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On the emergence of local flocking phenomena in Cucker-Smale ensemble

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On the emergence of local flocking phenomena in Cucker-Smale ensemble

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Abstract

We study the Cucker-Smale model which describes the flocking phenomena. In detail, we focus on the sufficient conditions to achieve the local flocking phenomena in various scenarios and models. The Lyapunov functional approach and bootstrapping argument play key roles to prove the asymptotic stability on emergence of flocking along time evolution. In the Cucker-Smale model, the dynamics of the particles are presented by the couplings proportional to the relative velocities between each pair of particles. The sufficient condition to the global flocking was suggested by Cucker and Smale, while necessary condition or local flocking was not analytically studied.
Our interests covers not only the Cucker-Smale particle model, but also the unit-speed model and the hydrodynamic model. The unit-speed model breaks the symmetry of equations and the hydrodynamic model needs the existence of solutions. The hydrodynamic equations are the macroscopic description through the mean-field limit process of the particle model. We avoid free boundary problems by describing Lagrangian variables.

Key words: Cucker-Smale model, critical coupling strength, hydrodynamic model, flocking, multi-cluster flocking, dynamical system
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Contents

Abstract **i**

1 Introduction **1**

2 Preliminaries **5**
2.1 Flocking phenomena 5
2.2 Two particle results 8
2.3 Review on the global flocking 12
2.4 Cucker-Smale model with unit speed constraint 13
2.4.1 A generalized J-K model 15
2.4.2 A multi-dimensional C-S type model 18
2.5 Hydrodynamic descriptions of Cucker-Smale model 20

3 Existence of bi-cluster flocking **24**
3.1 A framework for bi-cluster flocking 24
3.1.1 Reformulation of the C-S model 25
3.1.2 Derivation of a dissipative differential inequalities . . . 28
3.1.3 Well-prepared initial configurations 32
3.2 Analysis of the bi-cluster flocking phenomenon 33
3.2.1 Some elementary estimates in finite-time 33
3.2.2 Bi-cluster flocking estimates 35
3.3 Numerical simulations 39

4 Multi-cluster flocking in terms of coupling strength **42**
4.1 Necessary condition for mono-cluster flocking 43
4.1.1 Framework and main result 44

4.1.2 Dynamics of local averages and fluctuations 47
4.1.3 Non-existence of mono-cluster flocking 49
4.2 Emergence of multi-cluster flocking 57
4.2.1 Framework and main result 58
4.2.2 Dynamics of local averages and fluctuations 59
4.2.3 Proof on multi-cluster flocking 60
4.3 Numerical simulations 64
4.3.1 Non-flocking configurations 65
4.3.2 Emergence of multi-cluster flocking 65

5 Existence of bi-cluster flocking with unit speed constraint 68
5.1 A generalized two-dimensional J-K model 68
5.1.1 A reformulation of the J-K model 69
5.1.2 A class of admissible initial data 70
5.1.3 Time-evolution of functionals 71
5.1.4 Emergence of bi-cluster flocking 76
5.2 The multi-dimensional C-S model with the unit speed constraint 79
5.2.1 A reformulation of the C-S model 80
5.2.2 A class of admissible initial data 82
5.2.3 Time-evolution of functionals 83
5.2.4 Emergence of bi-cluster flocking 86
5.3 Numerical simulations 90

6 Multi-cluster flocking with unit speed constraint 93
6.1 A necessary condition for mono-cluster flocking 93
6.1.1 A framework and main result 94
6.1.2 Dynamics of local averages and fluctuations 96
6.1.3 Non-existence of mono-cluster flocking 97
6.2 Emergence of multi-cluster flocking 105
6.2.1 A framework and main result 105
6.2.2 Dynamics of local averages and fluctuations 107
6.2.3 Proof on multi-cluster flocking 110
6.3 Numerical simulations 114
6.3.1 Non-existence of mono-cluster flocking 115
6.3.2 Total separation of particles 115

6.3.3 Emergence of multi-cluster flockings 116

7 Local flocking scenarios with two ensemble coupling network 120

7.1 Emergence of mono-cluster flocking 122

7.1.1 Estimates on moments and functionals 122

7.1.2 Proof on the mono-cluster flocking pheonomena 126

7.2 Emergence of the local flocking phenomena 129

7.2.1 Description of bi-cluster flocking 130

7.2.2 Emergence of bi-cluster flocking 134

7.2.3 Description of partial flocking 142

7.2.4 Emergence of partial flocking 145

8 Bi-cluster flocking on hydrodynamic Cucker-Smale model 151

8.1 Lagrangian formulation and variables 153

8.1.1 Lagrangian formulation 153

8.1.2 Macroscopic quantities 156

8.2 Description of frameworks and main results 159

8.2.1 Description of mono-cluster flocking 160

8.2.2 Description of bi-cluster flocking 161

8.3 Dynamics of the coupled C-S system 163

8.3.1 Dynamics of mono-cluster flocking 164

8.3.2 Bi-cluster flocking 168

8.4 Global existence of classical solutions 175

8.4.1 Local existence of smooth solutions 176

8.4.2 A priori estimates 177

8.4.3 Proof on the global-time flocking phenomena 187

9 Conclusion and future works 188

Appendix A Gronwall type inequalities 190

Bibliography 191

Abstract (in Korean) 201

Acknowledgement (in Korean) 202

Chapter 1

Introduction

Emergence of coherent motions have been studied in various area to represent collective behaviors, such as the flocking of birds, aggregation of bacteria, and swarming of fish. These 'flocking' phenomena are often observed in complex biological systems, where self-propelled agents only use limited information and their independent decisions. Its individualistic coherent behaviors have been studied rigorously using various type of couplings and dynamics [11, 31, 28, 29, 30, 57, 73, 74, 75]. From the pioneering work of Winfree and Kuramoto [79, 54], it has been extensively studied in possible applications to mobile and sensor networks, in the control of robots and unmanned aerial vehicles [57, 66, 68]. Since the flocking phenomena occur in various context and properties, many agent-based models have been proposed and studied extensively both analytically and numerically. Among these types of equations, our interest lies on the Cucker-Smale flocking model.

Our main focus is on the flocking model introduced by Cucker and Smale [26], which describes couplings as nonlocally interacting N-body system. Let $\boldsymbol{z}_i = (\boldsymbol{x}_i, \boldsymbol{v}_i) \in \mathbb{R}^{2d}$ be the phase-space coordinate of the i-th Cucker-Smale (C-S) flocking agent. Then, the dynamics of C-S flocking particles is governed by the following system of ODEs:

$$\begin{aligned}
&\dot{\boldsymbol{x}}_i = \boldsymbol{v}_i, \quad t > 0, \quad i = 1, \cdots, N, \\
&\dot{\boldsymbol{v}}_i = \frac{K}{N} \sum_{j=1}^{N} \psi(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|)(\boldsymbol{v}_j - \boldsymbol{v}_i), \\
&(\boldsymbol{x}_i, \boldsymbol{v}_i)(0) = (\boldsymbol{x}_{i0}, \boldsymbol{v}_{i0}),
\end{aligned} \tag{1.0.1}$$

where K is the positive coupling strength, and the communication weight $\psi : \mathbb{R}_+ \to \mathbb{R}$ is the communication weight satisfying the positivity, analytic continuity, monotonicity. Here, we assume one more condition, integrability, and call it short-range communication weight. For simplicity, we sometimes assume it to take the algebraic decay form

$$\psi(s) := \frac{1}{(1+s)^{\beta}}, \quad \beta > 1, \tag{1.0.2}$$

or to be an analytic function from $\mathbb{R}$ to $\mathbb{R}$,

$$\psi(s) := \frac{1}{(1+s^2)^{\frac{\beta}{2}}}, \quad \beta > 1. \tag{1.0.3}$$

Since the velocities have symmetric structures with relative velocities, the mean velocity is constant for time t. Hence we may assume that the mean position and velocity are initially zero.

In [26], Cucker and Smale introduced this second-order Newtonian system supplemented by the weighted relaxation internal forces governing the spatial and velocity dynamics of particles. They also provided several sufficient conditions for admissible initial configurations leading to the global flocking. They found that the flocking condition mainly depends on the communication weight. The Cucker-Smale model and its variants have been extensively studied in previous literature [1, 2, 5, 11, 13, 14, 15, 18, 19, 22, 25, 26, 33, 35, 37, 46, 48, 49, 63, 72] from the viewpoint of mono-cluster flocking in terms of initial configurations and communication weights.

It is worth to mention that if $\beta \leq 1$ in (1.0.2), ψ is not integrable and called a long-range communication weight, and mono-cluster(global) flocking occurs for any initial data [26, 47, 49]. Thus, the emergence of multi-cluster(non-global) flocking is possible only for the case $\beta > 1$. When the communication weight is short-ranged, for example, it decays as the Coulomb potential, Cucker and Smale showed that global flocking (or mono-cluster flocking) is possible for some well-prepared initial configurations; furthermore, they suggested that local flocking (or multi-cluster flocking) configurations might emerge for properly chosen initial configurations. In this thesis, we focus on the sufficient conditions and properties to the local flocking phenomena. Our main interest lies on the fact that even only a part of particles can flock while others does not flock. It will be another story to show that not all the particles flock.

We also treat variant models of C-S flocking model. One is the C-S model with unit speed constraint, which has different properties on the law of conservations. We use similar approach as in the C-S model to get the flocking result. In numerical simulations, we can see local flocking phenomena occur easier than original model. However, it is hard to show that they do flock since this model is more difficult to get general flocking estimates. The other is the hydrodynamic description of C-S model for a large number of particles through the kinetic equations. Using the mean-field limit process, the equation (1.0.1) can be approximately described as partial differential equations with mass density and bulk velocity. To avoid hydrodynamic difficulties including the free boundary problem, we assumed sufficient regularity and density while using the energy method. We focused on the use of arguments from the particle-level system to show the existence of locally flocking smooth solutions.

The rest of this thesis consists of eight chapters. In Chapter 2, we briefly explain C-S model and the mathematical definition of flocking in this model. We also give some elementary estimates for the C-S model and review the global flocking result. In addition, we introduce other flocking models treated in this thesis. In Chapter 3, as a starting point, we prove the existence of bi-cluster flocking by suggesting a sufficient condition leads to the bi-cluster flocking phenomena. The approaches of Chapter 3 are adopted as main methods in other chapters with technical modifications. In Chapter 4, we analyze the multi-cluster flocking configurations in terms of the coupling strength K. We also suggest some properties of multi-cluster flocking to see the difference and difficulties compared to the global flocking results. In Chapter 5 and 6, we do the similar arguments on the Cucker-Smale model with unit speed constraint. The existence of bi-cluster flocking is proved in Chapter 5, and then we do further analysis on the coupling strength and other properties in Chapter 6. After that, in Chapter 7, we consider two ensemble system of original Cucker-Smale model with different network coupling strength. Using two different ensembles, we focused on various scenarios of local flocking phenomena. In Chapter 8, we generalize flocking results of particles to the hydrodynamic system of Cucker-Smale ensembles. Using the Lagrangian variables, we present a method to adopt the proofs of particle models in hydrodynamic models. Finally, Section 9 is devoted to a brief summary of the thesis and further open topics.

Notation: Throughout the thesis, we use superscripts to denote the components of a vector; for example $\boldsymbol{x} := (x^1, \cdots, x^d) \in \mathbb{R}^d$. Superscripts also stand for powers when it can be easily distinguished. Subscripts are used to represent the ordering of particles. In particular, for vectors $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^d$, its ℓ_2-norm and the inner product are defined as follows:

$$\|\boldsymbol{x}\| := \left(\sum_{i=1}^{d} (x^i)^2 \right)^{\frac{1}{2}}, \qquad \langle \boldsymbol{x}, \boldsymbol{v} \rangle := \sum_{i=1}^{d} x^i v^i,$$

where x^i and v^i are the i-th components of $\boldsymbol{x}$ and $\boldsymbol{v}$, respectively.

Chapter 2

Preliminaries

In this chapter, we introduce basic properties of C-S model and review previous results on the flocking phenomena. We also mention modified models treated in this thesis.

2.1 Flocking phenomena

First, we introduce definitions for the type of flockings in the C-S model which will be used throughout the thesis.

Definition 2.1.1. [26, 47] *Let $\mathcal{G} := \{\big(\boldsymbol{x}_i(t), \boldsymbol{v}_i(t)\big)\}_{i=1}^N$ be an ensemble of a C-S flocking group.*

1. *The configuration $\mathcal{G}$ tends to a mono-cluster (global) flocking configuration asymptotically if and only if the following two conditions hold:*

$$\sup_{t\geq 0} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| < \infty, \quad \lim_{t\to\infty} \|\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)\| = 0, \text{ for all } i, j.$$

2. *A sub-configuration $\mathcal{I} := \{\big(\boldsymbol{x}_i(t), \boldsymbol{v}_i(t)\big)\}_{i\in I}$ for $I \subset \{1, 2, \cdots, N\}$ tends to a flocking configuration asymptotically if and only if the following two conditions hold:*

$$\sup_{t\geq 0} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| < \infty, \quad \lim_{t\to\infty} \|\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)\| = 0, \text{ for all } i, j \in I.$$

3. *The configuration $\mathcal{G}$ tends to a multi-cluster(local) flocking configuration asymptotically if and only if there exist subclasses* $\mathcal{G}_\alpha = \{(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t))\}_{i=1}^{N_\alpha}$, $\alpha = 1, 2, \cdots, n$, *such that*

$$
\begin{aligned}
&(i)\ |\mathcal{G}_\alpha| \geq 1, \quad |\mathcal{G}_\alpha| = N_\alpha, \quad \sum_{\alpha=1}^{n} |\mathcal{G}_\alpha| = N, \quad \mathcal{G} = \bigcup \mathcal{G}_\alpha, \\
&\quad \sup_{t\geq 0} \|\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\alpha j}(t)\| < \infty, \quad \lim_{t\to\infty} \|\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\alpha j}(t)\| = 0, \\
&\quad \text{for any} \ \ \alpha \in \{1, 2, \cdots, n\}, \ \text{and} \ \ 1 \leq i \neq j \leq N_\alpha. \\
&(ii)\ \sup_{t\geq 0} \|\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\beta j}(t)\| = \infty, \ \text{for any} \ \alpha \neq \beta, \ i, j.
\end{aligned}
$$

In the Cucker-Smale equations with a positive communication weight, we usually consider the emergent behaviors as $t \to \infty$. With this concept, we can classify the final relative positions into the two cases, bounded or not. Hence the relative velocities play key roles to see the flocking phenomena. Note that if the configuration $\mathcal{G}$ does not tend to the mono-cluster flocking, then it goes to a multi-cluster flocking configuration asymptotically except for measure zero critical cases. We can also see that the boundedness of relative positions is closely related to the velocity cohesion by comparison with the heat equation as follows.

The most important feature of the C-S model is the interaction given by relative velocities. For a function u of one space variable x with time variable t, the heat equation can be described as follows:

$$\partial_t u = \alpha \partial_x^2 u.$$

Let us consider the one dimensional space discretization $x_i = ih$ for some $h > 0$. Then the equation becomes semi-discretized form,

$$\frac{du_i(t)}{dt} = \frac{\alpha}{h}\left(\frac{u_{i+1} - u_i}{h} + \frac{u_{i-1} - u_i}{h}\right), \ \text{for all } i.$$

Note that the right hand side consists of relative value of u. Hence the nonlocal version of one dimensional semi-discretized heat equation with bounded domain can be described in a similar form of velocity dynamics in the C-S model,

$$\frac{du_i(t)}{dt} = \sum_{j=1}^{N} K\psi(j,i)\left(u_j - u_i\right), \text{ for all } i = 1, \cdots, N,$$

where we used K instead of thermal diffusivity α, and the communication weight $\psi(j,i)$ is nonnegative and bounded l^1 norm. Therefore, we can conclude that velocities in the C-S model follows basic rules of heat dissipation, which tend to stabilize to the same mean value.

This dissipative velocity gives us important properties to treat regularity problems. We study the time-evolution of the first and second velocity momenta along the dynamics of (1.0.1):

$$M_1(t) := \sum_{i=1}^{N} \boldsymbol{v}_i, \quad M_2(t) := \sum_{i=1}^{N} \|\boldsymbol{v}_i\|^2. \tag{2.1.1}$$

Lemma 2.1.1. *Let* $(\boldsymbol{x}_i, \boldsymbol{v}_i)$ *be a solution to* (1.0.1)-(1.0.2)*. Then*

$$M_1(t) = M_1(0), \qquad M_2(t) \le M_2(0), \quad t \ge 0.$$

Proof. (i) The conservation of the total momentum follows from the anti-symmetry of $\psi(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|)(\boldsymbol{v}_j - \boldsymbol{v}_i)$ in the change of i and j.

(ii) Taking the inner product to the second equation in (1.0.1) with $2\boldsymbol{v}_i$ and summing the resulting relation over all i yields

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{N}\|\boldsymbol{v}_i\|^2 &= \frac{2K}{N}\sum_{i,j=1}^{N}\psi(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|)\langle \boldsymbol{v}_i, \boldsymbol{v}_j - \boldsymbol{v}_i\rangle \\ &= -\frac{K}{N}\sum_{i,j=1}^{N}\psi(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|)\|\boldsymbol{v}_j - \boldsymbol{v}_i\|^2 \le 0.\end{aligned}$$

Thus $M_2(t) \le M_2(0)$ for $t \ge 0$. □

Note that the decreasing second momentum forces to control maximal speed of particles. From now on, we will use $\sqrt{M_2(0)}$ only for the upper bound of particle speed $\|v_i\|$.

Before we start to look over global flocking, it is worth to note that the definition of bi-cluster flocking is not an equilibrium point.

Remark 2.1.1. *Note that mono-cluster flocking is a solution to the C-S model, whereas a bi-cluster flocking configuration can not be a solution to the C-S model, even if the communication weight is given by* (1.0.2)*. For example, suppose that the bi-cluster flocking configuration*

$$\boldsymbol{v}_1, \ \cdots, \ \boldsymbol{v}_k = \mathbf{u_1}, \quad \boldsymbol{v}_{k+1}, \cdots, \boldsymbol{v}_N = \mathbf{u_2}, \quad \mathbf{u_1} \neq \mathbf{u_2},$$

satisfies the C-S system with (1.0.2) *at* $t_0 \in (0, \infty)$*. Then, it follows from the C-S model that for* $i \leq k$*,*

$$\begin{aligned} 0 = \Big| \frac{d\boldsymbol{v}_i(t_0)}{dt} \Big| &= \Bigg| \frac{K}{N} \sum_{j=1}^{N} \psi(\|\boldsymbol{x}_j(t_0) - \boldsymbol{x}_i(t_0)\|)(\boldsymbol{v}_j(t_0) - \boldsymbol{v}_i(t_0)) \Bigg| \\ &= \frac{K\|\mathbf{u_2} - \mathbf{u_1}\|}{N} \sum_{j=k+1}^{N} \psi(\|\boldsymbol{x}_j(t_0) - \boldsymbol{x}_i(t_0)\|) > 0. \end{aligned}$$

Thus, we have a contradiction unless $\psi(\|\boldsymbol{x}_j(t_0) - \boldsymbol{x}_i(t_0)\|) = 0$*. However, a bi-cluster flocking configuration can emerge asymptotically if*

$$\lim_{t \to \infty} \sum_{j=k+1}^{N} \psi(\|\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)\|) = 0.$$

2.2 Two particle results

In this section, we discuss a two-particle system that has a different asymptotic flocking state depending on the initial conditions.

