by the different techniques used. First of all, we consider 2 natural numbers $(N, M) \in \mathbb{N}^2$ and one strictly positive real number $T \in (0, +\infty)$. Moreover, we take into account a quadruple of matrices $(A, B, C, W) \in M(N, N; \mathbb{R}) \times M(N, M; \mathbb{R}) \times M(N, N; \mathbb{R}) \times M(M, M; \mathbb{R})$. We suppose (A, B) Kalman-controllable, $C \in Sym(N; \mathbb{R})$ positive definite and $W \in Sym(M; \mathbb{R})$ positive definite. Furthermore, we consider a state-target $z \in \mathbb{R}^M$, a control-target $v \in \mathbb{R}^M$ and an initial data $x_0 \in \mathbb{R}^N$. By Theorem 1.3, for every control function $u \in L^2((0, T); \mathbb{R}^M)$ there exists a unique state solution of:

$$\begin{cases} x_t + Ax = Bu & \text{a.e. } t \in (0, T) \\ x(0) = x_0 \end{cases} \tag{3.3}$$

Finally, $(OCP)^T$ consists in minimizing the functional:

$$J^T : L^2((0, T); \mathbb{R}^M) \longmapsto \mathbb{R} \tag{3.4}$$

$$u \longrightarrow \frac{1}{2} \int_0^T \left[\|Cx - z\|^2 + \|Wu - v\|^2 \right] dt$$

On the other hand, we define the following vector subspace of $\mathbb{R}^N \times \mathbb{R}^M$

Definition 3.1.

$$M = \left\{ (x, u) \in \mathbb{R}^N \times \mathbb{R}^M \mid Ax = Bu \right\}$$

At this stage, let us define the stationary functional:

$$J^0 : M \longmapsto \mathbb{R}$$
$$(x, u) \longrightarrow \frac{1}{2} \left[\|Cx - z\|^2 + \|Wu - v\|^2 \right]$$

We will name $(OCP)^s$ the above functional minimization problem.

We highlight here the main differences between the setting of the 2^{nd} chapter and the setting of the present chapter.

1. In this chapter we consider an arbitrary control target $w \in \mathbb{R}^M$, whereas in the previous chapter it was fixed $w = 0$.

2. Here we have a generic weight matrix $W \in Sym(M; \mathbb{R})$ positive definite for the control term $\|Wu - v\|^2$ in the functionals, while in the 1$^{\text{st}}$ chapter it had to be the identity I_M.

3. In the first chapter we asked only (A, C) observable, whereas here we require C (strictly) positive definite. It is possible to check, by Kalman condition (2.4), that, whenever $C \in GL(N; \mathbb{R})$, the pair (A, C) is actually observable.

One can observe that the 2. is not a significative generalization. In fact, whenever $W \neq I_M$ it is possible to reduce to $W = I_M$, defining $\tilde{B} \overset{\text{def}}{=} BW^{-1}$.

As in the 2$^{\text{nd}}$ chapter, there exists a unique minimizer both for $(OCP)^T$ and $(OCP)^s$.

Furthermore, $\forall T \in (0, +\infty)$ the optimal triple $(x^T, p^T, u^T) \in AC([0,T]; \mathbb{R}^N) \times AC([0,T]; \mathbb{R}^N) \times L^2((0,T); \mathbb{R}^M)$ solves the state-adjoint state Pontryagin system:

$$\begin{cases} x_t^T + Ax^T = Bu^T & \text{a.e. } t \in (0,T) \\ -p_t^T + A^*p^T = C(Cx^T - z) & \text{a.e. } t \in (0,T) \\ u^T = W^{-1}v - W^{-2}B^*p^T & \text{a.e. } t \in (0,T) \\ x^T(0) = x_0 \\ p^T(T) = 0 \end{cases} \tag{3.5}$$

As in the previous chapter, using the above system of differential equations, we can show that $(x^T, p^T, u^T) \in C^\infty([0,T], \mathbb{R}^N) \times C^\infty([0,T], \mathbb{R}^N) \times C^\infty([0,T], \mathbb{R}^M)$.

Moreover, it is possible to prove, employing the techniques used in chapter 1 (namely Lemma 2.3 and Theorem 2.1) , that there exists a unique optimal triple $(\overline{x}, \overline{p}, \overline{u})$ for the stationary problem $(OCP)^s$, which is the unique solution of the system:

$$\begin{cases} A\overline{x} = B\overline{u} \\ A^*\overline{p} = C(C\overline{x} - z) \\ \overline{u} = W^{-1}v - W^{-2}B^*\overline{p}. \end{cases} \tag{3.6}$$

At this stage we will present a Lemma which is a key tool to show both the Global Turnpike Property for the Linear Quadratic Case and the Local

Turnpike Property for the NonLinear Convex Case. It relies mostly on the Algebraic Riccati Equation (ARE). We define the matrix:

$$M \stackrel{\text{def}}{=} \begin{pmatrix} -A & -BW^{-2}B^* \\ -C^2 & A^* \end{pmatrix}$$

Lemma 3.1. *Let $(A, B, W, C) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R}) \times Sym(N; \mathbb{R}) \times Sym(N; \mathbb{R})$ be such that W and C are positive definite. Then, the matrix $M \in sp(N, \mathbb{R})$. Moreover, if (A, B) is Kalman-Controllable, then*

1. *there exists a matrix $P \in GL(2N; \mathbb{R})$ such that:*

$$P^{-1}MP = \begin{pmatrix} -A - BW^{-2}B^*\widehat{E}_+ & 0 \\ 0 & -A - BW^{-2}B^*\widehat{E}_- \end{pmatrix}$$

where $Re(\sigma(-A - BW^{-2}B^*\widehat{E}_+)) \subset (-\infty, 0)$ and $Re(\sigma(-A - BW^{-2}B^*\widehat{E}_-)) \subset (0, +\infty)$;

2.
$$Re(\sigma(M)) \subset \mathbb{R} \setminus \{0\}\,;$$

3.
$$-\sigma(M) = \sigma(M).$$

In the next proof we follow the approach of [2] and [21].

Proof. First of all, thanks to Proposition 6 page 6 of [4], the matrix $M \in sp(N; \mathbb{R})$. If (A, B) is Kalman-Controllable, we apply Lemma 2.5 to (A, BW^{-1}, C). Therefore, there exists a unique symmetric positive definite matrix $\widehat{E}_+ \in Sym(N; \mathbb{R})$ satisfying the Algebraic Riccati Equation:

$$\widehat{E}A + A\widehat{E} + \widehat{E}BW^{-2}B^*\widehat{E} - C^2 = 0 \qquad \text{(ARE)} \tag{3.7}$$

Furthermore, there exists a unique symmetric negative definite matrix $\widehat{E}_- \in Sym(N; \mathbb{R})$ solution of the Algebraic Riccati Equation (3.7). At this point, one defines

$$P \stackrel{\text{def}}{=} \begin{pmatrix} I_N & I_N \\ \widehat{E}_+ & \widehat{E}_- \end{pmatrix}$$

Its inverse reads as follows:

$$P^{-1} = \begin{pmatrix} -(\widehat{E}_+ - \widehat{E}_-)^{-1}\widehat{E}_- & (\widehat{E}_+ - \widehat{E}_-)^{-1} \\ (\widehat{E}_+ - \widehat{E}_-)^{-1}\widehat{E}_+ & -(\widehat{E}_+ - \widehat{E}_-)^{-1} \end{pmatrix}$$

Then,

$$MP = \begin{pmatrix} -A - BW^{-2}B^*\widehat{E}_+ & -A - BW^{-2}B^*\widehat{E}_- \\ -C^2 + A^*\widehat{E}_+ & -C^2 + A^*\widehat{E}_- \end{pmatrix}$$

Hence, $P^{-1}MP$ is such that:

$(P^{-1}MP)_{1,1} = -(\widehat{E}_+ - \widehat{E}_-)^{-1}\widehat{E}_-(-A - BW^{-2}B^*\widehat{E}_+) + (\widehat{E}_+ - \widehat{E}_-)^{-1}(-C^2 + A^*\widehat{E}_+);$

$(P^{-1}MP)_{1,2} = -(\widehat{E}_+ - \widehat{E}_-)^{-1}\widehat{E}_-(-A - BW^{-2}B^*\widehat{E}_-) + (\widehat{E}_+ - \widehat{E}_-)^{-1}(-C^2 + A^*\widehat{E}_-);$

$(P^{-1}MP)_{2,1} = (\widehat{E}_+ - \widehat{E}_-)^{-1}\widehat{E}_+(-A - BW^{-2}B^*\widehat{E}_+) - (\widehat{E}_+ - \widehat{E}_-)^{-1}(-C^2 + A^*\widehat{E}_+);$

$(P^{-1}MP)_{2,2} = (\widehat{E}_+ - \widehat{E}_-)^{-1}\widehat{E}_+(-A - BW^{-2}B^*\widehat{E}_-) - (\widehat{E}_+ - \widehat{E}_-)^{-1}(-C^2 + A^*\widehat{E}_-).$

Using the fulfillment of the Algebraic Riccati Equation, one gets:

$$P^{-1}MP = \begin{pmatrix} -A - BW^{-2}B^*\widehat{E}_+ & 0 \\ 0 & -A - BW^{-2}B^*\widehat{E}_- \end{pmatrix}$$

By Lemma 2.5 (5), the differential linear system

$$\begin{cases} x_t + (A + BW^{-2}B^*\widehat{E}_+)x = 0 & \forall t \in (0, T) \\ x(0) = x_0 \end{cases} \tag{3.8}$$

is exponentially globally asymptotically stable. Employing the characterisation of exponentially globally asymptotically stable linear systems (e.g. Theorem 8 page 37 of [4]), one obtains that

$$Re(\sigma(-A - BW^{-2}B^*\widehat{E}_+)) \subset (-\infty, 0).$$

Furthermore, from

$$(\widehat{E}_+ - \widehat{E}_-)(-A - BW^{-2}B^*\widehat{E}_+) + (-A - BW^{-2}B^*\widehat{E}_-)^*(\widehat{E}_+ - \widehat{E}_-) = 0.$$

and the invertibility of the difference $\widehat{E}_+ - \widehat{E}_-$, one deduces that:

$$\sigma(-A - BW^{-2}B^*\widehat{E}_-) = -\sigma(-A - BW^{-2}B^*\widehat{E}_+) \subset (0, +\infty)$$

Then,

$$Re(\sigma(M)) \subset \mathbb{R} \setminus \{0\}$$

and

$$-\sigma(M) = \sigma(M)$$

as desired. □

We are going to prove the Global Turnpike Property for the Finite Dimensional Linear Quadratic Case, which, as announced, will be a guideline for the proof of the Local Turnpike Property for the NonLinear Convex Case.

Theorem 3.1. *Let* $(A, B, C, W) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R}) \times Sym(N; \mathbb{R}) \times Sym(M; \mathbb{R})$ *be such that* (A, B) *is Kalman-contollable,* C *and* W *are positive definite. Furthermore, take into account arbitrary targets* $(z, v) \in \mathbb{R}^N \times \mathbb{R}^M$ *and a arbitrary initial data* $x_0 \in \mathbb{R}^N$*. It follows that there exist* 2 *constants* $(C, \mu) \in (0, +\infty)^2$ *such that* $\forall T \in (0, +\infty)$*:*

$$\|x^T(t) - \overline{x}\| + \|p^T(t) - \overline{p}\| + \|u^T(t) - \overline{u}\| \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right] \qquad \forall t \in [0, T]. \tag{3.9}$$

Proof. First of all, $\forall T \in (0, +\infty)$ we define the perturbation functions as follows:

$$\delta x^T : [0, T] \longmapsto \mathbb{R}^N$$
$$t \longrightarrow x^T(t) - \overline{x},$$
$$\delta p^T : [0, T] \longmapsto \mathbb{R}^N$$
$$t \longrightarrow p^T(t) - \overline{p}$$

and

$$\delta u^T : [0, T] \longmapsto \mathbb{R}^M$$
$$t \longrightarrow u^T(t) - \overline{u}.$$

We remind that the nonstationary optimal pair (x^T, p^T) is solution of the system:

$$\begin{cases} x_t^T(t) + Ax^T(t) = BW^{-1}v - BW^{-2}B^*p^T(t) & \forall t \in (0, T) \\ -p_t^T(t) + A^*p^T(t) = C(Cx^T(t) - z) & \forall t \in (0, T) \end{cases} \tag{3.10}$$

On the other hand, the stationary optimal triple $(\overline{x}, \overline{p}, \overline{u})$ is the unique solution of:

$$\begin{cases} A\overline{x} = B\overline{u} \\ A^*\overline{p} = C(C\overline{x} - z) \\ \overline{u} = W^{-1}v - W^{-2}B^*\overline{p} \end{cases} \tag{3.11}$$

This yields that the optimal pair state-adjoint state $(\overline{x}, \overline{p})$ for $(OCP)^s$ is the unique solution of the algebraic linear system:

$$\begin{cases} A\overline{x} = BW^{-1}v - BW^{-2}B^*\overline{p} \\ A^*\overline{p} = C(C\overline{x} - z) \end{cases} \tag{3.12}$$

By subtracting the above systems, we get that $\forall T \in (0,+\infty)$ the perturbation pair $(\delta x^T, \delta p^T)$ is solution of the linear differential system:

$$\begin{cases} \delta x_t^T(t) + A\delta x^T(t) = -BW^{-2}B^*\delta p^T(t) & \forall t \in (0,T) \\ -\delta p_t^T(t) + A^*\delta p^T(t) = C^2 \delta x^T(t) & \forall t \in (0,T) \\ \delta x^T(0) = x_0 - \overline{x} \\ \delta p^T(T) = -\overline{p} \end{cases} \tag{3.13}$$

At this stage, let us define the function

$$Z : [0,T] \longmapsto \mathbb{R}^{2N}$$

$$t \longrightarrow (\delta x^T(t), \delta p^T(t))$$

and the terminal conditions

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N}$$

$$(x_1, y_1, x_2, y_2) \longrightarrow (x_1 - x_0 + \overline{x}, y_2 + \overline{p})$$

Hence, Z is a solution of the differential linear system:

$$\begin{cases} Z_t(t) = MZ(t) & \forall t \in (0,T) \\ R(Z(0), Z(T)) = 0 \end{cases} \tag{3.14}$$

In order to get the desired estimate for the perturbation functions, we will solve (3.14) by the shooting method. In the present case, thanks to the linearity of the dynamics, $\Omega = \Omega_0 = \mathbb{R}^{2N}$. Therefore, our aim is to show that the set

$$GS(R) = \left\{ Z_0 \in \mathbb{R}^{2N} \mid R(Z_0; Z(T; Z_0)) = 0 \right\}$$

is actually a singleton. Preliminarily, we observe that, for every $Z_0 = (x_1, y_1) \in GS(R)$, it follows that:

$$R((x_1, y_1), Z(T; (x_1, y_1))) = 0$$

which yields

$$x_1 = x_0 - \overline{x}.$$

Then,

$$GS(R) = \left\{ Z_0 \in \mathbb{R}^{2N} \mid R(Z_0; Z(T; Z_0)) = 0 \right\} \subset \{x_0 - \overline{x}\} \times \mathbb{R}^N.$$

Hence the only degree of freedom in the Shooting Method is $\delta p^T(0)$. At this step it turns out to be useful Lemma 3.1. At this stage, $\forall Z_0 \in \mathbb{R}^{2N}$, we define the function:

$$Z_1 : [0,T] \longmapsto \mathbb{R}^{2N}$$

$$t \longmapsto P^{-1}Z(t)$$

Therefore, $\forall t \in (0,T)$,

$$\frac{d}{dt}Z_1(t) = P^{-1}\frac{d}{dt}Z(t) = P^{-1}MZ(t) = P^{-1}MPP^{-1}Z(t) = P^{-1}MPZ_1(t)$$

which means

$$\frac{d}{dt}Z_1(t) = \begin{pmatrix} -A - BW^{-2}B^*\widehat{E}_+ & 0 \\ 0 & -A - BW^{-2}B^*\widehat{E}_- \end{pmatrix} Z_1(t) \tag{3.15}$$

Now, we define the projection operators:

$$p_1 : \mathbb{R}^{2N} \longmapsto \mathbb{R}^N$$

$$(x,y) \longrightarrow x$$

$$p_2 : \mathbb{R}^{2N} \longmapsto \mathbb{R}^N$$

$$(x,y) \longrightarrow y.$$

Employing these definitions, it is worth defining:

$$g : [0,T] \longmapsto \mathbb{R}^N$$

$$t \longrightarrow p_1(Z_1(t)) = p_1(P^{-1}Z(t))$$

and

$$h : [0,T] \longmapsto \mathbb{R}^N$$

$$t \longrightarrow p_2(Z_1(t)) = p_2(P^{-1}Z(t)).$$

Therefore the system (3.15) is equivalent to

$$\begin{cases} \frac{d}{dt}g(t) = (-A - BW^{-2}B^*\widehat{E}_+)g(t) & \forall t \in (0,T) \\ \frac{d}{dt}h(t) = (-A - BW^{-2}B^*\widehat{E}_-)h(t) & \forall t \in (0,T). \end{cases} \tag{3.16}$$

Therefore, we uncoupled the system. Furthermore, by Lemma 3.1, $Re(\sigma(-A-BW^{-2}B^*\widehat{E}_+)) \subset (-\infty,0)$ and $Re(\sigma(-A-BW^{-2}B^*\widehat{E}_-)) \subset (0,+\infty)$. Then, 0 is an hyperbolic equilibrium point for this differential linear system. Since $P^{-1}MP$ has both eigenvalues with positive real part and eigenvalues with negative real part, 0 is actually a saddle equilibrium point for system (3.16). Hence, the pair $(\tilde{g},\tilde{h}) = (g, h(T-\cdot))$ satisfies

$$\begin{cases} \frac{d}{dt}\tilde{g}(t) = (-A - BW^{-2}B^*\widehat{E}_+)\tilde{g}(t) & \forall t \in (0,T) \\ \frac{d}{dt}\tilde{h}(t) = (A + BW^{-2}B^*\widehat{E}_-)\tilde{h}(t) & \forall t \in (0,T). \end{cases} \tag{3.17}$$

By Theorem 8 page 37 of [4] together with Lemma (2.4), there exist 2 constants $(C,\mu) \in (0,+\infty)^2$ such that:

$$\|\tilde{g}(t)\| \le Ce^{-\mu t}\|\tilde{g}(0)\| \qquad \forall t \in [0,T]$$

$$\|\tilde{h}(t)\| \le Ce^{-\mu t}\|\tilde{h}(0)\| \qquad \forall t \in [0,T]$$

which in turn implies

$$\|g(t)\| = \|\tilde{g}(t)\| \le Ce^{-\mu t}\|g(0)\| \qquad \forall t \in [0,T] \tag{3.18}$$

$$\|h(t)\| = \|\tilde{h}(T-t)\| \le Ce^{-\mu(T-t)}\|\tilde{h}(0)\| = Ce^{-\mu(T-t)}\|h(T)\| \qquad \forall t \in [0,T] \tag{3.19}$$

At this stage, let us define, $\forall g_0 \in \mathbb{R}^N$, $g(\cdot;g_0)$ the unique solution of the Linear Cauchy Problem

$$\begin{cases} \frac{d}{dt}g(t) = (-A - BW^{-2}B^*\widehat{E}_+)g(t) \qquad \forall t \in (0,T) \\ g(0) = g_0 \end{cases} \tag{3.20}$$

and, $\forall h_T \in \mathbb{R}^{2N}$, $h(\cdot;h_T)$ the unique solution of the Linear Cauchy Problem

$$\begin{cases} \frac{d}{dt}h(t) = (-A - BW^{-2}B^*\widehat{E}_-)h(t) \qquad \forall t \in (0,T) \\ h(T) = h_T. \end{cases} \tag{3.21}$$

Finally, we name $Z_1(\cdot;(g_0,h_T)) \stackrel{\text{def}}{=} (g(\cdot;g_0),h(\cdot;h_T))$. At this point, we define

$$\tilde{R} : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N}$$

$$(z_1,z_2) \longrightarrow R(Pz_1,Pz_2).$$

Therefore, the Shooting Method for the system

$$\begin{cases} Z_t(t) = MZ(t) \qquad \forall t \in (0,T) \\ R(Z(0),Z(T)) = 0 \end{cases} \tag{3.22}$$

is well posed if and only if the corresponding Shooting Method for

$$\begin{cases} \frac{d}{dt}g(t) = (-A - BW^{-2}B^*\widehat{E}_+)g(t) \qquad \forall t \in (0,T) \\ \frac{d}{dt}h(t) = (-A - BW^{-2}B^*\widehat{E}_-)h(t) \qquad \forall t \in (0,T) \\ \tilde{R}((g(0;g_0),h(0;h_T)),(g(T;g_0),h(T;h_T))) = 0 \end{cases} \tag{3.23}$$

is well posed, namely

$$\#GS(R) = \#\left\{Z_0 \in \mathbb{R}^{2N} \mid R(Z_0;Z(T;Z_0)) = 0\right\} =$$

$$= \# \left\{ (g_0, h_T) \in \mathbb{R}^{2N} \mid \tilde{R}((g(0; g_0), h(0; h_T)), (g(T; g_0), h(T; h_T))) = 0 \right\}$$

Furthermore, we remember that

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N}$$

$$\begin{pmatrix} x_i \\ y_i \\ x_f \\ y_f \end{pmatrix} \longrightarrow \begin{pmatrix} x_i - x_0 + \overline{x} \\ y_f + \overline{p} \end{pmatrix}$$

Then, by definition of P,

$$\tilde{R} : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N}$$

$$\begin{pmatrix} v_i \\ w_i \\ v_f \\ w_f \end{pmatrix} \longrightarrow \begin{pmatrix} v_i + w_i - x_0 + \overline{x} \\ \widehat{E}_+ v_f + \widehat{E}_- w_f + \overline{p} \end{pmatrix}$$

At this stage, let us define the map

$$f : \mathbb{R}^{2N} \longmapsto \mathbb{R}^{2N}$$

$$\begin{pmatrix} g_0 \\ h_T \end{pmatrix} \longrightarrow \begin{pmatrix} x_0 - \overline{x} - h(0; h_T) \\ \widehat{E}_-^{-1}(-\overline{p} - \widehat{E}_+ g(T; g_0)) \end{pmatrix}$$

We observe that (g_0, h_T) is a fixed point for f if and only if:

$$\tilde{R}((g_0, h(0; h_T)), (g(T; g_0), h_T)) = 0.$$

For every $((g_0^1, h_T^1), (g_0^2, h_T^2)) \in \mathbb{R}^{4N}$, we estimate $\|f((g_0^2, h_T^2)) - f((g_0^1, h_T^1))\|$, i.e.:

$$\|p_1(f((g_0^2, h_T^2)) - f((g_0^1, h_T^1)))\| = \|x_0 - \overline{x} - h(0; h_T^2) - x_0 + \overline{x} + h(0; h_T^1)\| \leq$$

Thanks to (3.19),

$$\leq C\|h_T^2 - h_T^1\| e^{-\mu T}$$

and

$$\|p_2(f((g_0^2, h_T^2)) - f((g_0^1, h_T^1)))\| = \|\widehat{E}_-^{-1}(-\overline{p} - \widehat{E}_+ g(T; g_0^2)) - \widehat{E}_-^{-1}(-\overline{p} - \widehat{E}_+ g(T; g_0^1))\| \leq$$

by (3.18),

$$= \|\widehat{E}_-^{-1} \widehat{E}_+ (g(T; g_0^2) - g(T; g_0^1))\| \leq C\|\widehat{E}_-^{-1}\| \|\widehat{E}_+\| \|g_0^2 - g_0^1\| e^{-\mu T}$$

Therefore, $\forall((g_0^1,h_T^1),(g_0^2,h_T^2))\in\mathbb{R}^{4N}$,

$$\|f(g_0^2,h_T^2)-f(g_0^1,h_T^1)\|\leq C(\|\widehat{E}_-^{-1}\|\|\widehat{E}_+\|+1)e^{-\mu T}(\|g_0^2-g_0^1\|+\|h_T^1-h_T^2\|).$$

Hence, if $T\in(0,+\infty)$ is sufficiently large, there exists $\alpha\in[0,1)$ such that $\forall((g_0^1,h_T^1),(g_0^2,h_T^2))\in\mathbb{R}^{4N}$:

$$\|f(g_0^2,h_T^2)-f(g_0^1,h_T^1)\|\leq\alpha\|g_0^2-g_0^1\|$$

At this point, by Contractions-Fixed Point Theorem, there exists a unique $(g_0,h_T)\in\mathbb{R}^{2N}$ fixed point for f. By definition of f this means that:

$$\left\{(g_0,h_T)\in\mathbb{R}^{2N}\mid\tilde{R}((g(0;g_0),h(0;h_T)),(g(T;g_0),h(T;h_T)))=0\right\}$$

is a singleton. Then, the Shooting Method for the system

$$\begin{cases}Z_t(t)=MZ(t) & \forall t\in(0,T)\\ R(Z(0),Z(T))=0\end{cases}\tag{3.24}$$

is well posed. This in turn implies the existence and uniqueness of the solution for

$$\begin{cases}Z_t(t)=MZ(t) & \forall t\in(0,T)\\ R(Z(0),Z(T))=0\end{cases}\tag{3.25}$$

In order to complete the proof, we perform some inequalities employing (3.18) and (3.19). Moreover, we are going to use that $(g_0,h_T)\in\mathbb{R}^{2N}$ is a fixed point for f, namely:

$$\begin{cases}g_0=x_0-\overline{x}-h(0;h_T)\\ h_T=\widehat{E}_-^{-1}(-\overline{p}-\widehat{E}_+g(T;g_0)).\end{cases}\tag{3.26}$$

Firstly,

$$\|g_0-(x_0-\overline{x})\|=\|h(0;h_T)\|\leq C\|h_T\|e^{-\mu T}.\tag{3.27}$$

Secondly,

$$\|h_T+\widehat{E}_-^{-1}\overline{p}\|\leq\|\widehat{E}_-^{-1}\widehat{E}_+g(T;g_0))\|\leq Ce^{-\mu T}\|\widehat{E}_-^{-1}\widehat{E}_+\|\|g_0\|.\tag{3.28}$$

Furthermore, using (3.28) into (3.27), we obtain:

$$\|g_0-(x_0-\overline{x})\|\leq C\left[\|\widehat{E}_-^{-1}\overline{p}\|e^{-\mu T}+\|\widehat{E}_-^{-1}\widehat{E}_+\|\|g_0\|e^{-2\mu T}\right]\tag{3.29}$$

Therefore,

$$\|g_0\|\leq C\left[\|\widehat{E}_-^{-1}\overline{p}\|e^{-\mu T}+\|\widehat{E}_-^{-1}\widehat{E}_+\|\|g_0\|e^{-2\mu T}\right]+\|x_0-\overline{x}\|$$

Taking T sufficiently large, $C\|\widehat{E}_-^{-1}\widehat{E}_+\|e^{-2\mu T} \leq \frac{1}{2}$. Hence,

$$\|g_0\| \leq C\|\widehat{E}_-^{-1}\overline{p}\| + 2\|x_0 - \overline{x}\|. \tag{3.30}$$

Then, $\|g(0)\|$ is bounded uniformly in T. Similarly, employing (3.27) into (3.28), we get:

$$\|h_T + \widehat{E}_-^{-1}\overline{p}\| \leq C\|\widehat{E}_-^{-1}\widehat{E}_+\| \left[\|x_0 - \overline{x}\|e^{-\mu T} + \|h_T\|e^{-2\mu T}\right]. \tag{3.31}$$

Therefore,

$$\|h_T\| \leq C\|\widehat{E}_-^{-1}\widehat{E}_+\| \left[\|x_0 - \overline{x}\|e^{-\mu T} + \|h_T\|e^{-2\mu T}\right] + \|\widehat{E}_-^{-1}\overline{p}\|. \tag{3.32}$$

Then, choosing T big enough,

$$\|h_T\| \leq C\|\widehat{E}_-^{-1}\widehat{E}_+\|\|x_0 - \overline{x}\| + 2\|\widehat{E}_-^{-1}\overline{p}\|. \tag{3.33}$$

Hence, we have already deduced a bound of $\|h_T\|$ uniform on the time horizon T. Let us use (3.18) together with (3.30). We obtain $\forall t \in [0, T]$:

$$\|g(t; g_0)\| \leq C\left\{C\|\widehat{E}_-^{-1}\overline{p}\| + 2\|x_0 - \overline{x}\|\right\}e^{-\mu t} \leq Ce^{-\mu t}. \tag{3.34}$$

Moreover, employing (3.19) and (3.33),

$$\|h(t; h_T)\| \leq C\left\{C\|\widehat{E}_-^{-1}\widehat{E}_+\|\|x_0 - \overline{x}\| + 2\|\widehat{E}_-^{-1}\overline{p}\|\right\}e^{-\mu(T-t)} \leq Ce^{-\mu(T-t)}. \tag{3.35}$$

Since $(g(\cdot; g_0), h(\cdot; h_T)) = (P^{-1}\delta x^T, P^{-1}\delta p^T)$ is the unique solution of

$$\begin{cases} \frac{d}{dt}Z_1(t) = P^{-1}MPZ_1(t) & \forall t \in (0, T) \\ \tilde{R}(Z_1(0), Z_1(T)) = 0, \end{cases} \tag{3.36}$$

we can deduce the following inequalities for $\delta x^T = g(\cdot; g_0) + h(\cdot; h_T)$ and $\delta p^T = \widehat{E}_+ g(\cdot; g_0) + \widehat{E}_- h(\cdot; h_0)$

$$\|\delta x^T(t)\| = \|g(t; g_0) + h(t; h_T)\| \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right]. \tag{3.37}$$

and

$$\|\delta p^T(t)\| = \|\widehat{E}_+ g(\cdot; g_0) + \widehat{E}_- h(\cdot; h_0)\| \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right]. \tag{3.38}$$

Finally, recalling that $\delta u^T(\cdot; x_0) = u^T(\cdot; x_0) - \overline{u} = -W^{-2}B^*\delta p^T(\cdot; x_0)$, we obtain:

$$\|\delta u^T(t)\| = \| - W^{-2}B^*\delta p^T(\cdot; x_0)\| \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right]. \tag{3.39}$$

This concludes the proof.

□

3.2 Well Posedeness and Convergence of Averages for NonLinear Convex Case

We are going to introduce the setting for the NonLinear Convex Case.

First of all, for an arbitrary $q \in \mathbb{N}$, let us define

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^q \in C^2(\mathbb{R}^{4N}, \mathbb{R}^q)$$

R will represent the initial-terminal condition for the pair (state,adjoint state)$= (x^T, p^T)$. $\forall\ (x_0, x_1) \in \mathbb{R}^N \times \mathbb{R}^N$, R can have 2 different shapes:

<u>$(TC1)$</u>

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N} \in C^2(\mathbb{R}^{4N}, \mathbb{R}^{2N})$$

$$\begin{pmatrix} x_i \\ y_i \\ x_f \\ y_f \end{pmatrix} \longrightarrow \begin{pmatrix} x_i - x_0 \\ y_f \end{pmatrix}$$

which corresponds to the case where the initial point of the state is fixed, while the final point of the state is left free;

<u>$(TC2)$</u>

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N} \in C^2(\mathbb{R}^{4N}, \mathbb{R}^{2N})$$

$$\begin{pmatrix} x_i \\ y_i \\ x_f \\ y_f \end{pmatrix} \longrightarrow \begin{pmatrix} x_i - x_0 \\ x_f - x_1 \end{pmatrix}$$

which corresponds to the case where both the initial and the final point of the state are fixed.

Furthermore, we consider a pair of matrices $(A, B) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R})$ Kalman-Controllable. Then, $\forall u \in L^2((0,T); \mathbb{R}^M)$, we take into account $x_u \in AC([0,T], \mathbb{R}^N)$ the unique solution of:

$$\begin{cases} x_t(t) + Ax(t) = Bu(t) & \text{a.e. } t \in (0,T) \\ x(0) = x_0 \end{cases} \tag{3.40}$$

Then, let us define, for $(TC1)$

$$\mathcal{U} \stackrel{\text{def}}{=} L^2((0,T); \mathbb{R}^M)$$

and for $(TC2)$

$$\mathcal{U} \stackrel{\text{def}}{=} \left\{ u \in L^2((0,T); \mathbb{R}^M) \mid x_u(T) = x_1 \right\}.$$

We remind that, since the pair (A, B) is Kalman-Controllable, $\mathcal{U} \neq \varnothing$. $\mathcal{U}$ will be called the set of admissible control functions. Moreover, it will be useful during the proof of the well posedeness, the definition of the 0-controls space. To this extent, $\forall v \in L^2((0,T);\mathbb{R}^M)$, we take into account the unique $\varphi_v \in AC([0,T];\mathbb{R}^N)$ solution of:

$$\begin{cases} \varphi_t(t) + A\varphi(t) = Bv(t) & \text{a.e. } t \in (0,T) \\ \varphi(0) = 0. \end{cases} \tag{3.41}$$

Then, the set of 0-controls reads as:

$$\mathcal{U}_0 \stackrel{\text{def}}{=} \left\{ v \in L^2((0,T);\mathbb{R}^M) \mid \varphi_v(T) = 0 \right\}. \tag{3.42}$$

We take into account

$$F : \mathbb{R}^N \longmapsto \mathbb{R} \in C^2(\mathbb{R}^N;\mathbb{R})$$

and

$$L : \mathbb{R}^M \longmapsto \mathbb{R} \in C^2(\mathbb{R}^M;\mathbb{R})$$

such that there exist $(\alpha, \beta) \in (0, +\infty)^2$ such that $\forall u \in \mathbb{R}^M \quad \alpha I_M \leq L_{uu}(u) \leq \beta I_M$ and $\forall x \in \mathbb{R}^N \alpha I_N \leq F_{xx}(x)$.

Let us define $(OCP)^T$ and $(OCP)^s$. $(OCP)^T$ concerns the minimization of the following functional

$$J^T : \mathcal{U} \longmapsto \mathbb{R} \tag{3.43}$$

$$u \longrightarrow \int_0^T (L(u) + F(x))\, dt$$

As usual, the stationary problem is defined as follows. First of all, we define a vector subspace of $\mathbb{R}^N \times \mathbb{R}^M$.

Definition 3.2.

$$M = \left\{ (x, u) \in \mathbb{R}^N \times \mathbb{R}^M \mid Ax = Bu \right\}$$

The stationary control problem $(OCP)^s$ consists in minimizing the map:

$$J^s : M \longmapsto \mathbb{R} \tag{3.44}$$

$$(x, u) \longrightarrow L(u) + F(x).$$

We aim at proving the well posedeness of the above problems. We begin with the non stationary one.

Proposition 3.1. *We take into account* $(\alpha, \beta) \in (0, +\infty)^2$*, two functions*

$$F : \mathbb{R}^N \longmapsto \mathbb{R},$$

$$L : \mathbb{R}^M \longmapsto \mathbb{R}$$

and a pair of matrices $(A, B) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R})$ *Kalman-Controllable. We suppose that*

- $(F, L) \in C^2(\mathbb{R}^N, \mathbb{R}) \times C^2(\mathbb{R}^M, \mathbb{R})$;
- $\forall u \in \mathbb{R}^M \quad \alpha I_M \leq L_{uu}(u) \leq \beta I_M$ *and* $\forall x \in \mathbb{R}^N \alpha I_N \leq F_{xx}(x)$;
- $R \in C^2(\mathbb{R}^{4N}, \mathbb{R}^{2N})$ *representing* $(TC1)$ *or* $(TC2)$;
- $T \in (0, +\infty)$.

Then,

1. *there exists a unique* $u^T \in \mathcal{U}$ *such that*
$$\inf_{\mathcal{U}} J^T = J^T(u^T);$$
2. *there exists a unique optimal triple* $(x^T, p^T, u^T) \in AC([0, T], \mathbb{R}^N) \times AC([0, T], \mathbb{R}^N) \times AC([0, T]; \mathbb{R}^M)$ *which is the unique solution of*
$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = Bu^T(t) & a.e.\ t \in (0, T) \\ -\frac{d}{dt}p^T(t) + A^*p^T(t) = F_x(x^T(t)) & a.e.\ t \in (0, T) \\ L_u(u^T(t)) = -B^*p^T(t) & a.e.\ t \in (0, T) \\ R((x^T(0), p^T(0)), (x^T(T), p^T(T))) = 0. \end{cases} \tag{3.45}$$
3. $$(x^T, p^T, u^T) \in C^2([0, T]; \mathbb{R}^N) \times C^2([0, T]; \mathbb{R}^N) \times C^1([0, T]; \mathbb{R}^M),$$
then, the above system is satisfied $\forall t \in (0, T)$.

Proof. All throughout the proof, we consider $T \in (0, +\infty)$ fixed. We divide the proof of 1. into 3 steps.
1^{st} Step
In this first step we determine minimizer candidate $\hat{u} \in \mathcal{U}$. First of all, we deduce by the strong convexity of F and L, that there exist 2 positive constants $(C_1, C_2) \in [0, +\infty)^2$ such that

$$\forall x \in \mathbb{R}^N \qquad F(x) \geq \frac{\alpha}{2}\|x\|^2 - C_1$$

$$\forall u \in \mathbb{R}^M \qquad L(u) \geq \frac{\alpha}{2}\|u\|^2 - C_2.$$

Then, $\forall u \in L^2((0,T);\mathbb{R}^M)$

$$J^T(u) = \int_0^T [L(u(t)) + F(x(t))]\,dt \geq \int_0^T \left[\frac{\alpha}{2}\|u(t)\|^2 + \frac{\alpha}{2}\|x(t)\|^2 - C_1 - C_2\right] dt =$$

$$= \frac{\alpha}{2}\|u\|^2_{L^2((0,T);\mathbb{R}^M)} + \frac{\alpha}{2}\|x\|^2_{L^2((0,T);\mathbb{R}^N)} - C_1T - C_2T$$

Therefore, $\forall u \in \mathcal{U}$,

$$J^T(u) = \int_0^T [L(u(t)) + F(x(t))]\,dt \geq \frac{\alpha}{2}\|u\|^2_{L^2((0,T);\mathbb{R}^M)} + \frac{\alpha}{2}\|x\|^2_{L^2((0,T);\mathbb{R}^N)} - (C_1+C_2)T \tag{3.46}$$

This means that there exists $\gamma^T \in \mathbb{R}^+$ such that $Range(J^T) \subset [-\gamma^T, +\infty)$, which in turn implies that:

$$\inf_{\mathcal{U}} J^T \in \mathbb{R}.$$

At this stage, we take a minimizing sequence $\{u_n\}_{n\in\mathbb{R}} \subset \mathcal{U}$, i.e. such that:

$$J^T(u_n) \underset{n\to+\infty}{\longrightarrow} \inf_{\mathcal{U}} J^T \in \mathbb{R}.$$

Hence, $\left\{J^T(u_n)\right\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded. By (3.46), the sequences $\{u_n\}_{n\in\mathbb{N}} \subset L^2((0,T);\mathbb{R}^M)$ and $\{x_n\}_{n\in\mathbb{N}} \subset L^2((0,T);\mathbb{R}^N)$ are bounded too. This yields, using Banach-Alaoglu Theorem, up to taking a common subsequence, that there exist $(\hat{u},\hat{x}) \in L^2((0,T);\mathbb{R}^M) \times L^2((0,T);\mathbb{R}^N)$ such that

$$u_n \underset{n\to+\infty}{\rightharpoonup} \hat{u} \qquad \text{in } L^2((0,T);\mathbb{R}^M)$$

$$x_n \underset{n\to+\infty}{\rightharpoonup} \hat{x} \qquad \text{in } L^2((0,T);\mathbb{R}^N).$$

We are going to show that there exists a representative of $\hat{x}$ such that it is absolutely continuous and:

$$\begin{cases} \hat{x}_t(t) + A\hat{x}(t) = B\hat{u}(t) & \text{a.e. } t \in (0,T) \\ \hat{x}(0) = x_0 \end{cases} \tag{3.47}$$

In fact, $\forall n \in \mathbb{N}$ and for a.e. $t \in (0,T)$

$$\begin{cases} \frac{d}{dt}x_n(t) + Ax_n(t) = Bu_n(t) & \text{a.e. } t \in (0,T) \\ x_n(0) = x_0. \end{cases} \tag{3.48}$$

By the definition of the solution for the system above,

$$x_n(t) = x_0 + \int_0^t (-Ax_n(s) + bu_n(s))\, ds$$

By the definition of the weak convergence $\forall t \in (0, T)$,

$$\int_0^t (-Ax_n(s) + Bu_n(s))\, ds \underset{n\to+\infty}{\longrightarrow} \int_0^t (-A\hat{x}(s) + B\hat{u}(s))\, dt.$$

At this point, we define

$$\hat{y} : [0, T] \longmapsto \mathbb{R}^N$$
$$t \longrightarrow x_0 + \int_0^t [-A\hat{x}(s) + B\hat{u}(s)]\, ds$$

Then, $\forall t \in [0, T]$

$$x_n(t) \underset{n\to+\infty}{\longrightarrow} \hat{y}(t) \tag{3.49}$$

This convergence is bounded uniformly in $t \in [0, T]$ and $n \in \mathbb{N}$. Therefore, by the Dominated Convergence Theorem,

$$x_n \underset{n\to+\infty}{\longrightarrow} \hat{y} \qquad \text{in } L^2((0, T); \mathbb{R}^N)$$

On the other hand, we already know that

$$x_n \underset{n\to+\infty}{\rightharpoonup} \hat{x} \qquad \text{in } L^2((0, T); \mathbb{R}^N).$$

Hence, by the uniqueness of the weak limit, we get $\hat{x} = \hat{y}$. Therefore, there exists an absolutely continuous representative of $\hat{x}$ such that:

$$\begin{cases} \hat{x}_t(t) + A\hat{x}(t) = B\hat{u}(t) & \text{a.e. } t \in (0, T) \\ \hat{x}(0) = x_0. \end{cases} \tag{3.50}$$

and, moreover, $\hat{u} \in \mathcal{U}$, by (3.49).