Consider a two-particle system on the real line $\mathbb{R}$:

$$\begin{aligned} &\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \quad t > 0, \ x_i, v_i \in \mathbb{R}, \\ &\dot{v}_1 = \frac{K}{2} \psi(|x_2 - x_1|)(v_2 - v_1), \\ &\dot{v}_2 = \frac{K}{2} \psi(|x_1 - x_2|)(v_1 - v_2), \\ &(x_i, v_i)(0) = (x_{i0}, v_{i0}). \end{aligned} \tag{2.2.2}$$

To reduce the number of equations in (2.2.2), we consider the spatial and velocity differences:

$$x := x_1 - x_2, \quad v := v_1 - v_2.$$

Without loss of generality, we assume that

$$x_0 > 0, \quad v_0 > 0. \tag{2.2.3}$$

Then the differences of x and v satisfy

$$\dot{x} = v, \quad \dot{v} = -K\psi(|x|)v,$$

or equivalently,

$$dv = -K\psi(|x|)dx.$$

Integrating the above relation yields

$$v(t) = v_0 - K\int_{x_0}^{x(t)} \psi(|y|)dy. \tag{2.2.4}$$

Proposition 2.2.1. (Nonexistence of global flocking) *Let* (x, v) *be the solution to system* (2.2.2)-(2.2.3) *with initial data satisfying*

$$v_0 \geq K\int_{x_0}^{\infty} \psi(|y|)dy. \tag{2.2.5}$$

Then there is no global flocking.

Proof. Suppose

$$v_0 \geq K\int_{x_0}^{\infty} \psi(|y|)dy.$$

Then, it follows from (2.2.4) that

$$\begin{aligned} v(t) &= v_0 - K\int_{x_0}^{x(t)} \psi(|y|)dy \\ &\geq K\int_{x_0}^{\infty} \psi(|y|)dy - K\int_{x_0}^{x(t)} \psi(|y|)dy = K\int_{x(t)}^{\infty} \psi(|y|)dy. \end{aligned}$$

Thus, we have

$$v(t) \geq K \int_{x(t)}^{\infty} \psi(|y|)dy. \tag{2.2.6}$$

If global flocking occurs, then

$$\sup_{t \geq 0} |x(t)| < \infty, \quad \lim_{t \to \infty} v(t) = 0. \tag{2.2.7}$$

Note that the above two conditions (2.2.6) and (2.2.7) on x and v are not compatible, i.e., if $|x(t)| \leq x_\infty < \infty$, then

$$v(t) \geq K \int_{x(t)}^{\infty} \psi(|y|)dy \geq K \int_{x_\infty}^{\infty} \psi(|y|)dy,$$

which contradicts the fact that $\lim_{t \to \infty} v(t) = 0$.
Therefore, there is no global flocking. □

In the following two corollaries, we further analyze the long-time dynamics of the two-particle system under condition (2.2.5).

Corollary 2.2.1. (Slow divergence in position) *Let* (x, v) *be the solution to the system* (2.2.2)-(2.2.3) *with initial data* (x_0, v_0)*. Then the following assertions hold:*

1. *If* (x_0, v_0) *satisfies*
$$v_0 = K \int_{x_0}^{\infty} \psi(|y|)dy, \tag{2.2.8}$$
then the positions of the two particles diverge with the same asymptotic velocities.

2. *If* (x_0, v_0) *satisfies*
$$v_0 > K \int_{x_0}^{\infty} \psi(|y|)dy, \tag{2.2.9}$$
then the positions of the two particles diverge with different asymptotic velocities.

Proof. (i) In this proof of corollary, we adopt an explicit communication weight (1.0.2) for simplicity's sake. If not, it can be calculated using qualitative analysis on the integration of ψ. Suppose (x_0, v_0) satisfies

$$v_0 = K \int_{x_0}^{\infty} \psi(|y|)dy.$$

Using (2.2.4) and (2.2.8), we obtain

$$\begin{aligned} v(t) &= v_0 - K \int_{x_0}^{x(t)} \psi(|y|)dy \\ &= K \int_{x(t)}^{\infty} \psi(|y|)dy \\ &= \frac{K}{\beta - 1} \frac{1}{(1 + x(t))^{\beta - 1}}. \end{aligned}$$

Thus, we obtain a first-order equation:

$$\frac{dx}{dt} = \frac{K}{\beta - 1} \frac{1}{(1 + x(t))^{\beta - 1}}. \tag{2.2.10}$$

Directly integrating (2.2.10) yields

$$x(t) = \left(\frac{\beta K t}{\beta - 1} + (1 + x_0)^{\beta} \right)^{\frac{1}{\beta}} - 1, \quad v(t) = \frac{K}{\beta - 1} \left(\frac{\beta K t}{\beta - 1} + (1 + x_0)^{\beta} \right)^{\frac{1}{\beta} - 1}.$$

The above explicit formula implies

$$\lim_{t \to \infty} x(t) = \infty, \qquad \lim_{t \to \infty} v(t) = 0.$$

Note that the velocity difference of v goes to zero at the rate of $t^{-(1 - \frac{1}{\beta})}$.

(ii) Suppose (x_0, v_0) satisfies (2.2.9). It follows from (2.2.4) that

$$\begin{aligned} v(t) &= v_0 - K \int_{x_0}^{x(t)} \psi(|y|)dy \\ &= v_0 - K \int_{x_0}^{\infty} \psi(|y|)dy + K \int_{x(t)}^{\infty} \psi(|y|)dy. \end{aligned} \tag{2.2.11}$$

Note that (2.2.11) implies

$$v(t) \geq v_0 - K \int_{x_0}^{\infty} \psi(|y|)dy > 0, \quad t \geq 0.$$

Thus, the asymptotic velocities are not equal. On the other hand, if we set

$$v_\infty := v_0 - K \int_{x_0}^{\infty} \psi(|y|)dy,$$

then (2.2.11) implies

$$\frac{dx}{dt} = v_\infty + \frac{K}{\beta - 1} \left(1 + x(t)\right)^{1-\beta}.$$

Clearly, $x(t)$ increases faster than $v_\infty t$ by the comparison theorem. □

2.3 Review on the global flocking

In this section, we briefly review the sufficient conditions for the emergence of global flocking (mono-cluster flocking) for the C-S model in (1.0.1)-(1.0.2). Global flocking formation was first studied by Cucker and Smale [26]. They provided a sufficient condition that particles tends to flock globally for an algebraically decaying communication weight (1.0.3) with $\beta \geq 0$. For the long-ranged communication weight, they showed that mono-cluster flocking always occurs. Moreover, for the short-ranged communication weight, they showed that it occurs for initial configurations close to the flocking state. Later, Cucker and Smale's results were further generalized to general non-increasing communication weights using the energy method and Lyapunov functional approach, which was based on the ℓ^2-norm [47, 49]. In this case, the aforementioned results are dependent on the number of particles; thus, they can not be used in the mean-field limit, i.e., they do not provide corresponding results for the mean-field kinetic model. Later, a slightly modified approach based on ℓ^∞ was employed for the flocking analysis of C-S communication weights [14] and general nonincreasing communication weights [1]. The following theorem is most relevant result on the mono-cluster flocking configuration.

Theorem 2.3.1. [47]. *Suppose that the communication weight ψ is nonnegative, Lipschitz continuous, and nonincreasing, i.e.,*

$$\psi(r) \geq 0, \; r \geq 0, \quad \psi(\cdot) \in Lip(\mathbb{R}_+) \quad \text{and}$$
$$(\psi(r_2) - \psi(r_1))(r_2 - r_1) \leq 0, \quad r_1, r_2 \geq 0.$$

Let $(\boldsymbol{x}, \boldsymbol{v})$ be a solution to (1.0.1)-(1.0.2) with initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ satisfying the following condition:

$$\|\boldsymbol{x}_0\| > 0, \quad \|\boldsymbol{v}_0\| < \frac{K}{2} \int_{\|\boldsymbol{x}_0\|}^{\infty} \psi(2r) dr. \tag{2.3.12}$$

Then there exists a positive number x_M such that

$$\sup_{t \geq 0} \|\boldsymbol{x}(t)\| \leq x_M, \quad \|\boldsymbol{v}(t)\| \leq \|\boldsymbol{v}_0\| e^{-\psi(2x_M)t}, \quad t \geq 0.$$

Note that Theorem 2.1 yields a sufficient condition for mono-cluster flocking. The condition is quite similar to that of Proposition 2.2.1. The natural question is what phenomenon will appear if conditions in (2.3.12) are violated. In Chapter 3, we will study the case 2 of Corollary 2.2.1 for N-body system when the sufficient condition (2.3.12) for a global flocking conditions does not hold. The non-flocking property of two particle system is also similar to that of the N-body system.

2.4 Cucker-Smale model with unit speed constraint

Until the previous section, we discussed on the C-S flocking model. In the remaining part of this Chapter, we suggest two variant models of C-S model. Here, the C-S model with unit speed constraint will be our concern. It will be analyzed in the same level as the original C-S model in Chapter 5 and 6. We can get the similar result on flocking behavior with technical detours and more restricted conditions.

The Cucker-Smale model with unit speed constraint is one of the variant model inspired by Viscek's work. The modeling of flocking phenomena was first introduced by Vicsek's group [75] in the physics community and the unit speed constraint was employed in relation with the phase models for synchronization. In [20], they discussed the mono-cluster flocking for the Cucker-Smale model with unit speed constraint,

$$\begin{aligned}
&\dot{\boldsymbol{x}}_i = \boldsymbol{v}_i, \quad t > 0, \quad i = 1, \cdots, N, \\
&\dot{\boldsymbol{v}}_i = \frac{K}{N} \sum_{j=1}^{N} \psi(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|) \left(\boldsymbol{v}_j - \frac{\langle \boldsymbol{v}_j, \boldsymbol{v}_i \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle} \boldsymbol{v}_i \right), \\
&(\boldsymbol{x}_i, \boldsymbol{v}_i)(0) = (\boldsymbol{x}_{i0}, \boldsymbol{v}_{i0}), \quad \|\boldsymbol{v}_{i0}\| = 1,
\end{aligned} \tag{2.4.13}$$

where K and ψ are same parameters in (1.0.1).

The conservation of speed also is described in [20].

Lemma 2.4.1. [20] *We set*

$$\mathcal{A}(\boldsymbol{v}) := \min_{i \neq j} \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle.$$

Let $\big(\boldsymbol{x}_i(t), \boldsymbol{v}_i(t)\big)$ *be a global solution to system* (2.4.13) *with initial data with unit speed constraint:*

$$\|\boldsymbol{v}_{i0}\| = 1, \quad 1 \leq i \leq N, \qquad \mathcal{A}(\boldsymbol{v}_0) > 0.$$

Then, we have

$$\begin{aligned}
&(i)\ \|\boldsymbol{v}_i(t)\| = 1, \quad \textit{for all} \quad t \geq 0, \quad i = 1, \cdots, N, \\
&(ii)\ \mathcal{A}(\boldsymbol{v}(t)) \geq \mathcal{A}(\boldsymbol{v}_0), \quad t \geq 0.
\end{aligned}$$

Since the agent-based model (2.4.13) can be derived from the C-S model, we begin our discussion with the C-S model. In Section 2.3, we mentioned that the emergence of mono-cluster flocking in [26] depends on the far-field behavior of the communication rate $\psi(\cdot)$ and the initial configuration. Moreover, once mono-cluster flocking is guaranteed to occur, the asymptotic velocity of the agent can be determined a priori by the conservation of momentum:

$$\sum_{i=1}^{N} \boldsymbol{v}_i(t) = \sum_{i=1}^{N} \boldsymbol{v}_{i0}, \quad t \geq 0, \qquad \lim_{t \to \infty} \boldsymbol{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} \boldsymbol{v}_{j0}.$$

Due to the tradeoff, it does not preserve the speed of particles. On the other hand, the lack of symmetry in (2.4.13) makes the conservation of speed on each particle instead of that of momentum. This causes the major difficulty on the analysis of flocking parameters. In the following two subsections, we discuss our two agent-based models with unit speed constraints.

2.4.1 A generalized J-K model

In [39], Ha, Jeong, and Kang derived a generalized J-K model from the C-S model in (1.0.1):

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= (\cos\theta_i, \sin\theta_i), \quad i = 1, 2, \cdots, N, \\ \dot{\theta}_i &= \frac{K}{N}\sum_{k=1}^{N} \psi(\|\boldsymbol{x}_k - \boldsymbol{x}_i\|)\sin(\theta_k - \theta_i). \end{aligned} \tag{2.4.14}$$

In the control theory community, the planar J-K model is often used as a flocking model for self-propelled agents moving with unit speed. Because of the unit speed ansatz and planar nature of the model, the velocity can be rewritten by using the velocity phase θ_i:

$$\boldsymbol{v}_i = (\cos\theta_i, \sin\theta_i) = e^{\sqrt{-1}\theta_i}. \tag{2.4.15}$$

Substituting (2.4.15) into the C-S model with the all-to-all communication weight $\psi = 1$, the imaginary part of the velocity equation can be utilized to formally derive the agent-based model:

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= (\cos\theta_i, \sin\theta_i), \quad i = 1, 2, \cdots, N, \\ \dot{\theta}_i &= \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_i). \end{aligned} \tag{2.4.16}$$

Note that the subsystem for velocity phase θ_i in (2.4.16) is equivalent to the Kuramoto model for identical oscillators.

Emergence of mono-cluster flocking

In [39], they also showed that mono-flocking occurs if heading-angles of agents are confined to a half circle and satisfy some additional conditions. For comparison to our main results in Chapter 5 and 6, we briefly summarize the main result of [39]. First, we introduce velocity phase and spatial position diameters. For a given velocity phase Θ and position $\boldsymbol{x}$, set

$$D(\Theta(t)) := \max_{1\le i,j\le N} |\theta_i(t) - \theta_j(t)|, \quad D(\boldsymbol{x}(t)) := \max_{1\le i,j\le N} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|, \quad t \ge 0.$$

Theorem 2.4.1. *[39] Suppose that the initial data* $(\Theta_0, \boldsymbol{x}_0)$ *satisfy*

$$0 < D(\Theta_0) < \min\Big\{\pi, \frac{C(K,\Theta_0)}{2}\int_{D(\boldsymbol{x}_0)}^{\infty} \psi(s)ds\Big\}.$$

Then there exists a positive constant $D^\infty < \infty$ *such that*

$$\sup_{t\ge 0} D(\boldsymbol{x}(t)) \le D^\infty, \quad D(\Theta(t)) \le D(\Theta_0)\exp\Big[-C(K,\Theta_0)\psi(D^\infty)t\Big], \quad t \ge 0,$$

where $C(K,\Theta_0)$ *is a positive constant defined by*

$$C(K,\Theta_0) := \frac{K\sin D(\Theta_0)}{D(\Theta_0)}.$$

Emergence of bi-cluster flocking

Consider the generalized J-K system in (2.4.14) for two agents. Let $(\boldsymbol{x}_1, \boldsymbol{v}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{v}_2)$ be the states of the two agents governed by the following system:

$$\begin{cases} \dot{\boldsymbol{x}}_1 = (\cos\theta_1, \sin\theta_1), \quad \dot{\boldsymbol{x}}_2 = (\cos\theta_2, \sin\theta_2), \\ \dot\theta_1 = \dfrac{K}{2}\psi(\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|)\sin(\theta_2 - \theta_1), \\ \dot\theta_2 = \dfrac{K}{2}\psi(\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|)\sin(\theta_1 - \theta_2), \end{cases} \tag{2.4.17}$$

with well-prepared initial data:

$$\begin{aligned} &\boldsymbol{x}_1(0) = (\cos\theta_0, \sin\theta_0), \quad \theta_1(0) = \theta_0, \\ &\boldsymbol{x}_2(0) = (\cos\theta_0, -\sin\theta_0), \quad \theta_2(0) = -\theta_0. \end{aligned} \tag{2.4.18}$$

Below, we will show that system (2.4.17) with well-prepared initial data (2.4.18) exhibits asymptotic bi-cluster flocking. Note that it follows from the velocity equations in (2.4.17) that

$$\theta_1(t) + \theta_2(t) = \theta_1(0) + \theta_2(0) = 0, \quad \text{i.e.,} \quad \theta_1(t) = -\theta_2(t), \quad t \ge 0. \tag{2.4.19}$$

Then relation (2.4.19) and the equations for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ imply

$$x_1^1(t) = x_2^1(t), \quad x_1^2(t) = -x_2^2(t), \quad t \geq 0.$$

This yields

$$|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)| = |x_1^2(t) - x_2^2(t)| = 2|x_1^2(t)|, \quad t \geq 0. \tag{2.4.20}$$

From (2.4.19) and (2.4.20), we set x and θ as follows:

$$x(t) := x_1^2(t) \quad \text{and} \quad \theta(t) := \theta_1(t), \quad t \geq 0.$$

Lemma 2.4.2. *Let (x_i, θ_i) be a solution to system* (2.4.17) *and* (2.4.18) *satisfying*

$$x(0) = x_0 > 0 \quad \textit{and} \quad \theta(0) = \theta_0 \in [0, \frac{\pi}{2}).$$

Then $x(t)$ and $\theta(t)$ satisfy

$$\ln \frac{1 + \sin \theta(t)}{1 - \sin \theta(t)} = \ln \frac{1 + \sin \theta_0}{1 - \sin \theta_0} - K \int_{2x_0}^{2x(t)} \psi(\xi) d\xi, \qquad t \geq 0.$$

Proof. Note that x and θ satisfy

$$\dot{x} = \sin \theta, \quad \dot{\theta} = -K\psi(2x) \sin \theta \cos \theta, \quad t > 0. \tag{2.4.21}$$

This yields

$$\frac{d\theta}{\cos \theta} = -K\psi(2x) \sin \theta dt = -K\psi(2x) dx. \tag{2.4.22}$$

Integrating (2.4.22) yields the desired estimate. □

Next, we show that bi-cluster flocking can occur for some class of initial configurations.

Proposition 2.4.1. (Nonexistence of global flocking) *Let (x, θ) be the solution to system* (2.4.21) *with initial data satisfying*

$$x_0 > 0, \quad \theta_0 \in [0, \frac{\pi}{2}) \quad \textit{and} \quad \ln \frac{1 + \sin \theta_0}{1 - \sin \theta_0} \geq K \int_{2x_0}^{\infty} \psi(\xi) d\xi.$$

Then there is no global flocking. In particular,

$$\limsup_{t \to \infty} |x(t)| = \infty.$$

Proof. Suppose that initial data satisfy

$$\ln \frac{1+\sin\theta_0}{1-\sin\theta_0} \geq K \int_{2x_0}^{\infty} \psi(\xi) d\xi.$$

Then it follows from Lemma 2.4.2 that

$$\ln \frac{1+\sin\theta(t)}{1-\sin\theta(t)} = \ln \frac{1+\sin\theta_0}{1-\sin\theta_0} - K\Big(\int_{2x_0}^{\infty} \psi(\xi) d\xi - \int_{2x(t)}^{\infty} \psi(\xi) d\xi\Big).$$

This yields

$$\ln \frac{1+\sin\theta(t)}{1-\sin\theta(t)} \geq K \int_{2x(t)}^{\infty} \psi(y) dy.$$

Now, suppose global flocking occurs, i.e.,

$$\sup_{t>0} |x(t)| < \infty \quad \text{and} \quad \lim_{t\to\infty} \theta(t) = 0. \tag{2.4.23}$$

Then it follows from the first condition that there exists x_∞ such that

$$|x(t)| \leq x_\infty < \infty \quad \text{ for all } t > 0.$$

Thus, we have

$$\ln \frac{1+\sin\theta(t)}{1-\sin\theta(t)} \geq K \int_{2x_\infty}^{\infty} \psi(\xi) d\xi > 0, \quad t \geq 0,$$

which contradicts the second relation $\lim_{t\to\infty} \theta(t) = 0$ in (2.4.23). □

Remark 2.4.1. *Note that for the multi-dimensional C-S model, if the initial data satisfies a symmetry condition, it can also be reduced to the above situation. For example,*

$$v_1 = -v_2,\ x_2 = -x_1,\ x_1 = kv_1, \quad \textit{for some } k > 0.$$

2.4.2 A multi-dimensional C-S type model

In this subsection, we briefly discuss our multi-dimensional C-S model with unit speed. In [24], Choi and Ha studied how to couple C-S flocking dynamics and the unit speed constraint assumption using the idea of quantum synchronization introduced by Lohe [60].

Consider the one-dimensional line ℓ spanned by the unit vector $\frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|}$. Then the orthogonal projection of $\boldsymbol{v}_k$ onto the line ℓ is given by

$$\mathrm{Proj}_\ell \boldsymbol{v}_k = \left\langle \boldsymbol{v}_k, \frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|} \right\rangle \frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|} = \boldsymbol{v}_k^{\|}.$$

Thus, $\boldsymbol{v}_k$ can be decomposed as the sum of the tangential and orthogonal components with respect to ℓ as follows:

$$\boldsymbol{v}_k = \boldsymbol{v}_k^{\|} + \boldsymbol{v}_k^{\perp}, \qquad \boldsymbol{v}_k^{\|} := \left\langle \boldsymbol{v}_k, \frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|} \right\rangle \frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|}, \qquad \boldsymbol{v}_k^{\perp} := \boldsymbol{v}_k - \frac{\langle \boldsymbol{v}_k, \boldsymbol{v}_i \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle} \boldsymbol{v}_i.$$

The force F_{ik} of the i-agent acts on the k-th agent as follows:

$$\mathbf{F}_{ik} = \psi(\|\boldsymbol{x}_k - \boldsymbol{x}_i\|)\Big(\boldsymbol{v}_k - \frac{\langle \boldsymbol{v}_k, \boldsymbol{v}_i \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle} \boldsymbol{v}_i\Big). \tag{2.4.24}$$

Note that (2.4.24) yields

$$\langle \boldsymbol{v}_i, \mathbf{F}_{ik} \rangle = 0.$$

Finally, we combine (1.0.1) and (2.4.24) to obtain the desired C-S model in (2.4.13) with the unit speed constraint:

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= \boldsymbol{v}_i, \ i = 1, 2, \cdots, N, \\ \dot{\boldsymbol{v}}_i &= \frac{K}{N} \sum_{k=1}^{N} \psi(\|\boldsymbol{x}_k - \boldsymbol{x}_i\|) \left(\boldsymbol{v}_k - \frac{\langle \boldsymbol{v}_i, \boldsymbol{v}_k \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle} \boldsymbol{v}_i \right). \end{aligned} \tag{2.4.25}$$

It is easy to verify that total momentum is not conserved in system (2.4.13), but the speed of each particle is conserved, i.e.,

$$\sum_{i=1}^{N} \boldsymbol{v}_i(t) \neq \sum_{i=1}^{N} \boldsymbol{v}_{i0}, \qquad \frac{d\|\boldsymbol{v}_i\|}{dt} = 0, \quad 1 \le i \le N, \ t \ge 0.$$

Lemma 2.4.3. *Let* $(\boldsymbol{x}_i, \boldsymbol{v}_i)$ *be a solution to system* (2.4.25)*. Then the speed of the agents is constant along the flow of* (2.4.25)*, i.e.,*

$$\|\boldsymbol{v}_i(t)\| = \|\boldsymbol{v}_{i0}\|, \quad t \ge 0.$$

Proof. It follows from the equation of (2.4.25) that

$$\frac{1}{2}\frac{d\|\boldsymbol{v}_i\|^2}{dt} = \frac{K}{N}\sum_{k=1}^{N}\psi(\|\boldsymbol{x}_k-\boldsymbol{x}_i\|)\left(\langle\boldsymbol{v}_k,\boldsymbol{v}_i\rangle - \frac{\langle\boldsymbol{v}_k,\boldsymbol{v}_i\rangle}{\langle\boldsymbol{v}_i,\boldsymbol{v}_i\rangle}\langle\boldsymbol{v}_i,\boldsymbol{v}_i\rangle\right) = 0.$$

This implies

$$\|\boldsymbol{v}_i(t)\| = \|\boldsymbol{v}_i(0)\|, \quad t \geq 0.$$

□

We fix several functionals $D(\boldsymbol{x}), D(\boldsymbol{v})$ for a configuration $(\boldsymbol{x},\boldsymbol{v}) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$D(\boldsymbol{x}) := \max_{1\leq i,j\leq N}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|, \quad D(\boldsymbol{v}) := \max_{1\leq i,j\leq N}\|\boldsymbol{v}_i-\boldsymbol{v}_j\|.$$

Now, we briefly recall a result for the formation of mono-cluster flocking in [24]. Here $\mathcal{A}(\boldsymbol{v})$ is as in Lemma 2.4.1.

Theorem 2.4.2. [24] *Suppose that the communication weight ψ and initial configuration $(\boldsymbol{x}_0,\boldsymbol{v}_0)$ satisfy the following conditions:*

$$\begin{aligned}&(i)\ \|\boldsymbol{v}_{i0}\| = 1, \quad 1\leq i\leq N, \qquad \mathcal{A}(\boldsymbol{v}_0) > 0,\\ &(ii)\ 0 < D(\boldsymbol{v}_0) < KC_0\min\Big\{\int_{D(\boldsymbol{x}_0)}^{\infty}\psi(s)ds,\ \int_0^{D(\boldsymbol{x}_0)}\psi(s)ds\Big\}.\end{aligned}$$

Then there exists a unique solution $(x(t),v(t))$ to system (2.4.13) *satisfying asymptotic mono-cluster flocking:*

$$\sup_{t\geq 0}D(\boldsymbol{x}(t)) < D^\infty, \quad D(\boldsymbol{v}(t)) \leq D(\boldsymbol{v}_0)\exp\Big(-KC_0\psi(D^\infty)t\Big), \quad t\geq 0,$$

where D^∞ is a positive constant implicitly defined by the following relation:

$$D(\boldsymbol{v}_0) := KC_0\int_{D(\boldsymbol{x}_0)}^{D^\infty}\psi(s)ds.$$

2.5 Hydrodynamic descriptions of Cucker-Smale model

When the number of particles is sufficiently large and the whole ensemble is close to a flocking state, the dynamics of the ensemble of C-S flocking particles can be approximated by a hydrodynamic model for mass density and bulk velocity. More precisely, let $\rho = \rho(x,t)$ and $u = u(x,t)$ be the mass density and bulk velocity of the C-S ensemble at position $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}_+$. Here ρ and u play as the mono-kinetic ansatz. In this situation, the temporal-spatial evolution of (ρ, u) is governed by the following hydrodynamic C-S model:

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d,\ t > 0,\\
&\rho \partial_t u + \rho u \cdot \nabla u = -\kappa \rho \int_{\mathbb{R}^d} \psi(|y-x|)(u(x) - u(y))\rho(y)dy, \qquad (2.5.26)\\
&(\rho, u)(x, 0) = (\rho_0, u_0),
\end{aligned}$$

where κ is a nonnegative coupling strength and $\psi = \psi(|x-y|)$ is a bounded Lipschitz continuous communication weight satisfying the conditions

$$\psi \geq 0, \quad \|\psi\|_{L^\infty} + \|\psi\|_{\mathrm{Lip}} < \infty, \quad (\psi(r_1) - \psi(r_2))(r_1 - r_2) \leq 0, \quad r_1, r_2 > 0, \qquad (2.5.27)$$

which is already mentioned in Theorem 2.3.1.

Throughout the hydrodynamic part, we denote a spatial element x instead of $\boldsymbol{x}$ and the communication weight κ instead of K for a simple notation. Also, we suppress the t dependence in ρ and u, as long as there is no confusion;

$$\rho(x) := \rho(x,t), \qquad u(x) := u(x,t).$$

The system comprising (2.5.26) and (2.5.27) arises as a hydrodynamic model for the macroscopic description [38, 49] of the ensemble of C-S particles, when the continuum flocking group is close to a flocking configuration (see Section 2.1). Note that, in the formal zero coupling limit $\kappa \to 0$, this system reduces to pressureless gas dynamics, which has been extensively studied in the hyperbolic conservation law community [6, 7, 8, 9, 10, 16, 32, 64, 70, 76, 77, 78, 80]. For a mesoscopic description, we refer to the Vlasov-McKean type equation [3, 4, 33, 35, 49, 47, 52, 53].

In order to compare with the particle model, It is worth to see the limit process. From the particle to the ensemble, The kinetic description of Cucker-Smale model serves as an intermediate equation. The kinetic equation can be derived through mean-field limit from the particle model [48]. For a space-velocity density function $f(x,v,t)$,

$$\begin{aligned} &\partial_t f + v\cdot\nabla_x f + \nabla_v(fL[f]) = 0, \quad x,v\in\mathbb{R}^d,\ t>0,\\ &L[f] = \kappa\int_{\mathbb{R}^{2d}}\psi(|y-x|)(w-v)f(y,w,t)dydw,\\ &f(x,v,0) = f_0(x,v). \end{aligned} \tag{2.5.28}$$

We can get the hydrodynamic description (2.5.26) if the system is already locally flocked, in other words, the density function follows the mono-kinetic ansatz $f(x,v,t)=\rho(x,t)\delta_{u(x,t)}(x,t)$. In [48], they rigorously proved the global limit process from the particle model to the kinetic equation.

In Chapter 8, we consider the situation in which two homogeneous ensembles of C-S particles are interacting and then look for sufficient conditions leading to two scenarios: "*mono-cluster flocking*" and "*bi-cluster flocking*" asymptotically (see Definition 8.2.1 for their formal definitions). To describe such a situation, we adopt a coupled system of hydrodynamic models (2.5.26) for $(\rho_i,u_i),\ i=1,2$:

$$\begin{aligned} &\partial_t\rho_1 + \nabla\cdot(\rho_1u_1) = 0, \quad \partial_t\rho_2 + \nabla\cdot(\rho_2u_2) = 0, \quad (x,t)\in\mathbb{R}^d\times\mathbb{R}_+,\\ &\rho_1\partial_t u_1 + \rho_1u_1\cdot\nabla u_1\\ &\qquad = \kappa_{11}\int_{\Omega_1(t)}\rho_1(x)\rho_1(y)\psi(|y-x|)(u_1(y)-u_1(x))dy\\ &\qquad + \kappa_{12}\int_{\Omega_2(t)}\rho_1(x)\rho_2(y)\psi(|y-x|)(u_2(y)-u_1(x))dy,\\ &\rho_2\partial_t u_2 + \rho_2u_2\cdot\nabla u_2\\ &\qquad = \kappa_{22}\int_{\Omega_2(t)}\rho_2(x)\rho_2(y)\psi(|y-x|)(u_2(y)-u_2(x))dy\\ &\qquad + \kappa_{21}\int_{\Omega_1(t)}\rho_2(x)\rho_1(y)\psi(|y-x|)(u_1(y)-u_2(x))dy, \end{aligned} \tag{2.5.29}$$

subject to initial data

$$(\rho_i,u_i)(x,0) = (\rho_{i0},u_{i0}), \quad x\in\mathbb{R}^d,\ i=1,2. \tag{2.5.30}$$

Here the fluid regions $\Omega_1(t)$ and $\Omega_2(t)$ are the connected compact supports of the densities ρ_1 and ρ_2 at time t, respectively, and κ_{ii} and κ_{ij} are intra- and intercoupling strengths, which are assumed to be nonnegative.

Note that, when the intercoupling strengths κ_{ij} are turned off, system (2.5.29) becomes a juxtaposition of the identical C-S system (2.5.26). In our setting, one of the natural questions in relation to the asymptotic dynamics of the coupled system (2.5.29) is to identify the possible dynamic features depending on the relative strengths of the inter- and intracoupling strengths κ_{ij}, communication weight ψ, and initial configuration. There might be several plausible asymptotic scenarios, but, among others, we are mainly interested in two possible asymptotic pictures: merging of two local clusters or splitting of one cluster into several clusters. In Chapter 8, we will use the Lagrangian formulation in order to track the individual behaviors among the bulk density.

Chapter 3

Existence of bi-cluster flocking

In this chapter, we present a possible scenario on bi-cluster flocking by suggesting a set of initial data and coupling parameters, which tends to bi-cluster flocking configurations asymptotically. The possibility of multi-cluster flocking was already mentioned in [26], the first paper of Cucker and Smale on the flocking model. It is a natural consequence that the multi-cluster flocking configurations can emerge from non-flocking conditions. However, the existence and stability were not proved analytically. One of major difficulty is that the flocking configurations are naturally a property at infinite time, therefore, it seems too hard to distinguish local flocking groups from initial data. First, we suggest well-prepared initial configurations, and then we will prove that it tends to bi-cluster flocking configurations analytically and also see it numerically. Moreover, we prove that the convergence rate of speed is determined by the decay rate of communication weight, where we assumed algebraic decreasing. This chapter is based on the joint work in [18].

3.1 A framework for bi-cluster flocking

In this section, we present a formulation for the asymptotic formation of bi-cluster flocking configurations for a many-body system with N number of particles. At the end of the section, we present a non-trivial initial configurations of which solutions satisfy the desired bi-cluster flocking estimates.
Suppose that the initial configuration is close to the linear combination of

two diverging Dirac measures $\delta_{(\boldsymbol{x}_1,\boldsymbol{v}_1)} + \delta_{(\boldsymbol{x}_2,\boldsymbol{v}_2)}$:

$$\langle \boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{v}_2 - \boldsymbol{v}_1 \rangle < 0.$$

Furthermore, suppose that the coupling strength between two flocking groups is sufficiently weak. In this situation, it is reasonable to expect the emergence of bi-cluster flocking configurations. Under this scenario, we will prove that each agent's velocity converges to a constant velocity, and we analyze it properties such as the rate of the convergence.

3.1.1 Reformulation of the C-S model

In order to get simple notations, we reformulate the C-S model in (1.0.1) for the study of bi-flocking configurations. We need to derive a coupled system of ordinary differential equations in terms of micro fluctuations and macro local averages in position and velocity.

Consider a C-S flocking system composed of two local flocking groups $\mathcal{G}_1$ and $\mathcal{G}_2$ with $|\mathcal{G}_1| = N_1$ and $|\mathcal{G}_2| = N_2$. To register a membership and order in each group, we use a double subscript, $(x_{\alpha i}, v_{\alpha i})$, to denote the spatial-velocity position of the i-th member in the $\mathcal{G}_\alpha$-group ($\alpha = 1, 2$). In this setting, the interaction terms can be split into intra and inter interactions, and the original C-S model (1.0.1) can be rewritten as

$$\begin{aligned}
&\dot{\boldsymbol{x}}_{1i} = \boldsymbol{v}_{1i}, \quad \dot{\boldsymbol{x}}_{2j} = \boldsymbol{v}_{2j}, \quad i = 1, \cdots, N_1,\ j = 1, \cdots, N_2, \\
&\dot{\boldsymbol{v}}_{1i} = \frac{K}{N}\sum_{k=1}^{N_1} \psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{1k} - \boldsymbol{v}_{1i}) + \frac{K}{N}\sum_{k=1}^{N_2} \psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2k} - \boldsymbol{v}_{1i}), \\
&\dot{\boldsymbol{v}}_{2j} = \frac{K}{N}\sum_{k=1}^{N_2} \psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{2j}\|)(\boldsymbol{v}_{2k} - \boldsymbol{v}_{2j}) + \frac{K}{N}\sum_{k=1}^{N_1} \psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}\|)(\boldsymbol{v}_{1k} - \boldsymbol{v}_{2j}).
\end{aligned} \tag{3.1.1}$$

Next, we introduce local velocity averages and local fluctuations to decompose (3.1.1) into a coupled macro-micro system:

$$\boldsymbol{x}_{1c} := \frac{1}{N_1}\sum_{i=1}^{N_1} \boldsymbol{x}_{1i}, \quad \boldsymbol{x}_{2c} := \frac{1}{N_2}\sum_{j=1}^{N_2} \boldsymbol{x}_{2j},$$

$$
\begin{aligned}
\boldsymbol{v}_{1c} &:= \frac{1}{N_1}\sum_{i=1}^{N_1}\boldsymbol{v}_{1i}, \quad \boldsymbol{v}_{2c} := \frac{1}{N_2}\sum_{j=1}^{N_2}\boldsymbol{v}_{2j},\\
\hat{\boldsymbol{x}}_{\alpha i} &= \boldsymbol{x}_{\alpha i} - \boldsymbol{x}_{\alpha c}, \qquad \hat{\boldsymbol{v}}_{\alpha i} = \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha c}, \quad \alpha = 1, 2.
\end{aligned}
$$

We also study the relations between local fluctuations.

Lemma 3.1.1. *Let* $(\boldsymbol{x}, \boldsymbol{v})$ *be the solution to the system* (3.1.1) *and* (1.0.2). *Then, we have*

$$
\begin{aligned}
&(i)\ N_1\boldsymbol{v}_{1c}(t) + N_2\boldsymbol{v}_{2c}(t) = M_1(0), \quad t \ge 0.\\
&(ii)\ \|\hat{\boldsymbol{v}}_\alpha\|^2 + N_\alpha\|\boldsymbol{v}_{\alpha c}(t)\|^2 = \|\boldsymbol{v}_\alpha\|^2, \quad \alpha = 1, 2,
\end{aligned}
$$

where the following simplified notation is used:

$$
\|\hat{\boldsymbol{v}}_\alpha\|^2 := \sum_{i=1}^{N_\alpha}\|\hat{\boldsymbol{v}}_{\alpha i}(t)\|^2, \quad \|\boldsymbol{v}_\alpha\|^2 := \sum_{i=1}^{N_\alpha}\|\boldsymbol{v}_{\alpha i}(t)\|^2.
$$

Proof. (i) Note that

$$
N_1\boldsymbol{v}_{1c} + N_2\boldsymbol{v}_{2c} = \sum_{i=1}^{N_1}\boldsymbol{v}_{1i} + \sum_{i=1}^{N_2}\boldsymbol{v}_{2i} = M_1(t) = M_1(0).
$$

The first momentum $M_1(t)$ was defined in (2.1.1).
(ii) By definition of local fluctuations, it follows that for $\alpha = 1, 2$,

$$
\begin{aligned}
\sum_{i=1}^{N_\alpha}\|\hat{\boldsymbol{v}}_{\alpha i}\|^2 &= \sum_{i=1}^{N_\alpha}\|\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha c}\|^2\\
&= \sum_{i=1}^{N_\alpha}\Big(\|\boldsymbol{v}_{\alpha i}\|^2 + \|\boldsymbol{v}_{\alpha c}\|^2 - 2\langle\boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha c}\rangle\Big)\\
&= \sum_{i=1}^{N_\alpha}\|\boldsymbol{v}_{\alpha i}\|^2 + N_\alpha\|\boldsymbol{v}_{\alpha c}\|^2 - 2\Big\langle\sum_{i=1}^{N_\alpha}\boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha c}\Big\rangle\\
&= \sum_{i=1}^{N_\alpha}\|\boldsymbol{v}_{\alpha i}\|^2 - N_\alpha\|\boldsymbol{v}_{\alpha c}\|^2,
\end{aligned}
$$

which follows from the fact that $\sum_{i=1}^{N_\alpha}\boldsymbol{v}_{\alpha i} = N_\alpha\boldsymbol{v}_{\alpha c}$. □

Remark 3.1.1. *It follows from Lemma 4.1 (ii) that*

$$\|\boldsymbol{v}_{1c}\|^2 + \|\boldsymbol{v}_{2c}\|^2 \leq \frac{1}{\min\{N_1, N_2\}}\Big(\sum_{i=1}^{N_1}\|\boldsymbol{v}_{1i}\|^2 + \sum_{i=1}^{N_2}\|\boldsymbol{v}_{2i}\|^2\Big).$$