2nd Step

By the 2nd Step we want to conclude the proof of the existence of a minimizer. At this moment, we have shown the existence of $(\hat{x}, \hat{u}) \in AC([0, T]; \mathbb{R}^N) \times \mathcal{U}$ such that

$$u_n \underset{n\to+\infty}{\rightharpoonup} \hat{u} \qquad \text{in } L^2((0, T); \mathbb{R}^M)$$

$$x_n \underset{n\to+\infty}{\longrightarrow} \hat{x} \qquad \text{in } L^2((0, T); \mathbb{R}^N).$$

We aim at proving the lower semicontinuity of J^T with respect to the weak convergence

$$u_n \underset{n\to+\infty}{\rightharpoonup} \hat{u} \qquad \text{in } L^2((0,T);\mathbb{R}^M),$$

namely, we want to show that

$$J^T(\hat{u}) \leq \liminf_{n\to+\infty} J^T(u_n).$$

Indeed, let us consider the second addendum of J^T. Employing the convexity of F, we get:

$$\int_0^T F(x_n(t))dt \geq \int_0^T F(\hat{x}(t))dt + \int_0^T (F_x(\hat{x}(t)), x_n(t) - \hat{x}(t))_{\mathbb{R}^N}\, dt.$$

Since $\hat{x} \in C^0([0,T];\mathbb{R}^N)$, then $F_x \circ \hat{x} \in C^0([0,T];\mathbb{R}^N) \subset L^2((0,T);\mathbb{R}^N)$. Hence, by definition of weak convergence,

$$\int_0^T (F_x(\hat{x}(t)), x_n(t) - \hat{x}(t))_{\mathbb{R}^N}\, dt \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore,

$$\liminf_{n\to+\infty} \int_0^T F(x_n(t))dt \geq \int_0^T F(\hat{x}(t))dt.$$

After all, the reasoning for the second addendum is similar. Indeed, $\hat{u} \in L^2((0,T);\mathbb{R}^M)$. Then, recalling the hypothesis $L_{uu}(u) \leq \beta I_M \quad \forall u \in \mathbb{R}^M$, we obtain

$$\|L_u(u)\| \leq \|L_u(u) - L_u(0)\| + \|L_u(0)\| \leq \beta\|u\| + \|L_u(0)\|$$

Therefore,

$$\int_0^T \|L_u(\hat{u})\|^2 dt \leq \int_0^T \left[\beta\|\hat{u}\|^2 + C_3\right] dt < +\infty.$$

Hence, $L_u(\hat{u}) \in L^2((0,T);\mathbb{R}^M)$. Using the convexity of L, we get:

$$\int_0^T L(u_n(t))dt \geq \int_0^T L(\hat{u}(t))dt + \int_0^T (L_u(\hat{u}(t)), u_n(t) - \hat{u}(t))_{\mathbb{R}^M}\, dt$$

By definition of the weak convergence,

$$\int_0^T (L_u(\hat{u}(t)), u_n(t) - \hat{u}(t))_{\mathbb{R}^M}\, dt \underset{n\to+\infty}{\longrightarrow} 0$$

Then,

$$\liminf_{n\to+\infty} \int_0^T L(u_n(t))dt \geq \int_0^T L(\hat{u}(t))dt.$$

These arguments directly imply:

$$\liminf_{n\to+\infty} J^T(u_n) \geq \int_0^T F(\hat{x}(t))dt + \int_0^T L(\hat{u}(t))dt = J^T(\hat{u}).$$

Hence,

$$J^T(\hat{u}) \leq \liminf_{n\to+\infty} J^T(u_n) = \inf_{\mathcal{U}} J^T$$

which entails

$$J^T(\hat{u}) = \inf_{\mathcal{U}} J^T.$$

3^{rd} Step

The functional J^T is strictly convex. Then, by Proposition 1.2, the minimizer is unique. At this moment, we aim at proving 2. for $(TC1)$. To this extent, let us take into account the unique minimizer $\hat{u} \in \mathcal{U}$ of J^T. Moreover, we consider $\hat{x} \in AC([0,T];\mathbb{R}^N)$ the unique state such that:

$$\begin{cases} \hat{x}_t(t) + A\hat{x}(t) = B\hat{u}(t) & \text{a.e. } t \in (0,T) \\ \hat{x}(0) = x_0. \end{cases} \tag{3.51}$$

At this stage, we define $(x^T, u^T) \stackrel{\text{def}}{=} (\hat{x}, \hat{u})$. In the next part, $\forall v \in L^2((0,T);\mathbb{R}^M)$, we are going to take the Gateaux derivative of J^T in $\hat{u}$ with respect to v. To this purpose, let us define $\varphi \in AC([0,T];\mathbb{R}^N)$ the unique solution of the Cauchy Problem

$$\begin{cases} \varphi_t(t) + A\varphi(t) = Bv(t) & \text{a.e. } t \in (0,T) \\ \varphi(0) = 0. \end{cases} \tag{3.52}$$

Therefore, $\forall \eta \in \mathbb{R}$, $x^{T,\eta} \stackrel{\text{def}}{=} x^T + \eta\varphi \in AC([0,T],\mathbb{R}^N)$ is the unique solution of the cauchy Problem:

$$\begin{cases} x_t^{T,\eta}(t) + Ax^{T,\eta}(t) = B(u^T(t) + \eta v(t)) & \text{a.e. } t \in (0,T) \\ x^{T,\eta}(0) = x_0. \end{cases} \tag{3.53}$$

At this point, we notice that $\forall (\eta,t) \in \mathbb{R} \times [0,T]$

$$\exists \frac{d}{d\eta} x^{T,\eta}(t) \restriction_{\eta} = \varphi(t).$$

At this point, let us define the adjoint state $p^T \in AC([0,T];\mathbb{R}^N)$ the unique solution of:

$$\begin{cases} -p_t^T(t) + A^* p^T(t) = F_x(x^T(t)) & \text{a.e. } t \in (0,T) \\ p^T(T) = 0. \end{cases} \tag{3.54}$$

Then, we multiply the above equation by φ, obtaining:

$$\int_0^T -\left(p_t^T(t), \varphi(t)\right)_{\mathbb{R}^N} dt + \int_0^T \left(A^* p^T(t), \varphi(t)\right)_{\mathbb{R}^N} dt =$$

$$= \int_0^T \left(F_x(x^T(t)), \varphi(t)\right)_{\mathbb{R}^N} dt$$

Integrating by parts, we get:

$$-\left(p^T(T), \varphi(T)\right)_{\mathbb{R}^N} + \left(p^T(0), \varphi(0)\right)_{\mathbb{R}^N} + \int_0^T \left(p^T(t), \varphi_t(t)\right)_{\mathbb{R}^N} dt + \int_0^T \left(A^* p^T(t), \varphi(t)\right)_{\mathbb{R}^N} dt =$$

$$= \int_0^T \left(F_x(x^T(t)), \varphi(t)\right)_{\mathbb{R}^N} dt$$

Now, we employ the terminal conditions $p(T) = 0$ and $\varphi(0) = 0$ and the equation satisfied by φ, obtaining:

$$\int_0^T \left(p^T(t), Bv(t)\right)_{\mathbb{R}^N} dt =$$

$$= \int_0^T \left(F_x(x^T(t)), \varphi(t)\right)_{\mathbb{R}^N} dt$$

which is equivalent to

$$\int_0^T \left(B^* p^T(t), v(t)\right)_{\mathbb{R}^M} dt = \tag{3.55}$$

$$= \int_0^T \left(F_x(x^T(t)), \varphi(t)\right)_{\mathbb{R}^N} dt.$$

Now, we are in position to compute the required derivative. Indeed, we define

$$f : \mathbb{R} \longmapsto \mathbb{R}$$

$$\eta \longrightarrow \int_0^T \left[L(u^T(t) + \eta v(t)) + F(x^T(t) + \eta\varphi(t))\right] dt$$

By Theorem 1.10 of Differentiable Dependence on the Parameter, $\forall \eta \in \mathbb{R}$,

$$\exists \frac{d}{d\eta} f(\eta) = \int_0^T \left[\left(L_u(u^T(t) + \eta v(t)), v(t) \right)_{\mathbb{R}^M} + \left(F_x(x^T(t) + \eta \varphi(t)), \varphi(t) \right)_{\mathbb{R}^N} \right] dt.$$

Hence,

$$\exists \frac{d}{d\eta} f(0) = \int_0^T \left[\left(L_u(u^T(t), v(t) \right)_{\mathbb{R}^M} + \left(F_x(x^T(t)), \varphi(t) \right)_{\mathbb{R}^N} \right] dt.$$

Since 0 is a minimizer for f, then, by Fermat Theorem,

$$\frac{d}{d\eta} f(0) = 0$$

which in turn entails:

$$\int_0^T \left[\left(L_u(u^T(t), v(t) \right)_{\mathbb{R}^M} + \left(F_x(x^T(t)), \varphi(t) \right)_{\mathbb{R}^N} \right] dt = 0$$

At this moment, we use that

$$\begin{aligned} \int_0^T \left(B^* p^T(t), v(t) \right)_{\mathbb{R}^M} dt = \\ = \int_0^T \left(F_x(x^T(t)), \varphi(t) \right)_{\mathbb{R}^N} dt. \end{aligned} \tag{3.56}$$

This lead us to conclude that the first order conditions for J^T for u^T are equivalent to:

$$\forall v \in L^2((0,T); \mathbb{R}^M) \qquad \int_0^T \left[\left(L_u(u^T(t), v(t) \right)_{\mathbb{R}^M} + \left(B^* p^T(t), v(t) \right)_{\mathbb{R}^M} \right] dt = 0.$$

This integral condition is equivalent to:

$$B^* p^T(t) = -L_u(u^T(t)) \qquad \text{a.e. } t \in (0,T).$$

Hence, an equivalent version of the first order conditions for u^T is the following:

$$\begin{cases} x_t^T(t) + A x^T(t) = B u^T(t) & \text{a.e. } t \in (0,T) \\ -p_t^T(t) + A^* p^T(t) = F_x(x^T(t)) & \text{a.e. } t \in (0,T) \\ L_u(u^T(t)) = -B^* p^T(t) & \text{a.e. } t \in [0,T] \\ x^T(0) = x_0 \\ p^T(T) = 0. \end{cases} \tag{3.57}$$

Moreover, by the definition of p^T, since u^T and x^T are unique, then p^T is unique too. This entails that there exists and it is unique an optimal triple (x^T, p^T, u^T) for $(OCP)^T$. Furthermore, we can prove that the solution of the above system is unique. In fact, we take into account an arbitrary couple of solutions $((x^1, p^1, u^1), (x^2, p^2, u^2)) \in \left(AC([0,T];\mathbb{R}^N) \times AC([0,T];\mathbb{R}^N) \times L^2((0,T);\mathbb{R}^M)\right)^2$. Then both u^1 and u^2 fulfill the first order condition for J^T. Since J^T is convex, by Proposition 1.2, both u^1 and u^2 are minimizers for the functional J^T. Employing the uniqueness of the minimizer, we get $u^1 = u^2$. Hence, by definition $x^1 = x^2$ and $p^1 = p^2$. At this step, we are going to prove 2. for $(TC2)$. To this extent, we use the so called "penalization" method. $\forall \varepsilon > 0$ we take into account the functional

$$J_\varepsilon^T : L^2((0,T);\mathbb{R}^M) \longmapsto \mathbb{R}$$

$$u \longrightarrow \int_0^T [F(x) + L(u)]\,dt + \frac{\|x(T) - x_1\|^2}{\varepsilon}.$$

Using the techniques already employed for $(TC1)$, it is possible to prove that this functional admits a unique optimal triple $(x_\varepsilon, p_\varepsilon, u_\varepsilon)$, which satisfies:

$$\begin{cases} x_t(t) + Ax(t) = Bu(t) & \text{a.e. } t \in (0,T) \\ -p_t(t) + A^*p(t) = F_x(x(t)) & \text{a.e. } t \in (0,T) \\ L_u(u(t)) = -B^*p(t) & \text{a.e. } t \in (0,T) \\ x(0) = x_0 \\ p(T) = -\frac{x(T)-x_1}{\varepsilon}. \end{cases} \tag{3.58}$$

Since (A,B) is Kalman-Controllable, by Definition 1.6, there exists $\tilde{u} \in C^0([0,T],\mathbb{R}^M)$ which drives the system from x_0 to x_1. Furthermore, by definition of the minimizer, $\forall \varepsilon > 0$:

$$J_\varepsilon^T(u_\varepsilon) \leq J_\varepsilon^T(\tilde{u}) = \int_0^T [F(\tilde{x}) + L(\tilde{u})]\,dt. \tag{3.59}$$

By strong convexity of F and L, this yields the existence of a constant $C_T \in (0,+\infty)$ independent of $\varepsilon > 0$, such that $\forall \varepsilon > 0$:

$$\|u_\varepsilon\|^2_{L^2((0,T);\mathbb{R}^M)} \leq C_T \tag{3.60}$$

and

$$\|x_\varepsilon\|^2_{L^2((0,T);\mathbb{R}^N)} \leq C_T. \tag{3.61}$$

Employing the continuous dependence from the initial data for state equation, the sequence

$$\{x_\varepsilon\}_\varepsilon \subset W^{1,2}((0,T);\mathbb{R}^N) \tag{3.62}$$

is bounded. By Ascoli-Arzelà's Theorem, we infer that:

$$\{x_\varepsilon\}_\varepsilon \subset C^0([0,T],\mathbb{R}^N)$$

is relatively compact. Then, there exists $\hat{x} \in C^0([0,T];\mathbb{R}^N)$, such that, up to subsequences:

$$x_\varepsilon \underset{\varepsilon\to 0^+}{\longrightarrow} \hat{x}$$

strongly in $C^0([0,T],\mathbb{R}^N)$. Since $\{x_\varepsilon\}_\varepsilon \subset C^0([0,T],\mathbb{R}^N)$ is bounded,

$$\{F_x(x_\varepsilon)\}_\varepsilon \subset C^0([0,T],\mathbb{R}^N)$$

is bounded as well. Moreover, we will show that the sequence:

$$\left\{\frac{x_\varepsilon(T)-x_1}{\varepsilon}\right\}_\varepsilon \subset \mathbb{R}^N \tag{3.63}$$

is bounded. Indeed, by contradiction, if the above sequence is not bounded, there exists $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ approaching 0 such that

$$\frac{\|x_{\varepsilon_n}(T)-x_1\|}{\varepsilon_n} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Then, we define $\forall n \in \mathbb{N}$:

$$\tilde{p}_{\varepsilon_n} : [0,T] \longmapsto \mathbb{R}^N \tag{3.64}$$

$$t \longmapsto \frac{p_{\varepsilon_n}(t)}{\|p_{\varepsilon_n}(T)\|}.$$

We observe that, $\forall n \in \mathbb{N}$, p_{ε_n} solves the Cauchy Problem:

$$\begin{cases} -p_t(t) + A^*p(t) = \frac{1}{\|p_{\varepsilon_n}(T)\|}F_x(x_{\varepsilon_n}(t)) & \text{a.e. } t \in (0,T) \\ p(T) = -\frac{p_{\varepsilon_n}(T)}{\|p_{\varepsilon_n}(T)\|}. \end{cases} \tag{3.65}$$

At this moment, we notice that

$$\{\tilde{p}_{\varepsilon_n}(T)\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$$

is bounded. Furthermore, by the boundedness of the sequence (3.62), we infer:

$$\left\{\frac{1}{\|p_{\varepsilon_n}(T)\|}F_x(x_{\varepsilon_n})\right\}_{n\in\mathbb{N}} \subset C^0([0,T];\mathbb{R}^N)$$

is bounded as well. Then, by the continuous dependence from the initial data,

$$\{\tilde{p}_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset W^{1,2}((0,T);\mathbb{R}^N)$$

is bounded. By Ascoli-Arzelà's Theorem, up to subsequences, there exists $\tilde{p} \in C^0([0,T], \mathbb{R}^N)$ such that:

$$\tilde{p}_{\varepsilon_n} \underset{n\to+\infty}{\longrightarrow} \tilde{p} \tag{3.66}$$

in $C^0([0,T], \mathbb{R}^N)$. Since, $\forall n \in \mathbb{N}$, p_{ε_n} satisfies (3.65), the convergence (3.66) actually holds in $W^{1,2}((0,T); \mathbb{R}^N)$ and $\tilde{p}$ solves:

$$-\tilde{p}_t(t) + A^* \tilde{p}(t) = 0 \qquad \text{a.e. } t \in (0,T). \tag{3.67}$$

By the continuity of $\hat{p}$, one deduces that the above equality holds everywhere. Furthermore,

$$\|B^*(\tilde{p}_{\varepsilon_n})\|_{L^2((0,T);\mathbb{R}^M)} = \frac{1}{\|p_{\varepsilon_n}(T)\|} \|L_u(u_{\varepsilon_n})\|_{L^2((0,T);\mathbb{R}^M)} \leq$$

$$\leq \frac{1}{\|p_{\varepsilon_n}(T)\|} (\beta \|u_{\varepsilon_n}\|_{L^2((0,T);\mathbb{R}^M)} + C) \leq$$

by (3.60),

$$\leq C(\beta + 1) \frac{1}{\|p_{\varepsilon_n}(T)\|} \underset{n\to+\infty}{\longrightarrow} 0.$$

On the other hand,

$$B^*(\tilde{p}_{\varepsilon_n}) \underset{n\to+\infty}{\longrightarrow} B^*(\tilde{p})$$

in $L^2((0,T); \mathbb{R}^M)$. By uniqueness of the limit,

$$B^*(\tilde{p})(t) = 0 \qquad \text{a.e. } t \in (0,T).$$

By the continuity of $\tilde{p}$, the above equality holds everywhere. Therefore, $\tilde{p}$ satisfies

$$\begin{cases} -p_t(t) + A^* p(t) = 0 & \forall t \in (0,T) \\ B^* p(t) = 0 & \forall t \in [0,T]. \end{cases} \tag{3.68}$$

Hence, by the observability of (A^*, B^*), $\tilde{p}(T) = 0$. On the other hand, by definition of $\tilde{p}_{\varepsilon_n}$,

$$\|\tilde{p}_{\varepsilon_n}(T)\| = 1 \qquad \forall n \in \mathbb{N}.$$

This is a contradiction. Therefore, we have obtained (3.63). Hence, by using the continuous dependence from the data,

$$\{p_\varepsilon\}_\varepsilon \subset W^{1,2}((0,T); \mathbb{R}^N)$$

is bounded. By Ascoli-Arzelà's Theorem,

$$\{p_\varepsilon\}_\varepsilon \subset C^0([0,T]; \mathbb{R}^N)$$

is relatively compact. Moreover, by continuity of F_x and Heine-Cantor Theorem, up to subsequences:

$$F_x(x_\varepsilon) \underset{\varepsilon\to 0^+}{\longrightarrow} F_x(\hat{x})$$

in $C^0([0,T];\mathbb{R}^N)$. Furthermore, by the equation satisfied by p_ε,

$$\{p_\varepsilon\}_\varepsilon \subset W^{1,2}((0,T);\mathbb{R}^N)$$

is relatively compact and, up to subsequences, converge to some $\hat{p} \in W^{1,2}((0,T);\mathbb{R}^N)$, which solves:

$$-\frac{d}{dt}p + A^*p = F_x(\hat{x}) \qquad \text{a.e. } t \in (0,T).$$

Then, by (3.58), we know that

$$L_u(u_\varepsilon) = -B^*p_\varepsilon \underset{\varepsilon\to 0^+}{\longrightarrow} -B^*\hat{p}$$

strongly in $C^0([0,T];\mathbb{R}^M)$. This in turn implies that the sequence

$$\{L_u(u_\varepsilon)\}_\varepsilon \subset C^0([0,T],\mathbb{R}^M)$$

is a Cauchy sequence. By the strong convexity of L,

$$\{u_\varepsilon\}_\varepsilon \subset C^0([0,T],\mathbb{R}^M)$$

is a Cauchy sequence as well. Therefore, there exists $\hat{u} \in C^0([0,T];\mathbb{R}^M)$ such that, up to subsequences,

$$u_\varepsilon \underset{\varepsilon\to 0^+}{\longrightarrow} \hat{u}$$

strongly in $C^0([0,T];\mathbb{R}^M)$. At this point, we are going to show that $\hat{u} \in L^2((0,T);\mathbb{R}^M)$ is actually the minimizer of J^T over $\mathcal{U}$. First of all, taking the limit in the first equation of (3.58) as $\varepsilon \to 0^+$, we obtain:

$$\begin{cases} \hat{x}_t(t) + A\hat{x}(t) = B\hat{u}(t) & \forall t \in (0,T) \\ \hat{x}(0) = x_0. \end{cases} \tag{3.69}$$

Furthermore, by the above convergences, we observe that:

$$\int_0^T [L(u_\varepsilon) + F(x_\varepsilon)]\,dt \underset{\varepsilon\to 0^+}{\longrightarrow} \int_0^T [L(\hat{u}) + F(\hat{x})]\,dt.$$

Moreover, by (3.59), that there exists $D_T \in (0, +\infty)$ such that, $\forall \varepsilon > 0$, $J_\varepsilon^T(u_\varepsilon) \leq D_T$. Therefore,

$$\frac{\|x_\varepsilon(T) - x_1\|^2}{\varepsilon} \leq D_T \qquad \forall \varepsilon > 0,$$

which entails

$$x_\varepsilon(T) \underset{\varepsilon \to 0^+}{\longrightarrow} x_1. \tag{3.70}$$

Hence, $\hat{x}(T) = x_1$. Then, $\hat{u} \in \mathcal{U}$, the set of admissible control functions. Furthermore, by (3.63) and (3.70),

$$\frac{\|x_\varepsilon(T) - x_1\|^2}{\varepsilon} \underset{\varepsilon \to 0^+}{\longrightarrow} 0.$$

Hence,

$$J_\varepsilon^T(u_\varepsilon) \underset{\varepsilon \to 0^+}{\longrightarrow} J^T(\hat{u}). \tag{3.71}$$

Moreover, we notice that $\forall \varepsilon > 0$

$$J_\varepsilon^T \restriction_{\mathcal{U}} = J^T.$$

Therefore,

$$J_\varepsilon^T(u_\varepsilon) \leq J^T(u^T),$$

where u^T is the minimizer for J^T over $\mathcal{U}$. Hence, by (3.71),

$$J^T(\hat{u}) \leq J^T(u^T),$$

which in turn implies, by the uniqueness of the minimizer for J^T over $\mathcal{U}$, that $\hat{u}$ is the minimizer of J^T. Therefore, we have proved that there exists $p^T \in W^{1,2}((0,T); \mathbb{R}^N)$ such that (x^T, p^T, u^T) satisfies:

$$\begin{cases} x_t^T(t) + Ax^T(t) = Bu^T(t) & \text{a.e. } t \in (0,T) \\ -p_t^T(t) + A^* p^T(t) = F_x(x^T(t)) & \text{a.e. } t \in (0,T) \\ L_u(u^T(t)) = -B^* p^T(t) & \text{a.e. } t \in [0,T] \\ x^T(0) = x_0 \\ x^T(T) = x_1. \end{cases} \tag{3.72}$$

Moreover, using the observability of the pair $(-A^*, B^*)$, it is possible to prove the uniqueness for (3.72). Actually, we take into account an arbitrary pair $((x_1, p_1, u_1), (x_2, p_2, u_2))$ such that $\forall i \in \{1, 2\}$, (x_i, p_i, u_i) solves (3.72). By

Proposition 1.2, $u_1 = u_2 = u^T$. Therefore, $x_1 = x_2 = x^T$. Then, by (3.72), $p_2 - p_1 \in AC([0,T];\mathbb{R}^N)$ satisfies:

$$\begin{cases} (p_2 - p_1)_t(t) = A^*(p_2 - p_1)(t) & \text{a.e. } t \in (0,T) \\ B^*(p_2 - p_1) = 0 & \text{a.e. } t \in [0,T]. \end{cases} \tag{3.73}$$

First of all, this implies that actually $p_2 - p_1 \in C^1([0,T],\mathbb{R}^N)$ and that the above conditions hold $\forall t \in (0,T)$. This relation, by the observability (see Definition 1.5), directly entails that:

$$p_2(t) = p_1(t) \qquad \forall t \in [0,T].$$

Then, $(x_1, p_1, u_1) = (x_2, p_2, u_2)$. At this moment, we are in position to take the Gateaux derivative of J^T with respect to the admissible directions $v \in \mathcal{U}_0$. Repeating computations made for $(TC1)$, we get that the first order conditions for J^T are equivalent to:

$$\forall v \in \mathcal{U}_0 \qquad \int_0^T \left[\left(L_u(u^T(t), v(t)\right)_{\mathbb{R}^M} + \left(B^* p^T(t), v(t)\right)_{\mathbb{R}^M}\right] dt = 0.$$

Hence, first order conditions for u^T are equivalent to the requirement of the existence of p^T such that (x^T, p^T, u^T) satisfies (3.72). In order to prove the regularity property, we define

$$g : \mathbb{R}^M \times \mathbb{R}^N \longmapsto \mathbb{R}^M$$

$$(u,p) \longrightarrow L_u(u) + B^* p$$

The function $g \in C^1(\mathbb{R}^N \times \mathbb{R}^M, \mathbb{R}^M)$ and $\forall u \in \mathbb{R}^M$ $L_{uu}(u) \in GL(M;\mathbb{R})$. Then, by the Implicit Function Theorem, $\forall t_0 \in [0,T]$, there exists $(r_1, r_2) \in (0,+\infty)^2$ such that there exists a function

$$h : B^{\mathbb{R}^N}(p^T(t_0), r_1) \longmapsto B^{\mathbb{R}^M}(u^T(t_0), r_2) \in C^1(B^{\mathbb{R}^N}(p^T(t_0), r_1), B^{\mathbb{R}^M}(u^T(t_0), r_2))$$

characterised by the set equality

$$\left\{(p,u) \in B^{\mathbb{R}^N}(p^T(t_0), r_1) \times B^{\mathbb{R}^M}(u^T(t_0), r_2) \mid g(u,p) = 0\right\} = \left\{(p, h(p)) \mid p \in B^{\mathbb{R}^N}(p^T(t_0), r_1)\right\}.$$

Now, we distinguish 3 cases. If $t_0 \in (0,T)$, by the definition of continuity there exists a positive number $r \in (0,+\infty)$ such that ${p^T}^{-1}(B^{\mathbb{R}^N}(p^T(t_0), r_1)) \cap {u^T}^{-1}(B^{\mathbb{R}^M}(u^T(t_0), r_2)) = (t_0 - r, t_0 + r)$. Hence, $\forall t \in (t_0 - r, t_0 + r)$

$$u^T(t) = h(p^T(t)).$$

Analogously, if $t_0 = 0$, employing the continuity there exists $r \in (0, +\infty)$ such that ${p^T}^{-1}(B^{\mathbb{R}^N}(p^T(0), r_1)) \cap {u^T}^{-1}(B^{\mathbb{R}^M}(u^T(0), r_2)) = [0, r)$. Then, $\forall t \in [0, r)$

$$u^T(t) = h(p^T(t)).$$

Similarly, if $t_0 = T$, there exists a positive radius $r \in (0, +\infty)$ such that

$$u^T(t) = h(p^T(t)) \qquad \forall t \in (T - r, T]$$

Therefore, since $h \in C^1(B^{\mathbb{R}^N}(p^T(t_0), r_1), B^{\mathbb{R}^M}(u^T(t_0), r_2))$ and $p^T \in C^0([0,T], \mathbb{R})$, we obtain $u^T \in C^0([0,T]; \mathbb{R}^M)$. This intermediate result together with the first equation of (3.45) enables us to affirm that $x^T \in C^1([0,T]; \mathbb{R}^N)$. This, employing the second equation of (3.45), in turn entails that $p^T \in C^1([0,T]; \mathbb{R}^N)$. By the previous results, this implies that $u^T \in C^1([0,T]; \mathbb{R})$. Therefore, going back to the first equation of (3.45), we obtain $x^T \in C^2([0,T]; \mathbb{R}^N)$. Finally, employing the second equation of (3.45) together with the above achievements, $p^T \in C^2([0,T], \mathbb{R}^M)$. In conclusion, this regularity implies that the equations of (3.45) are satisfied everywhere. This finishes the proof. □

Now, it is time to prove the well posedeness for the stationary problem.

Lemma 3.2. *Consider $(A, B) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R})$ Kalman-Controllable. Moreover, we take into account $F \in C^2(\mathbb{R}^N, \mathbb{R})$ and $L \in C^2(\mathbb{R}^M, \mathbb{R})$. We assume that there exists $(\alpha, \beta) \in (0, +\infty)^2$ such that $\alpha I_M \leq L_{uu} \leq \beta I_M$ and $\alpha I_N \leq F_{xx}$. There exists a unique $(\overline{x}, \overline{u})$ optimal pair for $(OCP)^s$ which is the unique solution in M of:*

$$\begin{cases} A\overline{x} = B\overline{u} \\ (L_u(\overline{u}), v)_{\mathbb{R}^M} + (F_x(\overline{x}), \varphi)_{\mathbb{R}^N} = 0 \qquad \forall (\varphi, v) \in \mathbb{R}^N \times \mathbb{R}^M \ : \ A\varphi = Bv \end{cases} \tag{3.74}$$

Moreover, there exists $\overline{p} \in \mathbb{R}^N$ such that

$$A^*\overline{p} = F_x(\overline{x}) \tag{3.75}$$

and therefore $(\overline{x}, \overline{u}) \in M$ solves

$$(L_u(\overline{u}), v)_{\mathbb{R}^M} + (\overline{p}, Bv)_{\mathbb{R}^N} = 0 \qquad \forall v \in \mathbb{R}^M \text{such that } \exists \varphi \in \mathbb{R}^N \ : \ A\varphi = Bv \tag{3.76}$$

Proof. As usual, in order to show the existence of the minimizer, we employ Weierstrass Theorem 1.8. Indeed, $J^s \in C^0(M, \mathbb{R})$. Furthermore, we take into account for every $k \in \mathbb{R}$ the level set

$$F_k = \left\{(x, u) \in \mathbb{R}^{N+M} \mid J^s(x, u) \leq k\right\}.$$

We remind that, by strong convexity of F and L, there exist 2 positive constants $(C_1, C_2) \in [0, +\infty)^2$ such that

$$\forall x \in \mathbb{R}^N \qquad F(x) \geq \frac{\alpha}{2}\|x\|^2 - C_1$$

$$\forall u \in \mathbb{R}^M \qquad L(u) \geq \frac{\alpha}{2}\|u\|^2 - C_2.$$

Then, $\forall (x, u) \in F_k$

$$\frac{\alpha}{2}\left[\|x\|^2 + \|u\|^2\right] - C_1 - C_2 \leq F(x) + L(u) = J^s(x, u) \leq k.$$

This yields that F_k is compact. Therefore, by Weierstrass Theorem 1.8, there exists $(\overline{x}, \overline{u}) \in \mathbb{R}^N$ minimizer for J^s. By the strict convexity of J^s, we obtain that the minimizer is unique. At this point, we aim at proving the first order condition, for each direction $(v, \varphi) \in M$, one defines the following map:

$$f : \mathbb{R} \longmapsto \mathbb{R}$$

$$h \longrightarrow J^s((\overline{x}, \overline{u}) + h(v, \varphi))$$

0 is a global minimizer for f. Hence, by Fermat Theorem, $\frac{d}{dt}f(0) = 0$. After all,

$$\frac{d}{dt}f(0) = (L_u(\overline{u}), v)_{\mathbb{R}^M} + (F_x(\overline{x}), \varphi)_{\mathbb{R}^N}.$$

This entails

$$(L_u(\overline{u}), v)_{\mathbb{R}^M} + (F_x(\overline{x}), \varphi)_{\mathbb{R}^N} = 0.$$

The uniqueness of the solution for the equation (3.74) is a consequence of the convexity of J^s and the uniqueness of the minimizer. Furthermore, $Ker(A) \times \{0\} \subset M$. Hence, for every $\varphi \in Ker(A)$

$$(F_x(\overline{x}), \varphi)_{\mathbb{R}^N} = 0$$

Since $Range(A^*) = Ker(A)^\perp$, $F_x(\overline{x}) \in Ker(A)^\perp = Range(A^*)$, which is equivalent to

$$\exists \overline{p} \in \mathbb{R}^N \quad : \quad A^*\overline{p} = F_x(\overline{x}).$$

Therefore,

$$(L_u(\overline{u}), v)_{\mathbb{R}^M} + (A^*\overline{p}, \varphi)_{\mathbb{R}^N} = 0 \qquad \forall (v, \varphi) \in M$$

Hence, reminding that $(A^*\overline{p}, \varphi)_{\mathbb{R}^N} = (\overline{p}, A\varphi)_{\mathbb{R}^N} = (\overline{p}, Bv)_{\mathbb{R}^N}$, the first order condition reads as follows:

$$(L_u(\overline{u}), v)_{\mathbb{R}^M} + (\overline{p}, Bv)_{\mathbb{R}^N} = 0 \qquad \forall v \in \mathbb{R}^M \text{ such that } \exists \varphi \in \mathbb{R}^N \ : \ A\varphi = Bv$$

□

At this stage, we are in position to prove the Convergence of Averages for the NonLinear Convex Case. First of all, we need the following Remark.

Remark 3.1. $\forall (A, C) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(M, N; \mathbb{R})$ Kalman-Observable. There exists $K \in (0, +\infty)$ such that $\forall (a, b) \in \mathbb{R}^2$ and $\forall f \in L^2((a, b); \mathbb{R}^N)$, an arbitrary $y \in AC([a, b]; \mathbb{R}^N)$ solving

$$\frac{d}{dt}y(t) + Ay(t) = f(t) \qquad \text{a.e. } t \in (a, b),$$

satisfies

$$\|y(t)\|^2 \leq K \left\{ \int_a^b \|Cy(s)\|^2 dt + \int_a^b \|f(s)\|^2 ds + \|y(a)\|^2 \right\} \qquad \forall t \in [a, b].$$

Proof. This task is accomplished by using the continuous dependence from the data whenever $|b - a| \in [0, 1]$ and Lemma 2.1 applied to the observable pair (A, C), $\forall (a, b) \in \mathbb{R}^2$ such that $|b - a| \in [1, +\infty)$. □

Furthermore, the next Remark will play a key role in the next proof.

Remark 3.2. Let $\forall (A, B) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R})$ be Kalman-Controllable. Moreover, we consider $(F, L) \in C^2(\mathbb{R}^M, \mathbb{R}) \times C^2(\mathbb{R}^N, \mathbb{R})$ such that there exists $(\alpha, \beta) \in (0, +\infty)^2$ such that $\alpha I_M \leq L_{uu} \leq \beta I_M$ and $\alpha I_N \leq F_{xx}$. Then, $\forall (x_0, T) \in \mathbb{R}^N \times (0, +\infty)$ we take into account u^T the optimal control function for $(OCP)^T$ with initial condition x_0. It turns out that, $\forall (t_1, t_2) \in [0, T]^2$, $t_1 \leq t_2$, $u^T(\cdot + t_1) \restriction_{(0, t_2 - t_1)}$ is the optimal control function for $(OCP)^{t_2 - t_1}$ with initial point fixed at $x^T(t_1)$ and final point fixed at $x^T(t_2)$.

Proof. By contradiction, if the control function $u^T(\cdot + t_1) \restriction_{(0, t_2 - t_1)}$ was not optimal, there would exists an optimal one $v \neq u^T(\cdot + t_1) \restriction_{(0, t_2 - t_1)}$. Then, defining a.e.:

$$u_{mod} : (0, T) \longmapsto \mathbb{R}^M \tag{3.77}$$

$$t \longrightarrow \begin{cases} u^T(t) & t \in (0, t_1) \\ v(t - t_1) & t \in (t_1, t_2) \\ u^T(t) & t \in (t_2, T). \end{cases} \tag{3.78}$$

one obtains

$$J^T(u_{mod}) < J^T(u^T)$$

i.e., a contradiction. □

Now, we state the Theorem about Convergence of Averages for the NonLinear Convex Case.

Theorem 3.2 (Convergence of Averages). *Let* $\forall(A,B) \in \mathcal{M}(N,N;\mathbb{R}) \times \mathcal{M}(N,M;\mathbb{R})$ *be Kalman-Controllable and* $(F,L) \in C^2(\mathbb{R}^N,\mathbb{R}) \times C^2(\mathbb{R}^M,\mathbb{R})$. *If there exists* $(\alpha,\beta) \in (0,+\infty)^2$ *such that* $F_{xx} \geq \alpha I_N$ *and* $\alpha I_M \leq L_{uu} \leq \beta I_M$, *then there exists a* T*-independent constant* $C \in (0,+\infty)$ *such that:*

$$\int_0^T \left[\|x^T(t) - \overline{x}\|^2 + \|u^T(t) - \overline{u}\|^2\right] dt \leq C. \tag{3.79}$$

Furthermore, the averages converge, namely: for every $(a,b) \in [0,1]^2$ *such that* $a \neq b$

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} x^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{x} \tag{3.80}$$

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} u^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{u} \tag{3.81}$$

Moreover, there exists a unique $\overline{p} \in \mathbb{R}^N$ *such that:*

$$\begin{cases} A^*\overline{p} = F_x(\overline{x}) \\ L_u(\overline{u}) = -B^*\overline{p} \end{cases} \tag{3.82}$$

and

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} p^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{p}. \tag{3.83}$$

Proof. 1^{st} Step

First of all, we define:

$$\tilde{F} : \mathbb{R}^N \longmapsto \mathbb{R}$$

$$x \longmapsto F(x) - F(\overline{x})$$

and

$$\tilde{L} : \mathbb{R}^M \longmapsto \mathbb{R}$$

$$u \longmapsto L(u) - L(\overline{u}).$$

Substituting $F+L$ with $\tilde{F}+\tilde{L}$ both in (3.43) and in (3.44), the nonstationary optimal triple and the stationary one remains the same.

2^{nd} Step

We aim now at proving that there exists a constant $C \in (0,+\infty)$ independent of $T \in (0,+\infty)$ such that:

$$\int_0^T \left[\alpha\|x^T(t) - \overline{x}\|^2 - \|F_x(\overline{x})\|\|x^T(t) - \overline{x}\| + \alpha\|u^T(t) - \overline{u}\|^2 - \|L_u(\overline{u})\|\|u^T(t) - \overline{u}\|\right] dt \leq C. \tag{3.84}$$

Since (A, B) is controllable, by Theorem 1.7, there exists a exponentially stabilizing feedback function $G \in B(\mathbb{R}^N, \mathbb{R}^M)$. This yields the existence of $(M, \omega) \in (0, +\infty)^2$ such that for every $y_0 \in \mathbb{R}^N$ the unique y^G solution of the system

$$\begin{cases} y_t(t) + Ay(t) = BGy(t) & \forall t \in (0, +\infty) \\ y(0) = y_0 \end{cases} \tag{3.85}$$

satisfies:

$$\|y^G(t)\| \leq M\|y_0\|e^{-\omega t} \qquad \forall t \in [0, +\infty). \tag{3.86}$$

We define $y_0 \stackrel{\text{def}}{=} x_0 - \overline{x} \in \mathbb{R}^N$ and we take into account the unique y^G solution of:

$$\begin{cases} y_t(t) + Ay(t) = BGy(t) & \forall t \in (0, +\infty) \\ y(0) = x_0 - \overline{x} \end{cases} \tag{3.87}$$

Furthermore, we define

$$u^G : [0, T] \longmapsto \mathbb{R}^M$$
$$t \longrightarrow Gy^G(t) + \overline{u}$$

and

$$x^G : [0, T] \longmapsto \mathbb{R}^N$$
$$t \longrightarrow y^G(t) + \overline{x}.$$

We observe that x^G is the state associated with the control function u^G, namely:

$$\begin{cases} \frac{d}{dt}x^G(t) + Ax^G(t) = Bu^G(t) & \forall t \in (0, T) \\ x^G(0) = x_0. \end{cases} \tag{3.88}$$

By (3.86),

$$\forall t \in [0, T] \qquad \|y^G(t)\| \leq M\|x_0 - \overline{x}\|e^{-\omega t}.$$

Therefore, $\forall T \in (0, +\infty)$

$$\int_0^T \left[\tilde{F}(x^G(t)) + \tilde{L}(u^G(t))\right] dt =$$

$$= \int_0^T \left[\tilde{F}(x^G(t)) - \tilde{F}(\overline{x}) + \tilde{L}(u^G(t)) - \tilde{L}(\overline{u})\right] dt \leq$$

$$\leq \int_0^T \left[\max_{B^{\mathbb{R}^M}(0, \|G\|[M\|x_0 - \overline{x}\|] + \|\overline{u}\|)} \left\|\tilde{L}_u\right\| \|u^G(t) - \overline{u}\| + \max_{B^{\mathbb{R}^N}(0, M\|x_0 - \overline{x}\| + \|\overline{x}\|)} \left\|\tilde{F}_x\right\| \|x^G(t) - \overline{x}\|\right] dt \leq$$

$$\leq C\frac{\|x_0 - \overline{x}\|M}{\omega}\left[1 - e^{-\omega T}\right] \leq C\frac{\|x_0 - \overline{x}\|M}{\omega}.$$

Then, there exists $C \in (0,+\infty)$ independent of $T \in (0,+\infty)$ such that $\forall T \in (0,+\infty)$:

$$\tilde{J}^T(u^T) \leq \tilde{J}^T(u^G) \leq C \qquad \forall T \in (0,+\infty). \tag{3.89}$$

At this point, we employ the strong convexity of $\tilde{F}$ and $\tilde{L}$. In fact, by Taylor's Theorem in several variables with Lagrange Reminder, $\forall (x,u) \in \mathbb{R}^N \times \mathbb{R}^M$, there exists $(\xi_x, \xi_u) \in \mathbb{R}^N \times \mathbb{R}^M$ such that:

$$\tilde{F}(x) + \tilde{L}(u) = \tilde{F}(x) - \tilde{F}(\overline{x}) + \tilde{L}(u) - \tilde{L}(\overline{u}) =$$
$$= \left(\tilde{F}_x(\overline{x}), x - \overline{x}\right)_{\mathbb{R}^N} + \left(\tilde{F}_{xx}(\xi_x)(x - \overline{x}), x - \overline{x}\right)_{\mathbb{R}^N} +$$
$$+ \left(\tilde{L}_u(\overline{u}), u - \overline{u}\right)_{\mathbb{R}^N} + \left(\tilde{L}_{uu}(\xi_u)(u - \overline{u}), u - \overline{u}\right)_{\mathbb{R}^N} \geq$$
$$\geq -\|\tilde{F}_x(\overline{x})\|\|x - \overline{x}\| - \|\tilde{L}_u(\overline{u})\|\|u - \overline{u}\| + \alpha\|x - \overline{x}\|^2 + \alpha\|u - \overline{u}\|^2.$$

This enables us to deduce a lower estimate for $\tilde{J}^T(u^T)$. Indeed, $\forall T \in (0,+\infty)$

$$\tilde{J}^T(u^T) = \int_0^T \left[\tilde{L}(u^T(t)) + \tilde{F}(x^T(t))\right] dt \geq$$
$$\geq \int_0^T \left[-\|\tilde{F}_x(\overline{x})\|\|x^T(t) - \overline{x}\| - \|\tilde{L}_u(\overline{u})\|\|u^T(t) - \overline{u}\| + \alpha\|x^T(t) - \overline{x}\|^2 + \alpha\|u^T(t) - \overline{u}\|^2\right] dt.$$

Then, since $\tilde{J}^T(u^T) \leq C \ \forall T \in (0,+\infty)$, we deduce that:

$$\int_0^T \left[\alpha\|x^T(t) - \overline{x}\|^2 + \alpha\|u^T(t) - \overline{u}\|^2 - \|\tilde{F}_x(\overline{x})\|\|x^T(t) - \overline{x}\| - \|\tilde{L}_u(\overline{u})\|\|u^T(t) - \overline{u}\|\right] dt \leq C. \tag{3.90}$$

<u>3rd Step</u>

In what follows, we will use similar arguments to show an analogue result for $(TC2)$. Take into account $\forall K \in (0,+\infty)$ and arbitrary $(x_0, x_1) \in \overline{B(0,K)}^2$ initial and final points.

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N} \in C^2(\mathbb{R}^{4N}, \mathbb{R}^{2N})$$
$$\begin{pmatrix} x_i \\ y_i \\ x_f \\ y_f \end{pmatrix} \longrightarrow \begin{pmatrix} x_i - x_0 \\ x_f - x_1 \end{pmatrix}$$

represents initial and final conditions. Furthermore, $\forall V \in (0,+\infty)$ time horizon, the functional to be minimized reads as follows:

$$\tilde{J}^V : \mathcal{U} \longmapsto \mathbb{R}$$

$$u \longrightarrow \int_0^V \left[\tilde{F}(x) + \tilde{L}(u)\right] dt.$$

We aim at proving that there exists $C \in (0, +\infty)$, independent of the time horizon $V \in (0, +\infty)$ and $(x_0, x_1) \in \overline{B(0, K)}^2$, such that $\forall V \in (0, +\infty)$:

$$\int_0^V \left[\alpha \|u^T(t) - \overline{u}\|^2 + \alpha \|x^T(t) - \overline{x}\|^2\right] dt + \tag{3.91}$$

$$+ \int_0^V \left[- \left\|\tilde{L}_u(\overline{u})\right\| \|u^T(t) - \overline{u}\| - \left\|\tilde{F}_x(\overline{x})\right\| \|x^T(t) - \overline{x}\|\right] dt \leq C.$$

In order to prove the assertion, we consider an arbitrary $K \in \mathbb{R}^+$ and $(x_0, x_1, V) \in \overline{B^{\mathbb{R}^N}(0, K)}^2 \times (0, +\infty)$. First of all, we seek for a bound of $\inf_{\mathcal{U}} \tilde{J}^V$ uniform on time horizon $V \in (0, +\infty)$. This bound is easily obtained for $V \in (0, 4]$. We are going to deduce it $\forall V \in (4, +\infty)$. To this extent, since (A, B) is Kalman-Controllable, by Theorem 1.7, there exists a exponentially stabilizing feedback function $G \in B(\mathbb{R}^N, \mathbb{R}^M)$. Let us define $x(\cdot; x_0)$ the unique solution of

$$\begin{cases} x_t(t) + Ax(t) = BGx(t) & \forall t \in (0, +\infty) \\ x(0) = x_0 - \overline{x} \end{cases} \tag{3.92}$$

and $x(\cdot; x_1)$ the unique solution of

$$\begin{cases} x_t(t) + Ax(t) = BGx(t) & \forall t \in (0, +\infty) \\ x(0) = x_1 - \overline{x}. \end{cases} \tag{3.93}$$

By the definition of G,

$$\|x(t; x_1)\| \leq C\|x_0 - \overline{x}\| e^{-\omega t} \qquad \forall t \in \mathbb{R}^+$$

and

$$\|x(t; x_2)\| \leq C\|x_1 - \overline{x}\| e^{-\omega t} \qquad \forall t \in \mathbb{R}^+.$$

At this stage, by the controllability of the pair (A, B), there exists a control function $u \in C^0([\frac{V}{2} - 1, \frac{V}{2} + 1], \mathbb{R}^M)$ driving the system from $x(\frac{V}{2} - 1; x_0)$ to $x(\frac{V}{2} - 1; x_1)$. At this point, let us define:

$$u_{new} : (0, V) \longmapsto \mathbb{R}^M$$

$$t \longrightarrow \begin{cases} G(x(t; x_0)) + \overline{u} & t \in (0, \frac{V}{2} - 1] \\ u(t) + \overline{u} & t \in (\frac{V}{2} - 1, \frac{V}{2} + 1] \\ G(x(V - t; x_1)) + \overline{u} & t \in (\frac{V}{2} + 1, V). \end{cases} \tag{3.94}$$

Now, we consider the state associated to u_{new}, namely the unique solution x_{new} of the Cauchy Problem:

$$\begin{cases} x_t(t) + Ax(t) = Bu_{new}(t) & \text{a.e. } t \in (0, V) \\ x(0) = x_0. \end{cases} \tag{3.95}$$

By the definition of u_{new}, we have:

- $$\forall t \in [0, \frac{V}{2} - 1] \qquad x_{new}(t) = x(t; x_0) + \overline{x};$$
- $$\forall t \in [\frac{V}{2} + 1, V] \qquad x_{new}(t) = x(V - t; x_1) + \overline{x}.$$

Moreover, by the definitions, there exists $D \in (0, +\infty)$ such that $\forall V \in (0, +\infty)$:

$$\|u_{new} - \overline{u}\|_{L^\infty((0,V);\mathbb{R}^M)} \leq D$$

and

$$\|x_{new} - \overline{x}\|_{L^\infty((0,V);\mathbb{R}^N)} \leq D.$$

Therefore, $\forall\ V \in (4, +\infty)$

$$\int_0^V \left[\tilde{F}(x_{new}(t)) + \tilde{L}(u_{new}(t))\right] dt =$$

$$= \int_0^V \left[\tilde{F}(x_{new}(t)) - \tilde{F}(\overline{x}) + \tilde{L}(u_{new}(t)) - \tilde{L}(\overline{u})\right] dt \leq$$

$$\leq \int_0^V \left[\max_{B^{\mathbb{R}^M}(0,D+\|\overline{u}\|)} \left\|\tilde{L}_u\right\| \|u_{new}(t) - \overline{u}\| + \max_{B^{\mathbb{R}^N}(0,D+\|\overline{x}\|)} \left\|\tilde{F}_x\right\| \|x_{new}(t) - \overline{x}\|\right] dt \leq$$

$$\leq C \int_0^{\frac{V}{2}-1} \left[\|Gx(t; x_0)\| + \|x(t; x_0)\|\right] dt +$$

$$+ C \int_{\frac{V}{2}-1}^{\frac{V}{2}+1} \left[\|u_{new}(t) - \overline{u}\| + \|x_{new}(t) - \overline{x}\|\right] dt +$$

$$+ C \int_{\frac{V}{2}+1}^{V} \left[\|Gx(V - t; x_1)\| + \|x(V - t; x_1)\|\right] dt \leq$$

$$\leq C \int_0^{\frac{V}{2}-1} \left[\|x_0 - \overline{x}\| e^{-\omega t}\right] dt +$$

$$+C\int_{\frac{V}{2}-1}^{\frac{V}{2}+1} C dt+$$

$$+C\int_{\frac{V}{2}+1}^{V}\left[\|x_1-\overline{x}\|e^{-\omega(V-t)}\right]dt\leq$$

$$\leq C\frac{\|x_0-\overline{x}\|}{\omega}+C\frac{\|x_1-\overline{x}\|}{\omega}+C\leq C.$$

Then, there exists $C\in(0,+\infty)$ independent of $V\in(0,+\infty)$ such that $\forall V\in(0,+\infty)$:

$$\tilde{J}^V(u^V)\leq\tilde{J}^V(u_{new})\leq C.$$

At this point, the strong convexity of $\tilde{F}$ and $\tilde{L}$ enables us to deduce a lower estimate for $\tilde{J}^V(u^V)$. Indeed, $\forall V\in(0,+\infty)$

$$\tilde{J}^V(u^V)=\int_0^V\left[\tilde{L}(u^V(t))+\tilde{F}(x^V(t))\right]dt\geq$$

$$\geq\int_0^V\left[-\|\tilde{F}_x(\overline{x})\|\|x^V(t)-\overline{x}\|-\|\tilde{L}_u(\overline{u})\|\|u^V(t)-\overline{u}\|+\alpha\|x^V(t)-\overline{x}\|^2+\alpha\|u^V(t)-\overline{u}\|^2\right]dt.$$

Then, $\forall V\in(0,+\infty)$

$$\int_0^V\left[-\|\tilde{F}_x(\overline{x})\|\|x^V(t)-\overline{x}\|-\|\tilde{L}_u(\overline{u})\|\|u^V(t)-\overline{u}\|+\alpha\|x^V(t)-\overline{x}\|^2+\alpha\|u^V(t)-\overline{u}\|^2\right]dt\leq$$

$$\leq\tilde{J}^V(u^V)\leq C.$$

Hence, $\forall K\in\mathbb{R}^+$ there exists $C\in(0,+\infty)$ such that $\forall V\in(0,+\infty)$ and for any pair $(x_0,x_1)\in\overline{B(0,K)}^2$ defining the initial and final conditions

$$\int_0^V\left[-\|\tilde{F}_x(\overline{x})\|\|x^T(t)-\overline{x}\|-\|\tilde{L}_u(\overline{u})\|\|u^V(t)-\overline{u}\|+\alpha\|x^V(t)-\overline{x}\|^2+\alpha\|u^V(t)-\overline{u}\|^2\right]dt\leq C.$$

4^{th} Step

Henceforth, we aim at proving that there exists $M\in(0,+\infty)$ such that:

$$\left\{x^T(t)-\overline{x}\mid T\in(0,+\infty),\ t\in[0,T]\right\}\subset\overline{B^{\mathbb{R}^N}(0,M)}.$$

At this step, we define, $\forall(M,T)\in\mathbb{R}^+\times(0,+\infty)$,

$$F_T(M)\stackrel{\text{def}}{=}\left\{t\in[0,T]\mid\|x^T(t)-\overline{x}\|+\|u^T(t)-\overline{u}\|<M\right\}. \tag{3.96}$$

Since $(x^T, u^T) \in C^0([0,T], \mathbb{R}^N) \times C^0([0,T], \mathbb{R}^M)$, $\forall (M,T) \in \mathbb{R}^+ \times (0,+\infty)$ $F_T(M)$ is open in the relative topology of $[0,T]$. By definition this implies the existence of a family of nonempty intervals

$$\{I_\alpha\}_{\alpha \in \mathcal{A}} \subset [0,T]$$

such that

$$F_T(M) = \bigcup_{\alpha \in \mathcal{A}} I_\alpha.$$

It follows that there exists a family of closed intervals $\{J_\gamma\}_{\gamma \in \mathcal{B}} \subset [0,T]$ such that

$$[0,T] \setminus F_T(M) = \bigcup_{\gamma \in \mathcal{B}} J_\gamma.$$

By definition, $\forall T \in (0,+\infty)$:

$$\|x^T(t) - \overline{x}\| < M \qquad \forall t \in F_T(M).$$

In what follows, we are going to study the behaviour of $\|x^T(t) - \overline{x}\|$, $\forall t \in F_T(M)$. At this stage, we define an appropriate M. Indeed, if we take into account $M \in \mathbb{R}^+$ sufficiently large, $\forall b \in [M,+\infty)$:

$$\alpha b^2 - [\|F_x(\overline{x})\| + \|L_u(\overline{u})\|]\, b \geq \frac{\alpha}{2} b^2.$$

Since $x^T(\inf J_\gamma) \in \overline{B(0, M + \|\overline{x}\|)}$ and $x^T(\sup J_\gamma) \in \overline{B(0, M + \|\overline{x}\|)}$, by Remark 3.2 and the previous result, we know that there exists $C \in (0,+\infty)$ independent of T and γ such that $\forall T \in (0,+\infty)$, $\forall \gamma \in \mathcal{B}$

$$\int_{J_\gamma} \left[\alpha \|u^T(t) - \overline{u}\|^2 + \alpha \|x^T(t) - \overline{x}\|^2\right] dt + \tag{3.97}$$

$$+ \int_{J_\gamma} \left[- \left\|\tilde{L}_u(\overline{u})\right\| \|u^T(t) - \overline{u}\| - \left\|\tilde{F}_x(\overline{x})\right\| \|x^T(t) - \overline{x}\|\right] dt \leq C$$

By the definition of M and $F_T(M)$, $\forall T \in (0,+\infty)$, $\forall \gamma \in \mathcal{B}$:

$$\int_{J_\gamma} \left[\frac{\alpha}{2} \|u^T(t) - \overline{u}\|^2 + \frac{\alpha}{2} \|x^T(t) - \overline{x}\|^2\right] dt \leq \tag{3.98}$$

$$\leq \int_{J_\gamma} \left[\alpha \|u^T(t) - \overline{u}\|^2 + \alpha \|x^T(t) - \overline{x}\|^2\right] dt +$$

$$+ \int_{J_\gamma} \left[- \left\|\tilde{L}_u(\overline{u})\right\| \|u^T(t) - \overline{u}\| - \left\|\tilde{F}_x(\overline{x})\right\| \|x^T(t) - \overline{x}\|\right] dt \leq C.$$

Then, $\forall \gamma \in \mathcal{B}$

$$\int_{J_\gamma} \left[\|u^T(t)-\overline{u}\|^2 + \|x^T(t)-\overline{x}\|^2\right] dt \leq C. \tag{3.99}$$

At this point we take an arbitrary $t_0 \in [0,T] \setminus F_T(M)$. Therefore, there exists $\gamma \in \mathcal{B}$ such that $t_0 \in J_\gamma \subset [0,T] \setminus F_T(M)$. At this stage, since $x^T - \overline{x}$ satisfies

$$\frac{d}{dt}(x^T - \overline{x})(t) + A(x^T(t) - \overline{x}) = B(u^T(t) - \overline{u}) \qquad \text{a.e. } t \in (0,T),$$

by Remark 3.1 , there exists $C \in (0,+\infty)$ independent of $\gamma \in \mathcal{B}$ such that:

$$\|x^T(t)-\overline{x}\|^2 \leq C\left\{\int_{\inf J_\gamma}^{\sup J_\gamma} \|x^T(s)-\overline{x}\|^2 ds + \int_{\inf J_\gamma}^{\sup J_\gamma} \|u^T(s)-\overline{u}\|^2 ds + \|x^T(\inf J_\gamma)-\overline{x}\|^2\right\} \leq C. \tag{3.100}$$

Therefore there exists $M_1 \in \mathbb{R}^+$ independent of $T \in (0,+\infty)$ and $\gamma \in \mathcal{B}$ such that:

$$\|x^T(t) - \overline{x}\| \leq M_1 \qquad \forall t \in [0,T] \setminus F_T(M).$$

Hence, up to define M bigger than M_1, there exists a positive constant $M \in \mathbb{R}^+$ such that:

$$\left\{x^T(t) - \overline{x} \mid T \in (0,+\infty),\ t \in [0,T]\right\} \subset \overline{B^{\mathbb{R}^N}(0,M)}. \tag{3.101}$$

<u>5^{th} Step</u>

At this step, we are going to show that

$$\frac{1}{T}\int_0^T \left[\|x^T(t)-\overline{x}\|^2 + \|u^T(t)-\overline{u}\|^2\right] dt \underset{T\to+\infty}{\longrightarrow} 0. \tag{3.102}$$

As a matter of fact, subtracting the first order conditions for the non stationary problem and the stationary one, one gets

$$\begin{cases} (x^T - \overline{x})_t + A(x^T - \overline{x}) = B(u^T - \overline{u}) & \forall t \in (0,T) \\ -(p^T - \overline{p})_t + A^*(p^T - \overline{p}) = F_x(x^T) - F_x(\overline{x}) & \forall t \in (0,T). \end{cases} \tag{3.103}$$

Moreover, we have $L_u(u^T) = -B^* p^T$ and $A^*\overline{p} = F_x(\overline{x})$. Multiplying the equation of $p^T - \overline{p}$ by $x^T - \overline{x}$, integrating it and using the Integration by Parts, one gets:

$$\int_0^T \left(\left(L_u(u^T) - L_u(\overline{u}), u^T - \overline{u}\right)_{\mathbb{R}^M} + \left(F_x(x^T) - F_x(\overline{x}), x^T - \overline{x}\right)_{\mathbb{R}^N}\right) dt = \tag{3.104}$$

$$= \left(x_0 - \overline{x}, p^T(0) - \overline{p}\right)_{\mathbb{R}^N} + \left(x^T(T) - \overline{x}, \overline{p}\right)_{\mathbb{R}^N} - \int_0^T \left((L_u(\overline{u}) + B^*\overline{p}), u^T\right)_{\mathbb{R}^N} dt$$

First of all, we aim at proving:

$$\frac{1}{T}\left\{\left(x_0 - \overline{x}, p^T(0) - \overline{p}\right)_{\mathbb{R}^N} + \left(x^T(T) - \overline{x}, \overline{p}\right)_{\mathbb{R}^N} - \int_0^T \left((L_u(\overline{u}) + B^*\overline{p}), u^T\right)_{\mathbb{R}^N} dt\right\} \underset{T\to+\infty}{\longrightarrow} 0.$$

In order to achieve this result, we make some computations. By the strong convexity of F and L and (3.89), we have $\forall T \in (0, +\infty)$:

$$\int_0^T \left(\|x^T(t) - \overline{x}\|^2 + \|u^T(t) - \overline{u}\|^2\right) dt \leq C(1 + T). \tag{3.105}$$

This implies, using Jensen Theorem, that the generalised sequences

$$\left\{\frac{1}{T}\int_0^T x^T dt\right\}_{T\in(0,+\infty)} \subset \mathbb{R}^N$$

and

$$\left\{\frac{1}{T}\int_0^T u^T dt\right\}_{T\in(0,+\infty)} \subset \mathbb{R}^M$$

are bounded. Therefore, by Bolzano-Weierstrass Theroem, they are relatively compact. At this point, we take the average of the state equation, getting:

$$A\left(\frac{1}{T}\int_0^T x^T dt\right) = \frac{1}{T}\int_0^T Bu^T dt - \frac{x^T(T) - x_0}{T} \tag{3.106}$$

This equation enables us to carry on an analysis on the accumulation points of the above generalised sequences. In fact, from (3.106), one obtains that for every $(\varphi, v) \in \mathbb{R}^N \times \mathbb{R}^M$ such that there exists $\{T_k\}_{k\in\mathbb{N}} \subset \{T_n\}_{n\in\mathbb{N}}$ such that:

$$\frac{1}{T_k}\int_0^{T_k} x^{T_k} dt \underset{k\to+\infty}{\longrightarrow} \varphi$$
$$\frac{1}{T_k}\int_0^{T_k} u^{T_k} dt \underset{k\to+\infty}{\longrightarrow} v$$

it happens that

$$A\varphi = Bv.$$

This result enables us to affirm

$$\frac{1}{T_k}\int_0^{T_k} \left(u^{T_k}, L_u(\overline{u}) + B^*\overline{p}\right)_{\mathbb{R}^M} dt =$$

$$= \left(L_u(\overline{u}) + B^*\overline{p}, \frac{1}{T_k}\int_0^{T_k} u^{T_k} dt \right)_{\mathbb{R}^M} \underset{k\to+\infty}{\longrightarrow} (v, L_u(\overline{u}) + B^*\overline{p})_{\mathbb{R}^M} = 0.$$

Therefore, the sequence $\left\{ \frac{1}{T}\int_0^T \left(u^T, L_u(\overline{u}) + B^*\overline{p}\right)_{\mathbb{R}^M} dt \right\}_{T\in(0,+\infty)}$ is such that for each subsequence exists a subsubsequence converging to 0. This, thanks to Proposition 1.1, allows us to prove the stronger result

$$\frac{1}{T}\int_0^T \left(u^T, L_u(\overline{u}) + B^*\overline{p}\right)_{\mathbb{R}^M} dt \underset{T\to+\infty}{\longrightarrow} 0 \tag{3.107}$$

The above convergence is true for each $\overline{p}$ satisfying $A^*\overline{p} = F_x(\overline{x})$.

By (3.101), $\forall T \in (0, +\infty)$

$$\|x^T(T) - \overline{x}\| \leq M. \tag{3.108}$$

At this point, we aim at estimating $\|p^T(0) - \overline{p}\|$. To this extent, since (A^*, B^*) is Kalman-Observable, we employ Remark 3.1 applied to $p^T(T - \ \cdot) - \overline{p} \in AC([0,T];\mathbb{R}^N)$, obtaining:

$$\|p^T(0) - \overline{p}\|^2 \leq K \left[\int_0^T \|B^*p^T(t) - B^*\overline{p}\|^2 dt + \int_0^T \|F_x(x^T(t)) - F_x(\overline{x})\|^2 dt + \|\overline{p}\|^2 \right] \leq$$

by $L_u(u^T) = -B^*p^T$,

$$\leq K \left[\int_0^T \|L_u(u^T(t)) - L_u(\overline{u})\|^2 dt + \int_0^T \|F_x(x^T(t)) - F_x(\overline{x})\|^2 dt + \right. \tag{3.109}$$

$$\left. + \int_0^T \|L_u(\overline{u}) + B^*\overline{p}\|^2 dt + \|\overline{p}\|^2 \right] \leq$$

employing Mean Value Theorem, $L_{uu}(u) \leq \beta I_M \quad \forall u \in \mathbb{R}^M$ and (3.101),

$$\leq K \left[\int_0^T \beta\|u^T(t) - \overline{u}\|^2 dt + \int_0^T \sup_{\overline{B^{\mathbb{R}^N}(0, M+\|\overline{x}\|)}} \|F_{xx}\|^2 \|x^T(t) - \overline{x}\|^2 dt + \|\overline{p}\|^2 + CT \right] \leq C(T+1).$$

Therefore,

$$\|p^T(0) - \overline{p}\| \leq C(\sqrt{T} + 1) \qquad \forall T \in (0, +\infty). \tag{3.110}$$

Finally, going back to (3.104), using (3.105), (3.108) and (3.110), one gets

$$\frac{1}{T}\int_0^T \left(F_x(x^T) - F_x(\overline{x}), x^T - \overline{x}\right)_{\mathbb{R}^N} dt + \frac{1}{T}\int_0^T \left(L_u(u^T) - L_u(\overline{u}), u^T - \overline{u}\right)_{\mathbb{R}^N} dt \leq \tag{3.111}$$

$$\leq \frac{1}{T}\left[\|p^T(0)-\overline{p}\|\|x_0-\overline{x}\|+\|\overline{p}\|\|x^T(T)-\overline{x}\|\right]-\frac{1}{T}\int_0^T\left(L_u(\overline{u})+B^*\overline{p},u^T\right)_{\mathbb{R}^M}dt\leq$$

$$\leq C\frac{1}{T}\left[\sqrt{T}+1\right]-\frac{1}{T}\int_0^T\left(L_u(\overline{u})+B^*\overline{p},u^T\right)_{\mathbb{R}^M}dt$$

Since both F and L are strongly convex, by Definition 1.12,

$$\frac{1}{T}\int_0^T\|u^T(t)-\overline{u}\|^2dt+\frac{1}{T}\int_0^T\|x^T(t)-\overline{x}\|^2dt\leq$$

$$\leq C\frac{1}{T}\left[\sqrt{T}+1\right]-C\frac{1}{T}\int_0^T\left(L_u(\overline{u})+B^*\overline{p},u^T(t)\right)_{\mathbb{R}^M}dt\underset{T\to+\infty}{\to}0.$$

<u>6^{th} Step</u>

At this stage, we prove the convergence of averages. Jensen Theorem yields the conclusion for (x^T,u^T). As regards $\left\{\frac{1}{T}\int_0^T p^T(t)dt\right\}_{T\in(0,+\infty)}$, we take the average of the equation satisfied by $p^T-\overline{p}$ obtaining

$$\frac{p^T(0)}{T}+A^*\left(\frac{1}{T}\int_0^T\left(p^T(t)-\overline{p}\right)dt\right)=\frac{1}{T}\int_0^T\left(F_x(x^T(t))-F_x(\overline{x})\right)dt$$

By estimate (3.110),

$$\frac{p^T(0)}{T}\underset{T\to+\infty}{\longrightarrow}0$$

Since we have already proved that

$$\frac{1}{T}\int_0^T\left\|x^T(t)-\overline{x}\right\|dt\underset{T\to+\infty}{\longrightarrow}0,$$

By the Mean Value Theorem and (3.101),

$$\frac{1}{T}\int_0^T\left\|F_x(x^T(t))-F_x(\overline{x})\right\|dt\leq C\frac{1}{T}\int_0^T\|x^T(t)-\overline{x}\|dt$$

Hence,

$$\frac{1}{T}\int_0^T\left\|F_x(x^T(t))-F_x(\overline{x})\right\|dt\underset{T\to+\infty}{\longrightarrow}0 \tag{3.112}$$

The averaged equation of $p^T-\overline{p}$, implies that the sequence

$$\left\{\frac{1}{T}\int_0^T A^*(p^T(t))dt\right\}_{T\in(0,+\infty)}\subset\mathbb{R}^N$$

is actually bounded. Moreover, since $L_u(u^T) = -B^*p^T$, $\forall T \in (0,+\infty)$ and $\forall t \in [0,T]$

$$\|B^*(p^T(t)-\overline{p})\| = \|L_u(u^T(t))+B^*\overline{p}\| = \|L_u(u^T(t))-L_u(0)+L_u(0)+B^*\overline{p}\| \leq$$

$$\leq \|L_u(u^T(t))-L_u(0)\|+\|L_u(0)+B^*\overline{p}\| \leq \sup_{\mathbb{R}^M}\|L_{uu}\|\|u^T(t)\|+\|L_u(0)+B^*\overline{p}\|$$

$$\leq \beta\|u^T(t)\|+C$$

Then,

$$\left\|B^*\left(\frac{1}{T}\int_0^T\left(p^T(t)-\overline{p}\right)dt\right)\right\| \leq$$

$$\leq \frac{1}{T}\int_0^T\|B^*(p^T(t)-\overline{p})\|dt \leq \frac{1}{T}\int_0^T\left(C\|u^T(t)\|+C\right)dt \leq C$$

This implies the sequence

$$\left\{B^*\left(\frac{1}{T}\int_0^T\left(p^T(t)-\overline{p}\right)dt\right)\right\}_{T\in(0,+\infty)}$$

is bounded. Then, by Lemma 2.2 and by the observability of the pair (A^*, B^*), we obtain the boundedness of the generalised sequence

$$\left\{\frac{1}{T}\int_0^T p^T(t)dt\right\}_{T\in(0,+\infty)} \subset \mathbb{R}^N.$$

This yields that there exists a subsequence of it converging to a certain $\overline{p} \in \mathbb{R}^N$. Thanks to the averaged equation satisfied by p^T and by (3.112), we have that $A^*(\overline{p}) = F_x(\overline{x})$. Furthermore, for every $T \in (0,+\infty)$, $L_u(u^T) = -B^*p^T$. Taking the average,

$$\frac{1}{T}\int_0^T L_u(u^T(t))dt = -B^*\left(\frac{1}{T}\int_0^T p^T dt\right). \tag{3.113}$$

Moreover,

$$\frac{1}{T}\int_0^T\|L_u(u^T(t))-L_u(\overline{u})\|dt \leq$$

$$\leq \frac{1}{T}\int_0^T\sup_{v\in\mathbb{R}^M}\|L_{uu}(v)\|\|u^T(t)-\overline{u}\|dt \leq$$

$$\leq \beta\frac{1}{T}\int_0^T\|u^T(t)-\overline{u}\|dt \underset{T\to+\infty}{\longrightarrow} 0.$$

Then, up to subsequences, we can take the limit of the equation (3.113) and get:

$$L_u(\overline{u}) = -B^*\overline{p}. \tag{3.114}$$

By the observability of the pair (A^*, B^*) and by Lemma 2.2, we deduce the uniqueness of such a $\overline{p}$. Therefore, we have proved the existence and the uniqueness of a $\overline{p} \in \mathbb{R}^N$ such that

$$\begin{cases} A^*\overline{p} = F_x(\overline{x}) \\ L_u(\overline{u}) = -B^*\overline{p} \end{cases} \tag{3.115}$$

Hence, we have that for every subsequence of $\left\{\frac{1}{T}\int_0^T p^T(t)dt\right\}_{T\in(0,+\infty)}$ there exists a subsubsequence converging to the unique $\overline{p}$ satisfying the above conditions. This, together with to Proposition 1.1, yields:

$$\frac{1}{T}\int_0^T p^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{p}.$$

<u>6^{th} Step</u>

Furthermore, we aim at showing (3.79). In fact, we choose $\overline{p} \in \mathbb{R}^N$ such that:

$$\begin{cases} A^*\overline{p} = F_x(\overline{x}) \\ L_u(\overline{u}) = -B^*\overline{p} \end{cases} \tag{3.116}$$

Then, we substitute it into (3.104), getting

$$\begin{aligned} \int_0^T \left(\left(L_u(u^T) - L_u(\overline{u}), u^T - \overline{u}\right)_{\mathbb{R}^M} + \left(F_x(x^T) - F_x(\overline{x}), x^T - \overline{x}\right)_{\mathbb{R}^N}\right) dt = \\ = \left(x_0 - \overline{x}, p^T(0) - \overline{p}\right)_{\mathbb{R}^N} + \left(x^T(T) - \overline{x}, \overline{p}\right)_{\mathbb{R}^N} \end{aligned} \tag{3.117}$$

By (3.101), (3.109) and strong convexity, we deduce that there exists $C \in (0,+\infty)$ such that $\forall T \in (0,+\infty)$

$$\alpha \int_0^T \left[\|x^T(t) - \overline{x}\|^2 + \|u^T(t) - \overline{u}\|^2\right] dt \leq$$

$$\leq C\left\{\left[\int_0^T \left(\|x^T(t) - \overline{x}\|^2 + \|u^T(t) - \overline{u}\|^2\right) dt\right]^{\frac{1}{2}} + 1\right\}.$$

Then, there exists a T-independent constant $C \in (0,+\infty)$ such that $\forall T \in (0,+\infty)$:

$$\int_0^T \left[\|x^T(t) - \overline{x}\|^2 + \|u^T(t) - \overline{u}\|^2\right] dt \leq C \tag{3.118}$$

□

As a consequence of this Theorem, is possible to define $\overline{p} \in \mathbb{R}^N$ as the unique solution of:

$$\begin{cases} A^*\overline{p} = F_x(\overline{x}) \\ L_u(\overline{u}) = -B^*\overline{p}. \end{cases} \tag{3.119}$$

At this point, we take into account the unique optimal pair $(\overline{x}, \overline{u}) \in M$ for $(OCP)^s$. Henceforth, we will name $(\overline{x}, \overline{p}, \overline{u})$ the optimal triple for $(OCP)^s$. $(\overline{x}, \overline{p}, \overline{u})$ is the unique solution of:

$$\begin{cases} A\overline{x} = B\overline{u} \\ A^*\overline{p} = F_x(\overline{x}) \\ L_u(\overline{u}) = -B^*\overline{p}. \end{cases} \tag{3.120}$$

3.3 Local Turnpike Property for NonLinear Convex Case

We aim now at showing the Local Turnpike Property for the NonLinear Convex Case. To this purpose, let us illustrate the following Lemmas. The first 2 Lemmas are a priori bounds for solutions of some differential systems.

Lemma 3.3. *Let $A \in \mathcal{M}(q, q; \mathbb{R})$ be such that its spectrum $\sigma(A) \subset (-\infty, 0)$ and assume that*

$$\Lambda : \mathbb{R}^q \longmapsto \mathbb{R}^q \in C^1(\mathbb{R}^q, \mathbb{R}^q)$$

satisfies

$$\frac{\Lambda(a)}{\|a\|} \underset{a \to 0}{\longrightarrow} 0.$$

Then, for every $\mu \in (0, -\sup(Re(\sigma(A))))$, there exists a pair $(C_\mu, \overline{r}_\mu) \in (0, +\infty)^2$ such that, $\forall T \in (0, +\infty)$ and for every $y \in C^0([0, T], \mathbb{R}^q)$ solution of

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + \Lambda(y(t)) & \forall t \in [0, T] \\ \|y(t)\| \leq \overline{r}_\mu & \forall t \in [0, T], \end{cases} \tag{3.121}$$

the following inequality holds:

$$\|y(t)\| \leq C_\mu e^{-\mu t} \|y(0)\| \qquad \forall t \in [0, T]. \tag{3.122}$$

Proof. First of all, we employ Proposition 2.9 page 120 of [3] due to Triggiani together with Corollary 2.1 page 92 of [3]. Hence, $\forall \mu \in (0, -\sup(\sigma(A)))$ there exists $\tilde{C}_\mu \in (0, +\infty)$ such that $\forall y_0 \in \mathbb{R}^q$ the unique solution $y \in C^1([0,T], \mathbb{R}^q)$ of the linear Cauchy Problem

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) & \forall t \in (0,T) \\ y(0) = y_0. \end{cases} \tag{3.123}$$

is such that

$$\|y(t)\| \leq \tilde{C}_\mu e^{-\mu t}\|y_0\| \qquad \forall t \in [0,T].$$

Let us define $\tilde{\mu} \stackrel{\text{def}}{=} -\sup(\sigma(A))$. Henceforth, we will work with an arbitrary $\mu \in (0, \tilde{\mu})$ fixed. We define $\mu_1 \stackrel{\text{def}}{=} \mu + \frac{\tilde{\mu}-\mu}{2}$. Moreover, by the hypothesis

$$\frac{\Lambda(a)}{\|a\|} \underset{a \to 0}{\longrightarrow} 0.$$

Which implies $\forall \varepsilon > 0$ the existence of $\overline{r}_\mu \in (0, +\infty)$ such that:

$$\frac{\Lambda(a)}{\|a\|} < \varepsilon \qquad \forall a \in B(0, \overline{r}_\mu) \setminus \{0\}.$$

ε is a degree of freedom we will use later. Hence, for every $y \in C^1([0,T]; \mathbb{R}^q)$ solution of (3.121), by the Duhamel's formula stated in Proposition 1.3:

$$y(t) = e^{At}y(0) + \int_0^t e^{A(t-s)}\Lambda(y(s))ds \qquad \forall t \in [0,T]$$

Therefore,

$$\|y(t)\| \leq \|e^{At}y(0)\| + \int_0^t \|e^{A(t-s)}\Lambda(y(s))\|ds \leq$$

$$\leq \tilde{C}_{\mu_1}e^{-\mu_1 t}\|y(0)\| + \int_0^t \tilde{C}_{\mu_1}e^{-\mu_1(t-s)}\|\Lambda(y(s))\|ds \leq$$

choosing $\varepsilon = \frac{\tilde{\mu}-\mu}{2\tilde{C}_{\mu_1}}$,

$$\leq \tilde{C}_{\mu_1}e^{-\mu_1 t}\|y(0)\| + \int_0^t e^{-\mu_1(t-s)}\frac{\tilde{\mu}-\mu}{2}\|y(s)\|ds.$$

By Gronwall's Lemma 1.3,

$$\|y(t)\| \leq C_\mu e^{-\mu t}\|y(0)\|.$$

□

The next 2 Lemma are based on a common setting, we are going to establish.

Setting

We take into account:

- 2 matrices $(D_1, D_2) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, N; \mathbb{R})$ such that

$$\sigma(D_1) \subset (-\infty, 0) \quad \wedge \quad \sigma(D_2) \subset (0, +\infty); \tag{3.124}$$

- $T \in (0, +\infty)$ time horizon;
- a generic radius $r \in (0, +\infty)$;
- a couple of reminder functions:

$$\Lambda_1 : \overline{B^{\mathbb{R}^{2N}}(0, r)} \longmapsto \mathbb{R}^N \in C^1(\overline{B^{\mathbb{R}^{2N}}(0, r)}, \mathbb{R}^N)$$

$$\frac{\Lambda_1(a_1, a_2)}{\|(a_1, a_2)\|} \underset{(a_1,a_2)\to 0}{\longrightarrow} 0 \tag{3.125}$$

and

$$\forall \Lambda_2 : \overline{B^{\mathbb{R}^{2N}}(0, r)} \longmapsto \mathbb{R}^N \in C^1(\overline{B^{\mathbb{R}^{2N}}(0, r)}, \mathbb{R}^N)$$

such that

$$\frac{\Lambda_2(a_1, a_2)}{\|(a_1, a_2)\|} \underset{(a_1,a_2)\to 0}{\longrightarrow} 0. \tag{3.126}$$

Then, we face the problem of analysing the Differential System

$$\begin{cases} \frac{d}{dt} g(t) = D_1 g(t) + \Lambda_1(g(t), h(t)) & \forall t \in (0, T) \\ \frac{d}{dt} h(t) = D_2 h(t) + \Lambda_2(g(t), h(t)) & \forall t \in (0, T). \end{cases} \tag{3.127}$$

Lemma 3.4. *We consider $(D_1, D_2) \in \mathcal{M}(N, N; \mathbb{R})^2$ fulfilling (3.124). For every $(\Lambda_1, \Lambda_2) \in C^1(\overline{B^{\mathbb{R}^{2N}}(0, r)}, \mathbb{R}^N)^2$ satisfying (3.125)-(3.126). For every $\mu \in (0, \min\{-\max(Re(\sigma(D_1))), \min(Re(\sigma(D_2)))\})$, there exist $\overline{r}_\mu \in (0, +\infty)$ independent of $T \in (0, +\infty)$,*

$$\Theta_1 : [0, \overline{r}_\mu] \longmapsto \mathbb{R}^+$$

such that

$$\Theta_1(\beta) \underset{\beta\to 0^+}{\longrightarrow} 0$$

and
$$\Theta_2 : [0, \overline{r}_\mu] \longmapsto \mathbb{R}^+$$
such that
$$\Theta_2(\beta) \underset{\beta \to 0^+}{\longrightarrow} 0,$$
which enables us to define the following estimate.
$$\forall (g,h) : [0,T] \longmapsto \mathbb{R}^{2N} \in C^1([0,T];\mathbb{R}^{2N})$$
satisfying
$$\|g(t)\| + \|h(t)\| \leq \overline{r}_\mu \qquad \forall t \in [0,T] \tag{3.128}$$
and the system of Ordinary Differential Equations (3.127)*, it holds:* $\forall t \in [0,T]$
$$\begin{cases} \|g(t)\| \leq C_\mu \left[\|g(0)\|e^{-\mu t} + e^{-\mu(T-t)}\|h(T)\|\Theta_1(\|h\|_{C^0})\right] \\ \|h(t)\| \leq C_\mu \left[\|h(T)\|e^{-\mu(T-t)} + e^{-\mu t}\|g(0)\|\Theta_2(\|g\|_{C^0})\right]. \end{cases} \tag{3.129}$$

Proof. First of all, we name $\tilde{\mu} \overset{\text{def}}{=} \min\left\{-\max(Re(\sigma(D_1))), \min(Re(\sigma(D_2)))\right\}$. Employing Proposition 2.9 page 120 of [3] and Corollary 2.1 page 92 of [3], $\forall \mu \in (0,\tilde{\mu})$ there exists $C_\mu \in (0,+\infty)$ such that $\forall (y_{1,0}, y_{2,T}) \in \mathbb{R}^{2N}$ the unique solution $(y_1,y_2) \in C^1([0,T],\mathbb{R}^N)^2$ of the linear Cauchy Problem
$$\begin{cases} \frac{d}{dt}y_1(t) = D_1 y_1(t) & \forall t \in (0,T) \\ \frac{d}{dt}y_2(t) = D_2 y_2(t) & \forall t \in (0,T) \\ y_1(0) = y_{1,0} \\ y_2(T) = y_{2,T} \end{cases} \tag{3.130}$$
fulfills the following inequalities:
$$\|y_1(t)\| \leq C_\mu e^{-\mu t}\|y_{1,0}\| \qquad \forall t \in [0,T]$$
and
$$\|y_2(t)\| \leq C_\mu e^{-\mu(T-t)}\|y_{2,T}\| \qquad \forall t \in [0,T].$$
To carry on the proof, we show that whenever we consider $r \in (0,+\infty)$ and $\Lambda \in C^1(B^{\mathbb{R}^p}(0,r),\mathbb{R}^q)$ such that:
$$\frac{\Lambda(a)}{\|a\|} \underset{a \to 0}{\longrightarrow} 0,$$
then,
$$Jac(\Lambda)(0) = 0.$$

Indeed, by the hypothesis, $\forall \varepsilon > 0$ there exists $h_\varepsilon \in (0, +\infty)$ such that $\forall (h, j) \in (-h_\varepsilon, h_\varepsilon) \setminus \{0\} \times \{1, \ldots, p\}$

$$\left\| \frac{\Lambda(he_j) - \Lambda(0)}{h} \right\| < \varepsilon$$

which in turn implies

$$\Lambda_{x_j}(0) = 0 \qquad \forall j \in \{1, \ldots, p\}$$

Then,

$$Jac(\Lambda)(0) = 0.$$

as desired. This together with $\Lambda \in C^1(B^{\mathbb{R}^p}(0, r), \mathbb{R}^q)$ entails that $\forall \varepsilon > 0$ there exists $\rho_\varepsilon \in (0, +\infty)$ such that

$$\|Jac(\Lambda)(a)\| < \varepsilon \qquad \forall a \in B^{\mathbb{R}^p}(0, \rho_\varepsilon).$$

This enables us to deduce an interesting inequality. In fact, we consider an arbitrary radius $r \in (0, +\infty)$ and

$$\Lambda : \overline{B^{\mathbb{R}^{2N}}(0, r)} \longmapsto \mathbb{R}^N \in C^1(\overline{B^{\mathbb{R}^{2N}}(0, r)}; \mathbb{R}^N)$$

such that

$$\frac{\Lambda(a_1, a_2)}{\|(a_1, a_2)\|} \underset{(a_1, a_2) \to 0}{\longrightarrow} 0.$$

Then, there exist 2 nondecreasing functions

$$\Psi_1 : [0, r] \longmapsto \mathbb{R}^+$$

$$\Psi_2 : [0, r] \longmapsto \mathbb{R}^+$$

fulfilling the conditions

$$\Psi_1(\beta) \underset{\beta \to 0^+}{\longrightarrow} 0$$

$$\Psi_2(\beta) \underset{\beta \to 0^+}{\longrightarrow} 0$$

leading to the inequality

$$\|\Lambda(a_1, a_2)\| \leq \Psi_1(\|a_1\|)\|a_1\| + \Psi_2(\|a_2\|)\|a_2\| \qquad \forall (a_1, a_2) \in \overline{B^{\mathbb{R}^{2N}}(0, r)}. \tag{3.131}$$

The proof relies on the previous computations and Lagrange's Theorem. In fact, we name

$$\Psi_1 : [0, r] \longmapsto \mathbb{R}^+$$

$$\beta \longrightarrow \sup_{\overline{B^{\mathbb{R}^N}(0,\beta)\times B^{\mathbb{R}^N}(0,\beta)}} \|\frac{\partial}{\partial a_1}\Lambda\|$$

and

$$\Psi_2 : [0,r] \longmapsto \mathbb{R}^+$$

$$\beta \longrightarrow \sup_{\overline{B^{\mathbb{R}^N}(0,\beta)\times B^{\mathbb{R}^N}(0,\beta)}} \|\frac{\partial}{\partial a_2}\Lambda\|$$

First of all, both Ψ_1 and Ψ_2 are nondecreasing and, by previous computations,

$$\Psi_1(\beta) \underset{\beta\to 0^+}{\longrightarrow} 0,$$

$$\Psi_2(\beta) \underset{\beta\to 0^+}{\longrightarrow} 0.$$

Moreover, $\forall (a_1, a_2) \in \overline{B^{\mathbb{R}^N}(0,r)}^2$ such that $\|a_1\| \leq \|a_2\|$,

$$\|\Lambda(a_1,a_2) - \Lambda(0,0)\| \leq \|\Lambda(a_1,0) - \Lambda(0,0)\| + \|\Lambda(a_1,a_2) - \Lambda(a_1,0)\| \leq$$

$$\leq \sup_{\overline{B^{\mathbb{R}^N}(0,\|a_1\|)\times\{0\}}} \|\frac{\partial}{\partial a_1}\Lambda\|\|a_1\| + \sup_{\overline{B^{\mathbb{R}^N}(0,\|a_2\|)}^2} \|\frac{\partial}{\partial a_2}\Lambda\|\|a_2\| =$$

$$= \Psi_1(\|a_1\|)\|a_1\| + \Psi_2(\|a_2\|)\|a_2\|$$

as required. Up to permutating permutating a_1 and a_2, we have obtained (3.131). Hence, in our situation $\forall i \in \{1,2\}$ there exists

$$\Psi_{i,1} : [0,r] \longmapsto \mathbb{R}^+$$

and

$$\Psi_{i,2} : [0,r] \longmapsto \mathbb{R}^+$$

satisfying

$$\Psi_{i,1}(\beta) \underset{\beta\to 0^+}{\longrightarrow} 0$$

$$\Psi_{i,2}(\beta) \underset{\beta\to 0^+}{\longrightarrow} 0$$

leading to the estimates

$$\|\Lambda_1(a_1,a_2)\| \leq \Psi_{1,1}(\|a_1\|)\|a_1\| + \Psi_{1,2}(\|a_2\|)\|a_2\| \qquad \forall (a_1,a_2) \in \overline{B^{\mathbb{R}^{2N}}(0,r)}$$

$$\|\Lambda_2(a_1,a_2)\| \leq \Psi_{2,1}(\|a_1\|)\|a_1\| + \Psi_{2,2}(\|a_2\|)\|a_2\| \qquad \forall (a_1,a_2) \in \overline{B^{\mathbb{R}^{2N}}(0,r)}.$$

We are now in the position to complete the proof. We consider $\forall \mu \in (0,\tilde{\mu})$, $\mu_1 \overset{\text{def}}{=} \mu + \frac{\tilde{\mu}-\mu}{2}$. Then, we take into account $r_{\mu_1} \in (0,+\infty)$ determined in

Lemma 3.3. First of all, let us use the Duhamel's Formula (Proposition 1.3), getting $\forall\ t \in [0,T]$

$$g(t) = e^{D_1 t} g(0) + \int_0^t e^{D_1(t-s)} \Lambda_1(g(s), h(s)) ds.$$

Therefore, for every $t \in [0,T]$

$$\|g(t)\| \leq \|e^{D_1 t} g(0)\| + \left\| \int_0^t e^{D_1(t-s)} \Lambda_1(g(s), h(s)) ds \right\| \leq$$

$$\leq C_{\mu_1} e^{-\mu_1 t} \|g(0)\| + \int_0^t \|e^{D_1(t-s)}\| \|\Lambda_1(g(s), h(s))\| ds \leq$$

$$\leq C_{\mu_1} e^{-\mu_1 t} \|g(0)\| + \int_0^t C_{\mu_1} e^{-\mu_1(t-s)} \|\Lambda_1(g(s), h(s))\| ds \leq$$

$$\leq C_{\mu_1} e^{-\mu_1 t} \|g(0)\| + \int_0^t C_{\mu_1} e^{-\mu_1(t-s)} (\Psi_{1,1}(\|g(s)\|)\|g(s)\| + \Psi_{1,2}(\|h(s)\|)\|h(s)\|) ds \leq$$

$$\leq C_{\mu_1} e^{-\mu_1 t} \|g(0)\| + \int_0^t C_{\mu_1} e^{-\mu_1(t-s)} \Psi_{1,1}(\|g(s)\|)\|g(s)\| ds +$$

$$+ \int_0^t C_{\mu_1} e^{-\mu_1(t-s)} \Psi_{1,2}(\|h(s)\|)\|h(s)\| ds \leq$$

by Lemma 3.3 applied to $z \stackrel{\text{def}}{=} (g, h(T-\cdot))$,

$$\leq C_{\mu_1} e^{-\mu_1 t} \|g(0)\| + \int_0^t C_{\mu_1} e^{-\mu_1(t-s)} e^{-\mu_1(T-s)} \Psi_{1,2}(\|h(s)\|)\|h(T)\| ds +$$

$$+ \int_0^t C_{\mu_1} e^{-\mu_1(t-s)} \Psi_{1,1}(\|g(s)\|)\|g(s)\| ds \leq$$

since for every $t \in [0,T]$, $\forall s \in [0,t]$ $\quad T-s \geq T-t$ and remembering that both $\Psi_{1,1}$ and $\Psi_{1,2}$ are nondecreasing,

$$\leq C_{\mu_1} e^{-\mu_1 t} \|g(0)\| + e^{-\mu_1(T-t)} \int_0^t C_{\mu_1} e^{-\mu_1(t-s)} \Psi_{1,2}(\|h\|_{C^0})\|h(T)\| ds +$$

$$+ \int_0^t C_{\mu_1} e^{-\mu_1(t-s)} \Psi_{1,1}(\|g\|_{C^0})\|g(s)\| ds \leq$$

At this stage, we compute $\forall t \in [0,T]$

$$\int_0^t e^{-\mu_1(t-s)} ds \leq \frac{1}{\mu_1}.$$

Therefore, we get $\forall t \in [0,T]$

$$\|g(t)\| \leq C_{\mu_1} e^{-\mu_1 t}\|g(0)\| + C_{\mu_1} e^{-\mu_1(T-t)}\frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\| +$$

$$+\int_0^t C_{\mu_1} e^{-\mu_1(t-s)}\Psi_{1,1}(\|g\|_{C^0})\|g(s)\|ds.$$

At this point, we employ generalised Gronwall's Lemma 1.3. To this extent, we define:

$$\beta : [0,T] \longmapsto \mathbb{R}^+$$

$$t \longmapsto C_{\mu_1} e^{-\mu_1 t}\|g(0)\| + C_{\mu_1} e^{-\mu_1(T-t)}\frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\|,$$

$$\gamma : [0,T] \longmapsto (0,+\infty)$$

$$t \longmapsto C_{\mu_1} e^{-\mu_1 t}$$

and

$$w : [0,T] \longmapsto \mathbb{R}^+$$

$$t \longmapsto e^{\mu_1 t}\Psi_{1,1}(\|g\|_{C^0}).$$

By these definitions, one can recognise that:

$$\|g(t)\| \leq \beta(t) + \int_0^t \gamma(t)w(s)\|g(s)\|ds \qquad \forall t \in [0,T].$$

Moreover, we compute $\forall t \in [0,T]$:

$$\int_0^t \gamma(\xi)w(\xi)d\xi = \int_0^t C_{\mu_1} e^{-\mu_1\xi}e^{\mu_1\xi}\Psi_{1,1}(\|g\|_{C^0})d\xi = tC_{\mu_1}\Psi_{1,1}(\|g\|_{C^0}).$$

At this step, we take into account the following integral:

$$\int_0^t e^{-\int_0^s \gamma(\xi)w(\xi)d\xi}\beta(s)w(s)ds.$$

First of all, if $\Psi_{1,1}(\|g\|_{C^0}) = 0$, then, $\forall t \in [0,T]$:

$$\|g(t)\| \leq C_{\mu_1} e^{-\mu_1 t}\|g(0)\| + e^{-\mu_1(T-t)}C_{\mu_1}\frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\|$$

as desired. On the other hand, whenever $\Psi_{1,1}(\|g\|_{C^0}) > 0$, we compute the integral:

$$\int_0^t e^{-\int_0^s \gamma(\xi)w(\xi)d\xi}\beta(s)w(s)ds =$$

$$= \int_0^t e^{-C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})s} C_{\mu_1} e^{-\mu_1 s}\|g(0)\| e^{\mu_1 s}\Psi_{1,1}(\|g\|_{C^0}) ds +$$

$$+ \int_0^t e^{-C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})s} C_{\mu_1} e^{-\mu_1(T-s)} \frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\| e^{\mu_1 s}\Psi_{1,1}(\|g\|_{C^0}) ds =$$

$$= \int_0^t e^{-C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})s} C_{\mu_1}\|g(0)\|\Psi_{1,1}(\|g\|_{C^0}) ds +$$

$$+ \int_0^t e^{-C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})s} e^{-\mu_1(T-2s)} C_{\mu_1}\frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\|\Psi_{1,1}(\|g\|_{C^0}) ds =$$

$$= \int_0^t e^{-C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})s} C_{\mu_1}\|g(0)\|\Psi_{1,1}(\|g\|_{C^0}) ds +$$

$$+ \int_0^t e^{-\mu_1(T-s)} e^{(\mu_1 - C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0}))s} C_{\mu_1}\frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\|\Psi_{1,1}(\|g\|_{C^0}) ds \leq$$

$$\leq \frac{1}{C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})} C_{\mu_1}\|g(0)\|\Psi_{1,1}(\|g\|_{C^0}) +$$

$$+ \frac{e^{-\mu_1 T} e^{(2\mu_1 - C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0}))t}}{2\mu_1 - C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})} C_{\mu_1}\frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\|\Psi_{1,1}(\|g\|_{C^0}).$$

At this stage, we are ready to apply the generalised Gronwall's Lemma "forward" 1.3, getting $\forall t \in [0,T]$:

$$\|g(t)\| \leq \gamma(t) e^{\int_0^t \gamma(\xi)w(\xi)d\xi} \int_0^t e^{-\int_0^s \gamma(\xi)w(\xi)d\xi}\beta(s)w(s) ds + \beta(t) =$$

$$= C_{\mu_1} e^{-\mu_1 t} e^{tC_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})} \Big[\|g(0)\| +$$

$$+ \frac{e^{-\mu_1 T} e^{(2\mu_1 - C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0}))t} C_{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\|\Psi_{1,1}(\|g\|_{C^0})}{(2\mu_1 - C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0}))\mu_1}\Big] + \beta(t) =$$

If we take $\overline{r}_{\mu_1} \in (0,+\infty)$ small enough, $\forall (g,h) \in C^1([0,T],\mathbb{R}^{2N}) \cap \overline{B^{C^0}(0,\overline{r}_{\mu_1})}^{C^0}$

$$\Psi_{1,1}(\|g\|_{C^0}) \in \left(0, \frac{\tilde{\mu} - \mu}{C_{\mu_1} 2}\right).$$

With this choice of $\overline{r}_\mu \in (0,+\infty)$,

$$\leq e^{-\mu t} C_{\mu_1}\|g(0)\| +$$

$$+ e^{-\mu_1(T-t)} \frac{1}{2\mu_1 - C_{\mu_1}\Psi_{1,1}(\|g\|_{C^0})} C_{\mu_1}^2 \frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0}\Psi_{1,1}(\|g\|_{C^0}))\|h(T)\| + \beta(t) \leq$$

$$\leq e^{-\mu t} C_{\mu_1}\|g(0)\| +$$

$$+e^{-\mu_1(T-t)}\frac{1}{2\mu_1-\Psi_{1,1}(\|g\|_{C^0})}C_{\mu_1}^2\frac{1}{\mu_1}\Psi_{1,2}(\|h\|_{C^0})\Psi_{1,1}(\|g\|_{C^0}))\|h(T)\|+\beta(t)\leq$$
$$\leq C_\mu\left[\|g(0)\|e^{-\mu t}+\Psi_{1,2}(\|h\|_{C^0})\|h(T)\|e^{-\mu_1(T-t)}\right].$$

Therefore there exists a pair of positive constants $(C_\mu, r_\mu)\in(0,+\infty)^2$ such that

$$\|g(t)\|\leq C_\mu\left[e^{-\mu t}\|g(0)\|+e^{-\mu(T-t)}\Psi_{1,2}(\|h\|_{C^0})\|h(T)\|\right]$$

Now, we aim at obtaining a similar estimate for h $\forall\mu\in(0,\min\{-\max(Re(\sigma(D_1))),\min(Re(\sigma(D_2)))\})$. This task is accomplished considering $(\tilde{g},\tilde{h})\overset{\text{def}}{=}(h(T-\cdot),g(T-\cdot))$. Indeed,

$$(\tilde{g},\tilde{h})\in C^1([0,T];\mathbb{R}^{2N})\cap\overline{B^{C^0}(0,\overline{r}_\mu)}^{C^0}$$

solves of the system of Ordinary Differential Equations:

$$\begin{cases}\frac{d}{dt}g(t)=-D_2g(t)-\Lambda_2(h(t),g(t)) & \forall t\in(0,T)\\ \frac{d}{dt}h(t)=-D_1h(t)-\Lambda_1(h(t),g(t)) & \forall t\in(0,T).\end{cases}\tag{3.132}$$

Applying the previous arguments to $\tilde{g}$ one obtains that there exists a couple $(C_\mu, r_\mu)\in(0,+\infty)^2$ such that $\forall t\in[0,T]$

$$\|h(t)\|\leq C_\mu\left[e^{-\mu(T-t)}\|h(T)\|+e^{-\mu t}\Psi_{2,1}(\|g\|_{C^0})\|g(0)\|\right].$$

Now, we name $\forall\mu\in(0,+\infty)$ C_μ the bigger between that defined for g and that defined for h. This concludes the proof.

□

The next Lemma face the problem of local existence and local uniqueness for the system (3.127) for small initial data.