Lemma 3.1.2. *Let* $(\boldsymbol{x}, \boldsymbol{v})$ *be the solution to the system* (3.1.1) *and* (1.0.2)*. Then, the local averages and fluctuations satisfy*

$$\begin{aligned}
&\dot{\boldsymbol{x}}_{1c} = \boldsymbol{v}_{1c}, \quad \dot{\boldsymbol{x}}_{2c} = \boldsymbol{v}_{2c}, \quad t > 0,\\
&N_1\dot{\boldsymbol{v}}_{1c} = -\frac{K}{N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\psi(\|\boldsymbol{x}_{2j} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{1i} - \boldsymbol{v}_{2j}),\\
&N_2\dot{\boldsymbol{v}}_{2c} = -\frac{K}{N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\psi(\|\boldsymbol{x}_{2j} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2j} - \boldsymbol{v}_{1i}),
\end{aligned} \tag{3.1.2}$$

and

$$\begin{cases}
\dot{\hat{\boldsymbol{x}}}_{1i} = \hat{\boldsymbol{v}}_{1i}, \qquad \dot{\hat{\boldsymbol{x}}}_{2j} = \hat{\boldsymbol{v}}_{2j},\\
\dot{\hat{\boldsymbol{v}}}_{1i} = -\dot{\boldsymbol{v}}_{1c} + \dfrac{K}{N}\displaystyle\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}\|)(\hat{\boldsymbol{v}}_{1k} - \hat{\boldsymbol{v}}_{1i})\\
\qquad + \dfrac{K}{N}\displaystyle\sum_{k=1}^{N_2}\psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)(\hat{\boldsymbol{v}}_{2k} - \hat{\boldsymbol{v}}_{1i}) + \dfrac{K}{N}\displaystyle\sum_{k=1}^{N_2}\psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2c} - \boldsymbol{v}_{1c}),\\
\dot{\hat{\boldsymbol{v}}}_{2j} = -\dot{\boldsymbol{v}}_{2c} + \dfrac{K}{N}\displaystyle\sum_{k=1}^{N_2}\psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{2j}\|)(\hat{\boldsymbol{v}}_{2k} - \hat{\boldsymbol{v}}_{2j})\\
\qquad + \dfrac{K}{N}\displaystyle\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}\|)(\hat{\boldsymbol{v}}_{1k} - \hat{\boldsymbol{v}}_{2j}) + \dfrac{K}{N}\displaystyle\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}\|)(\boldsymbol{v}_{1c} - \boldsymbol{v}_{2c}).
\end{cases} \tag{3.1.3}$$

Proof. (i) (Derivation of (3.1.2)): Summing (3.1.1) for $i = 1, \cdots, N_1$ yields

$$\begin{aligned}
\sum_{i=1}^{N_1}\dot{\boldsymbol{v}}_{1i} =& \frac{K}{N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}\psi(\|\boldsymbol{x}_{1j} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{1j} - \boldsymbol{v}_{1i})\\
&+ \frac{K}{N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\psi(\|\boldsymbol{x}_{2j} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2j} - \boldsymbol{v}_{1i}).
\end{aligned}$$

The first term on the right-hand side becomes zero by the skew-symmetry property. Interchanging i and j and using the fact that

$$\sum_{i=1}^{N_1} \boldsymbol{v}_{1i} = N_1 \boldsymbol{v}_{1c},$$

we conclude

$$N_1 \dot{\boldsymbol{v}}_{1c} = -\frac{K}{N} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \psi(\|\boldsymbol{x}_{2j} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{1i} - \boldsymbol{v}_{2j}).$$

Similarly, we have its counterpart of $\dot{\boldsymbol{v}}_{2c}$.
(ii) (Derivation of (3.1.3)): It follows from the equations (3.1.1).

□

3.1.2 Derivation of a dissipative differential inequalities

In this subsection, we introduce two ℓ_2-type functionals to describe the local flocking process:

$$\mathcal{X} := \|\hat{\boldsymbol{x}}_1\| + \|\hat{\boldsymbol{x}}_2\|, \quad \mathcal{V} := \|\hat{\boldsymbol{v}}_1\| + \|\hat{\boldsymbol{v}}_2\|.$$

The functionals $\mathcal{X}$ and $\mathcal{V}$ measure the spatial and velocity fluctuations around the local averages, respectively. We denote Δ_v as the average velocity difference between the two local groups,

$$\Delta_v := \boldsymbol{v}_{1c} - \boldsymbol{v}_{2c}.$$

Since this is not a scalar, we usually use its inner product with some fixed unit vector $\boldsymbol{e}$,

$$\Delta_v \cdot \boldsymbol{e} := (\boldsymbol{v}_{1c} - \boldsymbol{v}_{2c}) \cdot \boldsymbol{e}.$$

Next, we derive differential inequalities for the above quantities.

Lemma 3.1.3. *Let* $(\boldsymbol{x}_{\alpha c}, \boldsymbol{v}_{\alpha c})$, $(\hat{\boldsymbol{x}}_{\alpha i}, \hat{\boldsymbol{v}}_{\alpha i})$ $\alpha = 1, 2$ *be the solution to system* (3.1.2)-(3.1.3) *with* $\mathcal{V}(0) > 0$. *Then the functionals* $(\mathcal{X}, \mathcal{V}, \Delta_v)$ *satisfy the following coupled system of dissipative differential inequalities (SDDI):*

$$(i)\ |\dot{\mathcal{X}}| \le \mathcal{V},$$

$$(ii)\ \dot{\mathcal{V}} \le -\frac{K\min\{N_1,N_2\}}{N}\psi(\sqrt{2}\mathcal{X})\mathcal{V} + 2\sqrt{2}K\sqrt{M_2(0)}\psi_M,$$
$$(iii)\dot{\Delta}_v \cdot \boldsymbol{e} \ge -K\sqrt{\frac{2M_2(0)}{\min\{N_1,N_2\}}}\psi_M,$$

where $\boldsymbol{e}$ *is a fixed unit vector and the maximal communication weights* ψ_M *represent the maximal value of communication between different groups, which is defined as follows:*

$$\psi_M(t) := \max_{i,j}\psi(\|\boldsymbol{x}_{2j} - \boldsymbol{x}_{1i}\|).$$

Proof. (i) The estimates for $\dot{\mathcal{X}}$ follows directly from Cauchy-Schwarz's inequality.

(ii) For the time-evolution of $\|\hat{\boldsymbol{v}}_1\|$, we multiply (3.1.3) by $2\hat{\boldsymbol{v}}_{1i}$ and sum over $i = 1, \cdots, N_1$ to obtain

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{N_1}\|\hat{\boldsymbol{v}}_{1i}\|^2 =& \frac{2K}{N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}\psi(\|\boldsymbol{x}_{1j}-\boldsymbol{x}_{1i}\|)\langle\hat{\boldsymbol{v}}_{1i}, \hat{\boldsymbol{v}}_{1j} - \hat{\boldsymbol{v}}_{1i}\rangle \\ &+ \frac{2K}{N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\psi(\|\boldsymbol{x}_{2j}-\boldsymbol{x}_{1i}\|)\langle\hat{\boldsymbol{v}}_{1i}, \hat{\boldsymbol{v}}_{2j} - \hat{\boldsymbol{v}}_{1i}\rangle \\ &+ \frac{2K}{N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\psi(\|\boldsymbol{x}_{2j}-\boldsymbol{x}_{1i}\|)\langle\hat{\boldsymbol{v}}_{1i}, \boldsymbol{v}_{2c} - \boldsymbol{v}_{1c}\rangle \\ =&: \mathcal{I}_{01} + \mathcal{I}_{02} + \mathcal{I}_{03}.\end{aligned} \tag{3.1.4}$$

Next, we estimate the terms $\mathcal{I}_{01}$, $\mathcal{I}_{02}$, and $\mathcal{I}_{03}$, separately.

• (Estimate of $\mathcal{I}_{01}$): We use the standard interchanging trick ($i \leftrightarrow j$) and $\|\boldsymbol{x}_{1j} - \boldsymbol{x}_{1i}\| \le \sqrt{2}\mathcal{X}$ to obtain

$$\mathcal{I}_{01} = -\frac{K}{N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}\psi(\|\boldsymbol{x}_{1j}-\boldsymbol{x}_{1i}\|)\|\hat{\boldsymbol{v}}_{1j} - \hat{\boldsymbol{v}}_{1i}\|^2 \le -\frac{2KN_1\psi(\sqrt{2}\mathcal{X})}{N}\|\hat{\boldsymbol{v}}_1\|^2. \tag{3.1.5}$$

• (Estimate of $\mathcal{I}_{02}$): We use

$$\psi(\|\boldsymbol{x}_{2j}-\boldsymbol{x}_{1i}\|)\le\psi_M,$$
$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\|\hat{\boldsymbol{v}}_{1i}\|\|\hat{\boldsymbol{v}}_{2j}\|=\left(\sum_{i=1}^{N_1}\|\hat{\boldsymbol{v}}_{1i}\|\right)\left(\sum_{j=1}^{N_2}\|\hat{\boldsymbol{v}}_{2j}\|\right)\le\sqrt{N_1N_2}\|\hat{\boldsymbol{v}}_1\|\|\hat{\boldsymbol{v}}_2\|,$$

to obtain

$$\mathcal{I}_{02}\le\frac{2K\sqrt{N_1N_2}}{N}\psi_M\|\hat{\boldsymbol{v}}_1\|\|\hat{\boldsymbol{v}}_2\|. \tag{3.1.6}$$

• (Estimate of $\mathcal{I}_{03}$): We use $\displaystyle\sum_{i=1}^{N_1}\|\hat{\boldsymbol{v}}_{1i}\|\le\sqrt{N_1}\|\hat{\boldsymbol{v}}_1\|$ to estimate

$$\mathcal{I}_{03}\le\frac{2K}{N}\psi_M N_2\sqrt{N_1}\|\hat{\boldsymbol{v}}_1\|\|\boldsymbol{v}_{1c}-\boldsymbol{v}_{2c}\|. \tag{3.1.7}$$

In (3.1.4), estimates (3.1.5), (3.1.6), and (3.1.7) are combined to obtain

$$\frac{d\|\hat{\boldsymbol{v}}_1\|}{dt}\le-\frac{KN_1\psi(\sqrt{2}\mathcal{X})}{N}\|\hat{\boldsymbol{v}}_1\|+\frac{K\sqrt{N_1N_2}}{N}\psi_M\left(\|\hat{\boldsymbol{v}}_2\|+\sqrt{N_2}\|\boldsymbol{v}_{1c}-\boldsymbol{v}_{2c}\|\right). \tag{3.1.8}$$

Similarly,

$$\frac{d\|\hat{\boldsymbol{v}}_2\|}{dt}\le-\frac{KN_2\psi(\sqrt{2}\mathcal{X})}{N}\|\hat{\boldsymbol{v}}_2\|+\frac{K\sqrt{N_1N_2}}{N}\psi_M\left(\|\hat{\boldsymbol{v}}_1\|+\sqrt{N_1}\|\boldsymbol{v}_{1c}-\boldsymbol{v}_{2c}\|\right). \tag{3.1.9}$$

Combining (3.1.8) and (3.1.9) yields

$$\dot{\mathcal{V}}\le-\frac{K\min\{N_1,N_2\}}{N}\psi(\sqrt{2}\mathcal{X})\mathcal{V}+\frac{K}{N}\sqrt{N_1N_2}\psi_M\mathcal{P}_1, \tag{3.1.10}$$

where $\mathcal{P}_1$ indicates

$$\mathcal{P}_1=\left(\|\hat{\boldsymbol{v}}_1\|+\|\hat{\boldsymbol{v}}_2\|+(\sqrt{N_1}+\sqrt{N_2})\|\boldsymbol{v}_{1c}-\boldsymbol{v}_{2c}\|\right).$$

Furthermore, this term $\mathcal{P}_1$ can be treated as follows:

$$\begin{aligned}
&(\|\hat{\boldsymbol{v}}_1\| + \|\hat{\boldsymbol{v}}_2\| + (\sqrt{N_1} + \sqrt{N_2})\|\boldsymbol{v}_{1c} - \boldsymbol{v}_{2c}\|)^2 \\
&\qquad \le 4(\|\hat{\boldsymbol{v}}_1\|^2 + \|\hat{\boldsymbol{v}}_2\|^2 + (N_1 + N_2)\|\boldsymbol{v}_{1c} - \boldsymbol{v}_{2c}\|^2) \\
&\qquad \le 4(\|\hat{\boldsymbol{v}}_1\|^2 + \|\hat{\boldsymbol{v}}_2\|^2 + 2N\|v_{1c}\|^2 + 2N\|v_{2c}\|^2) \\
&\qquad \le 4(\frac{2N}{N_1}\|\boldsymbol{v}_1\|^2 + \frac{2N}{N_2}\|\boldsymbol{v}_2\|^2) \le \frac{8N}{\min\{N_1, N_2\}} M_2(t) \\
&\qquad \le \frac{8N^2}{\min\{N_1, N_2\}\max\{N_1, N_2\}} M_2(0).
\end{aligned} \tag{3.1.11}$$

Finally, (3.1.10) and (3.1.11) are combined to derive

$$\dot{\mathcal{V}} \le -\frac{K\min\{N_1, N_2\}}{N}\psi(\sqrt{2}\mathcal{X})\mathcal{V} + 2\sqrt{2}K\sqrt{M_2(0)}\psi_M.$$

Here we used the elementary identity:

$$\min\{N_1, N_2\}\max\{N_1, N_2\} = N_1 N_2. \tag{3.1.12}$$

(iii) We use the formulas for $N_1\dot{\boldsymbol{v}}_{1c}$ and $N_2\dot{\boldsymbol{v}}_{2c}$ in Lemma 3.2 to see

$$\frac{d}{dt}(\boldsymbol{v}_{1c} - \boldsymbol{v}_{2c}) \cdot \boldsymbol{e} = -\frac{K}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\psi(\|\boldsymbol{x}_{2j} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{1i} - \boldsymbol{v}_{2j}) \cdot \boldsymbol{e}. \tag{3.1.13}$$

In (3.1.13), we use the fact that $\psi(\|\boldsymbol{x}_{2j} - \boldsymbol{x}_{1i}\|) \le \psi_M$ to find

$$\frac{d}{dt}\Delta_v \cdot \boldsymbol{e} \ge -\frac{K}{N_1 N_2}\psi_M \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\left(|\boldsymbol{v}_{1i} \cdot \boldsymbol{e}| + |\boldsymbol{v}_{2j} \cdot \boldsymbol{e}|\right). \tag{3.1.14}$$

Then, we use (3.1.12) and

$$\begin{aligned}
\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(|\boldsymbol{v}_{1i} \cdot \boldsymbol{e}| + |\boldsymbol{v}_{2j} \cdot \boldsymbol{e}|) &\le \sum_{i=1}^{N_1}(\sqrt{N_2}\|\boldsymbol{v}_2\| + N_2\|\boldsymbol{v}_{1i}\|) \\
&\le N_1\sqrt{N_2}\|\boldsymbol{v}_2\| + N_2\sqrt{N_1}\|\boldsymbol{v}_1\| \\
&\le \sqrt{2N_1 N_2 \max\{N_1, N_2\} M_2(0)}
\end{aligned} \tag{3.1.15}$$

to obtain

$$\frac{d}{dt}\Delta_v \cdot \boldsymbol{e} \ge -K\sqrt{\frac{2M_2(0)}{\min\{N_1, N_2\}}}\psi_M.$$

□

3.1.3 Well-prepared initial configurations

In this subsection, we present a class of well-prepared initial configurations leading to bi-cluster flocking configurations. We now introduce parameters and delineate a class $\mathcal{S}$ of well-prepared initial data:

$$\mathcal{S} := \{(\boldsymbol{x}_0, \boldsymbol{v}_0) \in \mathbb{R}^{2dN} : \ (\mathcal{C}_0 0), (\mathcal{C}_0 1) \text{ and } (\mathcal{C}_0 2) \text{ hold}\}.$$

- $(\mathcal{C}_0 0)$(Parameters): For some fixed unit vector $\boldsymbol{e}$, the following parameters are chosen to be positive and satisfy:

$$\begin{aligned}&N \geq 3, \quad 2\alpha_0(\boldsymbol{e}) := \Delta_v(0) \cdot \boldsymbol{e}, \quad \mathcal{V}_0 := \mathcal{V}(0),\\ &\mathcal{X}_0 := \mathcal{X}(0), \quad r_0(\boldsymbol{e}) := \min_{i,j} \left\{ (\boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0)) \cdot \boldsymbol{e} \right\}.\end{aligned}$$

- $(\mathcal{C}_0 1)$(Small perturbations): The local velocity fluctuations are sufficiently small to satisfy

$$\mathcal{V}_0 < \frac{K \min\{N_1, N_2\}}{2N} \int_{\mathcal{X}_0}^{\infty} \psi(\sqrt{2}x) dx.$$

- $(\mathcal{C}_0 2)$(Close to bi-cluster): The initial configuration is close to the sum of two big clusters, which are expected to have asymptotic bi-cluster flocking; in particular,

$$\alpha_0(\boldsymbol{e}) > 4\sqrt{2}\mathcal{V}_0, \quad \int_{r_0(\boldsymbol{e})}^{\infty} \psi(x) dx < \frac{\alpha_0(\boldsymbol{e})\mathcal{V}_0}{2\sqrt{2}K\sqrt{M_2(0)}},$$

  where $\boldsymbol{e}$ is the same vector in $(\mathcal{C}_0 0)$.

Note that $\boldsymbol{e}$ is just a fixed parameter for convenience. Condition $(\mathcal{C}_0 0)$ says that two groups are distinguished by a plane and moving against that plane. Condition $(\mathcal{C}_0 1)$ implies that the initial velocity perturbations are sufficiently small, whereas condition $(\mathcal{C}_0 2)$ is introduced to guarantee that initially two local groups are close to the diverging two-particle system, which is the situation we mentioned at the beginning of this chapter. Moreover, note that

$(\mathcal{C}_0 1)$ only depends on $\mathcal{X}_0$ and $\mathcal{V}_0$. If we choose r_0 suitably large, $(\mathcal{C}_0 2)$ always holds. This implies that the initial data set is non-empty. Of course, the conditions $(\mathcal{C}_0 0) - (\mathcal{C}_0 2)$ are sufficient condition as in the global flocking analysis. In the sequel, for notational simplicity, we suppress h-dependence on $\alpha_0(\boldsymbol{e})$ and $r_0(\boldsymbol{e})$, i.e.,

$$\alpha_0 \equiv \alpha_0(\boldsymbol{e}), \qquad r_0 \equiv r_0(\boldsymbol{e}).$$

3.2 Analysis of the bi-cluster flocking phenomenon

In this section, we prove the non-trivial initial configurations stated in the last section satisfy the desired bi-cluster flocking estimates.

For the proof, we use the continuity argument to show that the global solutions satisfy the desired bi-cluster flocking estimates. More precisely, our strategy combines the estimates in Lemma 3.1.3 with the continuity argument, which can be summarized in the following three steps:

- Step A: Introduce an admissible set of initial configurations $\mathcal{S}$ leading to bi-cluster flocking configurations.
- Step B: Use the system of SDDI in Lemma 3.1.3 to show that the desired bi-cluster flocking estimates hold for some finite time interval $[0, T)$.
- Step C: By the continuity argument, T can be extended to infinity. Finally, conclude that the global solutions with initial data in $\mathcal{S}$ satisfy the desired bi-cluster flocking estimates.

Each step will be discussed in the following three subsections.

3.2.1 Some elementary estimates in finite-time

In this subsection, we provide estimates for the ℓ_2-type functionals $\mathcal{V}$ and Δ_v^k using the estimates given in Lemma 3.1.3.