Lemma 3.5. *Let* $(D_1,D_2)\in\mathcal{M}(N,N;\mathbb{R})^2$ *satisfying* (3.124). *We define*

$$K\overset{def}{=}4(\sup_{t\in\mathbb{R}^+}\|e^{D_1t}\|+\sup_{t\in\mathbb{R}^+}\|e^{-D_2t}\|+1).$$

We take into account $(\Lambda_1,\Lambda_2)\in C^1(\overline{B^{\mathbb{R}^{2N}}(0,r)},\mathbb{R}^N)^2$ *fulfilling* (3.125)-(3.126). *Then, there exists* $\rho\in(0,r)$ *such that*

1. $\forall\begin{pmatrix}g_0\\h_T\end{pmatrix}\in\overline{B^{\mathbb{R}^{2N}}(0,\frac{\rho}{K})}$

$$\exists!\begin{pmatrix}g(\cdot;(g_0,h_T))\\h(\cdot;(g_0,h_T))\end{pmatrix}:[0,T]\longmapsto\mathbb{R}^{2N}$$

$$\in C^1([0,T];\mathbb{R}^{2N})$$

solution of

$$\begin{cases} \frac{d}{dt}g(t) = D_1 g(t) + \Lambda_1(g(t),h(t)) & \forall t \in (0,T) \\ \frac{d}{dt}h(t) = D_2 h(t) + \Lambda_2(g(t),h(t)) & \forall t \in (0,T) \\ g(0) = g_0 \\ h(T) = h_T \\ \|g(t)\| + \|h(t)\| \leq \rho \qquad \forall t \in [0,T]. \end{cases} \tag{3.133}$$

2.

$$\Phi : (\overline{B^{\mathbb{R}^{2N}}(0,\frac{\rho}{K})}, d_{\mathbb{R}^{2N}}) \longmapsto (C^0([0,T],\mathbb{R}^{2N}), d_\infty)$$
$$\begin{pmatrix} g_0 \\ h_T \end{pmatrix} \longrightarrow \begin{pmatrix} g(\cdot;(g_0,h_T)) \\ h(\cdot;(g_0,h_T)) \end{pmatrix}$$

is continuous.

Proof. First of all, as usual, let us define $\tilde{\mu} \stackrel{\text{def}}{=} \min\left\{-\max(Re(\sigma(D_1))), \min(Re(\sigma(D_2)))\right\}$. Henceforth, we will work with an arbitrary $\mu \in (0,\tilde{\mu})$ fixed. At this stage, we remember that whenever we take into account $r \in (0,+\infty)$ and $\Lambda \in C^1(B^{\mathbb{R}^p}(0,r),\mathbb{R}^q)$ such that:

$$\frac{\Lambda(a)}{\|a\|} \underset{a\to 0}{\longrightarrow} 0,$$
$$Jac(\Lambda)(0) = 0.$$

Then, by the hypothesis (3.125)-(3.126), $\forall \varepsilon > 0$ there exists $\rho_{1,\varepsilon} \in (0,+\infty)$ such that

$$\|Jac(\Lambda_1)(a_1,a_2)\| < \varepsilon \qquad \forall (a_1,a_2) \in B^{\mathbb{R}^{2N}}(0,\rho_{1,\varepsilon})$$

and

$$\|Jac(\Lambda_2)(a_1,a_2)\| < \varepsilon \qquad \forall (a_1,a_2) \in B^{\mathbb{R}^{2N}}(0,\rho_{1,\varepsilon}).$$

ε is a degree of freedom we will use later. In order to prove 1., we consider an arbitrary initial-terminal data $(g_0,h_T) \in \mathbb{R}^{2N}$. Moreover, we take into account an arbitrary pair $(f_1,f_2) \in C^0([0,T];\mathbb{R}^N)^2$. By Picard's Theorem and sublinearity of the dynamics, there exists a unique global solution $\tilde{V}(f_1,f_2) \in C^1([0,T],\mathbb{R}^{2N})$ for the uncoupled Cauchy Problems:

$$\begin{cases} \frac{d}{dt}g(t) = D_1 g(t) + \Lambda_1(f_1(t),f_2(t)) & \forall t \in (0,T) \\ \frac{d}{dt}h(t) = D_2 h(t) + \Lambda_2(f_1(t),f_2(t)) & \forall t \in (0,T) \\ g(0) = g_0 \\ h(T) = h_T. \end{cases} \tag{3.134}$$

Therefore, we are able to define $\forall \rho \in (0, \min\{\rho_{1,\varepsilon}, r\})$, $\forall (g_0, h_T) \in B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})$:

$$V : (\overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty}, d_\infty) \longmapsto (C^0([0,T];\mathbb{R}^{2N}), d_\infty)$$

$$(f_1, f_2) \longrightarrow \tilde{V}(f_1, f_2).$$

First of all, we want to show:

$$V\left(\overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty}\right) \subset \overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty}.$$

To this extent, let us employ the Duhamel's Formula stated in Proposition 1.3. $\forall (f_1, f_2) \in \overline{B^{C^0}(0,\rho)}^{d_\infty\,2}$ and $\forall t \in [0,T]$

$$p_1(V(f_1, f_2))(t) = e^{D_1 t} g_0 + \int_0^t e^{D_1(t-s)} \Lambda_1(f_1(s), f_2(s)) ds.$$

For the following computations it is worth to define

$$K_0 \stackrel{\text{def}}{=} \sup_{t\in\mathbb{R}^+} \|e^{D_1 t}\| + \sup_{t\in\mathbb{R}^+} \|e^{-D_2 t}\|.$$

At this stage, let us estimate $\forall (f_1, f_2) \in \overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty\,2}$ and $\forall t \in [0,T]$:

$$\|p_1(V(f_1, f_2))(t)\| \leq \|e^{D_1 t} g_0\| + \left\|\int_0^t e^{D_1(t-s)} \Lambda_1(f_1(s), f_2(s)) ds\right\| \leq$$

$$\leq \|e^{D_1 t} g_0\| + \int_0^t \|e^{D_1(t-s)}\| \|\Lambda_1(f_1(s), f_2(s))\| ds \leq$$

$$\leq K_0 \|g_0\| + \int_0^t C_\mu e^{-\mu(t-s)} \varepsilon \|(f_1(s), f_2(s))\| ds \leq$$

$$\leq K_0 \|g_0\| + C_\mu e^{-\mu t} \frac{1}{\mu} \left[-1 + e^{\mu t}\right] \varepsilon \|(f_1, f_2)\|_{C^0} \leq$$

$$\leq K_0 \|g_0\| + C_\mu \frac{1}{\mu} \varepsilon \|(f_1, f_2)\|_{C^0} <$$

$$< K_0 \frac{1}{K} \rho + C_\mu \frac{1}{\mu} \varepsilon \rho \leq$$

defining $\varepsilon = \frac{\mu}{4C_\mu}$,

$$\leq \frac{1}{2} \rho$$

Furthermore, one can work similarly with $p_2(V(f_1, f_2))(T-\cdot)$, deducing $\forall t \in [0, T]$:

$$\|p_2(V(f_1, f_2))(t)\| \leq \frac{\rho}{2}.$$

Then,

$$V\left(\overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty}\right) \subset \overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty}.$$

Therefore,

$$V : (\overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty}, d_\infty) \longmapsto (\overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty}, d_\infty)$$

Now,

$$(C^0([0,T];\mathbb{R}^{2N}), d_\infty)$$

is a complete metric space and

$$\overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty} \subset C^0([0,T];\mathbb{R}^{2N})$$

is closed in the topology induced by the distance d_∞. Then,

$$(\overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty}, d_\infty)$$

is a complete metric space. We aim now at proving that V is a contraction.
To accomplish this task, we make the following computations. $\forall((f_{1,1}, f_{1,2}), (f_{2,1}, f_{2,2})) \in \overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty\,2}$ and $\forall t \in [0, T]$

$$\|p_1(V(f_{2,1}, f_{2,2})(t) - V(f_{1,1}, f_{1,2})(t))\| =$$

$$= \|e^{D_1 t}g_0 + \int_0^t e^{D_1(t-s)}\Lambda_1(f_{2,1}(s), f_{2,2}(s))ds - e^{D_1 t}g_0 - \int_0^t e^{D_1(t-s)}\Lambda_1(f_{1,1}(s), f_{1,2}(s))ds\| =$$

$$= \|\int_0^t \left[e^{D_1(t-s)}(\Lambda_1(f_{2,1}(s), f_{2,2}(s)) - \Lambda_1(f_{1,1}(s), f_{1,2}(s)))\right] ds\| \leq$$

$$\leq \int_0^t C_\mu \left[e^{-\mu(t-s)}\|\Lambda_1(f_{2,1}(s), f_{2,2}(s)) - \Lambda_1(f_{1,1}(s), f_{1,2}(s))\|\right] ds \leq$$

by the definition of ε,

$$\leq \int_0^t \left[e^{-\mu(t-s)}\frac{\mu}{4}\|(f_{2,1}(s), f_{2,2}(s)) - (f_{1,1}(s), f_{1,2}(s))\|\right] ds \leq$$

$$\leq \int_0^t \left[e^{-\mu(t-s)}\frac{\mu}{4}\|(f_{2,1}, f_{2,2}) - (f_{1,1}, f_{1,2})\|_{C^0}\right] ds =$$

$$= \frac{\mu}{4}\frac{1}{\mu}\left[1 - e^{-\mu t}\right] \|(f_{2,1}, f_{2,2}) - (f_{1,1}, f_{1,2})\|_{C^0} \leq$$
$$\leq \frac{1}{4}\|(f_{2,1}, f_{2,2}) - (f_{1,1}, f_{1,2})\|_{C^0([0,T];\mathbb{R}^{2N})}.$$

Again, one can use the same arguments applied to $p_2(V(f_1, f_2))(T-\cdot)$. Hence, $\forall((f_{1,1}, f_{1,2}), (f_{2,1}, f_{2,2})) \in \overline{B^{C^0([0,T];\mathbb{R}^{2N})}(0,\rho)}^{d_\infty\,2}$ and $\forall t \in [0, T]$

$$\|V(f_{2,1}, f_{2,2}) - V(f_{1,1}, f_{1,2})\|_{C^0} \leq \frac{1}{2}\|(f_{2,1}, f_{2,2}) - (f_{1,1}, f_{1,2})\|_{C^0}.$$

Therefore, V is a contraction. By Contraction Mapping Theorem 1.1,

$$\exists!(\hat{f}_1, \hat{f}_2) \in B^{C^0([0,T];\mathbb{R}^{2N})}(0, \rho)^2$$

such that:

$$V((\hat{f}_1, \hat{f}_2)) = (\hat{f}_1, \hat{f}_2).$$

Which is equivalent to:

$$\forall (g_0, h_T) \in \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})} \quad \exists! \begin{pmatrix} g(\cdot; (g_0, h_T)) \\ h(\cdot; (g_0, h_T)) \end{pmatrix} \in C^1([0, T], \mathbb{R}^{2N})$$

which satisfies (3.133). At this moment, we want to show that:

$$\Phi : (\overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}, d_{\mathbb{R}^{2N}}) \longmapsto (C^0([0, T], \mathbb{R}^{2N}), d_\infty)$$
$$\begin{pmatrix} g_0 \\ h_T \end{pmatrix} \longrightarrow \begin{pmatrix} g(\cdot; (g_0, h_T)) \\ h(\cdot; (g_0, h_T)) \end{pmatrix}$$
$$\in C^0\left(\overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}, C^0([0, T], \mathbb{R}^{2N})\right)$$

namely, the continuous dependence on initial-terminal conditions. First of all, by the previous step, Φ is well defined. In order to prove the continuous dependence on the initial-terminal conditions, let us take an arbitrary sequence

$$\{(g_{n,0}, h_{n,T})\}_{n\in\mathbb{N}} \subset \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$$

such that there exists a limit $(g_0, h_T) \in \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$, i.e.:

$$(g_{n,0}, h_{n,T}) \underset{n\to+\infty}{\longrightarrow} (g_0, h_T).$$

We aim at proving that

$$(g(\cdot; (g_{n,0}, h_{n,T})), h(\cdot; (g_{n,0}, h_{n,T}))) \underset{n\to+\infty}{\longrightarrow} (g(\cdot; (g_0, h_T)), h(\cdot; (g_0, h_T)))$$

in $(C^0([0,T],\mathbb{R}^{2N}),d_\infty)$. To this purpose, firstly we remind that, by definition:

$$\|g(t;(g_{n,0},h_{n,T}))\|+\|h(t;(g_{n,0},h_{n,T}))\|\leq\rho\qquad\forall t\in[0,T].$$

Hence, the sequence

$$\{(g(\cdot;(g_{n,0},h_{n,T})),h(\cdot;(g_{n,0},h_{n,T})))\}_{n\in\mathbb{N}}\subset C^0([0,T];\mathbb{R}^{2N})$$

is bounded. Moreover, by the equations, we obtain that there exists a (T,t) independent constant $C\in(0,+\infty)$ such that $\forall n\in\mathbb{N}$ and $\forall t\in[0,T]$

$$\|\frac{d}{dt}g(t;(g_{0,n},h_{T,n}))\|\leq C\rho+\sup_{\overline{B^{\mathbb{R}^{2N}}(0,\rho)}}\Lambda_1$$

and

$$\|\frac{d}{dt}h(t;(g_{0,n},h_{T,n}))\|\leq C\rho+\sup_{\overline{B^{\mathbb{R}^{2N}}(0,\rho)}}\Lambda_2.$$

Therefore, by Ascoli-Arzelà's Theorem, we get that, for every subsequence

$$\{(g(\cdot;(g_{n_k,0},h_{n_k,T})),h(\cdot;(g_{n_k,0},h_{n_k,T})))\}_{k\in\mathbb{N}}\subset\{(g(\cdot;(g_{n,0},h_{n,T})),h(\cdot;(g_{n,0},h_{n,T})))\}_{n\in\mathbb{N}}$$

there exists a subsubsequence

$$\left\{(g(\cdot;(g_{n_{k_h},0},h_{n_{k_h},T})),h(\cdot;(g_{n_{k_h},0},h_{n_{k_h},T})))\right\}_{h\in\mathbb{N}}\subset\{(g(\cdot;(g_{n_k,0},h_{n_k,T})),h(\cdot;(g_{n_k,0},h_{n_k,T})))\}_{k\in\mathbb{N}}$$

converging to $(g_\infty,h_\infty)\in C^0([0,T],\mathbb{R}^{2N})$. By the equations, $(g_\infty,h_\infty)\in C^0([0,T],\mathbb{R}^{2N})$ is the solution of (3.133). By the uniqueness, the limit $(g_\infty,h_\infty)\in C^0([0,T],\mathbb{R}^{2N})$ is independent of the particular subsubsequence. Then, by Proposition 1.1 and by definition,

$$(g(\cdot;(g_{n,0},h_{n,T})),h(\cdot;(g_{n,0},h_{n,T})))\underset{n\to+\infty}{\longrightarrow}(g(\cdot;(g_0,h_T)),h(\cdot;(g_0,h_T))).$$

Hence, we have already proved that:

$$\Phi\in C^0\left(\overline{B^{\mathbb{R}^{2N}}(0,\frac{\rho}{K})},C^0([0,T],\mathbb{R}^{2N})\right).$$

This concludes the proof of Lemma 3.5. □

The setting coincides with that of the previous section. Hence, $R\in C^2(\mathbb{R}^{4N},\mathbb{R}^{2N})$ represents $(TC1)$ or $(TC2)$. We explain how R can be in different situations. To this extent, let us take into account $\forall(x_0,x_1)\in\mathbb{R}^{2N}$.

(TC2)

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N} \in C^2(\mathbb{R}^{4N}, \mathbb{R}^{2N})$$
$$\begin{pmatrix} x_i \\ y_i \\ x_f \\ y_f \end{pmatrix} \longrightarrow \begin{pmatrix} x_i - x_0 \\ y_f \end{pmatrix}$$

which correspond to the initial-terminal condition where the initial point of the state is fixed, while the final point of the state is free ($TC1$);

(TC2)

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N} \in C^2(\mathbb{R}^{4N}, \mathbb{R}^{2N})$$
$$\begin{pmatrix} x_i \\ y_i \\ x_f \\ y_f \end{pmatrix} \longrightarrow \begin{pmatrix} x_i - x_0 \\ x_f - x_1 \end{pmatrix}$$

when both the initial and the final point of the state are fixed ($TC2$).

As in [20], the proof relies on the Hyperbolicity of the Hamiltonian system related to the Linearised Pontryagin System.

Theorem 3.3 (Local Turnpike Property)**.** *Let* $(\alpha, \beta) \in (0, +\infty)^2$. *Furthermore, we consider two functions*

$$F : \mathbb{R}^N \longmapsto \mathbb{R},$$
$$L : \mathbb{R}^M \longmapsto \mathbb{R}$$

and a pair of matrices $(A, B) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R})$ *Kalman-Controllable. We assume that*

- $(F, L) \in C^2(\mathbb{R}^N, \mathbb{R}) \times C^2(\mathbb{R}^M, \mathbb{R})$*;*
- $\forall u \in \mathbb{R}^M \quad \alpha I_M \leq L_{uu}(u) \leq \beta I_M$ *and* $\forall x \in \mathbb{R}^N \alpha I_N \leq F_{xx}(x)$*;*
- *initial-terminal conditions are represented by* $R \in C^2(\mathbb{R}^{4N}, \mathbb{R}^{2N})$ *as above;*
- $T \in (0, +\infty)$.

Then, there exists a triple of positive constants $(\bar{\varepsilon}, C, \mu) \in (0, +\infty)^3$ *such that if*

case (TC1)

$$\|\bar{x} - x_0\| + \|\bar{p}\| \leq \bar{\varepsilon} \tag{3.135}$$

case (TC2)

$$\|\overline{x} - x_0\| + \|\overline{x} - x_1\| \leq \overline{\varepsilon} \tag{3.136}$$

the optimal triple (x^T, p^T, u^T) *satisfies the following inequality:*

$$\|x^T(t) - \overline{x}\| + \|p^T(t) - \overline{p}\| + \|u^T(t) - \overline{u}\| \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right] \qquad \forall t \in [0, T]. \tag{3.137}$$

The above estimate is uniform on terminal data fulfilling the smallness condition.

Proof. 1st Step

As in the Linear Quadratic Case, $\forall T \in (0, +\infty)$ we define the perturbation functions:

$$\begin{aligned} \delta x^T : [0, T] &\longmapsto \mathbb{R}^N \in C^2([0, T]; \mathbb{R}^N) \\ t &\longrightarrow x^T(t) - \overline{x}, \\ \delta p^T : [0, T] &\longmapsto \mathbb{R}^N \in C^2([0, T]; \mathbb{R}^N) \\ t &\longrightarrow p^T(t) - \overline{p} \end{aligned}$$

and

$$\begin{aligned} \delta u^T : [0, T] &\longmapsto \mathbb{R}^N \in C^1([0, T]; \mathbb{R}^M) \\ t &\longrightarrow u^T(t) - \overline{u}. \end{aligned}$$

We recall that $\forall T \in (0, +\infty)$, the optimal triple $(x^T, p^T, u^T) \in C^2([0, T]; \mathbb{R}^N) \times C^2([0, T]; \mathbb{R}^N) \times C^1([0, T], \mathbb{R}^M)$ is the unique solution of the Pontryagin System:

$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = Bu^T(t) & \forall t \in (0, T) \\ -\frac{d}{dt}p^T(t) + A^*p^T(t) = F_x(x^T(t)) & \forall t \in (0, T) \\ L_u(u^T(t)) = -B^*p^T(t) & \forall t \in [0, T] \\ R((x^T(0), p^T(0)), (x^T(T), p^T(T))) = 0. \end{cases} \tag{3.138}$$

On the other hand, by (3.120), the stationary optimal triple $(\overline{x}, \overline{p}, \overline{u})$ is the unique solution of:

$$\begin{cases} A\overline{x} = B\overline{u} \\ A^*\overline{p} = F_x(\overline{x}) \\ L_u(\overline{u}) = -B^*\overline{p}. \end{cases} \tag{3.139}$$

Then, the perturbation triple $(\delta x^T, \delta p^T, \delta u^T)$ is the unique solution of:

$$\begin{cases} \frac{d}{dt}\delta x^T(t) + A\delta x^T(t) = B\delta u^T(t) & \forall t \in (0, T) \\ -\frac{d}{dt}\delta p^T(t) + A^*\delta p^T(t) = F_x(x^T(t)) - F_x(\overline{x}) & \forall t \in (0, T) \\ L_u(u^T(t)) - L_u(\overline{u}) = -B^*\delta p^T(t) & \forall t \in [0, T] \\ R((\delta x^T(0) + \overline{x}, \delta p^T(0) + \overline{p}), (\delta x^T(T) + \overline{x}, \delta p^T(T) + \overline{p})) = 0. \end{cases} \tag{3.140}$$

2^{nd} Step

Now, the linearization come into play. For this reason, we highlight that, whenever we consider $\Omega \subset \mathbb{R}^N$ open, $y_0 \in \Omega$ and $f \in C^1(\Omega, \mathbb{R}^N)$, we can define the reminder:

$$\Lambda : \Omega - y_0 \longmapsto \mathbb{R}^N \in C^1(\Omega - y_0, \mathbb{R}^N)$$

$$y \longrightarrow f(y + y_0) - f(y_0) - Jac(f)(y_0)y.$$

Then, by Taylor's Theorem,

$$\frac{\Lambda(y)}{\|y\|} \underset{y\to 0}{\longrightarrow} 0.$$

First of all, we linearize F_x at $\overline{x}$. Since $F_x \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, there exists

$$\Lambda_1 : \mathbb{R}^N \longmapsto \mathbb{R}^N \in C^1(\mathbb{R}^N, \mathbb{R}^N)$$

such that

$$\frac{\Lambda_1(a)}{\|a\|} \underset{a\to 0}{\longrightarrow} 0$$

and

$$F_x(x) = F_x(\overline{x}) + F_{xx}(\overline{x})(x - \overline{x}) + \Lambda_1(x - \overline{x}) \qquad \forall x \in \mathbb{R}^N$$

Therefore, we can rewrite (3.140) as follows:

$$\begin{cases} \frac{d}{dt}\delta x^T(t) + A\delta x^T(t) = B\delta u^T(t) \qquad \forall t \in (0,T) \\ -\frac{d}{dt}\delta p^T(t) + A^*\delta p^T(t) = F_{xx}(\overline{x})(\delta x^T(t)) + \Lambda_1(\delta x^T(t)) \qquad \forall t \in (0,T) \\ L_u(u^T(t)) - L_u(\overline{u}) = -B^*\delta p^T(t) \qquad \forall t \in [0,T] \\ R((\delta x^T(0) + \overline{x}, \delta p^T(0) + \overline{p}), (\delta x^T(T) + \overline{x}, \delta p^T(T) + \overline{p})) = 0. \end{cases} \tag{3.141}$$

We remind that, in our situation, R is already an affine function.

At this point, we face the problem of determining u^T from the relation

$$L_u(u^T(t)) = -B^*p^T(t) \qquad \forall t \in [0,T].$$

To this extent, we want to use the Implicit Function Theorem. Let us define:

$$f : \mathbb{R}^N \times \mathbb{R}^M \longmapsto \mathbb{R}^M$$

$$(a,b) \longrightarrow L_u(b) + B^*a.$$

$f \in C^1(\mathbb{R}^N \times \mathbb{R}^M, \mathbb{R}^M)$, $f(\overline{p}, \overline{u}) = 0$ and $f_b(\overline{p}, \overline{u}) \in GL(M, \mathbb{R})$. Therefore, the hypotheses of the Implicit Function Theorem are fulfilled. Then, there exist 2 positive radii $(r_1, r_2) \in (0, +\infty)^2$ such that there exists

$$\varphi : B^{\mathbb{R}^N}(\overline{p}, r_1) \longmapsto B^{\mathbb{R}^M}(\overline{u}, r_2)$$

such that

- $$\varphi \in C^1(B^{\mathbb{R}^N}(\overline{p}, r_1), B^{\mathbb{R}^M}(\overline{u}, r_2));$$

- $$\Big\{(a,b) \in B^{\mathbb{R}^N}(\overline{p}, r_1) \times B^{\mathbb{R}^M}(\overline{u}, r_2) \mid L_u(b) + B^* a = 0\Big\} =$$
$$= \Big\{(a,b) \in B^{\mathbb{R}^N}(\overline{p}, r_1) \times B^{\mathbb{R}^M}(\overline{u}, r_2) \mid b = \varphi(a)\Big\};$$

- $\forall a \in B^{\mathbb{R}^N}(\overline{p}, r_1)$,
$$\varphi_a(a) = -f_b(a, \varphi(a))^{-1} f_a(a, \varphi(a)) =$$
$$= -L_{uu}(\varphi(a))^{-1} B^*.$$

At this point, we observe that $\varphi \in C^1(B^{\mathbb{R}^N}(\overline{p}, r_1), B^{\mathbb{R}^M}(\overline{u}, r_2))$. Then, we can apply Taylor's Theorem to φ at $\overline{p}$. Therefore, there exists

$$\Lambda_2 : B^{\mathbb{R}^N}(0, r_1) \longmapsto \mathbb{R}^M \in C^1(B^{\mathbb{R}^N}(0, r_1), \mathbb{R}^M)$$

such that
$$\frac{\Lambda_2(a)}{\|a\|} \underset{a \to 0}{\longrightarrow} 0$$

and $\forall a \in \mathbb{R}^N$

$$\varphi(a) = \varphi(\overline{p}) + \varphi_a(\overline{p})(a - \overline{p}) + \Lambda_2(a - \overline{p}) =$$

employing that by the previous computations and $L_u(\overline{u}) = -B^*\overline{p}$,

$$= \overline{u} - L_{uu}(\overline{u})^{-1} B^*(a - \overline{p}) + \Lambda_2(a - \overline{p}).$$

Hence, for every $t \in [0, T]$ such that

$$\|\delta p^T(t)\| < r_1 \wedge \|\delta u^T(t)\| < r_2$$

$$u^T(t) = \varphi(p^T(t)) = \overline{u} - L_{uu}(\overline{u})^{-1} B^*(p^T(t) - \overline{p}) + \Lambda_2(p^T(t) - \overline{p})$$

which means that for every $t \in [0, T]$ such that

$$\|\delta p^T(t)\| < r_1 \wedge \|\delta u^T(t)\| < r_2$$

$$\delta u^T(t) = -L_{uu}(\overline{u})^{-1} B^*(\delta p^T(t)) + \Lambda_2(\delta p^T(t)).$$

At this point, we define

$$\mathcal{AT} \overset{\text{def}}{=} \left\{t \in [0, T] \mid \|\delta p^T(t)\| < r_1 \wedge \|\delta u^T(t)\| < r_2\right\}.$$

Employing the equation satisfied by δx^T, we obtain $\forall t \in \mathcal{AT}$

$$\frac{d}{dt}\delta x^T(t) = -A\delta x^T(t) - BL_{uu}(\overline{u})^{-1}B^*(\delta p^T(t)) + B\Lambda_2(\delta p^T(t))$$

and

$$-\frac{d}{dt}\delta p^T(t) = -A^*\delta p^T(t) + F_{xx}(\overline{x})(\delta x^T(t)) + \Lambda_1(\delta x^T(t)) \qquad \forall t \in (0,T).$$

Therefore, overall, we have obtained that:

$$\begin{cases} \frac{d}{dt}\delta x^T(t) = -A\delta x^T(t) - BL_{uu}(\overline{u})^{-1}B^*(\delta p^T(t)) + B\Lambda_2(\delta p^T(t)) & \forall t \in \mathcal{AT} \\ \frac{d}{dt}\delta p^T(t) = A^*\delta p^T(t) - F_{xx}(\overline{x})(\delta x^T(t)) - \Lambda_1(\delta x^T(t)) & \forall t \in (0,T) \\ \delta u^T(t) = -L_{uu}(\overline{u})^{-1}B^*(\delta p^T(t)) + \Lambda_2(\delta p^T(t)) & \forall t \in \mathcal{AT} \\ R((\delta x^T(0) + \overline{x}, \delta p^T(0) + \overline{p}), (\delta x^T(T) + \overline{x}, \delta p^T(T) + \overline{p})) = 0. \end{cases} \tag{3.142}$$

At this stage, we define, as in the Linear Quadratic Case,

$$Z : [0,T] \longmapsto \mathbb{R}^{2N}$$
$$t \longrightarrow (\delta x^T(t), \delta p^T(t))$$

and

$$M \stackrel{\text{def}}{=} \begin{pmatrix} -A & -BL_{uu}(\overline{u})^{-1}B^* \\ -F_{xx}(\overline{x}) & A^* \end{pmatrix} \in \mathcal{M}(2N, 2N; \mathbb{R}).$$

Furthermore, we define the projection operators:

$$p_1 : \mathbb{R}^{2N} \longmapsto \mathbb{R}^N$$
$$(x,y) \longrightarrow x$$
$$p_2 : \mathbb{R}^{2N} \longmapsto \mathbb{R}^N$$
$$(x,y) \longrightarrow y.$$

These definitions together with the previous deductions allow us to deduce:

$$\frac{d}{dt}Z(t) = \begin{pmatrix} -A & -BL_{uu}(\overline{u})^{-1}B^* \\ -F_{xx}(\overline{x}) & A^* \end{pmatrix} Z(t) + \begin{pmatrix} B\Lambda_2(p_2(Z(t))) \\ -\Lambda_1(p_1(Z(t))). \end{pmatrix} \qquad \forall t \in \mathcal{AT}. \tag{3.143}$$

At this moment, in order to obtain some information about $M \in \mathcal{M}(N,N;\mathbb{R})$, we want to use Lemma 3.1. To this extent, we remember that, by Proposition 1.5, there exists a unique $W \in Sym(M;\mathbb{R})$ such that $W^2 = L_{uu}(\overline{u})$ and a unique $C \in Sym(N;\mathbb{R})$ such that $C^2 = F_{xx}(\overline{x})$. We highlight that $W \in Sym(M;\mathbb{R})$ and $C \in Sym(N;\mathbb{R})$ are positive definite and the pair

(A, B) is Kalman-Controllable. Hence, the hypotheses of Lemma 3.1 are fulfilled. Moreover, we remind that, during the proof of Lemma 3.1, we have defined $\widehat{E}_+ \in Sym(N;\mathbb{R})$ the unique positive definite solution of the Algebraic Riccati Equation:

$$\widehat{E}A + A\widehat{E} + \widehat{E}BL_{uu}(\overline{u})^{-1}B^*\widehat{E} - F_{xx}(\overline{x}) = 0 \qquad \text{(ARE)} \tag{3.144}$$

Furthermore, we named $\widehat{E}_- \in Sym(N;\mathbb{R})$ the unique negative definite solution of the Algebraic Riccati Equation, (3.144). Therefore, we were able to define

$$P \stackrel{\text{def}}{=} \begin{pmatrix} I_N & I_N \\ \widehat{E}_+ & \widehat{E}_- \end{pmatrix}$$

In this context, Lemma 3.1 entails:

1. $M \in sp(N, \mathbb{R})$;
2. there exists a matrix $P \in GL(2N;\mathbb{R})$ such that:

$$P^{-1}MP = \begin{pmatrix} -A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+ & 0 \\ 0 & -A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_- \end{pmatrix}$$

   where $Re(\sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+)) \subset (-\infty, 0)$ and $Re(\sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_-)) \subset (0, +\infty)$;
3. $$Re(\sigma(M)) \subset \mathbb{R} \setminus \{0\};$$
4. $$-\sigma(M) = \sigma(M).$$

At this stage, let us define:

$$Z_1 : [0, T] \longmapsto \mathbb{R}^{2N} \in C^2([0, T];\mathbb{R}^{2N})$$

$$t \longrightarrow P^{-1}Z(t).$$

Since $P \in GL(2N;\mathbb{R})$, the linear transformation

$$L_{P^{-1}} : \mathbb{R}^{2N} \longmapsto \mathbb{R}^{2N}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \longrightarrow P^{-1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

is an homeomorphism. Hence, $L_{P^{-1}}(B^{\mathbb{R}^{2N}}(0,r_1))$ is an open set containing 0. We define now

$$\Lambda_3 : L_{P^{-1}}(B^{\mathbb{R}^{2N}}(0,r_1)) \longmapsto \mathbb{R}^{2N}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \longrightarrow P^{-1}\begin{pmatrix} B\Lambda_2(\widehat{E}_+a_1 + \widehat{E}_-a_2) \\ -\Lambda_1(a_1+a_2) \end{pmatrix}$$

Furthermore, $\Lambda_3 \in C^1(L_{P^{-1}}(B^{\mathbb{R}^{2N}}(0,r_1)), \mathbb{R}^{2N})$ and

$$\frac{\Lambda_3(a_1,a_2)}{\|(a_1,a_2)\|} \underset{(a_1,a_2)\to(0,0)}{\longrightarrow} 0.$$

We can conclude this 1^{st} Step with the following achievement:

$$\frac{d}{dt}Z_1(t) = \begin{pmatrix} -A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+ & 0 \\ 0 & -A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_- \end{pmatrix} Z_1(t) + \Lambda_3(Z_1(t)) \qquad \forall t \in \mathcal{AT} \tag{3.145}$$

Moreover, we name

$$g : [0,T] \longmapsto \mathbb{R}^N$$

$$t \longrightarrow p_1(Z_1(t)) = p_1(P^{-1}Z(t))$$

and

$$h : [0,T] \longmapsto \mathbb{R}^N$$

$$t \longrightarrow p_2(Z_1(t)) = p_2(P^{-1}Z(t)).$$

Therefore, the system (3.145) reads as follows:

$$\begin{cases} \frac{d}{dt}g(t) = (-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+)g(t) + p_1(\Lambda_3(g(t),h(t))) & \forall t \in \mathcal{AT} \\ \frac{d}{dt}h(t) = (-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_-)h(t) + p_2(\Lambda_3(g(t),h(t))) & \forall t \in \mathcal{AT}. \end{cases} \tag{3.146}$$

Since $Re(\sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+)) \subset (-\infty, 0)$ and $Re(\sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_-)) \subset (0,+\infty)$, the intuition would suggest that the first vectorial equation is contracting, while the second equation is expanding. This will be the object of investigation of the next Step.

2^{nd} Step

First of all, we know that $L_{P^{-1}}(B^{\mathbb{R}^{2N}}(0,r_1))$ is an open set containing 0. This implies the existence of a radius $\overline{r_1} \in (0,r_1)$ such that

$$B^{\mathbb{R}^{2N}}(0,\overline{r_1}) \subset L_{P^{-1}}(B^{\mathbb{R}^{2N}}(0,r_1)).$$

Furthermore, by Lemma 3.5 applied to the reminders $p_1 \circ \Lambda_3$ and $p_2 \circ \Lambda_3$, $\forall r \in (o, \overline{r}_1)$, there exists $\rho \in (0,r)$ such that

$$\forall \begin{pmatrix} g_0 \\ h_T \end{pmatrix} \in \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$$

$$\exists! \begin{pmatrix} g(\cdot;(g_0,h_T)) \\ h(\cdot;(g_0,h_T)) \end{pmatrix} : [0,T] \longmapsto \mathbb{R}^{2N}$$

$$\in C^1([0,T];\mathbb{R}^{2N})$$

solution of

$$\begin{cases} \frac{d}{dt} g(t) = (-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_+) g(t) + p_1(\Lambda_3(g(t),h(t))) & \forall t \in (0,T) \\ \frac{d}{dt} h(t) = (-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_-) h(t) + p_2(\Lambda_3(g(t),h(t))) & \forall t \in (0,T) \\ g(0) = g_0 \\ h(T) = h_T. \\ \|g(t)\| + \|h(t)\| \leq \rho \qquad \forall t \in [0,T]. \end{cases} \tag{3.147}$$

At this point, we remind that, by Lemma 3.1:

$$-\sup \sigma(-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_+) = \inf \sigma(-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_-).$$

Therefore, by Lemma 3.4,

$$\forall \mu \in (0, -\sup \sigma(-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_+)),$$

there exists $\overline{r}_\mu \in (0, \overline{r}_1)$,

$$\Theta_1 : [0, \overline{r}_\mu] \longmapsto \mathbb{R}^+$$

with

$$\Theta_1(\beta) \underset{\beta \to 0^+}{\longrightarrow} 0$$

$$\Theta_2 : [0, \overline{r}_\mu] \longmapsto \mathbb{R}^+$$

with

$$\Theta_2(\beta) \underset{\beta \to 0^+}{\longrightarrow} 0,$$

such that

$$\forall (g,h) : [0,T] \longmapsto \mathbb{R}^{2N} \in C^1([0,T];\mathbb{R}^{2N})$$

which satisfies:

$$\begin{cases} \frac{d}{dt} g(t) = (-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_+) g(t) + p_1(\Lambda_3(g(t),h(t))) & \forall t \in (0,T) \\ \frac{d}{dt} h(t) = (-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_-) h(t) + p_2(\Lambda_3(g(t),h(t))) & \forall t \in (0,T) \\ \|g(t)\| + \|h(t)\| \leq \overline{r}_\mu \qquad \forall t \in [0,T], \end{cases} \tag{3.148}$$

the inequality below holds true $\forall t \in [0,T]$:

$$\begin{cases} \|g(t)\| \le C_\mu \left[\|g(0)\|e^{-\mu t} + e^{-\mu(T-t)}\|h(T)\|\Theta_1(\|h\|_{C^0})\right] \\ \|h(t)\| \le C_\mu \left[\|h(T)\|e^{-\mu(T-t)} + e^{-\mu t}\|g(0)\|\Theta_2(\|g\|_{C^0})\right]. \end{cases} \tag{3.149}$$

Henceforth, the couple $(\overline{r}_\mu, \overline{\varepsilon}_\mu) \in (0,+\infty)^2$ will be a degree of freedom we will determine later. Moreover, we will use $\rho \in (0, \min\{\overline{r}_\mu, \overline{r}_1, r_1\})$.

At this point, initial-terminal condition come into play. Therefore, we call

$$\tilde{R} : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N}$$
$$(z_1, z_2) \longrightarrow R(Pz_1, Pz_2).$$

First of all we work with initial-terminal conditions where the initial point is fixed, while the final point is left free, namely $(TC1)$. $(TC1)$ in the new coordinate system correspond to:

$$\begin{cases} g_0 + h(0;(g_0,h_T)) = p_1(L_P((g(0), h(0;(g_0,h_T))))) = x_0 - \overline{x} \\ \widehat{E}_+ g(T;(g_0,h_T)) + \widehat{E}_- h_T = p_2(L_P((g(T;(g_0,h_T)), h_T))) = -\overline{p}. \end{cases} \tag{3.150}$$

Therefore, we look for $(g_0, h_T) \in B^{\mathbb{R}^N}(0, \frac{\rho}{K})^2$ such that $(g(\cdot;(g_0,h_T)), h(\cdot;(g_0,h_T)))$ satisfies the above conditions. To this extent, we use a fixed point argument. In fact, let us define

$$f : \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})} \longmapsto \mathbb{R}^{2N}$$
$$\begin{pmatrix} g_0 \\ h_T \end{pmatrix} \longrightarrow \begin{pmatrix} x_0 - \overline{x} - h(0;(g_0,h_T)) \\ -\widehat{E}_-^{-1}\overline{p} - \widehat{E}_-^{-1}\widehat{E}_+ g(T;(g_0,h_T)) \end{pmatrix}.$$

First of all, we show that $R(f) \subset \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$. Indeed, $\forall (g_0, h_T) \in \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$

$$\|f(g_0,h_T)\| \le \|x_0 - \overline{x} - h(0;(g_0,h_T))\| + \|-\widehat{E}_-^{-1}\overline{p} - \widehat{E}_-^{-1}\widehat{E}_+ g(T;(g_0,h_T))\| \le$$

$$\le \|x_0 - \overline{x}\| + \|h(0;(g_0,h_T))\| + \|\widehat{E}_-^{-1}\overline{p}\| + \|\widehat{E}_-^{-1}\widehat{E}_+ g(T;(g_0,h_T))\| \le$$

thanks to (3.149)

$$\le \overline{\varepsilon}_\mu + C_\mu \left[\|h_T\|e^{-\mu T} + \|g_0\|\Theta_2(\|g(\cdot;(g_0,h_T))\|_{C^0})\right] + \|\widehat{E}_-^{-1}\|\overline{\varepsilon}_\mu +$$

$$+\|\widehat{E}_-^{-1}\widehat{E}_+\|C_\mu \left[\|g_0\|e^{-\mu T} + \|h_T\|\Theta_1(\|h(\cdot;(g_0,h_T))\|_{C^0})\right] \le$$

At this moment, we determine $\overline{r}_\mu$ such that:

$$\begin{cases} \Theta_1(\overline{r}_\mu) < \frac{1}{6\|\widehat{E}_-^{-1}\widehat{E}_+\|C_\mu} \\ \Theta_2(\overline{r}_\mu) < \frac{1}{6C_\mu} \end{cases} \tag{3.151}$$

and

$$\overline{\varepsilon}_\mu = \frac{1}{6(1+\|\widehat{E}_-^{-1}\|)}\frac{\rho}{K}.$$

Furthermore, we suppose $T \in (0, +\infty)$ is sufficiently big. Hence,

$$\|f(g_0, h_T)\| \le \overline{\varepsilon}_\mu + \frac{1}{6}\|h_T\| + \frac{1}{6}\|g_0\| + \|\widehat{E}_-^{-1}\|\overline{\varepsilon}_\mu + \frac{1}{6}\|g_0\| + \frac{1}{6}\|h_T\| <$$

$$< \overline{\varepsilon}_\mu + \frac{1}{6}\frac{\rho}{K} + \frac{1}{6}\frac{\rho}{K} + \|\widehat{E}_-^{-1}\|\overline{\varepsilon}_\mu + \frac{1}{6}\frac{\rho}{K} + \frac{1}{6}\frac{\rho}{K} \le$$

by definition of $\overline{\varepsilon}_\mu$,

$$\le 6\frac{\rho}{6K} = \frac{\rho}{K}.$$

Therefore, $R(f) \subset \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$, i.e.:

$$f : \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})} \longmapsto \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}.$$

Moreover, by Lemma 3.5 (2.), $f \in C^0(\overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}, \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})})$. This enables us to apply Browder's Fixed Point Theorem, getting the existence of $(g_0, h_T) \in \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$ such that $f(g_0, h_T) = (g_0, h_T)$, which means

$$\begin{cases} x_0 - \overline{x} - h(0; (g_0, h_T)) = g_0 \\ -\widehat{E}_-^{-1}\overline{p} - \widehat{E}_-^{-1}\widehat{E}_+ g(T; (g_0, h_T)) = h_T. \end{cases} \tag{3.152}$$

which is equivalent to

$$\begin{cases} g_0 + h(0; (g_0, h_T)) = x_0 - \overline{x} \\ \widehat{E}_+ g(T; (g_0, h_T)) + \widehat{E}_- h_T = -\overline{p}. \end{cases} \tag{3.153}$$

We use a similar technique for initial-terminal condition where both the initial and the final point of the state are fixed. In the new coordinates, $(TC2)$ reads as follows:

$$\begin{cases} g(0) + h(0; (g_0, h_T)) = p_1(P(g(0; (g_0, h_T)), h(0; (g_0, h_T)))) = x_0 - \overline{x} \\ g(T; (g_0, h_T)) + h_T = p_1(P(g(T; (g_0, h_T)), h(T; (g_0, h_T)))) = x_1 - \overline{x}. \end{cases} \tag{3.154}$$

Indeed, we name

$$f : \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})} \longmapsto \mathbb{R}^{2N}$$

$$\begin{pmatrix} g_0 \\ h_T \end{pmatrix} \longrightarrow \begin{pmatrix} x_0 - \overline{x} - h(0; (g_0, h_T)) \\ x_1 - \overline{x} - g(T; (g_0, h_T)). \end{pmatrix}$$

First of all, we prove that $R(f) \subset \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$. In fact, $\forall (g_0, h_T) \in \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$

$$\|f(g_0, h_T)\| \leq \|x_0 - \overline{x} - h(0; (g_0, h_T))\| + \|x_1 - \overline{x} - g(T; (g_0, h_T))\| \leq$$

$$\leq \|x_0 - \overline{x}\| + \|h(0; (g_0, h_T))\| + \|x_1 - \overline{x}\| + \|g(T; (g_0, h_T))\| \leq$$

employing (3.149)

$$\leq \overline{\varepsilon}_\mu + C_\mu \left[\|h_T\| e^{-\mu T} + \|g_0\| \Theta_2(\|g(\cdot; (g_0, h_T))\|_{C^0})\right] +$$

$$+ \overline{\varepsilon}_\mu + C_\mu \left[\|g_0\| e^{-\mu T} + \|h_T\| \Theta_1(\|h(\cdot; (g_0, h_T))\|_{C^0})\right]$$

At this stage, we define $\overline{r}_\mu$ such that:

$$\begin{cases} \Theta_2(\overline{r}_\mu) < \frac{1}{6C_\mu} \\ \Theta_1(\overline{r}_\mu) < \frac{1}{6C_\mu} \end{cases} \tag{3.155}$$

and

$$\overline{\varepsilon}_\mu = \frac{1}{6} \frac{\rho}{K}.$$

Then, whenever $T \in (0, +\infty)$ is large enough,

$$\|f(g_0, h_T)\| \leq \overline{\varepsilon}_\mu + \frac{1}{6}\|h_T\| + \frac{1}{6}\|g_0\| + \overline{\varepsilon}_\mu + \frac{1}{6}\|g_0\| + \frac{1}{6}\|h_T\| \leq$$

$$\leq 6 \frac{\rho}{6K} = \frac{\rho}{K}.$$

Hence, $R(f) \subset \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$, i.e.:

$$f : \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})} \longmapsto \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}.$$

Furthermore, by Lemma 3.5 (2.), $f \in C^0(\overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}, \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})})$. Therefore, by Browder's Fixed Point Theorem, there exists a pair $(g_0, h_T) \in \overline{B^{\mathbb{R}^{2N}}(0, \frac{\rho}{K})}$ such that $f(g_0, h_T) = (g_0, h_T)$. Then,

$$\begin{cases} x_0 - \overline{x} - h(0; (g_0, h_T)) = g_0 \\ x_1 - \overline{x} - g(T; (g_0, h_T)) = h_T. \end{cases} \tag{3.156}$$

which in turn entails

$$\begin{cases} g(0) + h(0; (g_0, h_T)) = x_0 - \overline{x} \\ g(T; (g_0, h_T)) + h_T = x_1 - \overline{x}. \end{cases} \tag{3.157}$$

The above computation allows us to carry on the proof for both $(TC1)$ and $(TC2)$. In fact, let us define $(\delta\tilde{x}^T, \delta\tilde{p}^T) \overset{\text{def}}{=} L_P(g(\cdot;(g_0,h_T)), h(\cdot;(g_0,h_T)))$, we recognise that it solves

$$\frac{d}{dt}Z(t) = \begin{pmatrix} -A & -BL_{uu}(\overline{u})^{-1}B^* \\ -F_{xx}(\overline{x}) & A^* \end{pmatrix} Z(t) + \begin{pmatrix} B\Lambda_2(p_2(Z(t))) \\ -\Lambda_1(p_1(Z(t))). \end{pmatrix} \qquad \forall t \in (0,T). \tag{3.158}$$

Moreover, naming $\forall T \in (0,+\infty)$, $\tilde{u}^T \overset{\text{def}}{=} \varphi \circ (\delta\tilde{p}^T + \overline{p})$, the triple $(\delta\tilde{x}^T + \overline{x}, \delta\tilde{p}^t + \overline{p}, \tilde{u}^T) \in C^1([0,T],\mathbb{R}^N) \times C^1([0,T],\mathbb{R}^N) \times C^1([0,T],\mathbb{R}^M)$ satisfies the Pontryagin system:

$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = Bu^T(t) & \forall t \in (0,T) \\ -\frac{d}{dt}p^T(t) + A^*x^T(t) = F_x(x^T(t)) & \forall t \in (0,T) \\ L_u(u^T(t)) = -B^*p^T(t) & \forall t \in [0,T] \\ R((x^T(0),p^T(0)),(x^T(T),p^T(T))) = 0. \end{cases} \tag{3.159}$$

Therefore, by Proposition 3.1, $\forall T \in (0,+\infty)$

$$(x^T, p^T, u^T) = (\delta\tilde{x}^T + \overline{x}, \delta\tilde{p}^t + \overline{p}, \tilde{u}^T).$$

Hence,

$$\mathcal{AT} = (0,T),$$

namely the system (3.142) is satisfied $\forall t \in (0,T)$. Furthermore, $(\delta x^T, \delta p^T)$ is such that

$$\|\delta x^T(t)\| = \|g(t;(g_0,h_T))\| + \|h(t;(g_0,h_T))\| \leq \tag{3.160}$$

$$\leq C_\mu\left[\|g_0\|e^{-\mu t} + e^{-\mu(T-t)}\|h_T\|\Theta_1(\|h(\cdot;(g_0,h_T))\|_{C^0})\right] +$$
$$+C_\mu\left[\|h_T\|e^{-\mu(T-t)} + e^{-\mu t}\|g_0\|\Theta_2(\|g(\cdot;(g_0,h_T))\|_{C^0})\right]$$
$$\|\delta p^T(t)\| = \|\widehat{E}_+g(t;(g_0,h_T)) + \widehat{E}_-h(t;(g_0,h_T))\| \leq$$
$$\leq C_\mu(\|\widehat{E}_+\| + \|\widehat{E}_-\|)\left[\|g_0\|e^{-\mu t} + e^{-\mu(T-t)}\|h_T\|\Theta_1(\|h(\cdot;(g_0,h_T))\|_{C^0})\right] +$$
$$+C_\mu(\|\widehat{E}_+\| + \|\widehat{E}_-\|)\left[\|h_T\|e^{-\mu(T-t)} + e^{-\mu t}\|g_0\|\Theta_2(\|g(\cdot;(g_0,h_T))\|_{C^0})\right]. \tag{3.161}$$

which yields,

$$\|\delta x^T(t)\| \leq C_\mu\left[\|g_0\|e^{-\mu t} + e^{-\mu(T-t)}\|h_T\|\Theta_1(\rho)\right] + \tag{3.162}$$

$$+C_\mu\left[\|h_T\|e^{-\mu(T-t)} + e^{-\mu t}\|g_0\|\Theta_2(\rho)\right]$$
$$\|\delta p^T(t)\| \leq C_\mu(\|\widehat{E}_+\| + \|\widehat{E}_-\|)\left[\|g_0\|e^{-\mu t} + e^{-\mu(T-t)}\|h_T\|\Theta_1(\rho)\right] +$$

$$+C_\mu(\|\widehat{E}_+\| + \|\widehat{E}_-\|)\left[\|h_T\|e^{-\mu(T-t)} + e^{-\mu t}\|g_0\|\Theta_2(\rho)\right]. \tag{3.163}$$

At this step, we remind that

$$\|g_0\| + \|h_T\| \le \overline{r}_\mu.$$

Then, up to define $C_\mu \in (0,+\infty)$ a bigger T-independent constant,

$$\begin{cases} \|\delta x^T(t)\| \le C_\mu\left[e^{-\mu t} + e^{-\mu(T-t)}\right] \\ \|\delta p^T(t)\| \le C_\mu\left[e^{-\mu t} + e^{-\mu(T-t)}\right]. \end{cases} \tag{3.164}$$

Moreover,

$$\delta u^T = u^T - \overline{u} = \varphi(p^T) - \varphi(\overline{p})$$

Hence,

$$\|\delta u^T(t)\| \le \|\varphi(p^T) - \varphi(\overline{p})\| \le \sup_{B^{\mathbb{R}^M}(0,\rho)} \|Jac(\varphi)\|\|p^T(t) - \overline{p}\| \le \tag{3.165}$$

$$\le \sup_{B^{\mathbb{R}^M}(0,\rho)} \|Jac(\varphi)\|C_\mu\left[e^{-\mu t} + e^{-\mu(T-t)}\right]. \tag{3.166}$$

This concludes the proof of the Local Turnpike Property.

□

Remark 3.3. We observe that, by the previous proof,
$\forall \mu \in (0, -\sup\sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+))$ there exists a couple of positive constants $(C_\mu, \overline{\varepsilon}_\mu) \in (0,+\infty)^2$ such that the Local Turnpike Property holds with $(\overline{\varepsilon}_\mu, C_\mu, \mu)$.

3.4 Global Turnpike Property for NonLinear Convex Case

In this section, we will present a Global Turnpike result for the NonLinear Convex Case with initial-terminal conditions $(TC1)$, which means that there exists $x_0 \in \mathbb{R}^N$ such that the initial-terminal conditions are represented by the map:

$$R : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N} \tag{3.167}$$

$$\begin{pmatrix} x_i \\ y_i \\ x_f \\ y_f \end{pmatrix} \longrightarrow \begin{pmatrix} x_i - x_0 \\ y_f \end{pmatrix}$$

We are going to state the announced Global Turnpike Property. In the next statement, $\widehat{E}_+$ and $\widehat{E}_-$ are respectively the positive definite and the negative definite solution of the Algebraic Riccati Equation:

$$\widehat{E}A + A\widehat{E} + \widehat{E}BL_{uu}(\overline{u})^{-1}B^*\widehat{E} - F_{xx}(\overline{x}) = 0 \qquad \text{(ARE)}. \tag{3.168}$$

Theorem 3.4 (Global Turnpike Property)**.** *We take into account* 2 *positive real numbers* $(\alpha, \beta) \in (0, +\infty)^2$, 2 *functions*

$$F : \mathbb{R}^N \longmapsto \mathbb{R},$$

$$L : \mathbb{R}^M \longmapsto \mathbb{R}$$

and a pair of matrices $(A, B) \in \mathcal{M}(N, N; \mathbb{R}) \times \mathcal{M}(N, M; \mathbb{R})$ *Kalman-Controllable. We suppose that:*

- $(F, L) \in C^2(\mathbb{R}^N, \mathbb{R}) \times C^2(\mathbb{R}^M, \mathbb{R})$;
- $\forall u \in \mathbb{R}^M \quad \alpha I_M \leq L_{uu}(u) \leq \beta I_M$ *and* $\forall x \in \mathbb{R}^N \alpha I_N \leq F_{xx}(x)$;
- *initial-terminal conditions* $(TC1)$;
- $T \in (0, +\infty)$.

Furthermore, we consider the unique positive definite $\widehat{E}_+$ *and negative definite* $\widehat{E}_-$ *solutions of the Algebraic Riccati Equation:*

$$\widehat{E}A + A\widehat{E} + \widehat{E}BL_{uu}(\overline{u})^{-1}B^*\widehat{E} - F_{xx}(\overline{x}) = 0 \qquad \textit{(ARE)}. \tag{3.169}$$

Then, $\forall \mu \in (0, -\sup \sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+))$, *there exists a positive real number* $C_\mu \in (0, +\infty)$ *such that the optimal triple* (x^T, p^T, u^T) *satisfies the following estimate:*

$$\|x^T(t) - \overline{x}\| + \|p^T(t) - \overline{p}\| + \|u^T(t) - \overline{u}\| \leq C\left[e^{-\mu t} + e^{-\mu(T-t)}\right] \qquad \forall t \in [0, T]. \tag{3.170}$$

Proof. In order to prove this Theorem, we remind that, by Theorem 3.2, there exists a T independent constant $C \in (0, +\infty)$ such that $\forall T \in (0, +\infty)$

$$\int_0^T \|x^T(t) - \overline{x}\|^2 + \|u^T(t) - \overline{u}\|^2 dt \leq C$$

At this stage, $\forall \mu \in (0, -\sup \sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+))$ we take into account $\overline{\varepsilon}_\mu$ defining the smallness condition (3.136) in Theorem 3.3. Then, we define, $\forall \mu \in (0, -\sup \sigma(-A - BL_{uu}(\overline{u})^{-1}B^*\widehat{E}_+))$, $T_\mu = \frac{4C}{\overline{\varepsilon}_\mu^2}$. First of all, by the Mean

Value Theorem for Integrals 1.11, $\forall T \in [T_\mu, +\infty)$ there exists $(t_1^T, t_2^T) \in [0, T_\mu] \times [T - T_\mu, T]$ such that:

$$\begin{cases} \frac{1}{T_\mu} \int_0^{T_\mu} \left\| x^T(s) - \overline{x} \right\|^2 ds = \left\| x^T(t_1^T) - \overline{x} \right\|^2 \\ \frac{1}{T_\mu} \int_{T-T_\mu}^{T} \left\| x^T(s) - \overline{x} \right\|^2 ds = \left\| x^T(t_2^T) - \overline{x} \right\|^2 . \end{cases} \tag{3.171}$$

Therefore, by the Theorem 3.2 of Convergence of Averages, we get:

$$\frac{1}{T_\mu} \int_0^{T_\mu} \left\| x^T(t) - \overline{x} \right\|^2 dt \leq \tag{3.172}$$

$$\leq \frac{1}{T_\mu} \int_0^{T} \left\| x^T(t) - \overline{x} \right\|^2 + \left\| u^T(t) - \overline{u} \right\|^2 dt \leq \frac{1}{T_\mu} C = \frac{\varepsilon_\mu^2}{4}.$$

Similarly, $\forall T \in [T_\mu, +\infty)$

$$\frac{1}{T_\mu} \int_{T-T_\mu}^{T} \| x^T(t) - \overline{x} \|^2 dt \leq \frac{\varepsilon_\mu^2}{4}.$$

By the above deductions, $\forall \mu \in (0, -\sup \sigma(-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_+))$ there exists $T_\mu \in (0, +\infty)$ such that $\forall T \in [T_\mu, +\infty)$:

$$\begin{cases} \left\| x^T(t_1^T) - \overline{x} \right\| \leq \frac{\overline{\varepsilon}_\mu}{2} \\ \left\| x^T(t_2^T) - \overline{x} \right\| \leq \frac{\overline{\varepsilon}_\mu}{2}. \end{cases} \tag{3.173}$$

We remind that in Theorem 3.3, $\forall \mu \in (0, -\sup \sigma(-A - BL_{uu}(\overline{u})^{-1} B^* \widehat{E}_+))$ the constant C_μ is independent of the initial-terminal condition fulfilling the smallness condition with $\overline{\varepsilon}_\mu$. Therefore, $\forall T \in [T_\mu, +\infty)$ we consider the terminal conditions $(TC2)$ defined by:

$$R_T : \mathbb{R}^{4N} \longmapsto \mathbb{R}^{2N} \in C^2(\mathbb{R}^{4N}, \mathbb{R}^{2N})$$

$$\begin{pmatrix} x_i \\ y_i \\ x_f \\ y_f \end{pmatrix} \longrightarrow \begin{pmatrix} x_i - x^T(t_{1,T}) \\ x_f - x^T(t_{2,T}). \end{pmatrix}$$

Then, we take into account, $\forall V \in (0, +\infty)$ time horizon, $(OCP)^V$ with terminal conditions $(TC2)$ defined by R_T above. Let us define $\forall V \in (0, +\infty)$ (y^V, q^V, w^V) the optimal triple for $(OCP)^V$ with terminal conditions defined by R_T. Hence, by Theorem 3.3, we obtain:

$$\|y^V(t) - \overline{x}\| + \|q^T(t) - \overline{p}\| + \|w^T(t) - \overline{u}\| \leq C_\mu \left[e^{-\mu t} + e^{-\mu(V-t)} \right] \qquad \forall t \in [0, V]. \tag{3.174}$$

Since, by Remark 3.2, $(x^T(\cdot+t_{1,T}), p^T(\cdot+t_{1,T}), u^T(\cdot+t_{1,T})) = (y^{t_{2,T}-t_{1,T}}, q^{t_{2,T}-t_{1,T}}, w^{t_{2,T}-t_{1,T}})$,

$$\|x^T(t;x_0)-\overline{x}\|+\|p^T(t;x_0)-\overline{p}\|+\|u^T(t;x_0)-\overline{u}\| \leq C_\mu \left[e^{-\mu(t-t_{1,T})} + e^{-\mu((t_{2,T}-t_{1,T})-t+t_{1,T})}\right] \leq \tag{3.175}$$
$$\leq C_\mu \left[e^{-\mu(t-T_\mu)} + e^{-\mu(T-T_\mu-t)}\right] \qquad \forall t \in [t_{1,T}, t_{2,T}].$$

At this point, by (3.101),

$$\left\{x^T(t) - \overline{x} \mid T \in (0,+\infty),\ t \in [0,T]\right\} \subset \overline{B^{\mathbb{R}^N}(0,M)}$$

Moreover, (3.109) and $L_{uu} \leq \beta I_M$ entail that there exists a T independent constant $M_1 \in (0,+\infty)$ such that:

$$\|p^T(0) - \overline{p}\|^2 \leq C \left[\int_0^T \|u^T(t) - \overline{u}\|^2 dt + \int_0^T \|x^T(t) - \overline{x}\|^2 dt + \|\overline{p}\|^2\right] \leq M_1^2. \tag{3.176}$$

Finally, $\forall T \in (0,+\infty)$ and $\forall t \in [0,T]$:

$$\alpha\|u^T(t) - 0\| \leq \|L_u(u^T(t)) - L_u(0)\| \leq \|L_u(u^T(t))\| + \|L_u(0)\| \leq \tag{3.177}$$
$$\leq \|B^*\|\|p^T(t)\| + \|L_u(0)\| \leq \|B^*\|M_1 + \|L_u(0)\|.$$

Then, there exists a positive constant $M_2 \in (0,+\infty)$ independent of $T \in (0,+\infty)$ such that:

$$\left\{u^T(t) - \overline{u} \mid T \in (0,+\infty),\ t \in [0,T]\right\} \subset \overline{B^{\mathbb{R}^N}(0,M_2)}$$

Therefore, defining

$$C_{\mu,new} \stackrel{\text{def}}{=} \max\left\{C_\mu e^{2T_\mu}, Me^{2T_\mu} + M_1 e^{2T_\mu} + M_2 e^{2T_\mu}\right\},$$

we have $\forall T \in (0,+\infty)$

$$\|x^T(t;x_0)-\overline{x}\|+\|p^T(t;x_0)-\overline{p}\|+\|u^T(t;x_0)-\overline{u}\| \leq C_{\mu,new} \left[e^{-\mu t} + e^{-\mu(T-t)}\right] \qquad \forall t \in [0,T]. \tag{3.178}$$

as required.

□

Chapter 4

Infinite Dimensional Linear Quadratic Case

In this chapter, we generalise the results of the Finite Dimensional Linear Quadratic Case to the Infinite Dimensional Linear Quadratic Case in a Parabolic Setting.

All throughout this chapter we will follow the dynamic approach of [16].

We consider $(X, (,)_X)$ and $(H, (,)_H)$ Hilbert spaces, such that there exists a dense inclusion, i.e.:

$$i : X \hookrightarrow H \in B(X, H)$$

one to one with $\overline{i(X)}^H = H$. As we have explained in Preliminaries, this inclusion leads to the scheme:

$$X \subseteq H \cong H' \subseteq X'.$$

Now, we take into account $(U, (,)_U)$ and $(V, (,)_V)$ Hilbert spaces. We will name U control space and V observation space. We are going now to introduce the operators involved in our analysis.

1. $A \in B(X, X')$ a bounded linear operator such that $R(A)$, the range of A, is closed into X';
2. $B \in B(U, X')$ a bounded linear operator called control operator;
3. $C \in B(X, V)$ a bounded linear operator named observation operator.

We are going to ask some hypotheses on A, which we will use all throughout the chapter.

Hypothesis 4.1 (Weakened coercivity of A).

$$\exists (\lambda, \mu) \in (0, +\infty)^2 \text{ such that } \qquad < Ax, x >_{(X', X)} + \mu \|x\|_H^2 \geq \lambda \|x\|_X^2$$

for all $x \in X$. This hypothesis is equivalent to ask that there exists $\mu \in (0, +\infty)$ such that the operator $A + \mu I_H$ is strongly coercive.

Using this hypothesis it is possible to solve a Banach space valued Cauchy Problem. We are going to find solutions to that problem in the Sobolev space $W^{1,2}((0,T);(X,X'))$ (see Definition 1.16). By Theorem 1.15, we get the inclusion

$$i : W^{1,2}((0,T);(X,X')) \hookrightarrow C^0([0,T],H) \in B(W^{1,2}((0,T);(X,X')), C^0([0,T],H)).$$

Then, whenever we work with a function $y \in W^{1,2}((0,T);(X,X'))$, we can choose the unique representative $y \in C^0([0,T],H)$. Therefore, for every $t \in [0,T]$ the expression $y(t)$ makes sense. Then, in the following Cauchy Problem, the initial condition $y(0) = y_0$ is well posed.

Definition 4.1. For every $(A, f, x_0) \in B(X,X') \times L^2((0,T);X') \times H$, $y \in W^{1,2}((0,T);(X,X'))$ is said to be a solution of the equation

$$\frac{d}{dt}y + Ay = f \qquad \text{in } (0,T)$$

if and only if $\forall \varphi \in W^{1,2}((0,T);(X,X'))$

$$(\varphi(T), y(T))_H - (\varphi(0), y(0))_H + \int_0^T < -\frac{d}{dt}\varphi(t) + A^*\varphi(t), y(t) >_{(X',X)} dt =$$

$$= \int_0^T < f(t), \varphi(t) >_{(X',X)} dt$$

Theorem 4.1. *Let us consider*

$$A : (0,T) \longmapsto B(X,X')$$

such that:

1. *for every* $(x_1, x_2) \in X^2$

$$< Ax_1, x_2 >_{(X',X)}: (0,T) \longmapsto \mathbb{R}$$

$$t \longrightarrow < A(t)x_1, x_2 >_{(X',X)}$$

   is measurable;

2. *there exist* 2 *constants* $(\mu, \lambda) \in (0, +\infty)^2$ *such that* $\forall t \in (0, +\infty)$

$$< A(t)x, x >_{(X',X)} + \mu\|x\|_H^2 \geq \lambda\|x\|_X^2 \qquad \forall x \in X. \tag{4.1}$$

Then, for every $f \in L^2((0,T);X')$ *and* $y_0 \in H$*, there exists a unique solution* $y \in W^{1,2}((0,T);(X,X'))$ *of the Initial Value Problem*

$$\begin{cases} \frac{d}{dt}y + Ay = f & \text{in } (0,T) \\ y(0) = y_0 \end{cases} \tag{4.2}$$

Moreover, a continuous dependence on the data holds, i.e.:

$$\Xi : H \times L^2((0,T);X') \longmapsto W^{1,2}((0,T),(X,X'))$$

$$(y_0, f) \longrightarrow y$$
$$\in B(H \times L^2((0,T);X'), W^{1,2}((0,T);(X,X')).$$

A proof for this theorem can be found in [14] at page 103. $\forall y_0 \in H$ the unique solution $y \in W^{1,2}((0,T);(X,X'))$ of the Cauchy Problem

$$\begin{cases} \frac{d}{dt}y + Ay = 0 & \text{in } (0,T) \\ y(0) = y_0 \end{cases} \tag{4.3}$$

is often indicated as $e^{-A(\cdot)}y_0$. By Theorem 4.1, for instance, whenever we consider an initial data $x_0 \in H$ and control function $u \in L^2((0,T);U)$, there exists a unique solution $x \in W^{1,2}((0,T);(X,X'))$ for the following Cauchy Problem

$$\begin{cases} \frac{d}{dt}x + Ax = Bu & \text{in } (0,T) \\ x(0) = x_0 \end{cases} \tag{4.4}$$

and the operator

$$R : H \times L^2((0,T);U) \longmapsto W^{1,2}((0,T),(X,X'))$$

$$(x_0, u) \longrightarrow x$$
$$\in B(H \times L^2((0,T);U), W^{1,2}((0,T);(X,X'))$$

Let us now present the Duhamel's representation formula in this framework. First of all, we give a "forward" version. We will use the notation of Theorem 4.1.

Proposition 4.1 (Duhamel's representation formula)**.** *We suppose:*

$$i : X \hookrightarrow H \in K(X,H).$$

Moreover, we consider:

$$A : X \longmapsto X' \in B(X,X')$$

such that hypothesis 4.1 holds. Then, for every $f \in L^\infty((0,T);H)$ *and* $y_0 \in H$*, the unique solution* $\Xi(y_0, f) \in W^{1,2}((0,T);(X,X'))$ *of the Initial Value Problem*

$$\begin{cases} \frac{d}{dt}y + Ay = f & \text{in } (0,T) \\ y(0) = y_0, \end{cases} \tag{4.5}$$

can be written as follows:

$$\Xi(y_0, f)(t) = \Xi(y_0, 0)(t) + \int_0^t \Xi(f(s), 0)(t-s)ds \qquad \forall t \in [0,T]$$

where the above integral is intended as an integral of functions in $L^1((0,T);H)$*.*

One can deduce this representation formula by reducing the above Cauchy Problem to a finite dimensional one. Then, one applies, in the finite dimensional framework, the Duhamel Formula as stated at page 488 of [18]. After that, passing to the limit, it is possible to show the required result. Moreover, we take into account some perturbations of the above Cauchy Problem.

Corollary 4.1. *Let* $T \in (0, +\infty)$*,*

$$A : (0,T) \longrightarrow B(X, X')$$

measurable and weakly coercive as in Theorem 4.1 and a perturbation operator

$$D : (0,T) \longrightarrow B(H, H)$$

such that $\forall (h_1, h_2) \in H^2$*, the map*

$$(Dh_1, h_2)_H : (0,T) \longrightarrow \mathbb{R}$$

is measurable and the set $\{D(t) \mid t \in (0,T)\}$ *is bounded in* $B(H,H)$*. Then,* $\forall (x_0, f) \in H \times L^2((0,T);X')$ *there exists a unique solution* $x \in W^{1,2}((0,T);(X,X'))$ *to the Cauchy Problem*

$$\begin{cases} x_t + (A + D)x = f & \text{in } (0,T) \\ x(0) = x_0 \end{cases} \tag{4.6}$$

Proof. In fact, we take into account the $B(X,X')$ valued function

$$A + D : (0,T) \longmapsto B(X, X'),$$

which is weakly coercive. Hence, employing Theorem 4.1, we get existence, uniqueness and continuous dependence from the data for the system above. □

Remark 4.1. We observe that whenever $f \in L^2_{loc}((0,+\infty);X')$ we can apply Corollary 4.1 for each $T \in (0,+\infty)$ and obtain the existence and the uniqueness of a solution $x \in W^{1,2}_{loc}((0,+\infty);(X,X'))$ of

$$\begin{cases} x_t + (A+D)x = f & \text{in } (0,+\infty) \\ x(0) = x_0 \end{cases} \tag{4.7}$$

Moreover, the following stability Lemma will be useful later.

Lemma 4.1. *Let* $T \in (0,+\infty)$,

$$A : (0,T) \longrightarrow B(X,X')$$

measurable and weakly coercive as in Theorem 4.1 and such that it exponentially stabilizes, i.e. there exist 2 constants $(C,\mu) \in (0,+\infty)^2$ *such that* $\forall x_0 \in H$ *the unique solution* $x \in W^{1,2}((0,T);(X,X'))$ *of*

$$\begin{cases} x_t + Ax = 0 & \textit{in } (0,T) \\ x(0) = x_0 \end{cases} \tag{4.8}$$

satisfies

$$\|x(t)\|_H \leq Ce^{-\mu t}\|x_0\|_H.$$

Then, A^* *exponentially stabilizes, namely* $\forall y_0 \in H$ *the unique solution* $y \in W^{1,2}((0,T);(X,X'))$ *of*

$$\begin{cases} y_t + A^*y = 0 & \textit{in } (0,T) \\ y(0) = y_0 \end{cases} \tag{4.9}$$

satisfies

$$\|y(t)\|_H \leq Ce^{-\mu t}\|x_0\|_H.$$

Proof. For every $y_0 \in H$, we consider the unique solution $y \in W^{1,2}((0,T);(X,X'))$ of (4.9). Furthermore, $\forall(\varphi_0,t) \in H \times (0,T)$ we take into account the unique solution φ of the problem

$$\begin{cases} -\varphi_t + A\varphi = 0 & \text{in } (0,t) \\ \varphi(t) = \varphi_0 \end{cases} \tag{4.10}$$

We multiply in (X',X) the equation of φ by y and integrating in $[0,t]$, we get:

$$0 = \int_0^t < -\varphi_t(s) + A\varphi(s), y(s) >_{(X',X)} ds =$$

integrating by parts

$$= - (\varphi(t), y(t))_H + (\varphi(0), y(0))_H + \int_0^t < \varphi(s), y_t(s) + A^* y(s) >_{(X,X')} ds =$$

$$= - (\varphi(t), y(t))_H + (\varphi(0), y(0))_H .$$

This yields

$$(\varphi(t), y(t))_H = (\varphi(0), y(0))_H \tag{4.11}$$

We notice that $\varphi(\cdot - t)$ is a solution of

$$\begin{cases} \tilde{\varphi}_t + A\tilde{\varphi} = 0 & \text{in } (0,t) \\ \tilde{\varphi}(0) = \tilde{\varphi}_0 \end{cases} \tag{4.12}$$

If we take into account $(C, \mu) \in (0, +\infty)^2$ characterising constants of the exponential stability for A, the fulfillment of the above system yields:

$$\|\varphi(0)\|_H \leq Ce^{-\mu t} \|\varphi_0\|_H$$

Coming back to the equation (4.11) and choosing $\varphi_0 = y(t)$, we obtain:

$$\|y(t)\|_H^2 = \|\varphi(0)\|_H \|y(0)\|_H \leq Ce^{-\mu t} \|y(t)\|_H \|y_0\|_H$$

This concludes the proof.

□

In the Finite Dimensional Linear Quadratic Case the Observability hypothesis was essential. A similar hypothesis on (A, C) will play a key role in the Infinite Dimensional Case too. We are going to formulate it in the following.

Hypothesis 4.2 (Observability Inequality). There exists $C \in (0, +\infty)$, independent of $T \in (0, +\infty)$, such that for every $(T, f, y_0, y) \in (0, +\infty) \times L^2((0,T); X') \times H \times W^{1,2}((0,T); (X, X'))$ satisfying

$$\begin{cases} \frac{d}{dt} y + Ay = f & \text{in } (0,T) \\ y(0) = y_0, \end{cases} \tag{4.13}$$

we have:

$$\|y(T)\|_H^2 \leq C \left[\int_0^T \left(\|f\|_{X'}^2 + \|Cy\|_V^2 \right) dt + \|y_0\|_H^2 \right] \tag{4.14}$$

We are going to show some situations where the above Observability Inequality holds. The first case is when A itself is coercive.

Remark 4.2. We assume the coercivity of A, namely:

$$\exists\lambda \in (0, +\infty) \text{ such that } \qquad < Ax, x >_{(X',X)} \geq \lambda \|x\|_X^2 \qquad \forall x \in X.$$

Then, there exists a positive constant C such that for every $(T, f, y_0, y) \in (0, +\infty) \times L^2((0,T); X') \times H \times W^{1,2}((0,T); (X, X'))$ satisfying (4.13), it holds:

$$\|y(T)\|_H^2 \leq C \left[\int_0^T \|f\|_{X'}^2 dt + \|y_0\|_H^2 \right]$$

Proof. Let us consider the dual pair (X, X'). For almost every $t \in (0,T)$ we multiply the equation with y and we integrate in $[0,T]$, getting:

$$\int_0^T < y_t(t), y(t) >_{(X',X)} dt + \int_0^T < Ay(t), y(t) >_{(X',X)} dt = \int_0^T < f(t), y(t) >_{(X',X)} dt$$

Integrating by parts and using the coercivity of A:

$$\frac{1}{2} \left[\|y(T)\|_H^2 - \|y(0)\|_H^2 \right] + \lambda \int_0^T \|y(t)\|_X^2 dt \leq \int_0^T < y_t(t) + Ay(t), y(t) >_{(X',X)} dt =$$

$$= \int_0^T < f(t), y(t) >_{(X',X)} dt \leq$$

By the definition of continuous functional, we have:

$$\leq \int_0^T \|f(t)\|_{X'} \|y\|_X dt \leq$$

Using the Young's inequality with ε,

$$\leq \int_0^T C \|f(t)\|_{X'}^2 dt + \frac{\lambda}{2} \int_0^T \|y(t)\|_X^2 dt$$

Then,

$$\frac{1}{2} \left[\|y(T)\|_H^2 - \|y(0)\|_H^2 \right] + \lambda \int_0^T \|y(t)\|_X^2 dt - \frac{\lambda}{2} \int_0^T \|y(t)\|_X^2 dt \leq C \int_0^T \|f(t)\|_{X'}^2 dt$$

Which yields

$$\|y(T)\|_H^2 \leq C \left[\int_0^T \|f(t)\|_{X'}^2 dt + \|y_0\|_H^2 \right]$$

as required. □

Another circumstance that can occur is the object of the next Remark.

Remark 4.3. Let $(A, C) \in B(X, X') \times B(X, V)$ be such that there exists $(C_0, T_0) \in (0, +\infty)$ such that for every $(y_0, y) \in H \times W^{1,2}((0, T_0); (X, X'))$ satisfying

$$\begin{cases} \frac{d}{dt}y + Ay = 0 & \text{in } (0, T_0) \\ y(0) = y_0, \end{cases} \tag{4.15}$$

we have:

$$\|y(T_0)\|_H^2 \leq C_0 \left[\int_0^{T_0} \|Cy\|_V^2 dt \right]. \tag{4.16}$$

Then, the hypothesis 4.2 holds true.

Proof. 1st Step

To achieve this result we use the same technique used in the proof of Lemma 2.1 in the finite dimensional case. For any $(z_0, h) \in H \times L^2((0, T_0); X')$, let us introduce $w_1 \in W^{1,2}((0, T_0); (X, X'))$ solution of the Cauchy Problem

$$\begin{cases} \frac{d}{dt}w + Aw = h & \text{in } (0, T_0) \\ w(0) = 0 \end{cases} \tag{4.17}$$

and $w_2 \in W^{1,2}((0, T_0); (X, X'))$ the unique solution of

$$\begin{cases} \frac{d}{dt}z + Az = 0 & \text{in } (0, T_0) \\ z(0) = z_0. \end{cases} \tag{4.18}$$

The continuous dependence from the data implies the existence of a positive constant C such that

$$\sup_{s \in [0, T_0]} \|w_1(s)\|_H^2 \leq C \int_0^{T_0} \|h(t)\|_{X'}^2 dt.$$

By the uniqueness, the solution for the Cauchy Problem (4.15) is given by $z = w_1 + w_2$. This enables us to make the following estimates:

$$\|z(T_0)\|_H^2 \leq 2\|w_1(T_0)\|_H^2 + 2\|w_2(T_0)\|_H^2 \leq$$

$$\leq C \int_0^{T_0} \|Cw_2(s)\|_V^2 ds + 2\|w_1(T_0)\|_H^2 =$$

$$= C \int_0^{T_0} \|C(z(s) - w_1(s))\|_V^2 ds + 2\|w_1(T_0)\|_H^2 \leq$$

$$\leq C \int_0^{T_0} \left[\|Cz(s)\|_V^2 + \|h(s)\|_{X'}^2 \right] ds.$$

Now, we go ahead with the proof.
2nd Step
Firstly, we take into account $T \in [T_0, +\infty)$. Then, for any $(f, y_0, y) \in L^2((0,T); X') \times H \times W^{1,2}((0,T); (X, X'))$ satisfying (4.15), we apply the 1st Step in $(T - T_0, T)$, obtaining:

$$\|y(T)\|_H^2 \leq C_0 \left[\int_{T-T_0}^{T} \left(\|f(t)\|_{X'}^2 + \|Cx\|_V^2 \right) dt \right] \leq$$

$$\leq C_0 \left[\int_0^T \left(\|f(t)\|_{X'}^2 + \|Cx\|_V^2 \right) dt \right].$$

3rd Step
Secondly, $\forall T \in (0, T_0)$, we apply the continuous dependence from the data.
□

In Controllability Theory, an inequality like (4.16) is often called "inverse inequality". For instance, see [12].

Remark 4.4 (Stationary Observability Inequality). Whenever hypothesis 4.2 holds true, there exists a constant $C \in (0, +\infty)$ such that:

$$\|x\|_X^2 \leq C \left[\|Ax\|_{X'}^2 + \|Cx\|_V^2 \right] \qquad \forall x \in X$$

Proof. For every $x \in X$, we define

$$\tilde{y} : \mathbb{R} \longmapsto X$$

$$t \longrightarrow tx$$

Applying to $\tilde{y}$ the hypothesis 4.2, we deduce the following inequality $\forall T \in (0, +\infty)$

$$T^2 \|x\|_H^2 \leq C \frac{T^3}{3} \left[\|Ax\|_{X'}^2 + \|Cx\|_V^2 \right] + CT \|x\|_{X'}^2 \leq$$

$$\leq C \frac{T^3}{3} \left[\|Ax\|_{X'}^2 + \|Cx\|_V^2 \right] + CT \|x\|_H^2.$$

It is sufficient to take $T \in (C, +\infty)$ to get the existence of a positive constant D such that:

$$\|x\|_H^2 \leq D \left[\|Ax\|_{X'}^2 + \|Cx\|_V^2 \right] \qquad \forall x \in X.$$

At this stage, by the hypothesis 4.1, there exists a constant $C \in (0, +\infty)$ such that:

$$\|x\|_X^2 \leq C \left[\|Ax\|_{X'} \|x\|_X + \|x\|_H^2 \right] \leq$$

thanks to Young's inequality with ε

$$\leq C\left[C\|Ax\|_{X'}^2 + \varepsilon\|x\|_X^2 + \|x\|_H^2\right]$$

which implies

$$\|x\|_X^2 \leq C\left[\|Ax\|_{X'}^2 + \|Cx\|_V^2\right].$$

This concludes the proof.

□

Now, we are interested to deal with the adjoint state equation. First of all, we notice that, if the hypothesis 4.1 is satisfied, the weakened coercivity holds even for the adjoint operator A^*, i.e.

$$\exists(\lambda,\mu) \in (0,+\infty)^2 \text{ such that } \qquad < A^*x, x >_{(X',X)} + \mu\|x\|_H^2 \geq \lambda\|x\|_X^2$$

Theorem 4.1 enables us to affirm that for any $p_0 \in H$ and for every $f \in L^2((0,T);X')$ there exists a unique $p \in W^{1,2}((0,T);(X,X'))$ solution of the Cauchy Problem

$$\begin{cases} -\frac{d}{dt}p + A^*p = f & \text{in } (0,T) \\ p(T) = p_0. \end{cases} \tag{4.19}$$

Furthermore, the continuous dependence from the data holds. Namely, as in the "forward" case, we define:

$$\Psi : H \times L^2((0,T);X') \longmapsto W^{1,2}((0,T);(X,X')) \tag{4.20}$$

$$(p_0, f) \longrightarrow p.$$

Then,

$$\Psi \in B(H \times L^2((0,T);X'), W^{1,2}((0,T);(X,X')).$$

We give now a "backward" version of Duhamel's representation formula.

Proposition 4.2 (Duhamel's representation formula)**.** *We assume*

$$i : X \hookrightarrow H \in K(X,H).$$

Furthermore, we take into account:

$$A : X \longmapsto X' \in B(X,X')$$

fulfilling Hypothesis 4.1. Then, for each $f \in L^\infty((0,T);H)$ and $p_T \in H$, the unique solution $p \in W^{1,2}((0,T);(X,X'))$ of the Final Value Problem

$$\begin{cases} -\frac{d}{dt}p + A^*p = f & \text{in } (0,T) \\ y(T) = p_T, \end{cases} \tag{4.21}$$

admits the following representation formula:

$$\Psi(p_T, f)(t) = \Psi(p_T, 0)(t) + \int_t^T \Psi(f(s), 0)(T-(s-t))ds \qquad \forall t \in [0,T]$$

where the above integral is intended as an integral of functions belonging to $L^1((0,T);H)$*.*

For the dual problem we need an Observability Inequality as well.

Hypothesis 4.3 (Observability Inequality for the dual problem)*.* There exists $C \in (0,+\infty)$, independent of $T \in (0,+\infty)$, such that for every $(T, f, p_0, p) \in (0,+\infty) \times L^2((0,T);X') \times H \times W^{1,2}((0,T);(X,X'))$ satisfying

$$\begin{cases} -\frac{d}{dt}p + A^*p = f & \text{in } (0,T) \\ p(T) = p_0, \end{cases} \tag{4.22}$$

it holds:

$$\|p(0)\|_H^2 \le C \left[\int_0^T \left(\|f\|_{X'}^2 + \|B^*p\|_{U'}^2 \right) dt + \|p_0\|_H^2 \right]. \tag{4.23}$$

As in the Cauchy Problem satisfied by the state, one can prove that whenever the operator A^* is coercive or an "inverse inequality" for (A^*, B^*) like that of Remark 4.3 holds, the hypothesis 4.3 turns out to be true. Moreover, we can establish a stationary inequality similar as in Remark 4.4. This is the object of the following Remark.

Remark 4.5 (Stationary Observability Inequality)*.* Under hypothesis 4.3, there exists a positive constant C such that:

$$\|x\|_X^2 \le C \left[\|A^*x\|_{X'}^2 + \|B^*x\|_{U'}^2 \right] \qquad \forall x \in X$$

If (A, B) is stabilizable, then hypothesis 4.3 holds true. This is the object of the next Lemma.

Lemma 4.2. *Suppose* (A,B) *is stabilizable, i.e. there exists* $F \in B(X,U)$ *a stabilizing feedback function and* $\delta \in (0,+\infty)$*, such that* $\forall (x_0, T) \in H \times (0,+\infty)$ *there exists a unique solution* $x \in W^{1,2}((0,T);(X,X'))$ *of the Cauchy Problem*

$$\begin{cases} \frac{d}{dt}x + Ax = BFx & \text{in } (0,T) \\ x(0) = x_0; \end{cases} \tag{4.24}$$

and this solution satisfies:

$$\int_0^T \|x(t)\|_X^2 dt \le \delta \|x_0\|_H^2. \tag{4.25}$$

Then, hypothesis 4.3 turns out to be true.

Proof. In order to get the desired result, one considers for each $(f, p_0) \in L^2((0,T);X') \times H$ an arbitrary $\varphi_0 \in H$ and the corresponding φ solution of the problem

$$\begin{cases} \frac{d}{dt}x + Ax = BFx & \text{in } (0,T) \\ x(0) = \varphi_0 \end{cases} \tag{4.26}$$

Now, one multiplies the equation of p by φ and integrates by parts, getting:

$$(p(0), \varphi(0))_H + \int_0^T < \varphi_t + A\varphi, p >_{(X',X)} dt = \int_0^T < f, \varphi >_{(X',X)} dt + (p_0, \varphi(T))_H$$

Therefore,

$$(p(0), \varphi_0)_H = -\int_0^T < B^* p, F\varphi >_{(U',U)} dt + \int_0^T < f, \varphi >_{(X',X)} dt + (p_0, \varphi(T))_H \leq$$

$$\leq C \left(\int_0^T \|\varphi\|_X^2 dt \right)^{\frac{1}{2}} \left\{ \int_0^T \|B^* p\|_{U'}^2 dt + \int_0^T \|f\|_{X'}^2 dt \right\}^{\frac{1}{2}} + (p_0, \varphi(T))_H \leq$$

using the hypothesis

$$\leq C\|\varphi_0\|_H \left\{ \int_0^T \|B^* p\|_{U'}^2 dt + \int_0^T \|f\|_{X'}^2 dt \right\}^{\frac{1}{2}} + (p_0, \varphi(T))_H .$$

By using (4.25),

$$\frac{1}{2}\|\varphi(T)\|_H^2 = \frac{1}{2}\|\varphi_0\|_H^2 + \int_0^T < BF\varphi - A\varphi, \varphi >_{(X',X)} dt \leq$$

$$\leq \frac{1}{2}\|\varphi_0\|_H^2 + C \int_0^T \|\varphi\|_X^2 dt \leq C\|\varphi_0\|_H^2$$

Then,

$$(p(0), \varphi_0)_H \leq C\|\varphi_0\|_H \left\{ \int_0^T \|B^* p\|_{U'}^2 dt + \int_0^T \|f\|_{X'}^2 dt \right\}^{\frac{1}{2}} + C\|p_0\|_H \|\varphi_0\|_H$$

Finally, defining $\varphi_0 = p(0)$, one obtains

$$\|p(0)\|_H \leq C \left\{ \int_0^T \|B^* p\|_{U'}^2 dt + \int_0^T \|f\|_{X'}^2 dt + \|p_0\|_H \right\}^{\frac{1}{2}}$$

□

We are going now to introduce the Optimal Control Problems in the functional framework already described and assuming hypotheses 4.1, 4.2 and 4.3. As we did in the finite dimensional case, for every initial data $x_0 \in H$ and for each control function $u \in L^2((0,T);U)$ we consider the unique $x \in W^{1,2}((0,T);(X,X'))$ solution of the Cauchy Problem

$$\begin{cases} \frac{d}{dt}x + Ax = Bu & \text{in } (0,T) \\ x(0) = x_0 \end{cases} \tag{4.27}$$

We will call, for every target $z \in V$, $(OCP)^T$ the problem of minimizing the functional

$$J^T : L^2((0,T);U) \longmapsto \mathbb{R} \tag{4.28}$$
$$u \longrightarrow \frac{1}{2}\int_0^T \left(\|u(t)\|_U^2 + \|Cx(t) - z\|_V^2\right) dt$$

This functional admits a unique minimizer and it satisfies the Pontryagin system, i.e. there exists a unique optimal triple $(x^T, p^T, u^T) \in W^{1,2}((0,T);(X,X')) \times W^{1,2}((0,T);(X,X')) \times L^2((0,T);U)$ such that:

$$\begin{cases} x_t^T + Ax^T = Bu^T & \text{in } (0,T) \\ -p_t^T + A^*p^T = C^*\Phi_V(Cx^T - z) & \text{in } (0,T) \\ u^T = -\Phi_U^{-1}B^*p^T \\ x^T(0) = x_0 \\ p^T(T) = 0 \end{cases} \tag{4.29}$$

Thanks to the third relation, if $B \in B(U,H)$, the optimal control $u^T \in C^0([0,T],U)$.

In order to introduce the stationary problem, we define the following closed vector subspace of $X \times U$.

Definition 4.2.

$$M = \{(x,u) \in X \times U \mid Ax = Bu\}$$

The problem consists in minimizing the map

$$J^s : M \longmapsto \mathbb{R}$$
$$(x,u) \longrightarrow \frac{1}{2}\left[\|u\|_U^2 + \|Cx - z\|_V^2\right]$$

We face now the problem of the well posedeness of the stationary problem, the existence and uniqueness of the minimizer and we find out some first order conditions satisfied by the minimizer. Whenever it exists and it is unique, the minimizer $(\overline{x}, \overline{u})$ is named the optimal pair for $(OCP)^s$.

Lemma 4.3. *Under our assumptions, there exists a unique* $(\overline{x}, \overline{u}) \in X \times U$ *optimal pair for* $(OCP)^s$ *satisfying the first order condition*

$$\begin{cases} A\overline{x} = B\overline{u} \\ (\overline{u}, v)_U + (C\overline{x} - z, C\varphi)_V = 0 \qquad \forall (\varphi, v) \in X \times U \ : \ A\varphi = Bv \end{cases} \tag{4.30}$$

Furthermore, there exists some $\overline{p} \in X$ *such that*

$$A^*\overline{p} = C^*\Phi_V(C\overline{x} - z) \tag{4.31}$$

and therefore

$$(\overline{u}, v)_U + (\overline{p}, \Phi_X^{-1}Bv)_X = 0 \qquad \forall v \in U \textit{such that} \ \exists \varphi \in X \ : \ A\varphi = Bv \tag{4.32}$$

Proof. First of all, we prove the existence of the minimizer. We will use a similar argument to that we used for the finite dimensional case. In fact, thanks to Remark 4.4, we have that for every $k \in \mathbb{R}$ the level sets

$$F_k = \{(x, u) \in M \mid J^s(x, u) \leq k\} \subset X \times U$$

are bounded. The positiveness of the functional J^s enables us to affirm that $\inf_M J^s \in \mathbb{R}$. Then, for an arbitrary $\{(x_n, u_n)\}_{n\in\mathbb{N}}$ minimizing sequence, the real sequence $\{J^s(x_n, u_n)\}_{n\in\mathbb{N}}$ is bounded. The boundedness of the level sets yields the boundedness of the sequence $\{(x_n, u_n)\}_{n\in\mathbb{N}}$ itself. Now, we use the Banach-Alaoglu Thorem 1.13 to justify the existence of a subsequence $\{(x_{n_k}, u_{n_k})\}_{k\in\mathbb{N}}$ and an element $(\overline{x}, \overline{u}) \in X \times U$ such that

$$(x_{n_k}, u_{n_k}) \underset{k\to+\infty}{\rightharpoonup} (\overline{x}, \overline{u})$$

weakly in $X \times U$. Since M is a closed subspace of $X \times U$, we deduce $(\overline{x}, \overline{u}) \in M$. The lower-semicontinuity of the norm with respect to the weak convergence (Proposition 1.6), implies that:

$$J^s(\overline{x}, \overline{u}) \leq \liminf_{k\to+\infty} J^s(x_{n_k}, u_{n_k}) = \inf_M J^s$$

This implies that $J^s(\overline{x}, \overline{u}) = \inf_M J^s$, i.e. there exists $(\overline{x}, \overline{u}) \in M$ minimizer for J^s. At this step, we investigate the uniqueness of minimizer. To this purpose, we show that J^s is strictly convex. Indeed, we remind that the square of each norm induced by a scalar product is strictly convex and we employ Remark 4.4 as we did in the finite dimensional case. This yields the strict convexity of J^s. Hence, by Proposition 1.2, for every couple of minimizers $\{(x_i, u_i)\}_{i\in\{1,2\}} \subset X \times U$, $u_1 = u_2$ and $x_1 = x_2$. We show now that the first order conditions above are actually satisfied. For every $(\overline{x}, \overline{u}) \in M$ minimizer and for each direction $(\varphi, v) \in M$, we define the following map:

$$g : \mathbb{R} \longmapsto \mathbb{R}$$

$$h \longrightarrow J^s((\overline{x}, \overline{u}) + h(\varphi, v))$$

This function belongs to the class $C^\infty(\mathbb{R}, \mathbb{R})$. We compute its derivative in 0, getting

$$\frac{d}{dh}g(0) = (\overline{u}, v)_U + (C\overline{x} - z, C\varphi)_V$$

Then, Fermat Theorem enables us to affirm that

$$\frac{d}{dt}g(0) = 0.$$

Therefore, the above equality reads as:

$$(\overline{u}, v)_U + (C\overline{x} - z, C\varphi)_V = 0 \qquad \forall (\varphi, v) \in M.$$

At this stage we notice that $ker(A) \times \{0\} \subset M$. Therefore, for each vector $\varphi \in Ker(A)$

$$< C^*\Phi_V(C\overline{x} - z), \varphi >_{(X',X)} = 0.$$

Moreover, $R(A)$ is closed into X', then, by the Closed Range Theorem 1.14, $R(A^*)$ is closed into X'. Then $C^*\Phi_V(C\overline{x} - z) \in Ker(A)^\perp = R(A^*)$. This is equivalent to:

$$\exists \overline{p} \in X \qquad \text{such that} \qquad C^*\Phi_V(C\overline{x} - z) = A^*\overline{p}$$

This means that for every $(\varphi, v) \in M$:

$$(C\overline{x} - z, C\varphi)_V =< A^*\overline{p}, \varphi >_{(X',X)} =< A\varphi, \overline{p} >_{(X',X)} =< Bv, \overline{p} >_{(X',X)}$$

This yields

$$(\overline{u}, v)_U + (\overline{p}, \Phi_X^{-1}Bv)_X = 0 \qquad \forall v \in U \text{ such that } \exists \varphi \in X \ : \ A\varphi = Bv.$$

as required. □

Remark 4.6. As in the finite dimensional case, we remark that, in general, $\overline{p} \in X$ such that

$$A^*\overline{p} = C^*\Phi_V(C\overline{x} - z)$$

is not unique. More precisely, it is unique if and only if $Ker(A^*) = \{0\}$.