First, note that for any i and j,

$$\min_{i,j} \|\boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0)\| \geq \min_{i,j} \{(\boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0)) \cdot \boldsymbol{e}\} > r_0 > 0.$$

By the continuity of solutions, there exists $T > 0$ such that

$$\min_{i,j} \|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\| > \alpha_0 t + r_0, \quad \text{for } t \in [0, T). \tag{3.2.1}$$

We now define T^* to be the supremum among all T satisfying (3.2.1). We will show that $T^* = \infty$ by the continuity argument.

Suppose not, i.e.,$T^* < \infty$. Then there exists i_0 and j_0 such that

$$\min_{i,j} \|\boldsymbol{x}_{1i}(T^*) - \boldsymbol{x}_{2j}(T^*)\| = \|\boldsymbol{x}_{1i_0}(T^*) - \boldsymbol{x}_{2j_0}(T^*)\| = \alpha_0 T^* + r_0. \tag{3.2.2}$$

By the non-increasing property of ψ, we have

$$\psi_M(t) = \max_{i,j} \psi(\|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\|) \leq \psi(\alpha_0 t + r_0), \quad \text{for any } t \in [0, T^*).$$

In the following two lemmas, we will give uniform upper and lower bounds for the functionals $\mathcal{V}$ and $\Delta_v \cdot \boldsymbol{e}$ in the time interval $[0, T^*]$, respectively.

Lemma 3.2.1. *Suppose that conditions* $(\mathcal{C}_0 0)$ *and* $(\mathcal{C}_0 2)$ *hold. Then we have*

$$\mathcal{V}(t) < 2\mathcal{V}_0 \quad \textit{for } t \in [0, T^*].$$

Proof. From Lemma 3.1.3 (ii), we have

$$\frac{d\mathcal{V}}{dt} + \frac{K \min\{N_1, N_2\}}{N} \psi(\sqrt{2}\mathcal{X})\mathcal{V} \leq 2\sqrt{2}K\sqrt{M_2(0)}\psi(\alpha_0 t + r_0) \quad \text{for } t \in [0, T^*].$$

Integrating the above inequality using Lemma 3.1.3 (i), we obtain

$$\begin{aligned} \mathcal{V}(t) + \frac{K \min\{N_1, N_2\}}{N} \Big| \int_{\mathcal{X}_0}^{\mathcal{X}(t)} \psi(\sqrt{2}x) dx \Big| \\ \leq \mathcal{V}(0) + 2\sqrt{2}K\sqrt{M_2(0)} \int_0^\infty \psi(\alpha_0 t + r_0) dt \end{aligned} \tag{3.2.3}$$

for $t \in [0, T^*]$. By condition $(\mathcal{C}_0 2)(ii)$, we know

$$2\sqrt{2}K\sqrt{M_2(0)}\int_0^\infty \psi(\alpha_0 t + r_0)dt = \frac{2\sqrt{2}K\sqrt{M_2(0)}}{\alpha_0}\int_{r_0}^\infty \psi(t)dt < \mathcal{V}_0.$$

Thus we have $\mathcal{V}(t) < 2\mathcal{V}_0$ for $t \in [0, T^*]$. □

Lemma 3.2.2. *Suppose that conditions* $(\mathcal{C}_0 0)$ *and* $(\mathcal{C}_0 2)$ *hold. Then*

$$(\boldsymbol{v}_{1i}(t) - \boldsymbol{v}_{2j}(t)) \cdot \boldsymbol{e} > \alpha_0, \quad \textit{for} \quad \forall i, j \textit{ and } t \in [0, T^*).$$

Proof. It follows from Lemma 3.1.3 (iii) that

$$\dot{\Delta}_v \cdot \boldsymbol{e} \geq -K\sqrt{\frac{2M_2(0)}{\min\{N_1, N_2\}}}\psi(\alpha_0 t + r_0) \quad \text{for } t \in [0, T^*].$$

This yields

$$\Delta_v(t) \cdot \boldsymbol{e} \geq 2\alpha_0 - K\sqrt{\frac{2M_2(0)}{\min\{N_1, N_2\}}}\int_0^\infty \psi(\alpha_0 t + r_0)dt \quad \text{for } t \in [0, T^*].$$

We now use the condition $(\mathcal{C}_0 2)$ to obtain

$$K\sqrt{\frac{2M_2(0)}{\min\{N_1, N_2\}}}\int_0^\infty \psi(\alpha_0 t + r_0)dt \leq \frac{K}{\alpha_0}\sqrt{\frac{2M_2(0)}{\min\{N_1, N_2\}}}\int_{r_0}^\infty \psi(x)dx < \frac{\alpha_0}{2}.$$

Then, this and the condition $(\mathcal{C}_0 0)$ yield

$$(\boldsymbol{v}_{1i}(t) - \boldsymbol{v}_{2j}(t)) \cdot \boldsymbol{e} = \Delta_v(t) \cdot \boldsymbol{e} + (\hat{\boldsymbol{v}}_{1i} - \hat{\boldsymbol{v}}_{2j}) \cdot \boldsymbol{e} > \frac{3}{2}\alpha_0 - \sqrt{2}\mathcal{V}(t) > \alpha_0$$

for $t \in [0, T^*]$, where we used Lemma 3.2.1. Thus we complete the proof. □

Therefore, on the time interval $[0, T^*]$, $\mathcal{V}(t)$ has an upper bound and $\Delta_v \cdot \boldsymbol{e}$ has a lower bound. Finally, we show that $T^* = \infty$ and obtain the bi-cluster flocking estimates.

3.2.2 Bi-cluster flocking estimates

In this subsection, we study the emergence of bi-cluster flocking groups in the C-S model.

Theorem 3.2.1. *Suppose that conditions ($\mathcal{C}_0 0$), ($\mathcal{C}_0 1$), and ($\mathcal{C}_0 2$) hold. Then, we have*

$$T^* = \infty \quad \textit{and} \quad \textit{bi-cluster flocking estimates.}$$

More precisely, the following estimates hold:

$$\begin{aligned}
&(i)\ \min_{i,j} \|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\| > \alpha_0 t + r_0, \quad t \in [0, \infty),\\
&(ii)\ \exists X_M > 0 \quad s.t. \quad \mathcal{X}(t) \le X_M, \quad t \in [0, \infty),\\
&(iii)\ \mathcal{V}(t) < \mathcal{O}(1)(1+t)^{-\beta} \quad as \quad t \to \infty.
\end{aligned}$$

Proof. (i) If $T^* < \infty$, by the definition of supremum there exist i_0, j_0 such that (3.2.2) holds. On the other hand, it follows from lemma 3.2.2 that

$$(\boldsymbol{v}_{1i}(t) - \boldsymbol{v}_{2j}(t)) \cdot \boldsymbol{e} > \alpha_0 \quad \text{for } t \in [0, T^*]. \tag{3.2.4}$$

Then, (3.2.4) and the condition ($\mathcal{H}0$) yield

$$\begin{aligned}
\|\boldsymbol{x}_{1i}(T^*) - \boldsymbol{x}_{2j}(T^*)\| &\ge |(\boldsymbol{x}_{1i}(T^*) - \boldsymbol{x}_{2j}(T^*)) \cdot \boldsymbol{e}|\\
&= \left| (\boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0)) \cdot \boldsymbol{e} + \int_0^{T^*} [(\boldsymbol{v}_{1i}(t) - \boldsymbol{v}_{2j}(t)) \cdot \boldsymbol{e}] dt \right|\\
&> \alpha_0 T^* + r_0,
\end{aligned}$$

for all i and j. This contracts (3.2.2). Thus $T^* = \infty$ and

$$\min_{i,j} \|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\| > \alpha_0 t + r_0 \quad \text{for } t \in [0, \infty).$$

(ii) From the above proof, we know

$$\psi_M(t) < \psi(\alpha_0 t + r_0) \quad \text{for } t \in [0, \infty).$$

Using the same method as in Lemma 3.2.1, we get

$$\begin{aligned}
&\mathcal{V}(t) + \frac{K \min\{N_1, N_2\}}{N} \int_{\mathcal{X}_0}^{\mathcal{X}(t)} \psi(\sqrt{2}x) dx\\
&\le \mathcal{V}_0 + 2\sqrt{2} K \sqrt{M_2(0)} \int_0^{\infty} \psi(\alpha_0 t + r_0) dt
\end{aligned} \tag{3.2.5}$$

for $t \in [0,\infty)$. Since condition $(\mathcal{C}_0 1)$ means that there exist $X_M > 0$ such that

$$2\mathcal{V}_0 = \frac{K\min\{N_1,N_2\}}{N}\int_{\mathcal{X}_0}^{X_M}\psi(\sqrt{2}x)dx.$$

Using (3.2.5) and condition $(\mathcal{C}_0 2)$, we obtain the following for any time $t \geq 0$,

$$\frac{K\min\{N_1,N_2\}}{N}\int_{\mathcal{X}_0}^{\mathcal{X}(t)}\psi(\sqrt{2}x)dx \leq \frac{K\min\{N_1,N_2\}}{N}\int_{\mathcal{X}_0}^{X_M}\psi(\sqrt{2}x)dx.$$

Thus $\mathcal{X}(t) \leq X_M$ for $t \in [0,\infty)$. It is worth to mention that this Lyapunov technique will play key roles to prove the boundedness in other chapters, too.

(iii) Define $\beta_0 := \dfrac{K\min\{N_1,N_2\}}{N}\psi(\sqrt{2}X_M)$. Using Lemma 3.1.3(ii), we get

$$\dot{\mathcal{V}} \leq -\beta_0\mathcal{V} + 2\sqrt{2}K\sqrt{M_2(0)}\psi(\alpha_0 t + r_0)$$

for $t \in [0,\infty)$. We use Lemma A.0.1 in appendix to conclude that

$$\begin{aligned}\mathcal{V}(t) &\leq \mathcal{V}_0 e^{-\beta_0 t} + \frac{2\sqrt{2}K\sqrt{M_2(0)}\psi(r_0)}{\beta_0}e^{-\frac{\beta_0}{2}t} + \frac{2\sqrt{2}K\sqrt{M_2(0)}}{\beta_0}\psi(\frac{\alpha_0}{2}t + r_0)\\ &\leq \mathcal{O}(1)(1+t)^{-\beta} \quad \text{as} \quad t \to \infty.\end{aligned} \tag{3.2.6}$$

□

Remark 3.2.1. *From the result of Theorem 3.2.1, the relative position between two clusters goes to ∞, and each cluster is bounded and the velocity differences in each group tend to zero asymptotically. This means that the bi-flocking phenomena occurs under conditions $(\mathcal{C}_0 0)$, $(\mathcal{C}_0 1)$, and $(\mathcal{C}_0 2)$. Furthermore, we can show each agent's velocity tends to a constant depending on the full history of solution.*

Next, we study the convergence rate of $\boldsymbol{v}_{1i}$ and $\boldsymbol{v}_{2j}$.

Theorem 3.2.2. *Suppose that conditions $(\mathcal{C}_0 0)$, $(\mathcal{C}_0 1)$, and $(\mathcal{C}_0 2)$ hold. Then there exist $\boldsymbol{v}_{1c}^\infty$, $\boldsymbol{v}_{2c}^\infty$ and positive constants C_1 and C_2 such that for any i,j,*

$$C_1(1+t)^{-(\beta-1)} \leq \|\boldsymbol{v}_{1i}(t)-\boldsymbol{v}_{1c}^\infty\|+\|\boldsymbol{v}_{2j}(t)-\boldsymbol{v}_{2c}^\infty\| \leq C_2(1+t)^{-(\beta-1)} \quad \textit{as } t \to \infty.$$

Proof. Note that $\boldsymbol{v}_{1c}$ satisfies

$$\frac{d\boldsymbol{v}_{1c}}{dt} = \frac{K}{N_1 N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\psi(\|\boldsymbol{x}_{2j}-\boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2j}-\boldsymbol{v}_{1i}). \tag{3.2.7}$$

Integrating (3.2.7) from 0 to t yields

$$\boldsymbol{v}_{1c}(t) = \boldsymbol{v}_{1c}(0) + \frac{K}{N_1 N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_0^t \psi(\|\boldsymbol{x}_{2j}(s)-\boldsymbol{x}_{1i}(s)\|)(\boldsymbol{v}_{2j}(s)-\boldsymbol{v}_{1i}(s))ds. \tag{3.2.8}$$

By letting $t \to \infty$ in (3.2.8), the asymptotic limit $\boldsymbol{v}_{1c}^{\infty}$ is given by

$$\boldsymbol{v}_{1c}^{\infty} := \boldsymbol{v}_{1c}(0) + \frac{K}{N_1 N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_0^\infty \psi(\|\boldsymbol{x}_{2j}(\tau)-\boldsymbol{x}_{1i}(\tau)\|)(\boldsymbol{v}_{2j}(\tau)-\boldsymbol{v}_{1i}(\tau))d\tau. \tag{3.2.9}$$

$\diamond$ Case A (upper bound estimate): Subtracting (3.2.9) from (3.2.8) and taking the ℓ^2-norm yields

$$\|\boldsymbol{v}_{1c}(t)-\boldsymbol{v}_{1c}^{\infty}\| \le \frac{K}{N_1 N}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_t^\infty \psi(\|\boldsymbol{x}_{2j}(\tau)-\boldsymbol{x}_{1i}(\tau)\|)\|\boldsymbol{v}_{2j}(\tau)-\boldsymbol{v}_{1i}(\tau)\|d\tau.$$

Again, we use

$$\psi(\|\boldsymbol{x}_{2j}-\boldsymbol{x}_{1i}\|) < \psi(\alpha_0 t + r_0), \quad \|\boldsymbol{v}_{2j}-\boldsymbol{v}_{1i}\| < \sqrt{2M_2(0)}$$

to obtain

$$\begin{aligned}
\|\boldsymbol{v}_{1c}(t)-\boldsymbol{v}_{1c}^{\infty}\| \;&\le\; \frac{KN_2\sqrt{2M_2(0)}}{N}\int_t^\infty \frac{1}{(1+\alpha_0\tau+r_0)^\beta}d\tau \\
&=\; \frac{KN_2\sqrt{2M_2(0)}}{\alpha_0(\beta-1)(1+r_0)^{\beta-1}N}\Big(1+\frac{\alpha_0}{1+r_0}t\Big)^{-(\beta-1)}.
\end{aligned}$$

Thus, we have

$$\|\boldsymbol{v}_{1c}(t)-\boldsymbol{v}_{1c}^{\infty}\| \le \mathcal{O}(1)(1+t)^{-(\beta-1)}. \tag{3.2.10}$$

Then, (3.2.6) and (3.2.10) yield the desired estimate:

$$\begin{aligned}
\|\boldsymbol{v}_{1i}(t)-\boldsymbol{v}_{1c}^{\infty}\| &\le \|\boldsymbol{v}_{1i}(t)-\boldsymbol{v}_{1c}(t)\| + \|\boldsymbol{v}_{1c}(t)-\boldsymbol{v}_{1c}^{\infty}\| \\
&\le \mathcal{V}(t) + \|\boldsymbol{v}_{1c}(t)-\boldsymbol{v}_{1c}^{\infty}\| \\
&\le \mathcal{O}(1)(1+t)^{-(\beta-1)} \quad \text{as } t\to\infty.
\end{aligned} \tag{3.2.11}$$

Similarly,

$$\|\boldsymbol{v}_{2j}(t) - \boldsymbol{v}_{2c}^{\infty}\| \le \mathcal{O}(1)(1+t)^{-(\beta-1)} \quad \text{as } t \to \infty. \tag{3.2.12}$$

Hence, by combining (3.2.11) and (3.2.12), we obtain the desired upper estimate.

$\diamond$ Case B (lower bound estimate): Define $R_0 := \max \|\boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0)\|$. It is easy to see

$$\begin{aligned} \|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\| &\le \|\boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0)\| + \int_0^t \|\boldsymbol{v}_{1i}(\tau) - \boldsymbol{v}_{2j}(\tau)\| d\tau \\ &\le \sqrt{2M_2(0)}t + R_0. \end{aligned} \tag{3.2.13}$$

We use (3.2.13) and Lemma 3.2.2 to obtain

$$\begin{aligned} \|\boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}^{\infty}\| &\ge \frac{KN_2\alpha_0}{N} \int_t^{\infty} \psi(\sqrt{2M_2(0)}\tau + R_0) d\tau \\ &\ge |\mathcal{O}(1)|(1+t)^{-(\beta-1)} \quad \text{as } t \to \infty. \end{aligned} \tag{3.2.14}$$

Again (3.2.14) yields

$$\begin{aligned} \|\boldsymbol{v}_{1i}(t) - \boldsymbol{v}_{1c}^{\infty}\| &\ge \|\boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}^{\infty}\| - \|\boldsymbol{v}_{1i}(t) - \boldsymbol{v}_{1c}(t)\| \\ &\ge \|\boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}^{\infty}\| - \mathcal{V}(t) \\ &\ge |\mathcal{O}(1)|(1+t)^{-(\beta-1)} \quad \text{as } t \to \infty. \end{aligned} \tag{3.2.15}$$

Similarly, we have

$$\|\boldsymbol{v}_{2j}(t) - \boldsymbol{v}_{2c}^{\infty}\| \ge |\mathcal{O}(1)|(1+t)^{-(\beta-1)} \quad \text{as } t \to \infty. \tag{3.2.16}$$

Finally, combining (3.2.15) and (3.2.16) yields desired lower bound estimate.

□

3.3 Numerical simulations

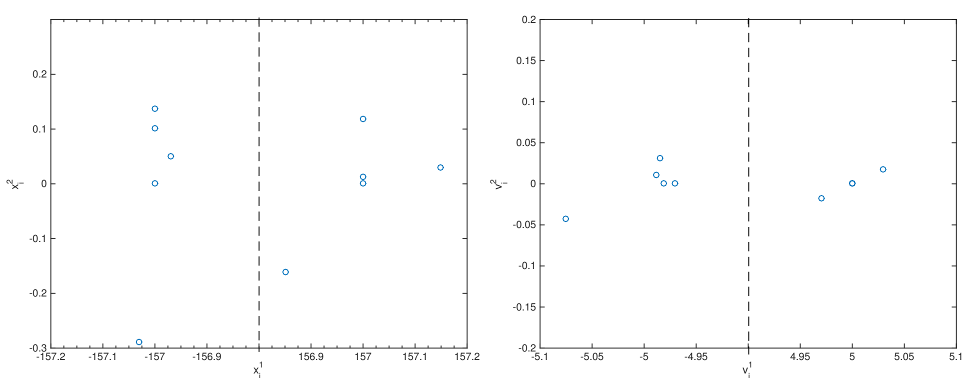
In this section, we provide several numerical simulations for the emergence of bi-cluster flocking and compare them to the analytical results in the previous section. For the numerical implementation, we employ the fourth order Runge-Kutta method. The parameter values in (3.1.1) are as follows:

$$\Delta t = 0.01, \quad N_1 = N_2 = 5, \quad \beta = 2, \quad d = 2,$$

and we set

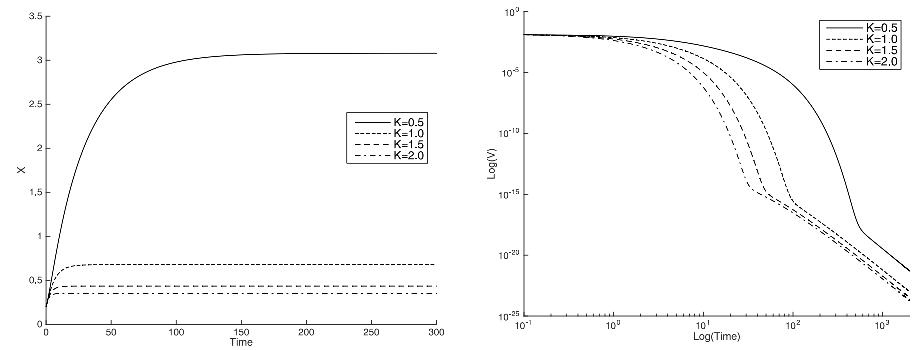
$$
\begin{aligned}
&\mathcal{X}_0 = 0.2, \quad \mathcal{V}_0 = 0.0125, \\
&\boldsymbol{e} = (1, 0), \quad r_0 = 313.9107, \quad \alpha_0 = 10, \quad M_2(0) = 2.5031.
\end{aligned}
$$

Under these conditions, $(\mathcal{C}_0 1)$ and $(\mathcal{C}_0 2)$ are satisfied. All initial positions and velocities in Figure 3.1 are generated randomly from the information r_0, $\mathcal{X}_0$, α_0, and $\mathcal{V}_0$.
In Figure 3.2, we simulate the dynamics of $\mathcal{X}$ and $\mathcal{V}$ measuring the local fluctuations around the local averages depending on different choices of the coupling strength. We plot four simulations in $X - t$ and $\log \mathcal{V} - \log t$ scale to show the algebraic decay toward zero. Clearly, the exponential decay rate is proportional to the coupling strength. In Figure 3.3, we simulate the dynamics of Δ_x and Δ_v measuring the differences between local averages depending on four different values. As noted in the analytical results in Theorem 3.2.1, Δ_x grows at least linearly, whereas Δ_v stays away from zero. The local velocity averages converge algebraically to constant values, as we can see Δ_v^1.



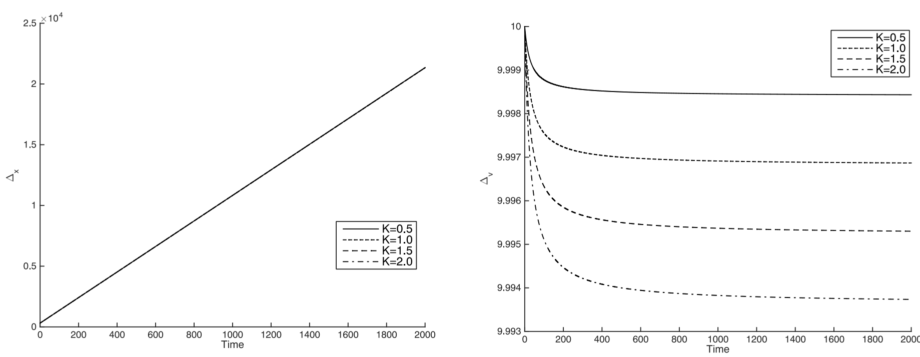
(a) Initial spatial distribution $\boldsymbol{x}_0$ (b) Initial velocity distribution $\boldsymbol{v}_0$

Figure 3.1: Initial position-velocity distribution



(a) Dynamics of $\mathcal{X}$ (b) Dynamics of $\mathcal{V}$ in $\log \mathcal{V} - \log t$ scale

Figure 3.2: Dynamics of local fluctuations for K=0.5, 1, 1.5, 2.



(a) Dynamics of Δ_x^1 (b) Dynamics of Δ_v^1

Figure 3.3: Dynamics of two separating groups for K=0.5, 1, 1.5, 2.

Chapter 4

Multi-cluster flocking in terms of coupling strength

In this chaper, we are interested in the existence of a positive critical coupling strength that can be formally defined as follows.

Definition 4.0.1. *For a given initial configuration $\boldsymbol{z}_0 := (\boldsymbol{x}_0, \boldsymbol{v}_0)$, a non-negative constant $K_c = K_c(\boldsymbol{z}_0)$ is the critical coupling strength for mono-cluster flocking if and only if the following two criteria hold.*

1. *If $K > K_c$, then the initial configuration $\boldsymbol{z}_0$ tends to a mono-cluster flocking configuration asymptotically.*

2. *If $K \leq K_c$, then the initial configuration $\boldsymbol{z}_0$ does not tend to a mono-cluster flocking configuration asymptotically.*

To see the role of the critical coupling strength, we first consider a two-particle system on the real line $\mathbb{R}$:

$$
\begin{aligned}
&\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \quad t > 0, \ x_i, v_i \in \mathbb{R}, \\
&\dot{v}_1 = \frac{K}{2}\psi(|x_2 - x_1|)(v_2 - v_1), \\
&\dot{v}_2 = \frac{K}{2}\psi(|x_1 - x_2|)(v_1 - v_2), \\
&(x_i, v_i)(0) = (x_{i0}, v_{i0}).
\end{aligned}
\tag{4.0.1}
$$

Then, Proposition 2.2.1 and Corollary 2.2.1 implies that if we define

$$K_c := \frac{|v_0|}{\int_{|x_0|}^{\infty} \psi(|y|)dy},$$

then $K > K_c$ implies flocking two particles, and $K \leq K_c$ implies separating. Therefore, we know the complete information on two particle system on the real line.

Note that Theorem 2.3.1 yields a sufficient condition for mono-cluster flocking. Through similar analysis, we present two coupling strength conditions for more general case; One is the condition for totally separating particles, and the other is the condition for multi-cluster flocking. These values mean lower bounds of the critical coupling strength, and will be displayed for example cases by numerical simulations. It looks quite difficult to cover the whole cases since the mono-cluster flocking states are already equilibrium state, so that they flocks even for $K = 0$. However, it has been shown that local flocking, especially bi-cluster flocking, can emerge from some well-prepared configurations close to bi-cluster configurations in the previous chapter. The use of infinite-norms is the main difference while the methods are similar as in Chapter 3, which allow us more general configuration on position using the time stamp T_0. This chapter is based on the joint work in [42].

4.1 Necessary condition for mono-cluster flocking

In this section, we provide a framework for the non-existence of mono-cluster flocking and state a *necessary condition* for the emergence of mono-cluster flocking. In order to achieve non-flocking configurations, we follow a scenario that every particle with different initial velocity does not flock each other. This is more restricted situation than non-flocking situation.

4.1.1 Framework and main result

First, we will introduce a framework for the non-existence of mono-cluster flocking. In the same way as in Chapter 3, let $\mathcal{G} := \big\{(\boldsymbol{x}_{i0}, \boldsymbol{v}_{i0})\big\}_{i=1}^{N}$ be an initial non-flocking configuration of an ensemble of C-S particles. To avoid technical difficulties, we group particles as its initial velocities. In detail, we set sub-ensembles $\mathcal{G}_1, \cdots, \mathcal{G}_n$ of the total ensemble $\mathcal{G}$ according to the initial data: for $\alpha = 1, \cdots, n$,

$$(\boldsymbol{x}_{\alpha i}, \boldsymbol{v}_{\alpha i}), (\boldsymbol{x}_{\alpha j}, \boldsymbol{v}_{\alpha j}) \in \mathcal{G}_\alpha \quad \Longleftrightarrow \quad \boldsymbol{v}_{\alpha i0} = \boldsymbol{v}_{\alpha j0}, \qquad |\mathcal{G}_\alpha| = N_\alpha. \tag{4.1.1}$$

We will use the simplified notation $\gamma_N := \frac{\min_\beta N_\beta}{N}$ for the number of particles.

Since we need the initial configuration which is not in a mono-cluster flocking state, we assume $n \geq 2$. Then, the original system (1.0.1) can be rewritten as

$$\begin{aligned}
&\dot{\boldsymbol{x}}_{\alpha i}(t) = \boldsymbol{v}_{\alpha i}(t), \quad t > 0, \quad i = 1, 2, \cdots, N_\alpha, \\
&\dot{\boldsymbol{v}}_{\alpha i}(t) = \frac{K}{N} \sum_{k=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\big(\boldsymbol{v}_{\alpha k}(t) - \boldsymbol{v}_{\alpha i}(t)\big) \\
&\qquad\qquad + \frac{K}{N} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\beta} \psi(\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\big(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\big), \\
&(\boldsymbol{x}_{\alpha i}(0), \boldsymbol{v}_{\alpha i}(0)) = (\boldsymbol{x}_{\alpha i0}, \boldsymbol{v}_{\alpha i0}).
\end{aligned} \tag{4.1.2}$$

We also introduce local averages and local fluctuations as in Chapter 3:

$$\begin{aligned}
&\boldsymbol{x}_{\alpha c} := \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} \boldsymbol{x}_{\alpha i}, \qquad \boldsymbol{v}_{\alpha c} := \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} \boldsymbol{v}_{\alpha i}, \\
&\hat{\boldsymbol{x}}_{\alpha i} := \boldsymbol{x}_{\alpha i} - \boldsymbol{x}_{\alpha c}, \qquad \hat{\boldsymbol{v}}_{\alpha i} := \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha c}.
\end{aligned} \tag{4.1.3}$$

Note that the relations (4.1.1) and (4.1.3) imply that

$$\|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty} = 0, \quad \alpha = 1, \cdots, n. \tag{4.1.4}$$

Next, we describe the geometry of the initial separation between the sub-ensembles. For a given initial configuration $(\boldsymbol{x}_0, \boldsymbol{v}_0)$, we set

$$\begin{aligned}
&\lambda_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \frac{1}{2}\min_{\beta\neq\alpha}\|\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\|, \quad D(\boldsymbol{x}_0) := \max_{\beta\neq\alpha,i,k}\|\boldsymbol{x}_{\beta k0} - \boldsymbol{x}_{\alpha i0}\|, \\
&T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \max_{\beta\neq\alpha,i,k}\left\{0, -\frac{\big(\boldsymbol{x}_{\beta k0} - \boldsymbol{x}_{\alpha i0}\big)\cdot\big(\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\big)}{\lambda_0^2}\right\}.
\end{aligned} \tag{4.1.5}$$

For notational simplicity, we suppress the $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ dependence of λ_0, T_0, K_0 as follows:

$$\lambda_0 := \lambda_0(\boldsymbol{x}_0, \boldsymbol{v}_0), \qquad T_0 := T_0(\boldsymbol{x}_0, \boldsymbol{v}_0).$$

Next, we introduce a coupling strength $K_0 := K_0(\boldsymbol{x}_0, \boldsymbol{v}_0)$ depending on the geometry of the initial configuration $(\boldsymbol{x}_0, \boldsymbol{v}_0)$.

- If the initial configuration satisfies

$$\min_{\beta\neq\alpha,i,k}\big(\boldsymbol{x}_{\beta k}(0) - \boldsymbol{x}_{\alpha i}(0)\big)\cdot\big(\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\big) < 0, \tag{4.1.6}$$

then, we set

$$\begin{aligned}
K_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \min\Bigg\{&\frac{\lambda_0^2}{12(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)}}, \\
&\frac{\lambda_0}{16(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0}, \\
&\frac{\lambda_0^2}{(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)\big(D(\boldsymbol{x}_0) + \sqrt{2M_2(0)}T_0\big)}\Bigg\}.
\end{aligned} \tag{4.1.7}$$

- If the initial configuration satisfies

$$\min_{\beta\neq\alpha,i,k}\big(\boldsymbol{x}_{\beta k}(0) - \boldsymbol{x}_{\alpha i}(0)\big)\cdot\big(\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\big) \geq 0, \tag{4.1.8}$$

then, we set

$$K_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \frac{\lambda_0^2}{6(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)}}. \tag{4.1.9}$$

Before we state our first main result, we comment on the conditions on (4.1.6) and (4.1.8) as follows. In the absence of flocking coupling, i.e., $K = 0$, the free flow (4.1.2) with $K = 0$ yields

$$\begin{aligned}&(\boldsymbol{v}_{\alpha i}(t), \boldsymbol{v}_{\beta i}(t)) = (\boldsymbol{v}_{\alpha c}(0), \boldsymbol{v}_{\beta c}(0)) \quad \text{and} \\ &(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{x}_{\beta i}(t)) = (\boldsymbol{x}_{\alpha i0} + t\boldsymbol{v}_{\alpha c}(0), \boldsymbol{x}_{\beta i0} + t\boldsymbol{v}_{\beta c}(0)), \quad t \geq 0.\end{aligned}$$

Thus, we have

$$\begin{aligned}&(\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)) \cdot (\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)) \\ &\qquad = (\boldsymbol{x}_{\beta i0} - \boldsymbol{x}_{\alpha i0}) \cdot (\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)) + t|\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)|^2.\end{aligned}$$

Note that since $\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0) \neq 0$, for sufficiently large $t \gg 1$, we have

$$(\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)) \cdot (\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)) > 0,$$

and we can reasonably estimate that a similar result will be obtained in a small-coupling-strength regime. This plausible scenario will be verified in Section 4.1.3.

We are now ready to state our main result as follows.

Theorem 4.1.1. *Let $(\boldsymbol{x}(t), \boldsymbol{v}(t))$ be a global solution of* (1.0.1) *with initial data satisfying the following condition:*

$$\max_{i \neq j} \|\boldsymbol{v}_{i0} - \boldsymbol{v}_{j0}\| > 0.$$

Let K_0 be a parameter defined in (4.1.7) *and* (4.1.9)*. If $K < K_0$, we have*

$$\min_{\alpha \neq \beta, i, k} \sup_{t \geq 0} \|\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\beta k}(t)\| = \infty, \qquad \min_{\alpha \neq \beta, i, k} \liminf_{t \to \infty} \|\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\beta k}(t)\| > 0.$$

In other words, mono-cluster flocking does not occur asymptotically. Moreover, all sub-ensembles are well separated.

Remark 4.1.1. *We now compare the results of Theorem 4.1.1 with the results on bi-cluster flocking formation presented in Chapter 3 as follows. Here, we will only concentrate on the condition related to the upper bound of K in Chapter 3:*

$$K < \frac{\lambda_0 \|\hat{\boldsymbol{v}}_{\alpha 0}\|}{2\sqrt{2M_2(0)} \int_{r_0}^{\infty} \psi(x)dx},$$

where r_0 is the initial spatial separation of two groups. First, because of the term $\|\hat{\boldsymbol{v}}_{\alpha 0}\|$ in K, Chapter 3 cannot include the $\|\hat{\boldsymbol{v}}_{\alpha 0}\| = 0$ case, which seems to be the most reasonable case.

Second, in Theorem 4.1.1, K_0 does not appear with r_0, since we do not want to restrict the initial condition to the case of "initially separating":

$$\min_{\beta\neq\alpha,i,k} \big(\boldsymbol{x}_{\beta k}(0) - \boldsymbol{x}_{\alpha i}(0)\big) \cdot \big(\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\big) \geq 0.$$

For this problem, we assign sufficient time T_0 to obtain well-organized separating data. Moreover, we can get a better value of K_0 in Theorem 4.1.1 when each group is separated initially.

4.1.2 Dynamics of local averages and fluctuations

In this subsection, we provide estimates of the local averages and fluctuations defined in (4.1.3).

Lemma 4.1.1. *Let $\big(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t)\big)$ be a solution of the system (1.0.1). Then, the local averages and fluctuations satisfy*

$$\begin{cases} \dot{\boldsymbol{x}}_{\alpha c}(t) = \boldsymbol{v}_{\alpha c}(t), \quad t \geq 0, \quad \alpha = 1, 2, \cdots, n, \\ \dot{\boldsymbol{v}}_{\alpha c}(t) = \dfrac{K}{NN_\alpha} \displaystyle\sum_{\beta\neq\alpha} \sum_{k=1}^{N_\beta} \sum_{i=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\big(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\big) \end{cases} \tag{4.1.10}$$

and

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}_{\alpha i}(t) = \hat{\boldsymbol{v}}_{\alpha i}(t), \quad t \geq 0, \quad \alpha = 1, \cdots, n, \quad i = 1, 2, \cdots, N_\alpha, \\ \dot{\hat{\boldsymbol{v}}}_{\alpha i}(t) = -\dot{\boldsymbol{v}}_{\alpha c}(t) + \dfrac{K}{N} \displaystyle\sum_{k=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\big(\hat{\boldsymbol{v}}_{\alpha k}(t) - \hat{\boldsymbol{v}}_{\alpha i}(t)\big) \\ \qquad\qquad + \dfrac{K}{N} \displaystyle\sum_{\beta\neq\alpha} \sum_{k=1}^{N_\beta} \psi(\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\big(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\big). \end{cases} \tag{4.1.11}$$

The proof of Lemma 4.1.1 is basically same as in Lemma 3.1.2.

In the following proposition, we derive the estimates of the time derivatives of $\|\boldsymbol{v}_{\alpha c}\|$ and $\|\hat{\boldsymbol{v}}_\alpha\|$.

Proposition 4.1.1. *Let* $\big(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t)\big), \alpha = 1, \cdots, n$ *be a solution of the system (4.1.2). Then, for any* α*, we have*

$$
\begin{aligned}
&(i)\ \Big\|\frac{d\boldsymbol{v}_{\alpha c}(t)}{dt}\Big\| \le K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t), \quad a.e.,\ t\in[0,\infty),\\
&(ii)\ \frac{d}{dt}\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} \le 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t),
\end{aligned}
$$

where $\psi_M(t) := \max_{\beta\neq\alpha,i,k} \psi(\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)$.

Proof. (i) We use the equation (4.1.10) in Lemma 4.1.1 and the relation

$$
\|\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\| \le \sqrt{2M_2(0)}. \tag{4.1.12}
$$

(ii) For any $t \ge 0$ and $\alpha \in \{1, \cdots, n\}$, by the analyticity of $\boldsymbol{x}_{\alpha i}(t)$ and $\boldsymbol{v}_{\alpha i}(t)$, the solution is analytic, so that $\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}$ is piecewise differentiable. There exists a set of times $\{t_k\}$ satisfying

$$
\begin{gathered}
0 = t_0 < t_1 < \cdots, < t_n < \cdots,\\
\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} \text{ is differentiable in } (t_{k-1}, t_k),\ k = 1, \cdots,
\end{gathered}
$$

where we can choose $i \in \{1, \cdots, N\}$ to satisfy

$$
\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} = \|\hat{\boldsymbol{v}}_{\alpha i}(t)\| \tag{4.1.13}
$$

in the time interval (t_{k-1}, t_k). We directly multiply the second equation of (4.1.11) in Lemma 4.1.1 by $2\hat{\boldsymbol{v}}_{\alpha i}(t)$ to obtain

$$
\begin{aligned}
\frac{d\|\hat{\boldsymbol{v}}_\alpha(t)\|^2_{2,\infty}}{dt} &= -2\langle \hat{\boldsymbol{v}}_{\alpha i}(t), \dot{\boldsymbol{v}}_{\alpha c}(t)\rangle\\
&\quad + \frac{2K}{N}\sum_{k=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\langle \hat{\boldsymbol{v}}_{\alpha i}(t), \hat{\boldsymbol{v}}_{\alpha k}(t) - \hat{\boldsymbol{v}}_{\alpha i}(t)\rangle\\
&\quad + \frac{2K}{N}\sum_{\beta\neq\alpha}\sum_{k=1}^{N_\beta} \psi(\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\langle \hat{\boldsymbol{v}}_{\alpha i}(t), \boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\rangle\\
&=: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.
\end{aligned} \tag{4.1.14}
$$

The remaining parts are similar to Lemma 3.1.3 as follows.
• (Estimate of $\mathcal{I}_{11}$): We use the estimate (i) of $\dot{\boldsymbol{v}}_{\alpha c}(t)$ to get

$$\begin{aligned}|\mathcal{I}_{11}(t)| &\le 2\|\hat{\boldsymbol{v}}_{\alpha i}(t)\|\|\dot{\boldsymbol{v}}_{\alpha c}(t)\| \\ &\le 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}.\end{aligned} \tag{4.1.15}$$

• (Estimate of $\mathcal{I}_{12}$): We use (4.1.13) to get

$$\begin{aligned}\langle \hat{\boldsymbol{v}}_{\alpha i}(t), \hat{\boldsymbol{v}}_{\alpha k}(t) - \hat{\boldsymbol{v}}_{\alpha i}(t)\rangle &= \langle \hat{\boldsymbol{v}}_{\alpha i}(t), \hat{\boldsymbol{v}}_{\alpha k}(t)\rangle - \|\hat{\boldsymbol{v}}_{\alpha i}(t)\|^2 \\ &\le \|\hat{\boldsymbol{v}}_{\alpha i}(t)\|(\|\hat{\boldsymbol{v}}_{\alpha k}(t)\| - \|\hat{\boldsymbol{v}}_{\alpha i}(t)\|) \\ &\le 0.\end{aligned}$$

This yields

$$\mathcal{I}_{12}(t) \le 0. \tag{4.1.16}$$

• (Estimate of $\mathcal{I}_{13}$): We have

$$\begin{aligned}|\mathcal{I}_{13}(t)| &\le \frac{2K}{N}(N-N_\alpha)\sqrt{2M_2(0)}\psi_M(t)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} \\ &\le 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}.\end{aligned} \tag{4.1.17}$$

Substituting (4.1.15), (4.1.16), and (4.1.17) in (4.1.14), we get

$$\frac{d\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}^2}{dt} \le 4K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}.$$

We now divide it by $2\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}$ to obtain the desired estimate. □

4.1.3 Non-existence of mono-cluster flocking

In this subsection, we will provide the proof of Theorem 4.1.1. First, we briefly outline our strategy as follows. The proof of our main results can be split into three stages. For a given initial configuration $(\boldsymbol{x}_0, \boldsymbol{v}_0)$,

- Initial stage (from mixed configuration to segregated configuration): there exists $T_0 \ge 0$ such that

$$(\boldsymbol{x}_{\beta k}(T_0) - \boldsymbol{x}_{\alpha i}(T_0)) \cdot \big(\boldsymbol{v}_{\beta c}(T_0) - \boldsymbol{v}_{\alpha c}(T_0)\big) > 0.$$

- Intermediate stage (from segregated configuration to close to non-mono-cluster flocking): there exists $T^* > T_0$ such that

$$\min_{\alpha\neq\beta,i,k} \Big\{(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)) \cdot \boldsymbol{e}_{\beta\alpha}\Big\} > \frac{\lambda_0}{2}, \text{for all } t \in [T_0, T^*),$$
$$\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| \geq \frac{\lambda_0}{2}(t - T_0),$$

  where $\boldsymbol{e}_{\beta\alpha}$ is the unit vector in the direction of $\boldsymbol{v}_{\beta c}(T_0) - \boldsymbol{v}_{\alpha c}(T_0)$.