Theorem 4.2 (Convergence of averages)**.** *We assume hypotheses 4.1, 4.2 and 4.3 hold true. Then*

$$\frac{1}{T}\min_{u\in L^2((0,T);U)} J^T \underset{T\to+\infty}{\longrightarrow} \min_{(x,u)\in M} J^s \tag{4.33}$$

$$\frac{1}{T}\int_0^T \left(\|u^T(t)-\overline{u}\|_U^2 + \|C\left(x^T(t)-\overline{x}\right)\|_V^2\right)dt = \underset{T\to+\infty}{O}\left(\frac{1}{T}\right) \underset{T\to+\infty}{\longrightarrow} 0 \tag{4.34}$$

Therefore, for every $(a,b)\in[0,1]^2$ *such that* $a\neq b$

$$\frac{1}{(b-a)T}\int_{aT}^{bT} x^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{x} \qquad \text{in } X \tag{4.35}$$

$$\frac{1}{(b-a)T}\int_{aT}^{bT} u^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{u} \qquad \text{in } U \tag{4.36}$$

Moreover, there exists a unique $\overline{p}\in X$ *satisfying*

$$\begin{cases} A^*\overline{p} = C^*\Phi_V(C\overline{x}-z) \\ \overline{u} = -\Phi_U^{-1}B^*\overline{p} \end{cases} \tag{4.37}$$

and

$$\frac{1}{(b-a)T}\int_{aT}^{bT} p^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{p} \qquad \text{in } X. \tag{4.38}$$

Proof. One subtracts the first order conditions for the non stationary and the stationary problem, obtaining:

$$\begin{cases} (x^T-\overline{x})_t + A(x^T-\overline{x}) = B(u^T-\overline{u}) & \text{in } (0,T) \\ -(p^T-\overline{p})_t + A^*(p^T-\overline{p}) = C^*\Phi_V C(x^T-\overline{x}) & \text{in } (0,T) \end{cases} \tag{4.39}$$

Furthermore, we have an explicit expression of the optimal control function in terms of the adjoint state $u^T = -\Phi_U^{-1}B^*p^T$. We aim at proving that there exists a T independent constant $C\in(0,+\infty)$ such that for every $T\in(0,+\infty)$

$$\int_0^T \left(\|u^T-\overline{u}\|_U^2 + \|C\left(x^T-\overline{x}\right)\|_V^2\right)dt \le C \tag{4.40}$$

From this estimate, the thesis follows.

First of all, we look for an alternative expression of (2.21) integrating the first order conditions, i.e.

$$\int_0^T \|C(x^T-\overline{x})\|_V^2 dt = \int_0^T \left\langle C^*\Phi_V C(x^T-\overline{x}), x^T-\overline{x}\right\rangle_{(X',X)} dt =$$

$$= \int_0^T < -(p^T - \overline{p})_t, x^T - \overline{x} >_{(X',X)} dt + \int_0^T < A(x^T - \overline{x}), (p^T - \overline{p}) >_{(X',X)} dt =$$

integrating by parts

$$= \big((p^T(0) - \overline{p}), (x^T(0) - \overline{x})\big)_H - \big((p^T(T) - \overline{p}), x^T(T) - \overline{x}\big)_H +$$

$$+ \int_0^T < p^T - \overline{p}, (x^T - \overline{x})_t >_{(X,X')} dt + \int_0^T < (p^T - \overline{p}), A(x^T - \overline{x}) >_{(X,X')} dt =$$

$$= \big(x_0 - \overline{x}, p^T(0) - \overline{p}\big)_H + \big(x^T(T) - \overline{x}, \overline{p}\big)_H + \int_0^T < p^T - \overline{p}, B(u^T - \overline{u}) >_{(X,X')} dt.$$

At this stage, $\forall t \in [0,T]$ we make the following computation:

$$< (p^T(t) - \overline{p}), B(u^T(t) - \overline{u}) >_{(X,X')} + \|u^T(t) - \overline{u}\|_U^2 =$$

using $u^T = -\Phi_U^{-1} B^* p^T$

$$= - \big((\overline{u} + \Phi_U^{-1} B^* \overline{p}), u^T(t)\big)_U$$

Hence,

$$\int_0^T \big(\|u^T - \overline{u}\|_U^2 + \|C\left(x^T - \overline{x}\right)\|_V^2\big)\, dt = \tag{4.41}$$

$$= \big(x_0 - \overline{x}, p^T(0) - \overline{p}\big)_H + \big(x^T(T) - \overline{x}, \overline{p}\big)_H - \int_0^T \big((\overline{u} + \Phi_U^{-1} B^* \overline{p}), u^T\big)_U\, dt.$$

Now, we are going to use the Observability Inequalities. We use firstly the inequality related to (A, C), i.e. hypothesis 4.2, to get:

$$\|x^T(T) - \overline{x}\|_H \leq C \left[\int_0^T \|B(u^T - \overline{u})\|_{X'}^2 dt + \int_0^T \|C(x^T - \overline{x})\|_V^2 dt + \|x_0 - \overline{x}\|_H^2\right]^{\frac{1}{2}}.$$

Using the boundedness of the operator B,

$$\|x^T(T) - \overline{x}\|_H \leq C \left[\int_0^T \|u^T - \overline{u}\|_U^2 dt + \int_0^T \|C(x^T - \overline{x})\|_V^2 dt + \|x_0 - \overline{x}\|_H^2.\right]^{\frac{1}{2}} \tag{4.42}$$

Now we deduce an estimate for $\|p^T(0) - \overline{p}\|_H$, by Hypothesis 4.3.

$$\|p^T(0) - \overline{p}\|_H \leq C \left[\int_0^T \|C(x^T - \overline{x})\|_V^2 dt + \int_0^T \|B^*(p^T - \overline{p})\|_{U'}^2 dt + \|\overline{p}\|_H^2\right]^{\frac{1}{2}} \tag{4.43}$$

which, by means of the boundedness of B^* and the equation $u^T = -\Phi_U^{-1} B^* p^T$, becomes:

$$\|p^T(0)-\overline{p}\|_H \leq C \left[\int_0^T \|C(x^T-\overline{x})\|_V^2 dt + \int_0^T \|u^T-\overline{u}\|_U^2 dt + \int_0^T \|\overline{u}+\Phi_U^{-1}B^*\overline{p}\|_U^2 dt + 1\right]^{\frac{1}{2}} \tag{4.44}$$

It is the right moment to prove a first estimate of (4.40). This estimate, will provide useful information concerning the boundedness of averages. In fact, from (4.41), we get:

$$\int_0^T \left(\|u^T-\overline{u}\|_U^2 + \|C\left(x^T-\overline{x}\right)\|_V^2\right) dt \leq$$

$$\leq C\left[\|p^T(0)-\overline{p}\|_H + \|x^T(T)-\overline{x}\|_H\right] + \left[\int_0^T \|u^T\|_U^2 dt\right]^{\frac{1}{2}} \left[\int_0^T \|\overline{u}+\Phi_U B^*\overline{p}\|_U^2 dt\right]^{\frac{1}{2}} \leq$$

$$\leq C(T^{\frac{1}{2}}+1)\left[\left(\int_0^T \|C(x^T-\overline{x})\|_V^2 dt + \int_0^T \|u^T-\overline{u}\|_U^2 dt\right)^{\frac{1}{2}} + T^{\frac{1}{2}}+1\right].$$

Which implies in turn, for T big enough,

$$\int_0^T \left(\|u^T-\overline{u}\|_U^2 + \|C\left(x^T-\overline{x}\right)\|_V^2\right) dt \leq CT \tag{4.45}$$

Therefore, the generalised sequences

$$\left\{\frac{1}{T}\int_0^T u^T dt\right\}_{T\in(0,+\infty)} \subset U$$

$$\left\{\frac{1}{T}\int_0^T Cx^T dt\right\}_{T\in(0,+\infty)} \subset V$$

are bounded. In order to show that $\left\{\frac{1}{T}\int_0^T x^T dt\right\}_{T\in(0,+\infty)}$ itself is bounded, we take the average of the state equation, obtaining:

$$A\left(\frac{1}{T}\int_0^T x^T dt\right) = \frac{1}{T}\int_0^T Bu^T dt - \frac{x^T(T)-x_0}{T} \tag{4.46}$$

We use the inequalities (4.42) and (4.45), to prove the following estimate:

$$\frac{\|x^T(T)-x_0\|}{T} \leq \frac{\sqrt{C}\sqrt{T}}{T}. \tag{4.47}$$

Then, the last addendum in the averaged equation vanishes as $T \to +\infty$. Therefore, the generalised sequence $\left\{\frac{1}{T}\int_0^T Ax^T dt\right\}_{T\in(0,+\infty)} \subset X'$ is bounded. We employ Remark 4.4 to get that the generalised sequence

$$\left\{\frac{1}{T}\int_0^T x^T dt\right\}_{T\in(0,+\infty)} \subset X$$

is bounded. Now, we are ready to apply the Banach-Alaoglu Theroem and obtain that each the above sequences admit subsequence converging weakly in U, V, X respectively to some limit. We need now more precise information about the (weak) limits themselves. First of all, we consider again the averaged equation (4.46), getting that for every $(\varphi, v) \in X \times U$ and for every subsequence such that:

$$\frac{1}{T_k}\int_0^{T_k} x^{T_k} dt \underset{k\to+\infty}{\rightharpoonup} \varphi \qquad \text{in } X$$

$$\frac{1}{T_k}\int_0^{T_k} u^{T_k} dt \underset{k\to+\infty}{\rightharpoonup} v \qquad \text{in } U,$$

$$A\varphi = Bv.$$

Hence, $(\varphi, v) \in M$. Using now the first order condition (4.32) and the definition of weak convergence, we obtain:

$$\frac{1}{T_k}\int_0^{T_k} \left(u^{T_k}, \overline{u} + \Phi_U^{-1}B^*\overline{p}\right)_U dt = \left(\overline{u} + \Phi_U^{-1}B^*\overline{p}, \frac{1}{T_k}\int_0^{T_k} u^{T_k} dt\right)_U \underset{k\to+\infty}{\longrightarrow} \left(v, \overline{u} + \Phi_U^{-1}B^*\overline{p}\right)_U = 0$$

Therefore, the sequence $\left\{\frac{1}{T}\int_0^T \left(u^T, \overline{u} + \Phi_U^{-1}B^*\overline{p}\right)_U dt\right\}_{T\in(0,+\infty)}$ is such that for each subsequence exists a subsubsequence converging to 0. Hence, Proposition 1.1 allows us to deduce the convergence of the whole sequence:

$$\frac{1}{T}\int_0^T \left(u^T, \overline{u} + \Phi_U^{-1}B^*\overline{p}\right)_{\mathbb{R}^M} dt \underset{T\to+\infty}{\longrightarrow} 0 \qquad \text{in } \mathbb{R}.$$

Notice that the above convergence is true for each $\overline{p} \in X$ satisfying $A^*\overline{p} = C^*\Phi_V\left(C\overline{x} - z\right)$. We are now ready to prove that

$$\frac{1}{T}\int_0^T \left(\|u^T - \overline{u}\|_U^2 + \|C\left(x^T - \overline{x}\right)\|_V^2\right) dt \underset{T\to+\infty}{\longrightarrow} 0 \tag{4.48}$$

Indeed, employing (4.41), (4.42), (4.44), we obtain:

$$\int_0^T \left(\|u^T - \overline{u}\|_U^2 + \|C\left(x^T - \overline{x}\right)\|_V^2\right) dt \leq$$

$$\leq C\left\{\left[\int_0^T\left(\|u^T-\overline{u}\|_U^2+\|C\left(x^T-\overline{x}\right)\|_V^2\right)dt\right]^{\frac{1}{2}}+\left[\int_0^T\|\overline{u}+\Phi_U^{-1}B^*\overline{p}\|_U^2dt\right]^{\frac{1}{2}}\right\}+$$

$$+C\left\{\|x_0-\overline{x}\|_H+\|\overline{p}\|_H-\int_0^T\left(u^T,\overline{u}+\Phi_U^{-1}B^*\overline{p}\right)_U dt\right\}$$

At this point, we employ the inequality (4.45),

$$\frac{1}{T}\int_0^T\left(\|u^T-\overline{u}\|_U^2+\|C(x^T-\overline{x})\|_V^2\right)dt\leq\frac{C}{\sqrt{T}}-\frac{1}{T}\int_0^T\left(u^T,\overline{u}+\Phi_U^{-1}B^*\overline{p}\right)_U dt$$

Which in turn implies

$$\frac{1}{T}\int_0^T\left(\|u^T-\overline{u}\|_U^2+\|C\left(x^T-\overline{x}\right)\|_V^2\right)dt\underset{T\to+\infty}{\longrightarrow}0$$

This entails, together with Jensen Theorem, the following strong convergence in V and U respectively

$$\frac{1}{T}\int_0^T Cx^Tdt\underset{T\to+\infty}{\longrightarrow}C\overline{x}\qquad\text{in } V$$

$$\frac{1}{T}\int_0^T u^Tdt\underset{T\to+\infty}{\longrightarrow}\overline{u}\qquad\text{in } U$$

Furthermore, we use the averaged equation (4.46), in order to get

$$\frac{1}{T}\int_0^T Ax^Tdt\underset{T\to+\infty}{\longrightarrow}A\overline{x}$$

strongly in X'. Finally, we apply Remark 4.4 to obtain

$$\frac{1}{T}\int_0^T x^Tdt\underset{T\to+\infty}{\longrightarrow}\overline{x}$$

strongly in X. We aim now at showing that, for a special $\overline{p}\in X$,

$$\frac{1}{T}\int_0^T p^Tdt\underset{T\to+\infty}{\longrightarrow}\overline{p}$$

strongly in X. To achieve this result, we take the average of the dual equation satisfied by p^T and we get:

$$\frac{p^T(0)}{T}+\frac{1}{T}\int_0^T A^*p^Tdt=\frac{1}{T}\int_0^T C^*\Phi_V(Cx^T-z)dt\tag{4.49}$$

$$(4.50)$$

This implies

$$\left\|\frac{1}{T}\int_0^T A^*p^T dt\right\|_{X'} \leq \frac{\|p^T(0)\|_H}{T} + \left[\frac{1}{T}\int_0^T \|C^*\Phi_V(Cx^T - z)\|_{X'}^2 dt\right]^{\frac{1}{2}} \leq$$

by the continuity of C^*,

$$\leq \frac{C + C\sqrt{T}}{T} + \frac{C\sqrt{T}}{\sqrt{T}} \leq C.$$

Now, it suffices to apply Remark 4.4, to get that

$$\left\{\frac{1}{T}\int_0^T p^T dt\right\}_{T\in(0,+\infty)} \subset X$$

is actually bounded. Banach-Alaoglu Theorem allows us to affirm that there exists a subsubsequence $\left\{\frac{1}{T_k}\int_0^{T_k} p^{T_k} dt\right\}_{k\in\mathbb{N}} \subset \left\{\frac{1}{T}\int_0^T p^T dt\right\}_{T\in(0,+\infty)}$ and one $\overline{p} \in X$ such that:

$$\frac{1}{T_k}\int_0^{T_k} p^T dt \underset{k\to+\infty}{\rightharpoonup} \overline{p} \qquad \text{in } X$$

We consider again the averaged equation of p^T

$$\frac{p^T(0)}{T} + \frac{1}{T}\int_0^T A^*p^T dt = \frac{1}{T}\int_0^T C^*\Phi_V(Cx^T - z)dt$$

and we deduce that $\overline{p}$ satisfies the equation $A^*\overline{p} = C^*\Phi_V(C\overline{x} - z)$. Furthermore, we can take the average of the equation $u^T = -B^*p^T$, and get a further property for the limit $\overline{p}$

$$\overline{u} = -\Phi_U^{-1}B^*\overline{p}.$$

Hence, we have shown that there exists a $\overline{p} \in X$ such that

$$\begin{cases} A^*\overline{p} = C^*\Phi_V(C\overline{x} - z) \\ \overline{u} = -\Phi_U^{-1}B^*\overline{p} \end{cases} \tag{4.51}$$

Moreover, employing Remark 4.5, we can prove that such a $\overline{p}$ is actually unique. We use this achievement to prove the strong convergence in X of $\left\{\frac{1}{T}\int_0^T p^T dt\right\}_{T\in(0,+\infty)}$ to the special $\overline{p}$ satisfying conditions(4.51). Thank to the averaged equation of $p^T - \overline{p}$,

$$\left\|\frac{1}{T}\int_0^T A^*(p^T(t) - \overline{p})dt\right\|_{X'} \leq \frac{\|p^T(0)\|_H}{T} + \left\|\frac{1}{T}\int_0^T C^*\Phi_V(Cx^T - \overline{x})dt\right\|_V \underset{T\to+\infty}{\longrightarrow} 0.$$

Then, employing again Remark 4.5,

$$\left\|\frac{1}{T}\int_0^T (p^T(t)-\overline{p})dt\right\|_X^2 \leq C\left[\left\|\frac{1}{T}\int_0^T A^*(p^T(t)-\overline{p})dt\right\|_{X'}^2 + \left\|\frac{1}{T}\int_0^T B^*(p^T-\overline{p})dt\right\|_{U'}^2\right] \underset{T\to+\infty}{\longrightarrow} 0$$

Hence,

$$\frac{1}{T}\int_0^T p^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{p} \qquad \text{in } X.$$

Furthermore, repeating the above arguments , for every $(a,b)\in[0,1]^2$ such that $a\neq b$, is possible to prove that:

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} x^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{x} \qquad \text{in } X$$

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} u^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{u} \qquad \text{in } U$$

$$\frac{1}{(b-a)\,T}\int_{aT}^{bT} p^T(t)dt \underset{T\to+\infty}{\longrightarrow} \overline{p} \qquad \text{in } X$$

Moreover, we take advantage of the existence of $\overline{p}\in X$ satisfying the conditions (4.51) employing it throughout the whole proof. Then, the equation (4.41) becomes:

$$\int_0^T \left(\|u^T-\overline{u}\|_U^2 + \|C\left(x^T-\overline{x}\right)\|_V^2\right)dt = \tag{4.52}$$

$$= \left(x_0-\overline{x}, p^T(0)-\overline{p}\right)_H + \left(x^T(T)-\overline{x},\overline{p}\right)_H.$$

Then, by (4.42) and (4.44), we obtain:

$$\int_0^T \|C(x^T-\overline{x})\|_V^2+\|u^T-\overline{u}\|_U^2 dt \leq C\left\{\left[\int_0^T \|C(x^T-\overline{x})\|_V^2+\|u^T-\overline{u}\|_U^2 dt\right]^{\frac{1}{2}} + \|x_0-\overline{x}\|_H + \|\overline{p}\|_H\right\}$$

Finally, this implies the existence of a constant $C\in(0,+\infty)$ such that:

$$\int_0^T \|C(x^T-\overline{x})\|_V^2+\|u^T-\overline{u}\|_U^2 dt \leq C \tag{4.53}$$

This, yields the conclusion.

□

In the next remark we highlight an interesting byproduct of the above proof.

Remark 4.7. A necessary condition for the turnpike property to be fulfilled is the existence of a constant $M \in \mathbb{R}^+$ such that

$$\left\{(x^T(t), p^T(t)) \mid T \in (0, +\infty),\ t \in [0, T]\right\} \subset B^{H\times H}(0, M). \tag{4.54}$$

Using hypotheses 4.2 and 4.3 together with estimate (4.53), it is possible to prove that (4.54) actually holds.

As we did in the finite dimensional case, we prepare the tools needed to describe more accurately the convergence of the non stationary optimal triple (x^T, p^T, u^T) to the corresponding stationary one $(\overline{x}, \overline{p}, \overline{u})$. For the moment, we focus on Problems where the target is $z = 0$. We will name the target 0 non stationary Optimal Control Problem $(OCP)_0^T$, the stationary one $(OCP)_0^s$. Moreover, we will call the corresponding functionals J_0^T and J_0^s respectively. We observe that the stationary optimal pair is $(\overline{x}, \overline{u}) = (0, 0)$ and the unique $\overline{p} \in X$ satisfying

$$\begin{cases} A^*\overline{p} = C^*\Phi_V C\overline{x} \\ \overline{u} = -\Phi_U^{-1} B^*\overline{p} \end{cases} \tag{4.55}$$

is actually $\overline{p} = 0$.

Definition of $\{\mathcal{E}(T)\}_{T\in(0,+\infty)}$ and $\widehat{E}$

First of all, we define a family of operators which resembles the family $\{\mathcal{E}(T)\}_{T\in(0,+\infty)}$ introduced the finite dimensional case, without using infinite dimensional Riccati's theory.

Definition 4.3. For every $x_0 \in H$ and $T \in (0, +\infty)$, we define $(x^T(\cdot; x_0), p^T(\cdot; x_0), u^T(\cdot; x_0))$ the optimal triple for $(OCP)_0^T$ with initial data x_0.

We are ready to define an infinite dimensional analogue of the family $\{\mathcal{E}(T)\}_{T\in(0,+\infty)}$.

Definition 4.4. For each $T \in (0, +\infty)$,

$$\mathcal{E}(T) : H \longmapsto H$$

$$x_0 \longrightarrow p^T(0; x_0)$$

We observe that for every $T \in (0, +\infty)$ the operator $\mathcal{E}(T)$ is linear. Now, we aim at proving its continuity. To this extent, we notice that, whenever we fix a bounded set B in H, all the constants in the proof of Theorem 4.2 become uniform on $x_0 \in B$. By hypothesis 4.3, there exists a constant $C \in (0, +\infty)$, independent of $(x_0, T) \in H \times (0, +\infty)$, such that for every $x_0 \in B$ and $T \in (0, +\infty)$:

$$\|p^T(0; x_0)\|_H^2 \leq C\left[\int_0^T \|B^* p^T(t; x_0)\|_{U'}^2 dt + \int_0^T \|C^*\Phi_V C x^T(t; x_0)\|_{X'}^2 dt\right] \leq$$

employing the continuity of the operators B^* and C^*

$$\leq C\left[\int_0^T \|u^T(t;x_0)\|_U^2 dt + \int_0^T \|C(x^T(t;x_0))\|_V^2 dt\right] \leq C$$

Hence,

$$\|p^T(0;x_0)\|_H^2 \leq C$$

where $C \in (0,+\infty)$ is independent of $x_0 \in B$ and of $T \in (0,+\infty)$. This implies that the set $\mathcal{E}(T)(B) \subset H$ is bounded. Which entails the continuity of the operator $\mathcal{E}(T)$ for all $T \in (0,+\infty)$. Furthermore, since C above is also independent of $T \in (0,+\infty)$, the set $\{\mathcal{E}(T)\}_{T\in(0,+\infty)} \subset B(H,H)$ is bounded. Moreover, $\forall T \in (0,+\infty)$ the operator $\mathcal{E}(T)$ is self-adjoint, namely:

$$(\mathcal{E}(T)x_0, x_1)_H = (x_0, \mathcal{E}(T)x_1)_H \qquad \forall T \in (0,+\infty)$$

To show this fact, we take into account $\forall x_0 \in H$, $(x^T(\cdot;x_0), p^T(\cdot;x_0)) \in W^{1,2}((0,T);(X,X'))^2$ the unique solution of the Cauchy Problem:

$$\begin{cases} x_t^T + Ax^T = -BB^*p^T & \text{in } (0,T) \\ -p_t^T + A^*p^T = C^*\Phi_V Cx^T & \text{in } (0,T) \\ x^T(0) = x_0 \\ p^T(T) = 0 \end{cases} \tag{4.56}$$

and, $\forall x_1 \in H$, $(x^T(\cdot;x_1), p^T(\cdot;x_1)) \in W^{1,2}((0,T);(X,X'))^2$ the unique solution of the Cauchy Problem:

$$\begin{cases} x_t^T + Ax^T = -BB^*p^T & \text{in } (0,T) \\ -p_t^T + A^*p^T = C^*\Phi_V Cx^T & \text{in } (0,T) \\ x^T(0) = x_1 \\ p^T(T) = 0 \end{cases} \tag{4.57}$$

At this step, we multiply in (X',X) the equation of $p^T(\cdot;x_0)$ with $x^T(\cdot;x_1)$ and we integrate in $[0,T]$, obtaining:

$$\int_0^T < -p^T(\cdot;x_0)_t(t) + A^*p^T(t;x_0), x^T(t;x_1) >_{(X',X)} dt =$$

$$= \int_0^T < C^*\Phi_V Cx^T(t;x_0), x^T(t;x_1) >_{(X',X)} dt.$$

Integrating by Parts, the above equality becomes:

$$\left(p^T(0;x_0), x^T(0;x_1)\right)_H + \int_0^T < p^T(t;x_0), x^T(\cdot;x_1)_t(t) + Ax^T(t;x_1) >_{(X,X')} dt =$$

$$= \int_0^T < C^*\Phi_V C x^T(t;x_0), x^T(t;x_1) >_{(X',X)} \, dt.$$

Which entails, employing the equation of $x^T(\cdot;x_1)$, that:

$$\left(p^T(0;x_0), x_1\right)_H + \int_0^T < p^T(t;x_0), -B\Phi_U^{-1}B^* p^T(t;x_1) >_{(X,X')} \, dt =$$

$$= \int_0^T < C^*\Phi_V C x^T(t;x_0), x^T(t;x_1) >_{(X',X)} \, dt.$$

Finally, this implies that:

$$\left(p^T(0;x_0), x_1\right)_H = \int_0^T \left(B^* p^T(t;x_0), B^* p^T(t;x_1)\right)_{U'} dt + \int_0^T \left(C x^T(t;x_0), C x^T(t;x_1)\right)_V dt.$$

Up to permutating $(x_0, x_1) \in H^2$, we get:

$$\left(p^T(0;x_1), x_0\right)_H = \int_0^T \left(B^* p^T(t;x_1), B^* p^T(t;x_0)\right)_{U'} dt + \int_0^T \left(C x^T(t;x_1), C x^T(t;x_0)\right)_V dt.$$

Which in turn implies:

$$\left(\mathcal{E}(T)x_0, x_1\right)_H = \left(p^T(0;x_0), x_1\right)_H = \left(p^T(0;x_1), x_0\right)_H = \left(\mathcal{E}(T)x_1, x_0\right)_H,$$

namely, $\forall T \in (0,+\infty)$, $\mathcal{E}(T)$ is self-adjoint. Furthermore, we want to show that for every $T \in (0,+\infty)$ the operator $\mathcal{E}(T)$ is positive, i.e.:

$$\left(\mathcal{E}(T)(x_0), x_0\right)_H \geq 0 \qquad \forall x_0 \in H \tag{4.58}$$

Indeed, we consider the optimal pair for $(OCP)_0^s$ $(\overline{x}, \overline{u}) = (0,0)$ and the unique $\overline{p}$ solution of

$$\begin{cases} A^*\overline{p} = C^*\Phi_V C\overline{x} \\ \overline{u} = -\Phi_U^{-1}B^*\overline{p}. \end{cases} \tag{4.59}$$

We know that $\overline{p} = 0$. Therefore, the equation (4.52) becomes

$$\int_0^T \left(\|u^T(t)\|_U^2 + \|Cx^T(t)\|_V^2\right) dt = \tag{4.60}$$

$$= \left(x_0, p^T(0;x_0)\right)_H \stackrel{\text{def}}{=} \left(\mathcal{E}(T)(x_0), x_0\right)_H$$

Hence,

$$\left(\mathcal{E}(T)x_0, x_0\right)_H = \min_{L^2((0,T);U)} J_0^T \tag{4.61}$$

Which in turn implies

$$(\mathcal{E}(T)x_0, x_0)_H = \min_{L^2((0,T);U)} J_0^T \geq 0,$$

i.e. the positiveness of the operator $\mathcal{E}(T)$. At this stage, we aim at proving the monotonicity of the function

$$\mathcal{E} : (0, +\infty) \longmapsto B(H, H)$$
$$T \longrightarrow \mathcal{E}(T).$$

The proof relies on the above characterization of the value function. In fact, for every $(t_1, t_2) \in (0, +\infty)^2$ such that $t_1 \leq t_2$, for each $x_0 \in H$:

$$(\mathcal{E}(t_1)x_0, x_0)_H = J_0^{t_1}(u^{t_1}) \leq J_0^{t_1}(u^{t_2}) \leq J_0^{t_2}(u^{t_2}) = (\mathcal{E}(t_2)x_0, x_0)_H .$$

Hence, we have proven the assertion. Therefore, for each $x_0 \in H$ the sequence $\{(\mathcal{E}(T)x_0, x_0)_H\}_{T\in(0,+\infty)}$ converges in $\mathbb{R}$. We are going to define an operator $\widehat{E} \in B(H, H)$ such that

$$(\mathcal{E}(T)x_0, x_0)_H \underset{T\to+\infty}{\longrightarrow} \left(\widehat{E}x_0, x_0\right)_H .$$

In order to define $\widehat{E}$, let us take an arbitrary sequence $\{T_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ such that $T_n \underset{n\to+\infty}{\longrightarrow} +\infty$ and $x_0 \in H$. We define:

$$\forall n \in \mathbb{N} \qquad (x_n, p_n, u_n) \stackrel{\text{def}}{=} (x^{T_n}(\cdot, x_0), p^{T_n}(\cdot, x_0), u^{T_n}(\cdot, x_0)).$$

Employing Remark 4.7, we get that, the set $\{(x_n(t), p_n(t)) \mid t \in [0, T_n],\ n \in \mathbb{N}\} \subset H \times H$ is bounded. Moreover, Theorem 4.1 entails for every $\tilde{T} \in (0, +\infty)$, the existence of $C \in (0, +\infty)$ possibly depending on $\tilde{T} \in (0, +\infty)$ such that for every $(x_0, T) \in H \times [\tilde{T}, +\infty)$

$$\int_0^{\tilde{T}} \|x^T(t; x_0)\|_X^2 + \|\frac{d}{dt}x^T(t; x_0)\|_{X'}^2 dt \leq C \left[\int_0^{\tilde{T}} \|Bu^T(t)\|_{X'}^2 dt + \|x_0\|_H^2\right] \leq$$

$$\leq C \left[\int_0^{\tilde{T}} \|u^T(t)\|_U^2 dt + \|x_0\|_H^2\right].$$

Hence, using estimate (4.53), it is possible to show the existence of a constant $C \in (0, +\infty)$, such that

$$\int_0^{\tilde{T}} \|x^T(t; x_0)\|_X^2 + \|\frac{d}{dt}x^T(t; x_0)\|_{X'}^2 dt \leq C \qquad \forall (x_0, T) \in H \times [\tilde{T}, +\infty)$$

Furthermore, applying Theorem 4.1 and using the above inequality to the adjoint state system, we obtain for all $\tilde{T} \in (0, +\infty)$ the existence of a constant $C \in (0, +\infty)$ such that:

$$\int_0^{\tilde{T}} \|p^T(t;x_0)\|_X^2 + \|\frac{d}{dt}p^T(t;x_0)\|_{X'}^2 dt \leq C \qquad \forall T \in [\tilde{T}, +\infty).$$

Therefore, for every $T \in (0, +\infty)$, the sequence $\{(x_n, p_n)\}_{n \in \mathbb{N}}$ is bounded in $W^{1,2}((0,T);(X,X'))^2$ too. Then, using Banach-Alaoglu Theorem, up to subsequences, there exists an element $(\hat{x}, \hat{p}) \in W^{1,2}_{loc}((0,+\infty);(X,X'))^2$ such that:

$$(x_n, p_n) \underset{n \to +\infty}{\rightharpoonup} (\hat{x}, \hat{p}) \tag{4.62}$$

weakly in $W^{1,2}((0,T);(X,X'))^2$ for every $T \in (0, +\infty)$. This implies that the limit $(\hat{x}, \hat{p}) \in C^0([0, +\infty); H)^2$. Furthermore, we have shown that the set $\{(x_n(t), p_n(t)) \mid t \in [0, T_n],\ n \in \mathbb{N}\} \subset H \times H$ is actually bounded. Hence, there exists $M \in \mathbb{R}^+$ such that $\forall n \in \mathbb{N}$:

$$\|x_n(t)\|_H + \|p_n(t)\|_H \leq M \qquad \forall t \in [0, T_n].$$

At this point, by Banach-Alaoglu Theorem and the lower semicontinuity of the norm with respect to the weak convergence, we obtain, up to subsequences,

$$(x_n(t), p_n(t)) \underset{n \to +\infty}{\rightharpoonup} (\hat{x}(t), \hat{p}(t)) \qquad \forall t \in [0, +\infty)$$

weakly in H^2, and

$$\|\hat{x}(t)\|_H + \|\hat{p}(t)\|_H \leq \liminf_{n \to +\infty} [\|x_n(t)\|_H + \|p_n(t)\|_H] \leq M \qquad \forall t \in [0, +\infty). \tag{4.63}$$

Therefore, $(\hat{x}, \hat{p}) \in L^\infty((0,+\infty); H)^2$. Moreover, by estimate (4.53), $\{u_n\}_{n \in \mathbb{N}} \subset L^2((0,+\infty); U)$ is bounded. Then, by Banach-Alaoglu Theorem, there exist $\hat{u} \in L^2((0,+\infty); U)$ such that, up to subsequences:

$$u_n \underset{n \to +\infty}{\rightharpoonup} \hat{u}$$

weakly in $L^2((0,+\infty); U)$. To go further, we assume some more hypotheses. Concerning the control operator, we suppose $B \in B(U, H)$. Moreover, we assume that the inclusion of X in H is compact, i.e.:

$$i : X \hookrightarrow H \in K(X, H).$$

This enables us to consider $B^* \in B(H, U)$ as the proper adjoint of B in H, i.e.:

$$B^* : H \longmapsto U \qquad \text{is the unique operator such that}$$

$$(h, Bu)_H = (B^* h, u)_U \qquad \forall (h,u) \in H \times U.$$

In what follows we show the independence of such $\hat{p}(0)$ of the particular sequence $\{T_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ such that $T_n \to +\infty$. In fact, let $(\{T_{1,n}\}_{n\in\mathbb{N}}, \{T_{2,n}\}_{n\in\mathbb{N}}) \subset (0,+\infty)^2$ be a pair of sequences tending to infinity. Then, up to subsequences,

$$(x_n(0), p_n(0)) \underset{n\to+\infty}{\rightharpoonup} (\hat{x}(0;x_0)), \hat{p}(0;x_0))$$

weakly in H^2. Now, we define a couple of operators:

$$\widehat{E}_1 : H \longrightarrow H$$

$$x_0 \longrightarrow \lim_{n\to+\infty}{}^{\sigma(H,H')} \left[p^{T_{n,1}}(0;x_0)\right]$$

and

$$\widehat{E}_2 : H \longmapsto H$$

$$x_0 \longrightarrow \lim_{n\to+\infty}{}^{\sigma(H,H')} \left[p^{T_{n,2}}(0;x_0)\right].$$

By the uniqueness of the weak limit, both $\widehat{E}_1$ and $\widehat{E}_2$ are linear. For the continuity, we take into account an arbitrary bounded set $B \subset H$. Then, the lower-semicontinuity of the norm with respect to the weak convergence and Remark 4.7 entails for all $(i, x_0) \in \{1,2\} \times B$:

$$\|\widehat{E}_i(x_0)\|_H \le \liminf_{n\to+\infty} \|p^{T_{i,n}}(0;x_0)\|_H \le M.$$

This yields $(\widehat{E}_1, \widehat{E}_2) \in B(H,H)^2$. Moreover, we know that $\forall x_0 \in H$:

$$\exists \lim_{T\to+\infty} (\mathcal{E}(T)x_0, x_0)_H = \lim_{T\to+\infty} \left(p^T(0;x_0), x_0\right)_H = l \in \mathbb{R}^+.$$

By the definition of weak limit, we deduce:

$$\exists \lim_{n\to+\infty} \left(p^{T_{n,1}}(0;x_0), x_0\right)_H = \left(\widehat{E}_1(x_0), x_0\right)_H = l$$

$$\exists \lim_{n\to+\infty} \left(p^{T_{n,2}}(0;x_0), x_0\right)_H = \left(\widehat{E}_2(x_0), x_0\right)_H = l$$

Hence,

$$\forall x_0 \in H \left(\widehat{E}_1(x_0), x_0\right)_H = \left(\widehat{E}_2(x_0), x_0\right)_H.$$

At the moment, since $\forall T \in (0,+\infty) \quad \mathcal{E}(T) = \mathcal{E}(T)^*$, we have both $\widehat{E}_1 = \widehat{E}_1^*$ and $\widehat{E}_2 = \widehat{E}_2^*$. Therefore, using Proposition 1.11, we get:

$$\|\widehat{E}_1(x_0) - \widehat{E}_2(x_0)\|_{B(H,H)} = \sup\left\{ \left| \left(\widehat{E}_1(x_0) - \widehat{E}_2(x_0), x_0\right)_H \right| \middle| x_0 \in S^H(0;1) \right\} = 0$$

Then, by the non degeneracy of the norm, $\widehat{E}_1 = \widehat{E}_2$. Hence $\forall x_0 \in H$, by Proposition 1.7,

$$\exists \lim_{T\to+\infty}{}^{\sigma(H,H')}\left[p^T(0;x_0)\right] \in H$$

Therefore, we are ready to define the operator:

$$\widehat{E} : H \longmapsto H \tag{4.64}$$

$$x_0 \longrightarrow \lim_{T\to+\infty}{}^{\sigma(H,H')}\left[p^T(0;x_0)\right].$$

We have already proved that $\widehat{E} \in B(H,H)$. At this stage, we are going to prove that for every $x_0 \in H$

$$\mathcal{E}(T)(x_0) \underset{T\to+\infty}{\longrightarrow} \widehat{E}(x_0) \tag{4.65}$$

strongly in H. As usual, we take into account an arbitrary sequence $\{T_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ such that $T_n \underset{n\to+\infty}{\longrightarrow} +\infty$. Then, we name $(x_n,p_n) = (x^{T_n}(\cdot;x_0), p^{T_n}(\cdot;x_0))$ and $(\hat{x},\hat{p})$ one of its weak accumulation points. The differences $(x_n-\hat{x}, p_n-\hat{p})$ satisfies:

$$\begin{cases} (x_n-\hat{x})_t + A(x_n-\hat{x}) = -BB^*(p_n-\hat{p}) & \text{in } (0,T_n) \\ -(p_n-\hat{p})_t + A^*(p_n-\hat{p}) = C^*\Phi_V C(x_n-\hat{x}) & \text{in } (0,T_n) \\ x_n(0)-\hat{x}(0) = 0 \end{cases} \tag{4.66}$$

We consider now an arbitrary $T \in (0,+\infty)$. We consider all $n \in \mathbb{N}$ such that $T_n \geq T$. We have proved yet that the sequences $\{x_n-\hat{x}\}_{n\in\mathbb{N}} \subset W^{1,2}((0,T);(X,X'))$ and $\{p_n-\hat{p}\}_{n\in\mathbb{N}} \subset W^{1,2}((0,T);(X,X'))$ are bounded. Employing Theorem 1.16, we obtain that there exist $\{x_{n_k}-\hat{x}\}_{k\in\mathbb{N}} \subset \{x_n-\hat{x}\}_{n\in\mathbb{N}}$ and $\{p_{n_k}-\hat{p}\}_{k\in\mathbb{N}} \subset \{p_n-\hat{p}\}_{n\in\mathbb{N}}$ such that they are strongly convergent in $L^2((0,T);H)$. By the previous deductions, their weak limit is 0. This yields:

$$x_{n_k} \underset{k\to+\infty}{\longrightarrow} \hat{x} \qquad \text{in } L^2((0,T);H)$$

$$p_{n_k} \underset{k\to+\infty}{\longrightarrow} \hat{p} \qquad \text{in } L^2((0,T);H).$$

Since $B \in B(U,H)$, from the continuous dependence from the data, $\forall k \in \mathbb{N}$

$$\|x_{n_k}-\hat{x}\|_{W^{1,2}((0,T);(X,X'))} \leq C\left[\|BB^*(p_{n_k}-\hat{p})\|_{L^2((0,T);X')}\right] \underset{k\to+\infty}{\longrightarrow} 0.$$

Therefore,

$$x_{n_k} \underset{k\to+\infty}{\longrightarrow} \hat{x} \qquad \text{in } W^{1,2}((0,T);(X,X'))$$

Again, by continuous dependence from the data:

$$p_{n_k} \underset{k\to+\infty}{\longrightarrow} \hat{p} \qquad \text{in } W^{1,2}((0,T);(X,X')).$$

These results implies

$$x_{n_k} \underset{k\to+\infty}{\longrightarrow} \hat{x} \qquad \text{in } C^0([0,T],H) \tag{4.67}$$

$$p_{n_k} \underset{k\to+\infty}{\longrightarrow} \hat{p} \qquad \text{in } C^0([0,T],H). \tag{4.68}$$

We remember that:

$$\exists \lim_{T\to+\infty}{}^{\sigma(H,H')} \left[p^T(0;x_0)\right] = \hat{p}(0;x_0) = \hat{E}(x_0).$$

Therefore, the sequence $\{\mathcal{E}(T)\}_{T\in(0,+\infty)}$ is compact and has a unique accumulation point $\hat{E}(x_0)$. Reasoning as Proposition 1.1, we obtain:

$$\mathcal{E}(T) \underset{T\to+\infty}{\longrightarrow} \widehat{E}(x_0) \qquad \text{in } H \tag{4.69}$$

i.e. the conclusion. To carry on our analysis, we will anticipate some arguments we will use to study the infinite horizon problem. We take into account an initial data $x_0 \in H$, and a sequence $\{T_n\}_{n\in\mathbb{N}}$ such that both

$$\left\{x^{T_n}(\cdot;x_0)\right\}_{n\in\mathbb{N}} \subset C^0([0,T],H)$$

and

$$\left\{p^{T_n}(\cdot;x_0)\right\}_{n\in\mathbb{N}} \subset C^0([0,T],H)$$

are convergent. We name their limit $\hat{x}(\cdot;x_0)$ and $\hat{p}(\cdot;x_0)$ respectively. This is possible thanks to the above deductions. We aim at proving that:

$$\widehat{E}(\hat{x}(t;x_0)) = \hat{p}(t;x_0) \qquad \forall t \in [0,+\infty). \tag{4.70}$$

First of all, by definition of weak convergence in $W^{1,2}((0,T);(X,X'))$ and continuity of the operators, $(\hat{x}(\cdot;x_0),\hat{p}(\cdot;x_0)) \in W^{1,2}_{loc}((0,+\infty);(X,X'))$ satisfies $\forall T \in (0,+\infty)$ the Differential System:

$$\begin{cases} \hat{x}_t + A\hat{x} = -BB^*\hat{x} & \text{in } (0,T) \\ -\hat{p}_t + A^*\hat{p} = C^*\Phi_V C\hat{x} & \text{in } (0,T) \\ \hat{x}(0) = x_0. \end{cases} \tag{4.71}$$

Moreover, by a semigroup argument, for every $s \in [t,T]$:

$$(x^T(s;x_0),p^T(s;x_0),u^T(s;x_0)) = (x^{T-t}(s-t;x^T(t;x_0)),p^{T-t}(s-t;x^T(t;x_0)),u^{T-t}(s-t;x^T(t;x_0))). \tag{4.72}$$

Therefore, $\forall t \in [0,+\infty)$, $\forall T \in (t,+\infty)$

$$p^T(t;x_0) = p^{T-t}(0;x^T(t;x_0)). \tag{4.73}$$

$\forall T \in (0,+\infty)$ since

$$p^{T_n}(\cdot;x_0) \underset{n\to+\infty}{\longrightarrow} \hat{p}(\cdot;x_0)$$

in $C^0([0,T],H)$, then, $\forall t \in [0,T]$,

$$p^{T_n}(t;x_0) \underset{n\to+\infty}{\longrightarrow} \hat{p}(t;x_0)$$

strongly in H. On the other hand, by (4.73),

$$p^{T_n}(t;x_0) = p^{T_n-t}(0;x^{T_n}(t,x_0)).$$

At this step we remember that, by (4.65),

$$\widehat{E}(\hat{x}(t;x_0)) = \lim_{n\to+\infty}{}^H \left[p^{T_n-t}(0;\hat{x}(t;x_0))\right].$$

Moreover, thanks to (4.65), $\forall n \in \mathbb{N}$:

$$\|\widehat{E}(\hat{x}(t;x_0)) - p^{T_n-t}(0;x^{T_n}(t;x_0))\|_H \le$$

$$\le \|\widehat{E}(\hat{x}(t;x_0)) - \mathcal{E}(T_n-t)(\hat{x}(t;x_0))\|_H + \|\mathcal{E}(T_n-t)(\hat{x}(t;x_0) - x^{T_n}(t;x_0))\|_H \le$$

$$\le \|\widehat{E}(\hat{x}(t;x_0)) - \mathcal{E}(T_n-t)(\hat{x}(t;x_0))\|_H + \|\mathcal{E}(T_n-t)\|_{B(H,H)}\|\hat{x}(t;x_0) - x^{T_n}(t;x_0)\|_H \le$$

thanks to the boundedness of the sequence $\{\mathcal{E}(T)\}_{T\in(0,+\infty)} \subset B(H,H)$,

$$\le \|\widehat{E}(\hat{x}(t;x_0)) - \mathcal{E}(T_n-t)(\hat{x}(t;x_0))\|_H + M\|\hat{x}(t;x_0) - x^{T_n}(t;x_0)\|_H \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence, $\forall (x_0,t) \in H \times [0,+\infty)$

$$\widehat{E}(\hat{x}(t;x_0)) = \lim_{n\to+\infty} p^{T_n-t}(0;x^{T_n}(t;x_0)) = \lim_{n\to+\infty} p^{T_n}(t;x_0) = \hat{p}(t;x_0).$$

Therefore $(\hat{x}(\cdot;x_0), \widehat{E}\hat{x}(\cdot;x_0)) \in W^{1,2}_{loc}((0,+\infty);(X,X'))^2$ satisfies the Differential System:

$$\begin{cases} \hat{x}_t + A\hat{x} = -BB^*\hat{p} & \text{in } (0,+\infty) \\ -\hat{p}_t + A^*\hat{p} = C^*\Phi_V C\hat{x} & \text{in } (0,+\infty) \\ \hat{x}(0) = x_0. \end{cases} \tag{4.74}$$

Hence, we are in position to affirm that $\hat{x} \in W^{1,2}_{loc}((0,+\infty);(X,X'))$ is the unique solution of the Cauchy Problem:

$$\begin{cases} \hat{x}_t + (A + BB^*\widehat{E})\hat{x} = 0 & \text{in } (0,+\infty) \\ \hat{x}(0) = x_0. \end{cases} \tag{4.75}$$

and $\widehat{E}\hat{x}(\cdot;x_0) \in W^{1,2}_{loc}((0,+\infty);(X,X'))$ solves the Differential Equation:

$$-\hat{p}_t + A^*\hat{p} = C^*\Phi_V C\hat{x} \qquad \text{in } (0,+\infty). \tag{4.76}$$

Now, as we did in the finite dimensional case, we show that the system

$$\begin{cases} \hat{x}_t + (A + BB^*\widehat{E})\hat{x} = 0 & \text{in } (0,+\infty) \\ \hat{x}(0) = x_0 \end{cases} \tag{4.77}$$

is exponentially globally asymptotically stable and the exponential convergence of $\mathcal{E}(t)$ to $\widehat{E}$. Furthermore, we show that the Hypotheses 4.1, 4.2 and 4.3 imply the existence of an exponentially stabilizing feedback $G \in B(H,H)$. The main difference with the finite dimensional case is that we do not use Riccati's theory.

Lemma 4.4. *We suppose Hypotheses 4.1, 4.2 and 4.3. Moreover we assume the inclusion*

$$i : X \hookrightarrow H \in K(X,H)$$

and $B \in B(U,H)$. Then:

1. *the differential linear system*

$$\begin{cases} \hat{x}_t + (A + BB^*\widehat{E})\hat{x} = 0 & \textit{in } (0,+\infty) \\ \hat{x}(0) = x_0 \end{cases} \tag{4.78}$$

   is exponentially globally asymptotically stable;

2. *$G \overset{def}{=} -B^*\widehat{E} \in B(H,H)$ is an exponentially stabilizing feedback function, i.e. there exist $(C_0,\omega) \in (0,+\infty)^2$ such that $\forall x_0 \in H$ the unique solution $\hat{x} \in W^{1,2}((0,T);(X;X'))$ of*

$$\begin{cases} \hat{x}_t + A\hat{x} = BG\hat{x} & \textit{in } (0,+\infty) \\ \hat{x}(0) = x_0, \end{cases} \tag{4.79}$$

   satisfies:

$$\|\hat{x}(t)\|_H \leq Ce^{-\omega t}\|x_0\|_H \qquad \forall t \in [0,+\infty).$$

3. there exist $(C,\mu) \in (0,+\infty)^2$ *such that*

$$\|\mathcal{E}(t) - \widehat{E}\|_{B(H,H)} \leq Ce^{-\mu t}. \tag{4.80}$$

Remark 4.8. We highlight that, in the next proof, we will show that the rate μ in (4.80) can be taken as any exponential rate of $A + BB^*\widehat{E}$.

Proof. We begin with the proof of (1.), namely the exponential stability. It is worth defining $L = A + BB^*\widehat{E}$. The unique solution of the system (4.78) is usually referred as $\hat{x}(\cdot;x_0)$ or $e^{-(\cdot)L}x_0$. First of all, we prove that hypothesis 4.2 implies that for every $z \in W^{1,2}((-\infty,0);(X,X'))$ satisfying the conditions below:

$$\begin{cases} z_t + Az = 0 & \text{in } (-\infty,0) \\ Cz \equiv 0 & \text{in } (-\infty,0) \\ z \in L^\infty((-\infty,0);H) \end{cases} \tag{4.81}$$

then, $z \equiv 0 \quad \text{in } (-\infty,0)$. To this extent, we consider $(\tau,t) \in (-\infty,0)^2$ such that $\tau < t$ and $\lambda \in (0,+\infty)$. τ and λ are degrees of freedom which we will use later. At this stage, we define

$$\tilde{z} : (-\infty,0) \longmapsto H$$

$$t \longrightarrow e^{\lambda t}z(t).$$

$\tilde{z} \in W^{1,2}_{loc}((-\infty,0);(X,X'))$ and it fulfills the following conditions:

$$\begin{cases} \tilde{z}_t + A\tilde{z} = \lambda\tilde{z} & \text{in } (-\infty,0) \\ C\tilde{z} \equiv 0 & \text{in } (-\infty,0) \end{cases} \tag{4.82}$$

Hence, we apply hypothesis 4.2 for $\tilde{z}$ in (τ,t), obtaining:

$$\|\tilde{z}(t)\|_H^2 \leq C\left[\lambda^2\int_\tau^t \|\tilde{z}\|_{X'}^2 ds + \|\tilde{z}(\tau)\|_H^2\right].$$

Employing the inclusion $H \subset X'$, we deduce that $\forall t \in (\tau,0)$

$$\|\tilde{z}(t)\|_H^2 \leq C_0\left[\lambda^2\int_\tau^t \|\tilde{z}\|_H^2 ds + \|\tilde{z}(\tau)\|_H^2\right].$$

By Gronwall's Lemma, for all $t \in (\tau,0)$:

$$\|\tilde{z}(t)\|_H^2 \leq C_0\|\tilde{z}(\tau)\|_H^2 e^{-C_0\lambda^2(\tau-t)}.$$

Then, for all $t \in (\tau, 0)$ taking the integral on $[\tau, t]$:

$$\int_\tau^t \|\tilde{z}(s)\|_H^2 ds \leq C_0 \|\tilde{z}(\tau)\|_H^2 \frac{1}{C_0\lambda^2} \left[e^{C_0\lambda^2(t-\tau)} - 1 \right] \leq$$

$$\leq \frac{1}{\lambda^2} \|\tilde{z}(\tau)\|_H^2 e^{C_0\lambda^2(t-\tau)}.$$

Employing now the definition of $\tilde{z}$ and the boundedness of z, we obtain $\forall(\tau, t) \in (0, +\infty)^2$ such that $t \in (\tau + 1, 0)$:

$$e^{2\lambda(t-1)} \int_{t-1}^t \|z(s)\|_H^2 ds \leq \int_{t-1}^t \|\tilde{z}(s)\|_H^2 ds \leq \int_\tau^t \|\tilde{z}(s)\|_H^2 ds \leq$$

$$\leq C \frac{1}{\lambda^2} e^{-C_0\lambda^2(\tau - t)} e^{2\lambda\tau}.$$

Hence,

$$\int_{t-1}^t \|z(s)\|_H^2 ds \leq C \frac{1}{\lambda^2} e^{(2\lambda - C_0\lambda^2)(\tau - t)}$$

At this stage, we employ our first degree of freedom choosing λ little enough such that $2\lambda - C_0\lambda^2 > 0$. We use now our second degree of freedom, taking the limit of the right hand side as $\tau \longrightarrow -\infty$, getting:

$$\int_{t-1}^t \|z(s)\|_H^2 ds = 0 \qquad \forall t \in (-\infty, 0)$$

Since $z \in C^0((-\infty, 0], H)$, this yields that $z(t) = 0 \; \forall t \in (-\infty, 0]$. The second step of the proof of (1.) consists in showing

$$\sup_{x_0 \in \overline{B^H(0,1)}} \left(\widehat{E}\hat{x}(t; x_0), \hat{x}(t; x_0) \right)_H \underset{t \to +\infty}{\longrightarrow} 0. \tag{4.83}$$

In order to prove this convergence, we consider $\forall x_0 \in H$ the map

$$l(\cdot) : (0, +\infty) \longmapsto \mathbb{R}$$

$$t \longrightarrow \sup_{x_0 \in \overline{B^H(0,1)}} \left(\widehat{E}\hat{x}(t; x_0), \hat{x}(t; x_0) \right)_H.$$

First of all, we compute $\forall x_0 \in H$ the derivative of $\left(\widehat{E}\hat{x}(\cdot; x_0), \hat{x}(\cdot; x_0) \right)_H$. By the formula of derivation of the product and (4.70), for almost every $t \in (0, +\infty)$

$$\frac{d}{dt} \left(\widehat{E}\hat{x}(t; x_0), \hat{x}(t; x_0) \right)_H = \frac{d}{dt} \left(\hat{p}(t; x_0), \hat{x}(t; x_0) \right)_H =$$

$$=< \frac{d}{dt}\hat{p}(t;x_0), \hat{x}(t;x_0) >_{(X',X)} + < \hat{p}(t;x_0), \frac{d}{dt}\hat{x}(t;x_0) >_{(X,X')}=$$

$$=< A^*\hat{p}(t;x_0)-C^*\Phi_V C\hat{x}(t;x_0), \hat{x}(t;x_0) >_{(X',X)} + < \hat{p}(t;x_0), -A\hat{x}(t;x_0)-BB^*\hat{p}(t;x_0) >_{(X,X')}=$$

$$= -\|B^*\widehat{E}\hat{x}(t;x_0)\|_U^2 - \|C\hat{x}(t;x_0)\|_H^2$$

Therefore, for a.e. $t \in (0,+\infty)$

$$\frac{d}{dt}\left(\widehat{E}\hat{x}(t;x_0), \hat{x}(t;x_0)\right)_H \le 0$$

This yields that the function $\left(\widehat{E}\hat{x}(\cdot;x_0), \hat{x}(\cdot;x_0)\right)_H$ is nonincreasing. This in turn implies that $l(\cdot;x_0)$ is nonincreasing. Moreover, since $\forall (x_0,T) \in H\times(0,+\infty)$ we have $(\mathcal{E}(T)x_0,x_0)_H \ge 0$, then for all $t\in[0,+\infty)$ we have $\left(\widehat{E}\hat{x}(t;x_0), \hat{x}(t;x_0)\right)_H \ge 0$. This entails that $l(t;x_0)\ge 0$ for every t. Then, there exists $l_\infty \in \mathbb{R}$ such that:

$$l(t) \underset{t\to+\infty}{\longrightarrow} l_\infty.$$

At this point, by the definition of the supremum, whenever is taken into account a sequence $\{t_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ such that $t_n \underset{n\to+\infty}{\longrightarrow} +\infty$, $\forall n\in\mathbb{N}$ there exists an $x_{0_n} \in \overline{B^H(0,1)}$ such that, defining $\hat{x}_n \overset{\text{def}}{=} \hat{x}(\cdot;x_{0_n})$,

$$\forall n \in \mathbb{N} \qquad l(t_n) \ge \left(\widehat{E}\hat{x}_n(t_n), \hat{x}_n(t_n)\right)_H \ge l(t_n) - \frac{1}{n}. \tag{4.84}$$

In the following steps, we are going to define a sequence of functions $\{z_n\}_{n\in\mathbb{N}}$, whose compactness will play a key role in the next part of the proof. To this extent, let us define for all $n\in\mathbb{N}$ $z_n = \hat{x}(t_n+\cdot;x_{0_n})$. It satisfies

$$\frac{d}{dt}z_n + Lz_n = 0 \qquad \text{in } (-t_n,+\infty)$$

At this stage, employing the estimate (4.63) and the hypothesis 4.2, we deduce the existence of $C\in(0,+\infty)$ independent of n and t such that:

$$\forall t \in (-t_n,+\infty) \qquad \|z_n(t)\|_H \le C.$$

Now, we develop some compactness arguments to deduce the desired result. The basic idea is to exploit the hypotheses $B\in B(U,H)$ and $i \in K(X,H)$. Let us consider each interval $(a,b)\subset\mathbb{R}$. We have already shown that $\{z_n\}_{n\in\mathbb{N}} \subset C^0([a,b],H)$ is bounded. This entails that the sequence

$\left\{BB^*\widehat{E}z_n\right\}_{n\in\mathbb{N}} \subset C^0([a,b],H)$ is bounded too. Hence it is bounded also in $L^2((a,b);X')$. Employing the continuous dependence from the data, we get:

$$\|z_n\|_{W^{1,2}((a,b);(X,X'))} \leq C\left[\|-BB^*\widehat{E}z_n\|_{L^2((a,b);X')} + \|z_n(a)\|_H\right] \leq CM$$

Which is equivalent to the boundedness of the sequence $\{z_n\}_{n\in\mathbb{N}} \subset W^{1,2}((a,b);(X,X'))$. Thanks to the compactness of the inclusion operator i, we can apply Theorem 1.16, which entails that the sequence $\{z_n\}_{n\in\mathbb{N}} \subset L^2([a,b];H)$ is relatively compact. Then, up to subsequences, it converges in $L^2([a,b];H)$. Again, up to subsequences, thanks to Theorem 1.3.4 page 16 of [9], $\{z_n\}_{n\in\mathbb{N}} \subset H$ converge almost everywhere. Moreover, we can take into account $(a',b') \in \mathbb{R}^2$ such that $a' < a < b < b'$. Then, the above a.e. convergence yields the existence of an $\overline{a} \in (a',a]$ such that $\{z_n(\overline{a})\}_{n\in\mathbb{N}} \subset H$ is convergent. Employing once more the continuous dependence from the data, we obtain:

$$\|z_n - z_m\|_{W^{1,2}((a,b);(X,X'))} \leq C\left[\|BB^*\widehat{E}(z_n - z_m)\|_{L^2((\overline{a},b);X')} + \|z_n(\overline{a}) - z_m(\overline{a})\|_H\right]$$

which entails that the sequence $\{z_n\}_{n\in\mathbb{N}} \subset W^{1,2}((a,b);(X,X'))$ is actually Cauchy, hence, by completeness, convergent. This yields the compactness of $\{z_n\}_{n\in\mathbb{N}} \subset C^0([a,b];H)$. Therefore, there exists an element $z \in W^{1,2}_{loc}(\mathbb{R};(X,X'))$ such that (up to subsequences):

$$z_n \underset{n\to+\infty}{\longrightarrow} z \qquad \text{in } W^{1,2}_{loc}(\mathbb{R};(X,X')).$$

Being the limit of the above sequence, $z \in W^{1,2}_{loc}(\mathbb{R};(X,X'))$ must satisfy the equation:

$$\frac{d}{dt}z + Lz = 0 \qquad \text{in } \mathbb{R}$$

We aim now at proving that $z \equiv 0$. To this extent we remind the definition of l and its monotonicity. This properties allow us to deduce the following inequalities:

$$\left(\widehat{E}\hat{x}_n(t_n), \hat{x}_n(t_n)\right)_H \leq \left(\widehat{E}\hat{x}_n(t+t_n), \hat{x}_n(t+t_n)\right)_H \leq l(t+t_n) \qquad \forall t \in (-\infty, 0),$$

which yields, using (4.84) and the compactness of $\{z_n\}_{n\in\mathbb{N}} \subset C^0([t,0];H)$

$$\left(\widehat{E}z(t), z(t)\right)_H = \lim_{s\to+\infty} l(s) = l_\infty \qquad \forall\, t \in (-\infty,0).$$

Since $\left(\widehat{E}z(\cdot), z(\cdot)\right)_H$ is a constant function, we use the computation of its derivative to obtain

$$B^*\widehat{E}z(t) = Cz(t) = 0 \qquad \forall\, t \in (-\infty,0). \tag{4.85}$$

As a matter of fact, on the one hand for any $t \in (-\infty, 0)$

$$\frac{d}{dt}\left(\widehat{E}\hat{x}_n(t_n+t), \hat{x}_n(t_n+t)\right)_H =$$

$$= -\|B^*\widehat{E}\hat{x}_n(t+t_n)\|_U^2 - \|C\hat{x}_n(t+t_n)\|_V^2 \underset{n\to+\infty}{\longrightarrow} -\|B^*\widehat{E}z(t)\|_U^2 - \|Cz(t)\|_V^2$$

on the other hand

$$\frac{d}{dt}\left(\widehat{E}\hat{x}_n(t_n+t), \hat{x}_n(t_n+t)\right)_H \underset{n\to+\infty}{\longrightarrow} \frac{d}{dt}\left(\widehat{E}z(t), z(t)\right)_H \equiv 0$$

This implies the required result, namely (4.85). Then, z satisfies the following conditions:

$$\begin{cases} \frac{d}{dt}z + Az = 0 \qquad \text{in } (-\infty, 0) \\ Cz = 0 \qquad \text{in } (-\infty, 0) \\ z \in L^\infty((-\infty, 0); H) \end{cases} \tag{4.86}$$

We use now the observability result proven in the first part of the proof to obtain:

$$z(t) = 0 \qquad \forall t \in (-\infty, 0).$$

This yields $\left(\widehat{E}z(t), z(t)\right)_H = 0 \qquad \forall t \in (-\infty, 0)$, which in turn implies:

$$\lim_{t\to+\infty} l(t) = 0$$

as required. Now, we aim at showing the convergence:

$$\sup_{x_0 \in \overline{B^H(0,1)}} \|\hat{x}(t; x_0)\|_H \underset{t\to+\infty}{\longrightarrow} 0 \tag{4.87}$$

To this extent, we reason by contradiction. In fact, if the above limit did not hold, there would exist an $\overline{\varepsilon} > 0$, a sequence $\{x_{0,n}\}_{n\in\mathbb{N}} \subset H$ and $\{t_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ such that $t_n \underset{n\to+\infty}{\longrightarrow} +\infty$ satisfying:

$$\|\hat{x}(t_n; x_{0,n})\|_H \geq \overline{\varepsilon} > 0 \qquad \forall n \in \mathbb{N}.$$

Defining, for every $n \in \mathbb{N}$, $w_n = \hat{x}(t_n + \cdot; x_{0_n})$ and using the previous reasonings, we can prove that there exists an element $w \in W^{1,2}_{loc}(\mathbb{R}; (X, X'))$ such that:

$$w_n \underset{n\to+\infty}{\longrightarrow} w \qquad \text{in } W^{1,2}_{loc}(\mathbb{R}; (X, X')).$$

At this point, employing our last achievement, we get:

$$\left(\widehat{E}\hat{x}(t+t_n; x_{0_n}), \hat{x}(t+t_n; x_{0_n})\right)_H \underset{n\to+\infty}{\longrightarrow} 0 \qquad \forall t \in (-\infty, 0)$$

which entails

$$\left(\widehat{E}w(t), w(t)\right)_H = 0 \qquad \forall t \in (-\infty, 0)$$

As before, this leads to:

$$B^*\widehat{E}w(t) = Cw(t) = 0 \qquad \forall\ t \in (-\infty, 0),$$

which, using the observability result (4.81), in turn implies

$$w(t) = 0 \qquad \forall t \in (-\infty, 0).$$

Hence, we have already proven the following convergence:

$$w_n \underset{n\to+\infty}{\longrightarrow} 0 \qquad \text{in } W^{1,2}_{loc}((-\infty, 0); (X, X'))$$

which yields

$$w_n \underset{n\to+\infty}{\longrightarrow} 0 \qquad \text{in } C^0_{loc}((-\infty, 0], H).$$

Therefore

$$\hat{x}(t_n; \hat{x}_{0,n}) \underset{n\to+\infty}{\longrightarrow} 0 \qquad \text{in } H.$$

This turns out to be a contradiction. Hence, we have proved (4.87). Our goal now is to show that there exist 2 constants $(C, \mu) \in (0, +\infty)^2$ such that

$$\|e^{-tL}x_0\|_H \le Ce^{-\mu t}\|x_0\|_H \qquad \forall (x_0, t) \in H \times (0, +\infty)$$

To this extent for all $t \in (0, +\infty)$, we define the operator

$$e^{-tL} : H \longmapsto H$$

$$x_0 \longrightarrow \hat{x}(t; x_0)$$

In previous steps we have shown:

$$\sup_{x_0 \in \overline{B(0,1)}} \|e^{-Lt}x_0\|_H \underset{t\to+\infty}{\longrightarrow} 0$$

which, by definition of the operatorial norm, is equivalent to:

$$\|e^{-tL}\|_{B(H,H)} \underset{t\to+\infty}{\longrightarrow} 0$$

This entails the existence of $t_1 \in (0, +\infty)$ such that:

$$\|e^{-Lt}\|_{B(H,H)} \le \frac{1}{2} \qquad \forall t \ge t_1$$

This yields, by induction on $n \in \mathbb{N}$,

$$\|e^{-Lnt_1}x\| \leq \frac{1}{2^n}$$

Indeed,
P(1) ($n = 1$) by definition of t_1.
Furthermore, if P($n-1$) holds true, does it imply P(n)? In order to show this implication, we can take into account an arbitrary $x_0 \in \overline{B^H(0,1)}$. Then,

$$\|e^{-nt_1L}x_0\|_H \leq \|e^{-(n-1)t_1L}\|_{B(H,H)}\|e^{-t_1L}x_0\|_H \leq \frac{1}{2^{n-1}}\frac{1}{2} = \frac{1}{2^n}$$

Hence, by the Principle of Induction, we get

$$\|e^{-Lnt_1}\|_{B(H,H)} \leq \frac{1}{2^n} \qquad \forall n \in \mathbb{N}.$$

In order to conclude, we consider $\sup_{t\in[0,t_1]} \|e^{-tL}\|_{B(H,H)}$. It is finite, thanks to the continuous dependence from initial data. We define $C_1 = \sup_{t\in[0,t_1]} \|e^{-tL}\|_{B(H,H)} \in \mathbb{R}^+$ and for each $t \in (0,+\infty)$ we set $n(t) = \lfloor \frac{t}{t_1} \rfloor$ Then,

$$\|e^{-tL}\| = \|e^{(-n(t)t_1-(\frac{t}{t_1}-n(t))t_1)L}\| \leq C_1\|e^{-n(t)t_1L}\| \leq C_1\frac{1}{2^{n(t)}}$$

$$\leq C_1e^{-n(t)\ln(2)} \leq C_1e^{-(\frac{t}{t_1}-1)\ln(2)} \leq C_1e^{\ln(2)}e^{-\frac{t}{t_1}\ln(2)}$$

At this stage, we define $C \stackrel{\text{def}}{=} 2\sup_{t\in[0,t_1]} \|e^{-tL}\|$ and $\mu = \frac{\ln(2)}{t_1}$, obtaining:

$$\|e^{-tL}\|_{B(H,H)} \leq Ce^{-\mu t} \qquad \forall t \in (0,+\infty)$$

This is equivalent to:

$$\|e^{-tL}x_0\|_H \leq Ce^{-\mu t}\|x_0\|_H \qquad \forall t \in (0,+\infty)$$

Therefore, the system is Exponentially Globally Asymptotically Stable. Our goal, at the moment, is to prove (2.), namely:

$$\|\mathcal{E}(t) - \widehat{E}\|_{B(H,H)} \leq Ce^{-\mu t} \qquad \forall t \in (0,+\infty).$$

To this extent, we recall that, $\forall(x_0,T) \in H \times (0,+\infty)$, $(x^T(\cdot;x_0), p^T(\cdot;x_0)) \in W^{1,2}((0,T);(X,X'))^2$ satisfies:

$$\begin{cases} x_t^T + Ax^T = -BB^*p^T & \text{in } (0,T) \\ -p_t^T + A^*p^T = C^*\Phi_V Cx^T & \text{in } (0,T) \\ x^T(0) = x_0 \\ p^T(T) = 0, \end{cases} \tag{4.88}$$

whereas $(\hat{x}(\cdot;x_0),\hat{p}(\cdot;x_0)) \in W^{1,2}_{loc}((0,+\infty);(X,X'))^2$ solves:

$$\begin{cases} \hat{x}(\cdot;x_0)_t + A\hat{x}(\cdot;x_0) = -BB^*\hat{p}(\cdot;x_0) & \text{in } (0,+\infty) \\ -\hat{p}(\cdot;x_0)_t + A^*\hat{p}(\cdot;x_0) = C^*\Phi_V C\hat{x}(\cdot;x_0) & \text{in } (0,+\infty) \\ \hat{x}(0;x_0) = x_0. \end{cases} \tag{4.89}$$

Hence, $(x^T(\cdot;x_0)-\hat{x}(\cdot;x_0), p^T(\cdot;x_0)-\hat{p}(\cdot;x_0)) \in W^{1,2}((0,T);(X,X'))^2$ solves

$$\begin{cases} (x^T(\cdot;x_0)-\hat{x}(\cdot;x_0))_t + A(x^T(\cdot;x_0)-\hat{x}(\cdot;x_0)) = -BB^*(p^T(\cdot;x_0)-\hat{p}(\cdot;x_0)) & \text{in } (0,T) \\ -(p^T(\cdot;x_0)-\hat{p}(\cdot;x_0))_t + A^*(p^T(\cdot;x_0)-\hat{p}(\cdot;x_0)) = C^*\Phi_V C(x^T(\cdot;x_0)-\hat{x}(\cdot;x_0)) & \text{in } (0,T) \\ x^T(0;x_0)-\hat{x}(0;x_0) = 0 \\ p^T(T)-\hat{p}(T) = -\hat{p}(T). \end{cases} \tag{4.90}$$

In the following computations, it is convenient to drop the dependence from x_0. At this stage, we multiply in (X',X) the equation of $p^T(\cdot;x_0)-\hat{p}(\cdot;x_0)$ by $x^T(\cdot;x_0)-\hat{x}(\cdot;x_0)$ and we integrate in $[0,T]$:

$$\int_0^T < -(p_t^T(t)-\hat{p}_t(t)) + A^*(p^T(t)-\hat{p}(t)), x^T(t)-\hat{x}(t) >_{(X',X)} dt =$$

$$= \int_0^T < C^*\Phi_V C(x^T(t)-\hat{x}(t)), x^T(t)-\hat{x}(t) >_{(X',X)} dt$$

Integrating by part this is equivalent to:

$$\big(p^T(0)-\hat{p}(0), x^T(0)-\hat{x}(0)\big)_H - \big(p^T(T)-\hat{p}(T), x^T(T)-\hat{x}(T)\big)_H +$$

$$+\int_0^T < (x_t^T(t)-\hat{x}_t(t))+A(x^T(t)-\hat{x}(t)), p^T(t)-\hat{p}(t) >_{(X',X)} dt = \int_0^T \|C(x^T(t)-\hat{x}(t))\|_V^2 dt$$

Hence, we are able to deduce:

$$\big(\hat{p}(T), x^T(T)-\hat{x}(T)\big)_H =$$

$$= \int_0^T \|B^*(p^T(t)-\hat{p}(t))\|_U^2 dt + \int_0^T \|C(x^T(t)-\hat{x}(t))\|_V^2 dt$$

Hence, we obtain the following estimate:

$$\int_0^T \|B^*(p^T(t)-\hat{p}(t))\|_U^2 dt + \int_0^T \|C(x^T(t)-\hat{x}(t))\|_V^2 dt \le \|\hat{p}(T)\|_H \|x^T(T)-\hat{x}(T)\|_H. \tag{4.91}$$

Since $x^T - \hat{x} \in W^{1,2}((0,T);(X,X'))^2$ satisfies

$$\begin{cases} (x^T - \hat{x})_t + A(x^T - \hat{x}) = -BB^*(p^T - \hat{p}) & \text{in } (0,T) \\ x^T(0) - \hat{x}(0) = x_0 - x_0 = 0 \end{cases} \tag{4.92}$$

we are in the position to apply hypothesis 4.2, getting:

$$\|x^T(T) - \hat{x}(T)\|_H^2 \le C\left[\int_0^T \|BB^*(p^T(t) - \hat{p}(t))\|_{X'}^2 + \|C(x^T(t) - \hat{x}(t))\|_V^2 dt\right] \le$$

employing $B \in B(U,H)$,

$$\le C\left[\int_0^T \|B^*(p^T(t) - \hat{p}(t))\|_U^2 + \|C(x^T(t) - \hat{x}(t))\|_V^2 dt\right]$$

namely, there exists a T-independent constant $C \in (0,+\infty)$ such that:

$$\|x^T(T) - \hat{x}(T)\|_H^2 \le C\left[\int_0^T \|B^*(p^T(t) - \hat{p}(t))\|_U^2 + \|C(x^T(t) - \hat{x}(t))\|_V^2 dt\right]. \tag{4.93}$$

Putting together inequalities (4.91) and (4.93), we show the existence of a positive constant C independent of $T \in (0,+\infty)$, such that:

$$\int_0^T \|B^*(p^T(t) - \hat{p}(t))\|_U^2 dt + \int_0^T \|C(x^T(t) - \hat{x}(t))\|_V^2 dt \le C\|\hat{p}(T)\|_H^2 \tag{4.94}$$

On the other hand, $p^T - \hat{p} \in W^{1,2}((0,T);(X,X'))^2$ is solution of:

$$-(p^T - \hat{p})_t + A^*(p^T - \hat{p}) = C^*\Phi_V C(x^T - \hat{x}) \qquad \text{in } (0,T) \tag{4.95}$$

At this point, we employ hypothesis 4.3, obtaining:

$$\|p^T(0) - \hat{p}(0)\|_H^2 \le C\left\{\int_0^T \left(\|B^*(p^T(t) - \hat{p}(t))\|_U^2 + \|C^*\Phi_V C(x^T(t) - \hat{x}(t))\|_{X'}^2\right) dt + \|\hat{p}(T)\|_H^2\right\} \le$$

Since both C and Φ_V are bounded linear operators,

$$\le C\left\{\int_0^T \left(\|B^*(p^T(t) - \hat{p}(t))\|_U^2 + \|C(x^T(t) - \hat{x}(t))\|_V^2\right) dt + \|\hat{p}(T)\|_H^2\right\} \le$$

employing the above estimate (4.94)

$$\le C\|\hat{p}(T)\|_H^2$$

Hence, there exists a constant $C \in (0,+\infty)$, independent of $T \in (0,+\infty)$ and $x_0 \in H$, such that for all $T \in (0,+\infty)$:

$$\|p^T(0;x_0) - \hat{p}(0;x_0)\|_H^2 \le C\|\hat{p}(T;x_0)\|_H^2 \tag{4.96}$$

The final step of the proof of 3. consists in using

$$\widehat{E}\hat{x}(t;x_0) = \hat{p}(t;x_0) \qquad \forall t \in [0,+\infty).$$

Indeed, employing the exponential stability already proved and the continuity of $\widehat{E}$, we get:

$$\|\hat{p}(t;x_0)\|_H = \|\widehat{E}\hat{x}(t;x_0)\|_H \le \|\widehat{E}\|_{B(H,H)}\|\hat{x}(t;x_0)\|_H \le C\|\widehat{E}\|_{B(H,H)}e^{-\mu t}\|x_0\|_H \qquad \forall t \in [0,+\infty).$$

which, from (4.96), yields the existence of a constant $C \in (0,+\infty)$ independent of T, such that:

$$\|p^T(0;x_0) - \hat{p}(0;x_0)\|_H^2 \le Ce^{-2\mu T}\|x_0\|_H^2$$

By the definition of $\mathcal{E}$ and $\widehat{E}$, we obtain:

$$\|\mathcal{E}(t) - \widehat{E}\|_{B(H,H)} \le Ce^{-\mu t} \qquad \forall t \in [0,+\infty)$$

namely, the requirement.

□

As we anticipated, we haven't used Riccati's theory. This leads us to a rougher estimate of $\|\mathcal{E}(t) - \widehat{E}\|_{B(H,H)}$ with respect to Corollary 2.1. At this point, we are going to investigate more accurately the nature of the limit $(\hat{x},\hat{p})$. To this extent, we take into account the stabilizing feedback $G \stackrel{\text{def}}{=} -B^*\widehat{E}$. At the moment, we point out a useful consequence of stability. We will show the existence of a constant $\delta \in (0,+\infty)$ such that $\forall \varphi_0 \in H$ the unique solution of:

$$\begin{cases} \varphi_t + A\varphi = BG\varphi & \text{in } (0,+\infty) \\ \varphi(0) = \varphi_0 \end{cases} \tag{4.97}$$

fulfills

$$\forall T \in (0,+\infty) \qquad \int_0^T \|\varphi(t)\|_X^2 dt \le \delta\|\varphi_0\|_H^2.$$

In fact $\forall T \in (0,+\infty)$, we just multiply in (X',X) the equation (4.97) by φ, getting:

$$\int_0^T < \varphi_t(t) + A\varphi(t), \varphi(t) >_{(X',X)} dt = \int_0^T < BG\varphi(t), \varphi(t) >_{(X',X)} dt$$

which implies

$$\frac{1}{2}\|\varphi(T)\|_H^2-\frac{1}{2}\|\varphi_0\|_H^2+\int_0^T < A\varphi(t),\varphi(t) >_{(X',X)} dt = \int_0^T < BG\varphi(t),\varphi(t) >_{(X',X)} dt$$

Thanks to this equality, hypothesis 4.1, we obtain the existence of a constant $C \in [0,+\infty)$ independent of $(x_0,T) \in H \times (0,+\infty)$ such that:

$$\lambda\int_0^T \|\varphi(t)\|_X^2 dt - \mu\int_0^T \|\varphi(t)\|_H^2 dt \leq$$

$$\leq C\|\varphi_0\|_H^2 + \left[\int_0^T \|BG\varphi(t)\|_{X'}^2 dt\right]^{\frac{1}{2}} \left[\int_0^T \|\varphi(t)\|_X^2 dt\right]^{\frac{1}{2}} \leq$$

by the continuity of $i^*\circ\Phi_H : H \hookrightarrow X'$ and since $(B,G) \in B(U,H)\times B(H,H)$,

$$\leq C\|\varphi_0\|_H^2 + C\left[\int_0^T \|\varphi(t)\|_H^2 dt\right]^{\frac{1}{2}} \left[\int_0^T \|\varphi(t)\|_X^2 dt\right]^{\frac{1}{2}} \leq$$

employing Young's inequality with ε,

$$\leq C\|\varphi_0\|_H^2 + C\int_0^T \|\varphi\|_H^2 dt + \varepsilon\int_0^T \|\varphi(t)\|_X^2 dt.$$

This, together with the definition of stability, implies the existence of a constant $\delta \in (0,+\infty)$ independent of $(T,x_0) \in (0,+\infty)\times H$ such that:

$$\int_0^T \|\varphi(t)\|_X^2 dt \leq C\int_0^T \|\varphi(t)\|_H^2 dt + C\|\varphi_0\|_H^2 \leq \delta\|\varphi_0\|_H^2$$

as required. A useful byproduct of this result is that $\forall\varphi_0 \in H$ the unique solution $\varphi \in W^{1,2}_{loc}((0,+\infty);(X,X'))$ of the differential system (**??**), actually lies in $W^{1,2}((0,+\infty);(X,X'))$. In order to carry on our analysis, we prove the following Remark.

Remark 4.9. There exists a constant $C \in (0,+\infty)$ independent of $s \in [0,+\infty)$ such that $\forall f \in L^2((0,+\infty);X')$ and $\forall\hat{p} \in W^{1,2}_{loc}((0,+\infty);(X,X')) \cap L^\infty((0,+\infty);H)$, such that $B^*\hat{p} \in L^2((0,+\infty);U)$, solution of the equation

$$-\hat{p}_t + A^*\hat{p} = f \qquad in\ (0,+\infty),$$

it holds, $\forall s \in [0,+\infty)$:

$$\|\hat{p}(s)\|_H^2 \leq C\left[\int_s^{+\infty} \|B^*\hat{p}(t)\|_{U'}^2 dt + \int_s^{+\infty} \|f(t)\|_{X'}^2 dt\right].$$

Therefore,

$$\|\hat{p}(t)\|_H \underset{t\to+\infty}{\longrightarrow} 0. \tag{4.98}$$

Proof. As we did to prove Lemma 4.2, we take an arbitrary $\varphi \in W^{1,2}((0,+\infty);(X,X'))$ solution of:

$$\varphi_t + A\varphi = BG\varphi \qquad \text{in } (0,+\infty). \tag{4.99}$$

$\forall (s,t) \in [0,+\infty)^2$, we multiply in (X,X') the equation of $\hat{p}(\cdot;x_0)$ by φ and we use the Formula of Integration by Parts, obtaining:

$$(\hat{p}(s),\varphi(s))_H + \int_s^t < \varphi_t(v) + A\varphi(v), \hat{p}(v) >_{(X',X)} dt = \int_s^t < f(v), \varphi(v) >_{(X',X)} dv + (\hat{p}(t),\varphi(t))_H$$

Then,

$$(\hat{p}(s),\varphi(s))_H = -\int_s^t (B^*\hat{p}(v), G\varphi(v))_U \, dv + \int_s^t < f(v),\varphi(v) >_{(X',X)} dv + (\hat{p}(t),\varphi(t))_H \leq$$

$$\leq C\left(\int_s^t \|\varphi(v)\|_X^2 dv\right)^{\frac{1}{2}} \left\{\int_s^t \|B^*\hat{p}(v)\|_U^2 dv + \int_s^t \|f(v)\|_{X'}^2 dv\right\}^{\frac{1}{2}} + (\hat{p}(t),\varphi(t))_H \leq$$

employing the hypothesis

$$\leq C\|\varphi(s)\|_H \left\{\int_s^t \|B^*\hat{p}(v)\|_U^2 dv + \int_s^t \|f(v)\|_{X'}^2 dv\right\}^{\frac{1}{2}} + (\hat{p}(t),\varphi(t))_H .$$

Hence,

$$(\hat{p}(s),\varphi(s))_H \leq C\|\varphi(s)\|_H \left\{\int_s^t \|B^*\hat{p}(v)\|_U^2 dt + \int_s^t \|f(v)\|_{X'}^2 dv\right\}^{\frac{1}{2}} + |(\hat{p}(t),\varphi(t))_H| \leq$$

$$\leq C\|\varphi(s)\|_H \left\{\int_s^{+\infty} \|B^*\hat{p}(v)\|_U^2 dt + \int_s^{+\infty} \|f(v)\|_{X'}^2 dv\right\}^{\frac{1}{2}} + |(\hat{p}(t),\varphi(t))_H|$$

As regards the last term:

$$|(\hat{p}(t),\varphi(t))_H| \leq M\|\varphi(t)\|_H \leq MCe^{-\mu t}\|x_0\|_H \underset{t\to+\infty}{\longrightarrow} 0$$

All in all, using our degree of freedom $\varphi(s) = \hat{p}(s)$ and taking the limit as $t \to +\infty$ on the right hand side, one obtains

$$\|\hat{p}(s)\|_H \leq C\left\{\int_s^{+\infty} \|B^*\hat{p}(v)\|_U^2 dt + \int_s^{+\infty} \|f(v)\|_{X'}^2 dv\right\}^{\frac{1}{2}}$$

This yields

$$0 \leq \limsup_{s\to+\infty} \|\hat{p}(s)\|_H \leq \limsup_{s\to+\infty} C\left\{\int_s^{+\infty} \|B^*\hat{p}(v)\|_U^2 dt + \int_s^{+\infty} \|f(v)\|_{X'}^2 dv\right\}^{\frac{1}{2}} = 0,$$

as desired. □

By (4.63) we obtain $\hat{p} \in L^\infty((0,+\infty);H)$ and (4.53) yields that $B^*\hat{p} \in L^2((0,+\infty);U)$. Then, by Remark 4.9, we get, $\forall x_0 \in H$,

$$\hat{p}(t;x_0) \underset{t\to+\infty}{\longrightarrow} 0$$

strongly in H. At this point, we have shown that $(\hat{x}(\cdot;x_0),\hat{p}(\cdot;x_0)) \in W^{1,2}((0,+\infty);(X,X'))^2$ satisfies the system below. For the moment, we drop the dependence from the initial data $x_0 \in H$.

$$\begin{cases} \hat{x}_t + A\hat{x} = -BB^*\hat{p}\hat{x} & \text{in } (0,+\infty) \\ -\hat{p}_t + A^*\hat{p} = C^*\Phi_V C\hat{x} & \text{in } (0,+\infty) \\ \hat{x}(0) = x_0 \\ \hat{p}(t) \underset{t\to+\infty}{\longrightarrow} 0 \text{ strongly in } H \end{cases} \tag{4.100}$$

First of all, we prove the uniqueness for such a system, i.e. we aim at showing that for every $x_0 \in H$ and $((\hat{x}_1(\cdot;x_0),\hat{p}_1(\cdot;x_0)),(\hat{x}_2(\cdot;x_0),\hat{p}_2(\cdot;x_0))) \in W^{1,2}((0,+\infty);(X,X'))^4$ pair of solutions of the above system, we have:

$$x_1(t;x_0) = x_2(t;x_0) \qquad \wedge \qquad p_1(t;x_0) = p_2(t;x_0) \qquad \forall t \in [0,+\infty).$$

We notice that since $\forall T \in (0,+\infty)$, $\forall i \in \{1,2\}$ and $\forall \varphi \in W^{1,2}((0,+\infty);(X,X'))$:

$$(\hat{p}_i(0),\varphi(0))_H - (\hat{p}_i(T),\varphi(T))_H + \int_0^T < \varphi_t + A\varphi, \hat{p} >_{(X',X)} dt = \int_0^T (C\hat{x}_i, C\varphi)_V \, dt$$

then, taking the limit as $T \to +\infty$,

$$(\hat{p}_i(0),\varphi(0))_H + \int_0^{+\infty} < \varphi_t + A\varphi, \hat{p}_i >_{(X',X)} dt = \int_0^{+\infty} (C\hat{x}_i, C\varphi)_V \, dt$$

Moreover, the pair $(\hat{x}_2 - \hat{x}_1, \hat{p}_2 - \hat{p}_1)$ satisfies

$$(\hat{x}_2 - \hat{x}_1)_t + A(\hat{x}_2 - \hat{x}_1) = -B\Phi_U^{-1}B^*(\hat{p}_2 - \hat{p}_1) \qquad \text{in } (0,+\infty)$$

and, setting a test function $\varphi = \hat{x}_2 - \hat{x}_1$, we obtain $\forall i \in \{1,2\}$

$$(\hat{p}_i(0), \hat{x}_2(0) - \hat{x}_1(0))_H + \int_0^{+\infty} < (\hat{x}_2 - \hat{x}_1)_t + A(\hat{x}_2 - \hat{x}_1), \hat{p}_i >_{(X',X)} dt =$$

$$= \int_0^{+\infty} (C\hat{x}_i, C(\hat{x}_2 - \hat{x}_1))_V \, dt$$

which in turn implies, by subtracting the above equalities:

$$(\hat{p}_2(0) - \hat{p}_1(0), \hat{x}_2(0) - \hat{x}_1(0))_H + \int_0^{+\infty} < (\hat{x}_2 - \hat{x}_1)_t + A(\hat{x}_2 - \hat{x}_1), \hat{p}_2 - \hat{p}_1 >_{(X',X)} dt =$$

$$= \int_0^{+\infty} (C(\hat{x}_2 - \hat{x}_1), C(\hat{x}_2 - \hat{x}_1))_V \, dt.$$

This expression is equivalent to:

$$(\hat{p}_2(0) - \hat{p}_1(0), x_0 - x_0)_H + \int_0^{+\infty} < -B\Phi_U^{-1}B^*(\hat{p}_2 - \hat{p}_1), \hat{p}_2 - \hat{p}_1 >_{(X',X)} dt =$$

$$= \int_0^{+\infty} \|C(\hat{x}_2 - \hat{x}_1)\|_V^2 dt.$$

Which yields

$$\int_0^{+\infty} \|B^*(\hat{p}_2 - \hat{p}_1)\|_{U'}^2 dt + \int_0^{+\infty} \|C(\hat{x}_2 - \hat{x}_1)\|_V^2 dt = 0$$

By measure theory,

$$B^*\hat{p}_2(t) = B^*\hat{p}_1(t) \qquad \text{a.e. } t \in (0, +\infty)$$

$$C\hat{x}_1(t) = C\hat{x}_2(t) \qquad \text{a.e. } t \in (0, +\infty).$$

Furthermore, since both $\hat{x}_i \in C^0([0, +\infty), H)$ and $\hat{p}_i \in C^0([0, +\infty), H)$

$$B^*\hat{p}_2(t) = B^*\hat{p}_1(t) \qquad \forall t \in [0, +\infty)$$

$$C\hat{x}_1(t) = C\hat{x}_2(t) \qquad \forall t \in [0, +\infty).$$

Hence, $\hat{x}_2 - \hat{x}_1$ satisfies:

$$\begin{cases} (\hat{x}_2 - \hat{x}_1)_t + A(\hat{x}_2 - \hat{x}_1) = -B\Phi_U^{-1}B^*(\hat{p}_2 - \hat{p}_1) = 0 & \text{in } (0, +\infty) \\ \hat{x}_2(0) - \hat{x}_1(0) = 0 \end{cases} \tag{4.101}$$

Using Theorem 4.1, this yields:

$$\hat{x}_1(t) = \hat{x}_2(t) \qquad \forall t \in [0, +\infty)$$

This enables us to affirm that $\hat{p}_2 - \hat{p}_1$ is a solution of the equation

$$-\hat{p}_t + A^*\hat{p} = C^*\Phi_V C\hat{x}_2 - C^*\Phi_V C\hat{x}_1 = 0 \qquad \text{in } (0, +\infty) \tag{4.102}$$

in the sense of the Definition (4.1). This entails that, using Remark 4.9, there exists a t-independent constant $C \in (0, +\infty)$ such that:

$$\forall t \in [0, +\infty) \qquad \|\hat{p}_2(t) - \hat{p}_1(t)\|_H^2 \leq$$

$$\leq C\left[\int_t^{+\infty} \|B^*(\hat{p}_2-\hat{p}_1)\|_U^2 ds + \int_t^{+\infty} \|C^*\Phi_V C(\hat{x}_2-\hat{x}_1)\|_{X'}^2 ds\right] = 0$$

Therefore, the uniqueness holds. Hence, $(\hat{x},\hat{p})$ is the unique solution of the system (4.100). The uniqueness for the above system allows us to be more precise about the convergences of the state and the adjoint state as the time horizon approaches infinity. Indeed, employing (4.67), (4.68) and Proposition 1.1, one obtains that $\forall x_0 \in H$ initial data:

$$x^T(\cdot;x_0) \underset{T\to+\infty}{\longrightarrow} \hat{x}(\cdot;x_0)$$

strongly in $C^0([0,T];H)$ and

$$p^T(\cdot;x_0) \underset{T\to+\infty}{\longrightarrow} \hat{p}(\cdot;x_0)$$

strongly in $C^0([0,T];H)$. Furthermore, by (4.62) and Proposition 1.7,

$$x^T(\cdot;x_0) \underset{T\to+\infty}{\rightharpoonup} \hat{x}(\cdot;x_0)$$

weakly in $L^2_{loc}((0,+\infty);X)$. Moreover, we remark that $\forall u \in L^2((0,+\infty);U)$ for the Cauchy Problem

$$\begin{cases} x_t + Ax = Bu & \text{in } (0,+\infty) \\ x(0) = x_0 \end{cases} \tag{4.103}$$

there exists and it is unique the solution $x \in W^{1,2}_{loc}((0,+\infty);(X,X'))$, thanks to an application of Theorem 4.1 $\forall T \in (0,+\infty)$. Now, we are going to prove the well posedeness of the infinite-horizon Optimal Control Problem $(OCP)_0^{+\infty}$. To this extent, thanks to the continuity of the operators C and B^*,

$$(Cx^T(\cdot;x_0), B^*p^T(\cdot;x_0)) \underset{T\to+\infty}{\rightharpoonup} (C\hat{x}, B^*\hat{p})$$

weakly in $L^2_{loc}((0,+\infty);V) \times L^2_{loc}((0,+\infty);U)$. By the lower-semicontinuity of the norm with respect to the weak convergence for every $T \in (0,+\infty)$,

$$\int_0^T \left(\|B^*\hat{p}\|_U^2 + \|C\hat{x}\|_V^2\right) dt \leq \liminf_{n\to+\infty} \int_0^T \left(\|B^*p_n\|_U^2 + \|Cx_n\|_V^2\right) dt \leq C.$$