- Final stage (emergence of non-mono-cluster configuration): finally, we show that

$$T^* = \infty$$

  and obtain the non-existence of mono-cluster flocking.

Emergence of segregated configurations

In this subsection, we will show that the configuration at time T_0 is well segregated:

$$\varDelta_{\beta k,\alpha i}(T_0) \geq 0, \tag{4.1.18}$$

where $\varDelta_{\beta k,\alpha i}(t) := (\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)) \cdot \big(\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\big)$.
Recall that

$$T_0 := \max_{\beta\neq\alpha,i,k} \Big\{ - \frac{\varDelta_{\beta k,\alpha i}(0)}{\lambda_0^2}, 0\Big\}.$$

In what follows, without loss of generality, we assume that

$$\varDelta_{\beta k,\alpha i}(0) < 0 \quad \text{so that} \quad T_0 > 0.$$

Otherwise, $T_0 = 0$ and the desired estimate (4.1.18) holds trivially.

Lemma 4.1.2. *Let $(\boldsymbol{x}(t), \boldsymbol{v}(t))$ be a global solution of* (1.0.1) *with non-flocking initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$. If the coupling strength K satisfies*

$$K \leq \frac{\lambda_0}{16(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0},$$

then, the following estimates hold: for $t \in [0, T_0]$,

$$\|\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\| \geq \frac{15\lambda_0}{8}, \quad \beta \neq \alpha, \quad \textit{and} \quad \|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} \leq \frac{\lambda_0}{8}.$$

Proof. (i) It follows from Proposition 4.1.1 and the non-increasing property of ψ that for any $\alpha \in \{1, \cdots, n\}$,

$$\left\| \frac{d\boldsymbol{v}_{\alpha c}(t)}{dt} \right\| \le K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t) \le K(1-\gamma_N)\sqrt{2M_2(0)}\psi(0).$$

Thus, for all $\beta \neq \alpha$, we have

$$\left\| \frac{d\big(\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\big)}{dt} \right\| \le 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi(0).$$

We integrate the above inequality directly from $t=0$ to $t=T_0$ to obtain

$$\left| \big(\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\big) - \big(\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\big) \right| \le 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0,$$

for $t \in [0, T_0]$.
By the assumption of K, we can obtain

$$\begin{aligned}
&\left\| \boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t) \right\| \\
&\qquad \ge \left\| \boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0) \right\| - \left\| \big(\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\big) - \big(\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\big) \right\| \\
&\qquad \ge \left\| \boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0) \right\| - 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0 \\
&\qquad \ge \min_{\alpha \neq \beta} \left\| \boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0) \right\| - \frac{\lambda_0}{8} \\
&\qquad \ge \frac{15\lambda_0}{8}, \quad t \in [0, T_0],
\end{aligned}$$

where we have used the relation (4.1.5).

(ii) By Proposition 4.1.1, for any $\alpha \in \{1, \cdots, n\}$,

$$\begin{aligned}
\frac{d}{dt}\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} &\le 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t) \\
&\le 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi(0).
\end{aligned}$$

We integrate the above relation over $[0, T_0]$ and use the assumption of K to get

$$\begin{aligned}
\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} &\le \|\hat{\boldsymbol{v}}_{\alpha 0}\|_{2,\infty} + 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0, \quad \text{for all} \ \ t \in [0, T_0] \\
&= 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0 \le \frac{\lambda_0}{8},
\end{aligned}$$

where we have used the relation (4.1.4). $\square$

Lemma 4.1.3. *Let* $(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t))$ *be a global solution of* (1.0.1) *with non-flocking initial data* $(\boldsymbol{x}_{\alpha i0}, \boldsymbol{v}_{\alpha i0})$. *If the coupling strength* K *satisfies*

$$
\begin{aligned}
K \le \min \Big\{ & \frac{\lambda_0}{16(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0}, \\
& \frac{\lambda_0^2}{(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)\Big(D(\boldsymbol{x}_0) + \sqrt{2M_2(0)}T_0\Big)} \Big\},
\end{aligned}
$$

then, we have

$$
\min_{\beta \neq \alpha, i, k} \Delta_{\beta k, \alpha i}(T_0) \ge 0.
$$

Proof. For the desired estimate, we claim that

$$
\min_{\beta \neq \alpha, i, k} \frac{d}{dt} \Delta_{\beta k, \alpha i}(T_0) \ge \lambda_0^2. \tag{4.1.19}
$$

Proof of claim (4.1.19): For $\forall\ t \in [0, T_0]$, $\alpha \neq \beta$, and i, k,

$$
\begin{aligned}
\big\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\big\| &= \Big\|\big(\boldsymbol{x}_{\beta k0} - \boldsymbol{x}_{\alpha i0}\big) + \int_0^t \big(\boldsymbol{v}_{\beta k}(s) - \boldsymbol{v}_{\alpha i}(s)\big) ds\Big\| \\
&\le \big\|\boldsymbol{x}_{\beta k0} - \boldsymbol{x}_{\alpha i0}\big\| + \sqrt{2M_2(0)}T_0 \\
&\le D(\boldsymbol{x}_0) + \sqrt{2M_2(0)}T_0,
\end{aligned}
$$

where we have used (4.1.12). By Lemma 4.1.2 and the assumption of K, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Delta_{\beta k, \alpha i}(t) \\
&\quad = \big(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\big) \cdot \big(\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\big) + \big(\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\big) \cdot \big(\dot{\boldsymbol{v}}_{\beta c}(t) - \dot{\boldsymbol{v}}_{\alpha c}(t)\big) \\
&\quad = \Big((\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\beta c}(t)) + (\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_\alpha^c(t)) + (\boldsymbol{v}_\alpha^c(t) - \boldsymbol{v}_{\alpha i}(t)) \Big) \cdot \big(\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\big) \\
&\qquad + \big(\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\big) \cdot \big(\dot{\boldsymbol{v}}_{\beta c}(t) - \dot{\boldsymbol{v}}_{\alpha c}(t)\big) \\
&\quad \ge \|\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\| \big(\|\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\alpha c}(t)\| - \|\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\beta c}(t)\| - \|\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\alpha c}(t)\| \big) \\
&\qquad - \|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| \|\dot{\boldsymbol{v}}_{\beta c}(t) - \dot{\boldsymbol{v}}_{\alpha c}(t)\| \\
&\quad \ge \frac{15}{8}\big(\frac{15}{8} - \frac{2}{8}\big)\lambda_0^2 - 2K\Big(D(\boldsymbol{x}_0) + \sqrt{2M_2(0)}T_0\Big)(1-\gamma_N)\sqrt{2M_2(0)}\psi(0) \\
&\quad \ge \lambda_0^2.
\end{aligned}
$$

We now integrate the relation (4.1.19) to obtain

$$\varDelta_{\beta k,\alpha i}(t) \geq \varDelta_{\beta k,\alpha i}(0) + \lambda_0^2 t, \quad t \in [0, T_0].$$

Then, the defining relation of T_0 in (4.1.5) implies that

$$\varDelta_{\beta k,\alpha i}(T_0) \geq \varDelta_{\beta k,\alpha i}(0) + \lambda_0^2 T_0 \geq 0.$$

We now take the minimum over α, β, i, and k to obtain the desired result. $\square$

Proof of Theorem 4.1.1

In this subsection, we provide the proof of Theorem 4.1.1. For simplicity, we set a normal vector in the direction of $\boldsymbol{v}_{\beta c}(T_0) - \boldsymbol{v}_{\alpha c}(T_0)$:

$$\boldsymbol{e}_{\beta\alpha} := \frac{\boldsymbol{v}_{\beta c}(T_0) - \boldsymbol{v}_{\alpha c}(T_0)}{\|\boldsymbol{v}_{\beta c}(T_0) - \boldsymbol{v}_{\alpha c}(T_0)\|}.$$

By Lemma 4.1.2, $e_{\beta\alpha}$ is well defined. Define

$$T_0^* := \sup \left\{ T \in [T_0, \infty) \,\middle|\, \min_{\alpha \neq \beta, i, k} \{ (\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)) \cdot \boldsymbol{e}_{\beta\alpha} \} > \frac{\lambda_0}{2}, \right. \\ \left. \text{for all } t \in [T_0, T) \right\}. \tag{4.1.20}$$

Lemma 4.1.4. *Let $(\boldsymbol{x}(t), \boldsymbol{v}(t))$ be a global solution of* (1.0.1) *with non-flocking initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$. If the coupling strength K satisfies*

$$K \leq \min \left\{ \frac{\lambda_0}{16(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0}, \right. \\ \left. \frac{\lambda_0^2}{(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)\Big(D(\boldsymbol{x}_0) + \sqrt{2M_2(0)}T_0\Big)} \right\},$$

then, we have

$$T_0^* > T_0 \quad \textit{and} \quad \psi_M(t) \leq \psi\Big(\frac{\lambda_0}{2}(t - T_0)\Big) \quad \textit{for} \quad t \in [T_0, T_0^*).$$

Proof. (i) It follows from Lemma 4.1.2 that

$$\begin{aligned}&\big(\boldsymbol{v}_{\beta k}(T_0)-\boldsymbol{v}_{\alpha i}(T_0)\big)\cdot \boldsymbol{e}_{\beta\alpha}\\ &\quad\geq \|\boldsymbol{v}_{\beta c}(T_0)-\boldsymbol{v}_{\alpha c}(T_0)\|-\|\hat{\boldsymbol{v}}_{\beta k}(T_0)\|-\|\hat{\boldsymbol{v}}_{\alpha i}(T_0)\|\\ &\quad\geq \left(\frac{15}{8}-\frac{1}{8}-\frac{1}{8}\right)\lambda_0>\frac{\lambda_0}{2}.\end{aligned}$$

Thus, there exists $\delta_0>0$ such that

$$\Big(\boldsymbol{v}_{\beta k}(t)-\boldsymbol{v}_{\alpha i}(t)\Big)\cdot \boldsymbol{e}_{\beta\alpha}>\frac{\lambda_0}{2}\quad \text{for all } t\in[T_0,T_0+\delta_0].$$

Hence, we have

$$T_0^*>T_0.$$

(ii) We use Lemma 4.1.2 to obtain

$$\begin{aligned}\|\boldsymbol{x}_{\beta k}(t)-\boldsymbol{x}_{\alpha i}(t)\| &\geq \big(\boldsymbol{x}_{\beta k}(t)-\boldsymbol{x}_{\alpha i}(t)\big)\cdot \boldsymbol{e}_{\beta\alpha}\\ &= \big(\boldsymbol{x}_{\beta k}(T_0)-\boldsymbol{x}_{\alpha i}(T_0)\big)\cdot \boldsymbol{e}_{\beta\alpha}\\ &\quad+\int_{T_0}^{t}\big(\boldsymbol{v}_{\beta k}(s)-\boldsymbol{v}_{\alpha i}(s)\big)\cdot \boldsymbol{e}_{\beta\alpha}ds\\ &\geq \frac{\lambda_0}{2}(t-T_0),\quad t\in[T_0,T_0^*).\end{aligned}$$

Thus, by the non-increasing property of $\psi(t)$, we have

$$\psi_M(t)\leq \psi\big(\frac{\lambda_0}{2}(t-T_0)\big),\ \text{for all}\ \ t\in[T_0,T_0^*).$$

Hence the conclusion holds. $\square$

Lemma 4.1.5. *Let $(\boldsymbol{x}(t),\boldsymbol{v}(t))$ be a global solution of (1.0.1) with non-flocking initial data $(\boldsymbol{x}_0,\boldsymbol{v}_0)$. If the coupling strength K satisfies*

$$\begin{aligned}K\leq \min\Bigg\{&\frac{\lambda_0}{16(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0},\\ &\frac{\lambda_0^2}{(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)\Big(D(\boldsymbol{x}_0)+\sqrt{2M_2(0)}T_0\Big)}\Bigg\},\end{aligned}$$

then, the following estimates hold: for $t \in [T_0, T_0^*)$,

$$(i) \quad \|\boldsymbol{v}_{\alpha c}(t) - \boldsymbol{v}_{\alpha c}(T_0)\| \le \frac{2K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)},$$

$$(ii) \quad \|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} \le \|\hat{\boldsymbol{v}}_\alpha(T_0)\|_{2,\infty} + \frac{4K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)}.$$

Proof. (i) By Proposition 4.1.1 and Lemma 4.1.4,

$$\left\|\frac{d\boldsymbol{v}_{\alpha c}(t)}{dt}\right\| \le K(1-\gamma_N)\sqrt{2M_2(0)}\psi\Big(\frac{\lambda_0}{2}(t-T_0)\Big).$$

We use the above relation to obtain

$$\|\boldsymbol{v}_{\alpha c}(t) - \boldsymbol{v}_{\alpha c}(T_0)\| \le \frac{2K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)}, \text{ for all } \ t \in [T_0, T_0^*).$$

(ii) By Proposition 4.1.1 and Lemma 4.1.4,

$$\frac{d\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}}{dt} \le 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi\Big(\frac{\lambda_0}{2}(t-T_0)\Big).$$

We use the above relation to obtain

$$\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} \le \|\hat{\boldsymbol{v}}_\alpha(T_0)\|_{2,\infty} + \frac{4K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)}, \ \forall t \in [T_0, T^*).$$

□

We are now ready to provide the proof of Theorem 4.1.1 as follows.

Proof. (Proof of Theorem 4.1.1) Let $(\boldsymbol{x}(t), \boldsymbol{v}(t))$ be a global solution of (1.0.1) with non-flocking initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$. If the coupling strength K satisfies

$$K < K_0,$$

then, we claim that

$$\begin{aligned} &\min_{\beta \ne \alpha, i, k} \Big(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\Big) \cdot \boldsymbol{e}_{\beta\alpha} > \frac{\lambda_0}{2}, \\ &\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| \ge \frac{\lambda_0}{2}(t-T_0), \quad t \in (T_0, \infty). \end{aligned} \tag{4.1.21}$$

For the proof of the above claim, we consider two cases:

$$\text{either} \quad T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) > 0, \quad \text{or} \quad T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) = 0.$$

• Case A: Suppose that

$$T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) > 0.$$

Then, it follows from the arguments in Lemma 4.1.4 that

$$T_0^* > T_0.$$

Suppose that

$$T_0^* < \infty.$$

Then, by definition in (4.1.20), there exist $\beta \neq \alpha, i, k$ such that

$$\big(\boldsymbol{v}_{\beta k}(T_0^*) - \boldsymbol{v}_{\alpha i}(T_0^*)\big) \cdot \boldsymbol{e}_{\beta\alpha} = \frac{\lambda_0}{2}. \tag{4.1.22}$$

On the other hand, it follows from Lemma 4.1.5 that

$$\begin{aligned}
&\big(\boldsymbol{v}_{\beta k}(T_0^*) - \boldsymbol{v}_{\alpha i}(T_0^*)\big) \cdot \boldsymbol{e}_{\beta\alpha} \\
&\quad = \Big[\big(\boldsymbol{v}_{\beta c}(T_0) - \boldsymbol{v}_{\alpha c}(T_0)\big) + \big(\boldsymbol{v}_{\beta k}(T_0^*) - \boldsymbol{v}_{\beta c}(T_0^*)\big) + \big(\boldsymbol{v}_{\beta c}(T_0^*) - \boldsymbol{v}_{\beta c}(T_0)\big) \\
&\qquad - \big(\boldsymbol{v}_{\alpha i}(T_0^*) - \boldsymbol{v}_{\alpha c}(T_0^*)\big) - \big(\boldsymbol{v}_{\alpha c}(T_0^*) - \boldsymbol{v}_{\alpha c}(T_0)\big)\Big] \cdot \boldsymbol{e}_{\beta\alpha} \\
&\quad \geq \|\boldsymbol{v}_{\beta c}(T_0) - \boldsymbol{v}_{\alpha c}(T_0)\| - \|\boldsymbol{v}_{\beta c}(T_0^*) - \boldsymbol{v}_{\beta c}(T_0)\| - \|\boldsymbol{v}_{\alpha c}(T_0^*) - \boldsymbol{v}_{\alpha c}(T_0)\| \\
&\qquad - \|\hat{\boldsymbol{v}}_{\beta k}(T_0^*)\| - \|\hat{\boldsymbol{v}}_{\alpha i}(T_0^*)\| \\
&\quad \geq \|\boldsymbol{v}_{\beta c}(T_0) - \boldsymbol{v}_{\alpha c}(T_0)\| - \|\hat{\boldsymbol{v}}_{\beta k}(T_0^*)\| - \|\hat{\boldsymbol{v}}_{\alpha i}(T_0^*)\| \\
&\qquad - \frac{12K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)} \\
&\quad \geq \frac{13\lambda_0}{8} - \frac{12K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)} \\
&\quad > \frac{13\lambda_0}{8} - \lambda_0 > \frac{\lambda_0}{2},
\end{aligned}$$

where we have used the assumption of K. This contradicts the inequality (4.1.22). Thus, we have $T_0^* = \infty$ and the desired estimate (4.1.21).