Since T was arbitrary,

$$\int_0^{+\infty} \left(\|B^*\hat{p}\|_U^2 + \|C\hat{x}\|_V^2\right) dt \leq C < +\infty$$

Therefore, we define $\hat{u} = -\Phi_U^{-1}B^*\hat{p}$. Then, there exists $\hat{u} = -\Phi_U^{-1}B^*\hat{p} \in L^2((0,+\infty);U)$ control function such that $\hat{x}$ is the unique solution of:

$$\begin{cases} \hat{x}_t + A\hat{x} = B\hat{u} & \text{in } (0,+\infty) \\ \hat{x}(0) = x_0 \end{cases} \tag{4.104}$$

and

$$\int_0^T \|\hat{u}\|_U^2 + \|C\hat{x}\|_V^2 dt < +\infty \tag{4.105}$$

the estimate being uniform on $x_0 \in B$, for every $B \subset H$ bounded set. This entails the well posedeness of the functional J_0^T. We will show now that $(\hat{x},\hat{p},\hat{u})$ is actually the unique optimal pair for $(OCP)_0^{+\infty}$. Indeed, whenever we take $(x,p,u) \in W^{1,2}_{loc}((0,+\infty);(X,X')) \times W^{1,2}_{loc}((0,+\infty);(X,X')) \times L^2((0,+\infty);U)$, for each $n \in \mathbb{N}$

$$\int_0^{T_n} \|u\|_U^2 + \|Cx\|_V^2 dt \geq \int_0^{T_n} \|B^*p_n\|_{U'}^2 + \|Cx_n\|_V^2 dt$$

Then, taking the limit as $n \to +\infty$ in the left hand side, we get:

$$\int_0^{+\infty} \|u\|_U^2 + \|Cx\|_V^2 dt \geq \int_0^{T_n} \|B^*p_n\|_{U'}^2 + \|Cx_n\|_V^2 dt$$

Finally, using the lower-semicontinuity of the norm with respect to the weak convergence, for every $T \in (0,+\infty)$

$$\int_0^T \|\hat{u}\|_U^2 + \|C\hat{x}\|_V^2 dt \leq \liminf_{n\to+\infty} \int_0^T \|B^*p_n\|_U^2 + \|Cx_n\|_V^2 dt \leq$$

$$\leq \liminf_{n\to+\infty} \int_0^{T_n} \|B^*p_n\|_U^2 + \|Cx_n\|_V^2 dt \leq \int_0^{+\infty} \|u\|_U^2 + \|Cx\|_V^2 dt$$

Which in turn implies:

$$\int_0^{+\infty} \|\hat{u}\|_U^2 + \|C\hat{x}\|_V^2 dt \leq \int_0^{+\infty} \|u\|_U^2 + \|Cx\|_V^2 dt$$

This yields the desired result. By Proposition 1.2, the minimizer $\hat{u}$ is also unique. Therefore, we have proved that, for every $x_0 \in H$, there exists a unique solution $(\hat{x},\hat{p}) \in W^{1,2}((0,+\infty),(X,X'))^2$ for the Cauchy Problem:

$$\begin{cases} \hat{x}_t + A\hat{x} = -B\Phi_U^{-1}B^*\hat{p} & \text{in } (0,+\infty) \\ -\hat{p}_t + A^*\hat{p} = C^*\Phi_V C\hat{x} & \text{in } (0,+\infty) \\ \hat{x}(0) = x_0 \\ \hat{p}(t) \underset{t\to+\infty}{\longrightarrow} 0 & \text{strongly in } H \end{cases} \tag{4.106}$$

which is also the unique optimal pair for $(OCP)_0^{+\infty}$. In what follows, we will call this solution $(\hat{x}(\cdot;x_0),\hat{p}(\cdot;x_0))$. At this stage, using the definition of $\mathcal{E}$ we are going to define an optimal feedback law for $(OCP)_0^T$ for every $T \in (0,+\infty)$. To this purpose, by (4.72), $\forall t \in [0,T]$:

$$u^T(t;x_0) = u^{T-t}(0;x^T(t;x_0)) = -B^* p^{T-t}(0;x^T(t;x_0)) \stackrel{\text{def}}{=} -B^*\mathcal{E}(T-t)(x^T(t;x_0)).$$

Hence, we have determined the optimal feedback law $\forall T \in (0,+\infty)$:

$$\begin{aligned} f^T : [0,T]\times H &\longmapsto U \\ (t,x_0) &\longrightarrow -B^*\mathcal{E}(T-t)(x^T(t;x_0)). \end{aligned} \tag{4.107}$$

We aim now at proving that for each $x_0 \in H$ the function

$$\mathcal{E}x_0 : \mathbb{R}^+ \longmapsto H$$

belongs to $\in C^0([0,+\infty),H)$. First of all, we show that $\forall \tilde{T} \in (0,+\infty)$ $\mathcal{E}(\cdot;x_0)$ is left continuous at $\tilde{T}$. To this purpose, we prove the following Remark

Remark 4.10. Under hypotheses 4.1, 4.2 and 4.3, $\forall T \in (0,+\infty)$ we define

$$\begin{aligned} \Theta_T : H &\longmapsto W^{1,2}((0,T);(X,X'))\times W^{1,2}((0,T);(X,X'))\times L^2((0,T);U) \\ x_0 &\longrightarrow (x^T(\cdot;x_0),p^T(\cdot;x_0),u^T(\cdot;x_0)) \end{aligned} \tag{4.108}$$

Then,

1. $\Theta_T \in B(H, W^{1,2}((0,T);(X,X'))\times W^{1,2}((0,T);(X,X'))\times L^2((0,T);U))$;
2. $\forall \tilde{T} \in (0,+\infty)$ there exists $K_{\tilde{T}} \in (0,+\infty)$ such that:

$$\|\Theta_T\| \le K_{\tilde{T}}.$$

Proof. $\forall T \in (0,+\infty)$ we observe that, since the optimal triple satisfies Pontryagin System (4.29), the map Θ is linear. At this step, $\forall \tilde{T} \in (0,+\infty)$ we aim at proving the boundedness of Θ_T uniform $\forall T \in (0,\tilde{T}]$. To this purpose, we take into account $\overline{B^H(0,1)}^H \subset H$. By the continuous dependence from the data there exists a constant $C_{\tilde{T}} \in (0,+\infty)$ such that $\forall (x_0,T) \in \overline{B^H(0,1)}^H \times (0,\tilde{T}]$:

$$\|x^T(\cdot;x_0)\|^2_{W^{1,2}((0,T);(X,X'))} \le C_{\tilde{T}} \left[\int_0^T \|-BB^*p^T(t)\|^2_{X'}dt + \|x_0\|^2_H\right].$$

Furthermore, by inequality (4.53), there exists a constant $C \in (0, +\infty)$ such that $\forall (x_0, T) \in \overline{B^H(0,1)}^H \times (0, +\infty)$

$$\int_0^T \left[\|Cx^T(t;x_0)\|_V^2 + \|u^T(t;x_0)\|_U^2\right] dt \leq C.$$

Hence, there exists a constant $K_{\tilde{T},1} \in (0, +\infty)$ such that $\forall (x_0, T) \in \overline{B^H(0,1)}^H \times (0, \tilde{T}]$

$$\|x^T(\cdot;x_0)\|^2_{W^{1,2}((0,T);(X,X'))} \leq K_{\tilde{T},1}.$$

On the other hand, since the continuous dependence from the data holds, there exists a constant $C_{\tilde{T}} \in (0, +\infty)$ such that $\forall (x_0, T) \in \overline{B^H(0,1)}^H \times (0, \tilde{T}]$:

$$\|p^T(\cdot;x_0)\|^2_{W^{1,2}((0,T);(X,X'))} \leq C_{\tilde{T}} \left[\int_0^T \|C^*\Phi_V Cx^T(\cdot;x_0)\|^2_{X'} dt + \|p^T(T;x_0)\|_H^2\right] \leq$$

Then, by (4.53), there exists a constant $K_{\tilde{T},2} \in (0, +\infty)$ such that $\forall (x_0, T) \in \overline{B^H(0,1)}^H \times (0, \tilde{T}]$

$$\|p^T(\cdot;x_0)\|^2_{W^{1,2}((0,T);(X,X'))} \leq K_{\tilde{T},2}.$$

Finally, once more by estimate (4.53), there exists $C \in (0, +\infty)$ such that $\forall (x_0, T) \in \overline{B^H(0,1)}^H \times (0, \tilde{T}]$:

$$\int_0^T \|u^T(t;x_0)\|_U^2 dt \leq C.$$

Therefore, we have proved that $\forall \tilde{T} \in (0, +\infty)$ $\forall T \in (0, \tilde{T}]$

$$\Theta_T \in B(H, W^{1,2}((0,T);(X,X')) \times W^{1,2}((0,T);(X,X')) \times L^2((0,T);U))$$

and there exists $K_{\tilde{T}} \in (0, +\infty)$ such that:

$$\|\Theta_T\| \leq K_{\tilde{T}} \qquad \forall T \in (0, \tilde{T}].$$

□

By this Remark, we can carry on the proof of the left continuity of $\mathcal{E}(\cdot)(x_0)$. Indeed, $\forall \tilde{T} \in (0, +\infty)$, $\forall T \in (0, \tilde{T}]$

$$\|p^{\tilde{T}}(0;x_0) - p^T(0;x_0)\|_H^2 \leq$$

$$\leq 2\|p^{\tilde{T}}(0;x_0) - p^{\tilde{T}}(\tilde{T} - T;x_0)\|_H^2 + 2\|p^{\tilde{T}}(\tilde{T} - T;x_0) - p^T(0;x_0)\|_H^2.$$

Since $p^{\tilde{T}}(\cdot; x_0) \in C^0([0,T]; H)$, the first addendum:

$$\|p^{\tilde{T}}(0; x_0) - p^{\tilde{T}}(\tilde{T} - T; x_0)\|_H \underset{T \to \tilde{T}^-}{\longrightarrow} 0.$$

In order to show the convergence to 0 of the second term, we observe that $(x^{\tilde{T}}(\cdot + \tilde{T} - T; x_0), p^{\tilde{T}}(\cdot + \tilde{T} - T; x_0), u^{\tilde{T}}(\cdot + \tilde{T} - T; x_0))$ is the optimal triple for $(OCP)_0^T$ with initial condition $x^{\tilde{T}}(\tilde{T} - T; x_0)$, namely the triple $(x^{\tilde{T}}(\cdot + \tilde{T} - T; x_0), p^{\tilde{T}}(\cdot + \tilde{T} - T; x_0), u^{\tilde{T}}(\cdot + \tilde{T} - T; x_0))$ solves:

$$\begin{cases} \frac{d}{dt}x + Ax = -B\Phi_U^{-1}B^*p & \text{in } (0,T) \\ x(0) = x^{\tilde{T}}(\tilde{T} - T; x_0) \\ -\frac{d}{dt}p + A^*p = C^*\Phi_V Cx & \text{in } (0,T) \\ p(T) = 0 \end{cases} \tag{4.109}$$

We are now in position to estimate $\|p^{\tilde{T}}(\tilde{T} - T; x_0) - p^T(0, x_0)\|_H^2$:

$$\|p^{\tilde{T}}(\tilde{T} - T; x_0) - p^T(0, x_0)\|_H^2 \leq \|p^{\tilde{T}}(\cdot + \tilde{T} - T; x_0) - p^T(\cdot, x_0)\|_{C^0([0,T],H)}^2$$

since the inclusion

$$i : W^{1,2}((0,T); (X, X')) \longmapsto C^0([0,T], H) \in B(W^{1,2}((0,T); (X, X')), C^0([0,T], H))$$

and there exists $D_{\tilde{T}} \in (0, +\infty)$ such that $\forall T \in (\frac{\tilde{T}}{2}, \frac{3\tilde{T}}{2})$:

$$\|i\| \leq D_{\tilde{T}}.$$

Then,

$$\|p^{\tilde{T}}(\tilde{T} - T; x_0) - p^T(0, x_0)\|_H^2 \leq \|p^{\tilde{T}}(\cdot + \tilde{T} - T; x_0) - p^T(\cdot, x_0)\|_{C^0([0,T],H)}^2 \leq$$

$$\leq C\|p^{\tilde{T}}(\cdot + \tilde{T} - T; x_0) - p^T(\cdot, x_0)\|_{W^{1,2}((0,T);(X,X'))} \leq$$

by Remark 4.10,

$$\leq C\|x^{\tilde{T}}(\tilde{T} - T; x_0) - x_0\|_H \underset{T \to \tilde{T}^-}{\longrightarrow} 0.$$

Hence, $\forall \tilde{T} \in (0, +\infty)$:

$$\|p^{\tilde{T}}(0; x_0) - p^T(0; x_0)\|_H^2 \leq$$

$$\leq 2\|p^{\tilde{T}}(0; x_0) - p^{\tilde{T}}(\tilde{T} - T; x_0)\|_H^2 + 2\|p^{\tilde{T}}(\tilde{T} - T; x_0) - p^T(0; x_0)\|_H^2 \underset{T \to \tilde{T}^-}{\longrightarrow} 0.$$

Therefore, $\forall (x_0, \tilde{T}) \in H \times (0, +\infty)$

$$\|\mathcal{E}(\tilde{T})(x_0) - \mathcal{E}(T)(x_0)\|_H = \|p^{\tilde{T}}(0; x_0) - p^T(0; x_0)\|_H^2 \underset{T \to \tilde{T}^-}{\longrightarrow} 0.$$

It remains to show the right continuity of $\mathcal{E}(\cdot; x_0)$. $\forall \tilde{T} \in [0, +\infty)$, we aim at proving that:

$$\|p^T(0; x_0) - p^{\tilde{T}}(0; x_0)\|_H \underset{T \to \tilde{T}^+}{\longrightarrow} 0.$$

To this purpose, we remind that $\forall x_0 \in H$, by Remark 4.7, there exists $M_1 \in \mathbb{R}^+$ such that:

$$\left\{x^T(t; x_0) \mid T \in (0, +\infty),\ t \in [0, T]\right\} \subset \overline{B^H(0, M_1)}.$$

Furthermore, by hypothesis 4.3 and inequality (4.53), there exists a positive constant $M_2 \in (0, +\infty)$ such that:

$$\left\{p^T(t; x_0) \mid T \in (0, +\infty),\ t \in [0, T]\right\} \subset \overline{B^H(0, M_2)}. \tag{4.110}$$

Then, by continuous dependence from the data,

$$\|x^T(\cdot; x_0)\|^2_{W^{1,2}((0,\tilde{T});(X,X'))} \leq C \left[\int_0^{\tilde{T}} \|BB^*(p^T(t; x_0))\|^2_{X'} dt + \|x_0\|^2_H\right]$$

and

$$\|p^T(\cdot; x_0)\|^2_{W^{1,2}((0,\tilde{T});(X,X'))} \leq C \left[\int_0^{\tilde{T}} \|C^*\Phi_V C x^T(t; x_0)\|^2_{X'} dt + \|p^T(\tilde{T}; x_0)\|^2_H\right].$$

By (4.53) and (4.110), we get that there exists a constant $M_3 \in (0, +\infty)$ such that $\forall T \in (0, +\infty)$

$$\left\{x^T(\cdot; x_0) \mid T \in (0, +\infty)\right\} \subset \overline{B^{W^{1,2}((0,\frac{3}{2}\tilde{T});(X,X'))}(0, M_3)}. \tag{4.111}$$

and

$$\left\{p^T(\cdot; x_0) \mid T \in (0, +\infty)\right\} \subset \overline{B^{W^{1,2}((0,\frac{3}{2}\tilde{T});(X,X'))}(0, M_3)}. \tag{4.112}$$

At this step, we consider an arbitrary sequence $\{T_n\}_{n\in\mathbb{N}} \subset [\tilde{T}, +\infty)$ such that:

$$T_n \underset{n \to +\infty}{\longrightarrow} \tilde{T}.$$

For every subsequence $\{T_{n_k}\}_{k\in\mathbb{N}} \subset \{T_n\}_{n\in\mathbb{N}}$

$$\left\{x^{T_{n_k}}(\cdot; x_0) \mid k \in \mathbb{N}\right\} \subset \overline{B^{W^{1,2}((0,\tilde{T});(X,X'))}(0, M_3)}. \tag{4.113}$$

and

$$\left\{p^{T_{n_k}}(\cdot;x_0) \mid k \in \mathbb{N}\right\} \subset \overline{B^{W^{1,2}((0,\tilde{T});(X,X'))}(0,M_3)}. \tag{4.114}$$

Hence, by Banach-Alaoglu Theorem, there exist

$$\hat{y}(\cdot;x_0) \in W^{1,2}((0,\tilde{T});(X,X'))$$

$$\hat{q}(\cdot;x_0) \in W^{1,2}((0,\tilde{T});(X,X'))$$

$$\left\{x^{T_{n_{k_h}}}(\cdot;x_0)\right\}_{h\in\mathbb{N}} \subset \left\{x^{T_{n_k}}(\cdot;x_0)\right\}_{k\in\mathbb{N}}$$

and

$$\left\{p^{T_{n_{k_h}}}(\cdot;x_0)\right\}_{h\in\mathbb{N}} \subset \left\{p^{T_{n_k}}(\cdot;x_0)\right\}_{k\in\mathbb{N}}$$

such that:

$$x^{T_{n_{k_h}}}(\cdot;x_0) \underset{h\to+\infty}{\rightharpoonup} \hat{y}(\cdot;x_0)$$

weakly in $W^{1,2}((0,\tilde{T});(X,X'))$ and

$$p^{T_{n_{k_h}}}(\cdot;x_0) \underset{h\to+\infty}{\rightharpoonup} \hat{q}(\cdot;x_0)$$

weakly in $W^{1,2}((0,\tilde{T});(X,X'))$. We are going to show that $(\hat{y},\hat{q})$ is actually independent of the particular subsequence taken into account. Since $A \in B(X,X')$, $B \in B(U,H)$ and $C \in B(X,V)$, $(\hat{y},\hat{q}) \in W^{1,2}((0,\tilde{T});(X,X'))^2$ satisfies the System of Differential Equations:

$$\begin{cases} \frac{d}{dt}\hat{y} + A\hat{y} = -B\Phi_U^{-1}B^*\hat{q} & \text{in } (0,\tilde{T}) \\ -\frac{d}{dt}\hat{q} + A^*\hat{q} = C^*\Phi_V C\hat{y} & \text{in } (0,\tilde{T}) \\ \hat{y}(0) = x_0. \end{cases} \tag{4.115}$$

At this moment, we point out, thanks to hypothesis 4.3 and (4.112), that $\forall T \in [\tilde{T}, \frac{3}{2}\tilde{T}]$

$$\|p^T(\tilde{T};x_0)\|_H \le C\left[\int_{\tilde{T}}^T \|B^*p^T(t;x_0)\|_{U'}^2 dt + \int_{\tilde{T}}^T \|C^*\Phi_V Cx^T(t;x_0)\|_{X'}^2 dt\right] \le$$

by (4.112),

$$\le \int_{\tilde{T}}^T M_4 dt = M_4(T-\tilde{T}).$$

Then, by Proposition 1.6,

$$\|\hat{q}(\tilde{T})\|_H \le \liminf_{h\to+\infty} \|p^{T_{n_{k_h}}}(\tilde{T};x_0)\|_H \le \liminf_{h\to+\infty} M_4(T_{n_{k_h}} - \tilde{T}) = 0.$$

Therefore,

$$\hat{q}(\tilde{T}; x_0) = 0.$$

Hence, $(\hat{y}, \hat{q}) \in W^{1,2}((0, \tilde{T}); (X, X'))^2$ solves:

$$\begin{cases} \frac{d}{dt}\hat{y} + A\hat{y} = -B\Phi_U^{-1}B^*\hat{q} & \text{in } (0, \tilde{T}) \\ -\frac{d}{dt}\hat{q} + A^*\hat{q} = C^*\Phi_V C\hat{y} & \text{in } (0, \tilde{T}) \\ \hat{y}(0) = x_0 \\ \hat{q}(T) = 0. \end{cases} \tag{4.116}$$

Therefore, $(\hat{y}, \hat{q}) = (x^{\tilde{T}}(\cdot; x_0), p^{\tilde{T}}(\cdot; x_0))$. Then, by Proposition 1.7:

$$x^T(\cdot; x_0) \underset{T \to \tilde{T}^+}{\rightharpoonup} x^{\tilde{T}}(\cdot; x_0)$$

weakly in $W^{1,2}((0, \tilde{T}); (X, X'))$ and

$$p^T(\cdot; x_0) \underset{T \to \tilde{T}^+}{\rightharpoonup} p^{\tilde{T}}(\cdot; x_0)$$

weakly in $W^{1,2}((0, \tilde{T}); (X, X'))$. At this stage, we aim at proving that:

$$p^T(\cdot; x_0) \underset{T \to \tilde{T}^+}{\longrightarrow} p^{\tilde{T}}(\cdot; x_0)$$

strongly in $C^0([0, \tilde{T}], H)$. We take an arbitrary sequence $\{T_n\}_{n \in \mathbb{N}} \subset [\tilde{T}, +\infty)$ such that

$$T_n \underset{n \to +\infty}{\longrightarrow} \tilde{T}^+.$$

For every subsequence $\{T_{n_k}\}_{k \in \mathbb{N}} \subset \{T_n\}_{n \in \mathbb{N}}$

$$\left\{p^{T_{n_k}}(\cdot; x_0)\right\}_{k \in \mathbb{N}} \subset W^{1,2}((0, \tilde{T}); (X, X'))$$

is bounded. By Simon's Theorem 1.16,

$$\left\{p^{T_{n_k}}(\cdot; x_0)\right\}_{k \in \mathbb{N}} \subset L^2((0, \tilde{T}); H)$$

is compact. We already know that,

$$p^T(\cdot; x_0) \underset{T \to \tilde{T}^+}{\longrightarrow} p^{\tilde{T}}(\cdot; x_0)$$

weakly in $L^2((0, \tilde{T}); H)$. Then,

$$\exists \left\{p^{T_{n_{k_h}}}(\cdot; x_0)\right\}_{h \in \mathbb{N}} \subset \left\{p^{T_{n_k}}(\cdot; x_0)\right\}_{k \in \mathbb{N}}.$$

such that

$$p^{T_{n_{k_h}}}(\cdot;x_0) \underset{h\to+\infty}{\longrightarrow} p^{\tilde{T}}(\cdot;x_0)$$

strongly in $L^2((0,\tilde{T});H)$. By Proposition 1.1,

$$p^T(\cdot;x_0) \underset{T\to\tilde{T}^+}{\longrightarrow} p^{\tilde{T}}(\cdot;x_0)$$

strongly in $L^2((0,\tilde{T});H)$. Employing $B\in B(U,H)$ and the continuous dependence from the data:

$$\|x^T(\cdot;x_0)-x^{\tilde{T}}(\cdot;x_0)\|^2_{W^{1,2}((0,\tilde{T});(X,X'))} \leq C\left[\int_0^T \|BB^*(p^T(t;x_0)-p^{\tilde{T}}(t;x_0))\|^2_{X'}dt+\|x_0-x_0\|^2_H\right]\leq$$

$$\leq C\left[\int_0^{\tilde{T}} \|p^T(t;x_0)-p^{\tilde{T}}(t;x_0)\|^2_H dt\right] \underset{T\to\tilde{T}^+}{\longrightarrow} 0.$$

Once more, by the continuous dependence from the data,

$$\|p^T(\cdot;x_0)-p^{\tilde{T}}(\cdot;x_0)\|^2_{W^{1,2}((0,\tilde{T});(X,X')} \leq C\left[\int_0^T \|C^*\Phi_V C(x^T(t;x_0)-x^{\tilde{T}}(t;x_0))\|^2_{X'}dt+\right.$$

$$\left.+\|p^T(\tilde{T};x_0)-p^{\tilde{T}}(\tilde{T},x_0)\|^2_H\right]\leq$$

$$\leq C\left[\int_0^{\tilde{T}} \|x^T(t;x_0)-x^{\tilde{T}}(t;x_0)\|^2_X dt+M_4(T-\tilde{T})\right] \underset{T\to\tilde{T}^+}{\longrightarrow} 0.$$

Then,

$$p^T(\cdot;x_0) \underset{T\to\tilde{T}^+}{\longrightarrow} p^{\tilde{T}}(\cdot;x_0)$$

strongly in $W^{1,2}((0,\tilde{T});(X,X'))$. Hence,

$$p^T(\cdot;x_0) \underset{T\to\tilde{T}^+}{\longrightarrow} p^{\tilde{T}}(\cdot;x_0)$$

strongly in $C^0([0,\tilde{T}];H)$. This in turn entails that:

$$p^T(0;x_0) \underset{T\to\tilde{T}^+}{\longrightarrow} p^{\tilde{T}}(0;x_0)$$

which is equivalent to:

$$\mathcal{E}(T)(x_0) \underset{T\to\tilde{T}^+}{\longrightarrow} \mathcal{E}(\tilde{T})(x_0)$$

strongly in H. Therefore, we have already shown the right continuity of $\mathcal{E}(\cdot;x_0)$. Then, we are able to assert that:

$$\mathcal{E}(\cdot)(x_0) : [0,+\infty) \longrightarrow H \in C^0([0,\infty),H).$$

At this point, we remind that, by (4.75), $\forall x_0 \in H$, $\hat{x}(\cdot;x_0)$ is the unique solution of the linear Cauchy Problem:

$$\begin{cases} \hat{x}_t + (A + BB^*\widehat{E})\hat{x} = 0 & \text{in } (0,+\infty) \\ \hat{x}(0) = x_0. \end{cases} \tag{4.117}$$

Then, we have uncoupled the optimality system for the infinite horizon problem. In the next proof, we will show how to use Lemma 4.4 to deduce the turnpike property in the Infinite Dimensional Linear Quadratic Case. Henceforth, we will work with $(OCP)^T$ starting at an arbitrary initial data $x_0 \in H$ with target $z \in V$. We will drop the dependence from x_0 both in the finite-horizon optimal triple (x^T, p^T, u^T) and in the target 0 infinite horizon optimal triple $(\hat{x}, \hat{p}, \hat{u})$. We will name $(\overline{x}, \overline{p}, \overline{u}) \in X \times X \times U$ the optimal triple for $(OCP)^S$, namely $(\overline{x}, \overline{u})$ is the unique minimizer of J^s and $\overline{p}$ is the unique solution of:

$$\begin{cases} A^*\overline{p} = C^*\Phi_V(C\overline{x} - z) \\ \overline{u} = -\Phi_U^{-1}B^*\overline{p} \end{cases} \tag{4.118}$$

Theorem 4.3. *We suppose hypotheses 4.1, 4.2 and 4.3 holds true. Furthermore, we assume the inclusion $i : X \hookrightarrow H \in K(X,H)$ and $B \in B(U,H)$. Then, $\forall (z,x_0) \in V \times H$*

1. *$\forall T \in (0,+\infty)$ there exists a unique solution $h^T \in W^{1,2}((0,T);(X,X'))$ of the Cauchy Problem:*

$$\begin{cases} -h_t^T + (A^* + \mathcal{E}(T-\cdot)BB^*)h^T = 0 & \text{in } (0,T) \\ h^T(T) = -\overline{p} \end{cases} \tag{4.119}$$

2. *$\forall T \in (0,+\infty)$*

$$f^T : [0,T] \times H \longmapsto H$$
$$(t,x) \longrightarrow \overline{u} - B^*\left[\mathcal{E}(T-t)(x-\overline{x}) + h^T(t)\right]$$

is an optimal feedback law for $(OCP)^T$.

3. *There exists 2 constants $(C,\mu) \in (0,+\infty)^2$ such that, for any time horizon T, $\forall t \in [0,T]$:*

$$\|x^T(t;x_0) - \overline{x}\|_H + \|p^T(t;x_0) - \overline{p}\|_H + \|u^T(t;x_0) - \overline{u}\|_U \leq C\left[\|x_0 - \overline{x}\|_H e^{-\mu t} + \|\overline{p}\|_H e^{-\mu(T-t)}\right]. \tag{4.120}$$

The constant $C \in (0,+\infty)$ depending only on the target z, the initial data x_0 and the operators A, B and C.

Proof. First of all, we remind that the family of operators $\{\mathcal{E}(T)\}_{T\in(0,+\infty)} \subset B(H,H)$ is bounded. Therefore, the family of operators $\{\mathcal{E}(T)BB^*\}_{T\in(0,+\infty)} \subset B(H,H)$ is bounded too. This enables us to use Corollary 4.1 in order to prove (1.). Therefore, there exists a unique solution $h^T \in W^{1,2}((0,T);(X,X'))$ of the Cauchy Problem

$$\begin{cases} h_t^T(t) = (A^* + \mathcal{E}(T-t)BB^*)h^T(t) & \text{in } (0,T) \\ h^T(T) = -\overline{p} \end{cases} \tag{4.121}$$

As usual, we define $L = A + BB^*\widehat{E}$. Since $\widehat{E}^* = \widehat{E}$, h^T satisfies

$$\begin{cases} h_t^T(t) = L^*h^T(t) + (\mathcal{E}(T-t) - \widehat{E})BB^*h^T(t) & \forall t \in (0,T) \\ h^T(T) = -\overline{p} \end{cases} \tag{4.122}$$

At this stage, we define

$$g : [0,T] \longmapsto X$$
$$t \longrightarrow (\mathcal{E}(T-t) - \widehat{E})BB^*h^T(t).$$

We recognise that $g \in C^0([0,T];H)$. This enables us to use Proposition 4.2, with the exponential notation, obtaining $\forall t \in [0,T]$:

$$h^T(t) = -e^{-(T-t)L^*}\overline{p} - \int_t^T e^{-(s-t)L^*}\left((\mathcal{E}(T-s) - \widehat{E})BB^*h^T(s)\right) ds$$

In the next computations, we will look for the desired estimate of $\|h^T\|_H$ employing Lemma 4.4, namely:

$$\|h^T(t)\|_H \leq \left\|-e^{-(T-t)L^*}\overline{p}\right\|_H + \left\|\int_t^T e^{-(s-t)L^*}\left((\mathcal{E}(T-s) - \widehat{E})BB^*h^T(s)\right) ds\right\|_H \leq$$

At this stage, it turns to be useful Lemma 4.4(1) and Lemma 4.1. It implies that L^* exponentially stabilizes. We will use this property in the next computations. Moreover, we will employ Lemma 4.4(3).

$$\leq \|\overline{p}\|_H e^{-\mu(T-t)} + C\int_t^T e^{-\mu(s-t)}e^{-\mu(T-s)}\|h^T(s)\|_H ds =$$

$$= \|\overline{p}\|_H e^{-\mu(T-t)} + C\int_t^T e^{-\mu(T-t)}\|h^T(s)\|_H ds =$$

We compute $t \in [0,T]$:

$$\int_t^T e^{-\mu(T-s)} ds = \frac{1}{\mu}\left[1 - e^{-\mu(T-t)}\right] \leq \frac{1}{\mu},$$

Gronwall's Lemma 1.4 entails for every $t \in [0,T]$

$$\|h^T(t)\|_H \leq C\|\overline{p}\|_H e^{-\mu(T-t)} \qquad \forall t \in [0,T] \tag{4.123}$$

the constant C being independent of $T \in (0,+\infty)$. At this point, we are going to prove the following explicit representation of the difference $p^T - \overline{p}$:

$$p^T(t) - \overline{p} = \mathcal{E}(T-t)(x^T(t) - \overline{x}) + h^T(t)$$

To this extent, we show that $\forall \varphi \in H$

$$\left(p^T(t) - \overline{p}, \varphi\right)_H = \left(x^T(t) - \overline{x}, \mathcal{E}(T-t)\varphi\right)_H + \left(h^T(t), \varphi\right)_H$$

In order to prove the above condition, we remind we have already shown in (4.72) that for every $(\varphi, t) \in H \times (0,T)$ and for each $s \in [t,T]$:

$$(x^T(s;\varphi), p^T(s;\varphi), u^T(s;\varphi)) = (x^{T-t}(s-t; x^T(t;\varphi)), p^{T-t}(s-t; x^T(t;\varphi)), u^{T-t}(s-t; x^T(t;\varphi))).$$

Therefore, for all $s \in [t,T]$

$$p^{T-t}(s-t; x^T(t;\varphi)) = p^T(s;\varphi). \tag{4.124}$$

Hence, taking $s = t$, $\forall t \in (0,T)$,

$$\mathcal{E}(T-t)x^T(t;\varphi) = p^T(t;\varphi).$$

Hence, $\forall (t,\varphi) \in (0,T) \times H$, if we define (w,q) the solution of

$$\begin{cases} w_t + Aw^T = -BB^* q & \text{in } (t,T) \\ -q_t + A^* q = C^* \Phi_V C w & \text{in } (t,T) \\ w(t) = \varphi \\ q(T) = 0 \end{cases} \tag{4.125}$$

then $(\tilde{w}, \tilde{q}) = (w(\cdot + t), q(\cdot + t))$ solves

$$\begin{cases} \tilde{w}_t + A\tilde{w}^T = -BB^* \tilde{q} & \text{in } (0,T-t) \\ -\tilde{q}_t + A^* \tilde{q} = C^* \Phi_V C \tilde{w} & \text{in } (0,T-t) \\ \tilde{w}(0) = \varphi \\ \tilde{q}(T-t) = 0. \end{cases} \tag{4.126}$$

Therefore, we have $\forall s \in (t,T)$

$$\mathcal{E}(T-s)w(s) \overset{\text{def}}{=} p^{T-s}(0; w(s)) = p^{T-s}(0; x^{T-t}(s-t;\varphi)) =$$

using (4.124), with $\tilde{T} = T - t$, $\tilde{t} = s - t$ and $\tilde{s} = \tilde{t}$

$$= p^{T-t}(\tilde{t}; \varphi) = p^{T-t}(s - t; \varphi) = \tilde{q}(s - t) = q(s)$$

Hence, $\forall (t, \varphi) \in (0, +\infty) \times H$,

$$\mathcal{E}(T - s)w(s) = q(s) \qquad \forall s \in [t, T] \tag{4.127}$$

At this stage, we multiply in (X', X) the equation of q by $x^T - \overline{x}$ and we integrate in $[t, T]$:

$$\int_t^T < -q_t(v) + A^* q(v), x^T(v) - \overline{x} >_{(X',X)} dv = \int_t^T < C^* \Phi_V C w(v), x^T(v) - \overline{x} >_{(X',X)} dv$$

By the Formula of Integration by Parts, we deduce:

$$\big(q(t), x^T(t) - \overline{x}\big)_H - \big(q(T), x^T(T) - \overline{x}\big)_H + \int_t^T < q(v), (x^T - \overline{x})_t(v) + A(x^T(v) - \overline{x}) >_{(X,X')} dv =$$

$$= \int_t^T < C^* \Phi_V C w(v), x^T(v) - \overline{x} >_{(X',X)} dv =$$

$$= \int_t^T < w(v), C^* \Phi_V C(x^T(v) - \overline{x}) >_{(X,X')} dv =$$

This is equivalent to:

$$\big(q(t), x^T(t) - \overline{x}\big)_H + \int_t^T < q(v), -BB^*(p^T(v) - \overline{p}) >_{(X,X')} dv =$$

$$= \int_t^T < w(v), -(p^T - \overline{p})_t(v) + A^*(p^T - \overline{p})(v) >_{(X,X')} dv =$$

again, integrating by parts,

$$= \big(w(t), p^T(t) - \overline{p}\big)_H - \big(w(T), p^T(T) - \overline{p}\big)_H +$$

$$+ \int_t^T < w_t(v) + Aw(v), p^T(v) - \overline{p} >_{(X',X)} dv =$$

$$= \big(\varphi, p^T(t) - \overline{p}\big)_H + (w(T), \overline{p})_H +$$

$$- \int_t^T < BB^* q(v), p^T(v) - \overline{p} >_{(X',X)} dv$$

It remains to prove that

$$\big(h^T(t), \varphi\big)_H = \tag{4.128}$$

$$= -\,(w(T),\overline{p})_H + \int_t^T < (x^T - \overline{x})_t(v) + A(x^T(v) - \overline{x}), q(v) >_{(X',X)} dv +$$

$$+ \int_t^T < BB^* q(v), p^T(v) - \overline{p} >_{(X',X)} dv$$

In order to prove this result, we verify:

$$(h(T), w(T))_H - (\varphi, h(t))_H = 0$$

To this extent, we multiply in (X, X') the equation of w by h^T and we integrate in $[t, T]$, getting:

$$\int_t^T < w_t(v) + Aw(v), h^T(v) >_{(X',X)} dv = \int_t^T < -BB^* q(v), h^T(v) >_{(X',X)} dv \tag{4.129}$$

We develop the left hand side integrating by parts:

$$\int_t^T < w_t(v) + Aw(v), h^T(v) >_{(X',X)} dv = \big(w(T), h^T(T)\big)_H - \big(w(t), h^T(t)\big)_H +$$

$$+ \int_t^T < -h_t^T(v) + A^* h^T(v), w(v) >_{(X',X)} dv =$$

$$= \big(w(T), h^T(T)\big)_H - \big(\varphi, h^T(t)\big)_H - \int_t^T < \mathcal{E}(T-v) BB^* h^T(v), w(v) >_{(X',X)} dv$$

On the other hand, the right hand side reads as follows:

$$\int_t^T < -BB^* q(v), h^T(v) >_{(X',X)} dv =$$

$$= -\int_t^T < q(v), BB^* h^T(v) >_{(X,X')} dv$$

Then (4.129) becomes:

$$\big(w(T), h^T(T)\big)_H - \big(\varphi, h^T(t)\big)_H - \int_t^T < \mathcal{E}(T-v) BB^* h^T(v), w(v) >_{(X',X)} dv =$$

$$= \int_t^T - < BB^* h^T(v), q(v) >_{(X',X)} dv =$$

$$= -\int_t^T < BB^* h^T(v), \mathcal{E}(T-v) w(v) >_{(X',X)} dv$$

Hence, we deduce:

$$\left(w(T), h^T(T)\right)_H - \left(\varphi, h^T(t)\right)_H = 0$$

Which in turn implies:

$$\left(\varphi, h^T(t)\right)_H = -\left(w(T), \overline{p}\right)_H . \qquad (4.130)$$

We aim now at showing that:

$$\int_t^T < (x^T - \overline{x})_t(v) + A(x^T(v) - \overline{x}), q(v) >_{(X',X)} dt + \int_t^T < BB^* q(v), p^T(v) - \overline{p} >_{(X',X)} dv = 0$$

Indeed,

$$\int_t^T < (x^T - \overline{x})_t(v) + A(x^T(v) - \overline{x}), q(v) >_{(X',X)} dv =$$

$$= \int_t^T < -BB^*(p^T(v) - \overline{p}), q(v) >_{(X',X)} dv =$$

$$= -\int_t^T < BB^* q(v), p^T(v) - \overline{p} >_{(X',X)} dv.$$

Therefore, we have proved that:

$$\left(h^T(t), \varphi\right)_H = \qquad (4.131)$$

$$= -\left(w(T), \overline{p}\right)_H + \int_t^T < (x^T - \overline{x})_t(v) + A(x^T(v) - \overline{x}), q(v) >_{(X',X)} dv +$$

$$+ \int_t^T < BB^* q(v), p^T(v) - \overline{p} >_{(X',X)} dv$$

This allow us to affirm that: $\forall \varphi \in H$

$$\left(p^T(t) - \overline{p}, \varphi\right)_H = \left(x^T(t) - \overline{x}, \mathcal{E}(T-t)\varphi\right)_H + \left(h^T(t), \varphi\right)_H$$

which entails

$$p^T(t; x_0) - \overline{p} = \mathcal{E}(T-t)(x^T(t; x_0) - \overline{x}) + h^T(t) \qquad \forall t \in (0, T) \qquad (4.132)$$

This is a key point for the whole proof, because we are ready to deduce the optimal affine feedback law as follows $\forall T \in (0, +\infty)$:

$$f^T : [0, T] \times H \longmapsto H \qquad (4.133)$$

$$(t,x) \longrightarrow \overline{u} - B^* \left[\mathcal{E}(T-t)(x-\overline{x}) + h^T(t)\right]$$

thus proving (2.). This means that we can find out a new version of the Cauchy Problem satisfied by $x^T(\cdot;x_0)-\overline{x}$. Indeed, $x^T(\cdot;x_0)-\overline{x} \in W^{1,2}((0,T);(X;X'))$ is the unique solution of the Cauchy Problem:

$$\begin{cases}(x^T-\overline{x})_t + A(x^T-\overline{x}) = -BB^*\left[\mathcal{E}(T-t)(x^T-\overline{x})+h^T\right] & \text{in } (0,T)\\ x^T(0)-\overline{x} = x_0-\overline{x}\end{cases} \tag{4.134}$$

Which yields $(x^T(\cdot;x_0)-\overline{x}) \in W^{1,2}((0,T);(X;X'))$ is the solution of the Cauchy Problem:

$$\begin{cases}(x^T-\overline{x})_t + (A+BB^*\widehat{E})(x^T-\overline{x})^T = BB^*(\widehat{E}-\mathcal{E}(T-\cdot))(x^T-\overline{x}) - BB^*h^T & \text{in } (0,T)\\ x^T(0)-\overline{x} = x_0-\overline{x}\end{cases} \tag{4.135}$$

Recognising that $BB^*(\widehat{E}-\mathcal{E}(T-\cdot))(x^T(\cdot;x_0)-\overline{x})-BB^*h^T \in C^0([0,T];H)$, we can apply the Proposition 4.1 in order to get the following representation formula for the difference $x^T(\cdot;x_0)-\overline{x}$, i.e. for every $t\in[0,T]$:

$$x^T(t;x_0)-\overline{x} = e^{-tL}(x_0-\overline{x}) + \int_0^t \left(e^{-(t-s)L}BB^*(\widehat{E}-\mathcal{E}(T-s))(x^T(s;x_0)-\overline{x}) - BB^*h^T(s)\right) ds.$$

Aiming at estimating $\|x^T(\cdot;x_0)-\overline{x}\|_H$, we employ Lemma 4.4 and (4.123). We obtain for each $t\in[0,T]$:

$$\|x^T(t,x_0)-\overline{x}\|_H \leq C\|x_0-\overline{x}\|_H e^{-\mu t} + C\int_0^t e^{-\mu(t-s)}e^{-\mu(T-s)}\|x^T(s;x_0)-\overline{x}\|_H ds +$$

$$+C\int_0^t \|\overline{p}\|_H e^{-\mu(t-s)}e^{-\mu(T-s)} ds \leq$$

$$\leq C\left(\|x_0-\overline{x}\|_H e^{-\mu t} + \|\overline{p}\|_H e^{-\mu(T-t)}\right) + C\int_0^t e^{-\mu(T-s)}e^{-\mu(t-s)}\|x^T(s;x_0)-\overline{x}\|_H ds.$$

At this point, we employ Gronwall's Lemma 1.3 getting:

$$\|x^T(t;x_0)-\overline{x}\|_H \leq C\left[\|x_0-\overline{x}\|_H e^{-\mu t} + \|\overline{p}\|_H e^{-\mu(T-t)}\right] \qquad \forall t\in[0,T].$$

We remind now that we have already proved the following representation formula for the difference $p^T(\cdot;x_0)-\overline{p}$:

$$p^T(t;x_0)-\overline{p} = \mathcal{E}(T-t)(x^T(t;x_0)-\overline{x}) + h^T(t) \qquad \forall t\in[0,T]$$

This, together with the previous achievements, entails for every $t\in[0,T]$:

$$\|p^T(t;x_0)-\overline{p}\|_H \leq C\|x^T(t;x_0)-\overline{x}\|_H + \|h^T(t)\|_H \leq C\left[\|x_0-\overline{x}\|_H e^{-\mu t} + \|\overline{p}\|_H e^{-\mu(T-t)}\right]$$

As regards the difference $u^T - \overline{u}$, we obtain:

$$\|u^T(t;x_0)-\overline{u}\|_U = \|B^*p^T(t;x_0)-B^*\overline{p}\|_U \leq C\left[\|x_0-\overline{x}\|_H e^{-\mu t} + \|\overline{p}\|_H e^{-\mu(T-t)}\right] \qquad \forall t\in[0,T].$$

This conclude the proof. $\square$

Now, we are going to give an example of the Turnpike Property with a Distributed Control Problem.

Example 4.1. Take into account an arbitrary open set $\Omega \subset \mathbb{R}^N$, such that $\partial\Omega \in C^2$. Moreover, consider

- $\omega \subset \Omega$, open;
- $\omega_0 \subset \Omega$, open;
- $M \in C^1(\Omega; \mathcal{M}(N,N;\mathbb{R}))$ bounded an uniformly coercive, i.e. $\exists(\alpha,\beta) \in (0,+\infty)^2\ \forall x\in\Omega\ \alpha I \leq M(x) \leq \beta I$;
- $$c \in C^1(\Omega;\mathbb{R})$$
  such that $c(x) \geq 0 \quad \text{a.e. } x\in\Omega$;
- $$b \in C^1(\Omega;\mathbb{R}^N);$$
- $(H, (,)_H) = \left(L^2(\Omega;\mathbb{R}), (,)_{L^2(\Omega;\mathbb{R})}\right)$;
- $(X, (,)_X) = \left(H_0^1(\Omega;\mathbb{R}), (,)_{H_0^1(\Omega;\mathbb{R})}\right)$;
- $$A : X \longmapsto X'$$
  $$u \longrightarrow (M\nabla u, \nabla(\cdot))_{H^N} + ((b,\nabla(u))_{\mathbb{R}^N}, \cdot)_H + (cu,\cdot)_H$$
- $V = L^2(\omega_0;\mathbb{R})$;
- $U = L^2(\omega;\mathbb{R})$;
- $$C : X \longmapsto V$$
  $$y \longrightarrow y\chi_{\omega_0};$$

-

$$B : U \longmapsto H$$

$$u \longrightarrow u\chi_\omega$$

One can observe that $A \in B(X, X')$, $C \in B(X, V)$ and $B \in B(U, H)$. Moreover, by Rellich-Kondrakhov Compactness Theorem, the inclusion

$$i : X \hookrightarrow H \in K(X, H).$$

Furthermore, $\forall (x_0, T) \in L^2(\Omega; \mathbb{R}) \times (0, +\infty)$, $\forall u \in L^2((0, T); U)$ the associated state is the solution of the Cauchy-Dirichlet Problem:

$$\begin{cases} \frac{d}{dt}x - div(M\nabla x) + cx + (b, \nabla x)_{\mathbb{R}^N} = u\chi_\omega \qquad \text{in } (0, T) \times \Omega \\ x = 0 \qquad \text{in } (0, T) \times \partial\Omega \\ x(0) = x_0 \qquad \text{in } \Omega \end{cases} \tag{4.136}$$

Employing Minty-Browder Theorem and Shaefer Fixed Point Theorem we can get:

$$A : X \longrightarrow X'$$

is an isomorphism. Furthermore, hypothesis 4.1 holds true. Moreover, by Maximum Principle 1.17 and Theorem 2.6 of [1], one can prove that A has only strictly positive eigenvalues. Hence, thanks to Hille-Yosida Theorem (see [3]) A is exponentially stabilizes. Then, by Lemma 4.1, A^* is exponentially stable too. Hence Hypotheses 4.2 and 4.3 hold. Moreover one can observe that they holds actually for every choice of $C \in B(X, V)$ and $B \in B(U, H)$. Therefore, the Turnpike Property holds true.

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