• Case B: Suppose that

$$T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) = 0.$$

In this case, recall that

$$K_0 = \frac{\lambda_0^2}{6(1-\gamma_N)\sqrt{2M_2(0)}\|\psi\|_{L^1(\mathbb{R}_+)}}.$$

Then, for $K < K_0$, we use arguments similar to those in Case A to obtain

$$\begin{aligned}
&\|\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\| > \lambda_0, \quad \text{for all} \quad t \geq 0, \\
&\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| > \lambda_0 t, \quad \text{for all} \quad t \geq 0.
\end{aligned}$$

Finally, it follows from Case A and Case B that the proof of Theorem 4.1.1 is complete. □

4.2 Emergence of multi-cluster flocking

In this section, we present the emergence of multi-cluster flocking for the C-S model (1.0.1). In the previous section, we divided the particles into n sub-ensembles, $\mathcal{G}_1, \cdots, \mathcal{G}_n$, according to their initial velocities, and we showed that for a small coupling strength $K < K_0$, any two different particles in different groups do not flock. Thus, it is natural to ask whether two different particles from the same group will flock in a small-coupling-strength regime. In what follows, we will focus on this question by allowing the initial velocities of different particles in the same group to be slightly different.

Consider a C-S flocking system with n sub-ensembles $\mathcal{G}_\alpha,\ \alpha = 1, 2, \cdots, n$:

$$\begin{aligned}
\dot{\boldsymbol{x}}_{\alpha i}(t) &= \boldsymbol{v}_{\alpha i}(t), \quad t \geq 0, \quad \alpha = 1, 2, \cdots, n, \quad i = 1, \cdots, N_\alpha, \\
\dot{\boldsymbol{v}}_{\alpha i}(t) &= \frac{K}{N}\sum_{k=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\big(\boldsymbol{v}_{\alpha k}(t) - \boldsymbol{v}_{\alpha i}(t)\big) \\
&\quad + \frac{K}{N}\sum_{\beta \neq \alpha}\sum_{k=1}^{N_\beta} \psi(\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\big(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\big).
\end{aligned} \tag{4.2.1}$$

4.2.1 Framework and main result

As in Section 4.1, we define parameters λ_0 and r_0 related to the separation of each sub-ensemble:

$$\lambda_0 := \frac{1}{2}\min_{\beta\neq\alpha}\|\boldsymbol{v}_{\beta c}(0)-\boldsymbol{v}_{\alpha c}(0)\|, \quad \text{and}$$
$$\int_{r_0}^{\infty}\psi(s)ds := \frac{\gamma_N\lambda_0}{8(1-\gamma_N)\sqrt{2M_2(0)}}\min_\alpha\int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{\infty}\psi(2x)dx.$$

Next, we state our framework $(\mathcal{C}_1)$ for multi-cluster flocking as follows. Note that we allow $\|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty}\neq 0$ for $\alpha=1,\cdots,n$.

- $(\mathcal{C}_1 1)$ (restriction on initial configurations): The initial configuration is well separated in the sense that

$$\max_\alpha\|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty}<\frac{\lambda_0}{4}, \qquad \min_{\beta\neq\alpha,i,k}\big(\boldsymbol{x}_{\beta k}(0)-\boldsymbol{x}_{\alpha i}(0)\big)\cdot\frac{\boldsymbol{v}_{\beta c}(0)-\boldsymbol{v}_{\alpha c}(0)}{\|\boldsymbol{v}_{\beta c}(0)-\boldsymbol{v}_{\alpha c}(0)\|}\geq r_0.$$

- $(\mathcal{C}_1 2)$ (restriction on coupling strengths): The coupling strength takes an intermediate value in the sense that

$$K_1<K<K_2,$$

where
$$K_1=\max_\alpha\Big\{\frac{2\|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty}}{\gamma_N\int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{\infty}\psi(2x)dx}\Big\},$$
$$K_2=\min_\alpha\Big\{\frac{2\lambda_0}{3\gamma_N\int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{\infty}\psi(2x)dx}\Big\}.$$

Theorem 4.2.1. *Suppose that the framework $(\mathcal{C}_1)$ holds, and let $(\boldsymbol{x}_{\alpha i}(t),\boldsymbol{v}_{\alpha i}(t))$ be a solution of the system (4.2.1) with initial configuration $(\boldsymbol{x}_{\alpha i0},\boldsymbol{v}_{\alpha i0})$. Then, we have the following estimates:*

(i) $\min_{\beta\neq\alpha,i,k}\|\boldsymbol{x}_{\beta k}(t)-\boldsymbol{x}_{\alpha i}(t)\|>r_0+\lambda_0 t,$

(ii) there exists x_α^∞ *such that* $\|\hat{\boldsymbol{x}}_\alpha(t)\|_\infty\leq x_\alpha^\infty,$

(iii) $\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}\leq C_\alpha\max\{e^{-\frac{K\psi(2x_\alpha^\infty)t}{2}},\psi\Big(r_0+\frac{\lambda_0 t}{2}\Big)\},$ *as* $t\to+\infty.$

In other words, multi-cluster flocking emerges.

Remark 4.2.1. *1. In Section 4.1, from the classification, we know that* $\|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty} = 0$ *for any* $\alpha \in \{1, \cdots, n\}$. *Thus, the initial assumption of velocities in the above theorem is satisfied naturally.*

2. Note that our assumption of the initial space difference r_0 *is crucial. We cannot predict the phenomenon of multi-cluster flocking without the assumption of distance since the communication weight* ψ *highly depend on the distance. This phenomena is well described in [42] with an explicit example.*

3. As in Remark 4.1.1, we compare the results of Theorem 4.2.1 with the results on bi-cluster flocking formation presented in Chapter 3 as follows. In Theorem 4.2.1, we discuss multi-cluster flocking, which can include the bi-cluster flocking result presented in Chapter 3 with similar requirements of $\lambda_0, \|\hat{\boldsymbol{v}}_\alpha(0)\|$, *and* K_1. *Moreover, we can get a better condition for* K_2, *as explained in Remark 4.1.1. Because of the term* $\|\hat{\boldsymbol{v}}_\alpha(0)\|$, *Chapter 3 cannot include the* $\|\hat{\boldsymbol{v}}_\alpha(0)\| = 0$ *case, which seems to be the most reasonable case.*

4.2.2 Dynamics of local averages and fluctuations

In this subsection, we study the time evolution of the local averages and fluctuations.

Proposition 4.2.1. *Let* $\big(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t)\big), \alpha = 1, \cdots, n$ *be the solution of the system (4.2.1). Then, we have*

$$(i)\ \left\|\frac{d\boldsymbol{v}_{\alpha c}(t)}{dt}\right\| \le K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t), \quad \frac{d\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}}{dt} \le \|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty},$$

$$(ii)\ \frac{d\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}}{dt} \le -K\gamma_N\psi\big(2\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}\big)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} + 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t),$$

where $\psi_M(t) := \max_{\beta\neq\alpha,i,k} \psi(\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)$.

Proof. (i) We can get the estimates from arguments similar to those in Proposition 4.1.1. The first estimate is obvious. By the analytic property of $\boldsymbol{x}_{\alpha i}(t)$

and $\boldsymbol{v}_{\alpha i}(t)$, we may assume and choose $i \in \{1, \cdots, N_\alpha\}$ with

$$\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty} = \|\hat{\boldsymbol{x}}_{\alpha i}(t)\|,$$

when we consider differential equations. By Lemma 4.1.1 and the Cauchy inequality, we have

$$\begin{aligned}
\frac{d\|\hat{\boldsymbol{x}}_{\alpha i}(t)\|^2}{dt} &= 2\langle \hat{\boldsymbol{x}}_{\alpha i}(t), \hat{\boldsymbol{v}}_{\alpha i}(t)\rangle \le 2\|\hat{\boldsymbol{x}}_{\alpha i}(t)\|\|\hat{\boldsymbol{v}}_{\alpha i}(t)\| \\
&\le 2\|\hat{\boldsymbol{x}}_{\alpha i}(t)\|_{2,\infty}\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}.
\end{aligned}$$

Thus, we have the second estimate.

(ii) In the same way, we can assume and choose $i \in \{1, \cdots, N_\alpha\}$ with

$$\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} = \|\hat{\boldsymbol{v}}_{\alpha i}(t)\|.$$

We use Lemma 4.1.1 and multiply $(4.2.1)_2$ by $2\hat{\boldsymbol{v}}_{\alpha i}(t)$ to get

$$\begin{aligned}
&\frac{d\|\hat{\boldsymbol{v}}_{\alpha i}(t)\|^2}{dt} \\
&= \frac{2K}{N}\sum_{k=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\langle \hat{\boldsymbol{v}}_{\alpha i}(t), \hat{\boldsymbol{v}}_{\alpha k}(t) - \hat{\boldsymbol{v}}_{\alpha i}(t)\rangle \\
&\quad + \frac{2K}{N}\sum_{\beta\neq\alpha}\sum_{k=1}^{N_\beta} \psi(\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\|)\langle \hat{\boldsymbol{v}}_{\alpha i}(t), \boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\rangle \\
&\quad - 2\langle \hat{\boldsymbol{v}}_{\alpha i}(t), \dot{\boldsymbol{v}}_{\alpha c}(t)\rangle \\
&\le -2K\gamma_N\psi\big(2\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}\big)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}^2 \\
&\quad + \ 4K(1-\gamma_N)\sqrt{2M_2(0)}\psi_M(t)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}.
\end{aligned}$$

Thus, we have the desired result. □

4.2.3 Proof on multi-cluster flocking

In this subsection, we prove Theorem 4.2.1, the emergence of multi-cluster flocking configurations for C-S dynamics.

Definition 4.2.1. *Define*

$$T_1^* := \sup\left\{T > 0 \middle| \min_{\beta\neq\alpha,i,k} \big(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\big)\cdot\overline{\boldsymbol{e}}_{\beta\alpha} > \lambda_0,\ t\in[0,T)\right\}, \tag{4.2.2}$$

where $\overline{\boldsymbol{e}}_{\beta\alpha}$ *is the unit vector in the direction of* $\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)$*:*

$$\overline{\boldsymbol{e}}_{\beta\alpha} := \frac{\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)}{\|\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\|}.$$

Lemma 4.2.1. *Let* $\big(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t)\big), \alpha = 1, \cdots, n$ *be the solution of the system (4.2.1) with initial data satisfying* $(\mathcal{C}_1 1)$*. Then, we have*

$$T_1^* > 0 \quad \textit{and} \quad \psi_M(t) \le \psi(r_0 + \lambda_0 t), \quad t\in[0,T_1^*).$$

Proof. (i) By the assumption of $(\mathcal{C}_1 1)$ for the initial data, we have

$$\big(\boldsymbol{v}_{\beta k}(0) - \boldsymbol{v}_{\alpha i}(0)\big)\cdot\overline{\boldsymbol{e}}_{\beta\alpha} \ge \|\boldsymbol{v}_{\beta k}(0) - \boldsymbol{v}_{\alpha i}(0)\| - \|\hat{\boldsymbol{v}}_{\beta k}(0)\| - \|\hat{\boldsymbol{v}}_{\alpha i}(0)\| \ge \frac{3}{2}\lambda_0 > \lambda_0.$$

Thus, we can conclude that $T_1^* > 0$.

(ii) By the initial assumptions, for any $\beta \neq \alpha$, $1 \le i \le N_\alpha$ and $1 \le k \le N_\beta$,

$$\begin{aligned}
\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| &\ge (\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t))\cdot\overline{\boldsymbol{e}}_{\beta\alpha} \\
&= (\boldsymbol{x}_{\beta k}(0) - \boldsymbol{x}_{\alpha i}(0))\cdot\overline{\boldsymbol{e}}_{\beta\alpha} + \int_0^t (\boldsymbol{v}_{\beta k}(s) - \boldsymbol{v}_{\alpha i}(s))\cdot\overline{\boldsymbol{e}}_{\beta\alpha} ds \\
&> r_0 + \lambda_0 t, \quad t\in[0,T_1^*).
\end{aligned}$$

Thus, by the non-increasing property of $\psi(t)$, we have the conclusion. □

Lemma 4.2.2. *Let* $\big(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t)\big), \alpha = 1, \cdots, n$ *be the solution of the system (4.2.1) with initial data satisfying* $(\mathcal{C}_1 1)$*. Then, we have*

$$(i)\ \|\boldsymbol{v}_{\alpha c}(t) - \boldsymbol{v}_{\alpha c}(0)\| \le \frac{K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\int_{r_0}^\infty \psi(s)ds, \quad t\in[0,T_1^*),$$

$$(ii)\ \|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} \le \|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty} + \frac{2K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\int_{r_0}^\infty \psi(s)ds.$$

The proof of Lemma 4.2.2 is similar to Lemma 4.1.4. We are now ready to prove Theorem 4.2.1.

Proof. (Proof of Theorem 4.2.1) Suppose that the framework $(\mathcal{C}_1)$ holds, and let $(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t))$ be a solution of the system (4.2.1) with initial configuration $(\boldsymbol{x}_{\alpha i0}, \boldsymbol{v}_{\alpha i0})$. Then, we claim that $T_1^* = +\infty$ and

$$
\begin{aligned}
&(i) \quad \min_{\beta\neq\alpha,i,k} \|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| > r_0 + \lambda_0 t,\\
&(ii) \quad \text{there exists } x_\alpha^\infty \text{ such that } \|\hat{\boldsymbol{x}}_\alpha(t)\|_\infty \le x_\alpha^\infty,\\
&(iii) \quad \|\hat{\boldsymbol{v}}_\alpha(t)\|_\infty \le C_\alpha \max\Big\{e^{-\frac{K\psi(2x_\alpha^\infty)t}{2}}, \psi\Big(r_0 + \frac{\lambda_0 t}{2}\Big)\Big\}, \text{ for some } C_\alpha > 0.
\end{aligned}
$$

If all of these are true, then Theorem 4.2.1 holds.

• (Estimate of (i)): It follows from Lemma 4.2.1 that

$$T_1^* > 0.$$

We now fix $\alpha \in \{1, \cdots, n\}$ and suppose that

$$T_1^* < +\infty.$$

Then, by definition in (4.2.2), there exists $\beta_0 \neq \alpha_0 \in \{1, \cdots, n\}$, $1 \le i_0 \le N_\alpha$ and $1 \le k_0 \le N_\beta$, such that

$$(\boldsymbol{v}_{\beta_0 k_0}(T_1^*) - \boldsymbol{v}_{\alpha_0 i_0}(T_1^*)) \cdot \overline{\boldsymbol{e}}_{\alpha\beta} = \lambda_0. \tag{4.2.3}$$

On the other hand, for any $\beta \neq \alpha$, $1 \le i \le N_\alpha$ and $1 \le k \le N_\beta$, we have

$$
\begin{aligned}
&\big(\boldsymbol{v}_{\beta k}(t) - \boldsymbol{v}_{\alpha i}(t)\big) \cdot \overline{\boldsymbol{e}}_{\alpha\beta}\\
&\quad \ge \|\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\| - \|\hat{\boldsymbol{v}}_{\alpha i}(t)\| - \|\hat{\boldsymbol{v}}_{\beta k}(t)\| - \|\boldsymbol{v}_{\beta c}(t) - \boldsymbol{v}_{\beta c}(0)\|\\
&\qquad - \|\boldsymbol{v}_{\alpha c}(t) - \boldsymbol{v}_{\alpha c}(0)\|\\
&\quad \ge \|\boldsymbol{v}_{\beta c}(0) - \boldsymbol{v}_{\alpha c}(0)\| - \|\hat{\boldsymbol{v}}_{\alpha i}(0)\| - \|\hat{\boldsymbol{v}}_{\beta k}(0)\|\\
&\qquad - \frac{6K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)} \int_{r_0}^\infty \psi(s)ds\\
&\quad \ge \frac{3\lambda_0}{2} - \frac{6K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)} \int_{r_0}^\infty \psi(s)ds\\
&\quad = \frac{3\lambda_0}{2} - \frac{3K}{4}\gamma_N \int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_\infty}^\infty \psi(2x)dx\\
&\quad > \lambda_0, \quad t \in [0, T_1^*],
\end{aligned}
$$

where we have used Lemma 4.2.2 and the upper bound of K. In particular, we have $(\boldsymbol{v}_{\beta_0 k_0}(T_1^*) - \boldsymbol{v}_{\alpha_0 i_0}(T_1^*)) \cdot \overline{\boldsymbol{e}}_{\alpha\beta} = \lambda_0$; this contradicts the relation (4.2.3). Thus, we have $T_1^* = +\infty$. Then, we apply the same arguments as those in Lemma 4.1.4 to derive the estimate

$$\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| > r_0 + \lambda_0 t, \quad t \geq 0.$$

(ii) It follows from Proposition 4.2.1 that

$$\begin{aligned} \frac{d\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}}{dt} &\leq \|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}, \\ \frac{d\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}}{dt} &\leq -K\gamma_N \psi\big(2\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}\big)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} \\ &\quad + 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi\Big(r_0 + \lambda_0 t\Big). \end{aligned} \tag{4.2.4}$$

We now define a Lyapunov functional $\mathcal{L}_\alpha(t)$:

$$\mathcal{L}_\alpha(t) := \|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} + K\gamma_N \int_0^{\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}} \psi(2x)dx. \tag{4.2.5}$$

Then, we use (4.2.4) and (4.2.5) to obtain

$$\begin{aligned} \frac{d\mathcal{L}_\alpha(t)}{dt} &= \frac{d\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}}{dt} + K\gamma_N \psi\big(2\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}\big)\frac{d\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}}{dt} \\ &\leq 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi(r_0 + \lambda_0 t). \end{aligned}$$

We integrate the above relation to obtain

$$\begin{aligned} &\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} + K\gamma_N \int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}} \psi(2x)dx \\ &\qquad \leq \|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty} + \frac{2K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)} \int_{r_0}^{+\infty} \psi(s)ds. \end{aligned}$$

In particular, we have

$$K\gamma_N \int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}} \psi(2x)dx \leq \|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty} + \frac{2K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)} \int_{r_0}^{+\infty} \psi(s)ds.$$

Then, by the assumption of K and choice of r_0, we have

$$\|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty} + \frac{2K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\int_{r_0}^{+\infty}\psi(s)ds < K\gamma_N\int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{\infty}\psi(2x)dx.$$

Thus, there exists x_α^∞ such that

$$\|\hat{\boldsymbol{v}}_\alpha(0)\|_{2,\infty} + \frac{2K}{\lambda_0}(1-\gamma_N)\sqrt{2M_2(0)}\int_{r_0}^{+\infty}\psi(s)ds = K\gamma_N\int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{x_\alpha^\infty}\psi(2x)dx,$$

or equivalently,

$$\int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty}}\psi(2x)dx < \int_{\|\hat{\boldsymbol{x}}_\alpha(0)\|_{2,\infty}}^{x_\alpha^\infty}\psi(2x)dx, \quad t\in[0,+\infty).$$

Therefore, for all $t\in[0,+\infty)$, we have

$$\|\hat{\boldsymbol{x}}_\alpha(t)\|_{2,\infty} \le x_\alpha^\infty. \tag{4.2.6}$$

(iii) By the inequality (4.2.6), we know that, for $t\in[0,+\infty)$,

$$\frac{d\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}}{dt} \le -K\gamma_N\psi\big(2x_\alpha^\infty\big)\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} + 2K(1-\gamma_N)\sqrt{2M_2(0)}\psi(r_0+\lambda_0 t).$$

We integrate the above inequality directly to yield

$$\begin{aligned}\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty} &\le \|\hat{\boldsymbol{v}}_{\alpha 0}\|_{2,\infty}e^{-\beta_0 t} + 2K(1-\gamma_N)\frac{\sqrt{2M_2(0)}}{\beta_0}\psi(r_0)e^{-\frac{\beta_0}{2}t}\\ &\quad + 2K(1-\gamma_N)\frac{\sqrt{2M_2(0)}}{\beta_0}\psi\Big(r_0+\frac{\lambda_0}{2}t\Big)\\ &\le C_\alpha\max\Big\{e^{-\frac{K\psi(2x_\alpha^\infty)t}{2}},\psi\Big(r_0+\frac{\lambda_0 t}{2}\Big)\Big\},\ \text{for some } C_\alpha>0,\end{aligned}$$

where $\beta_0 := K\gamma_N\psi(2x_\alpha^\infty)$. □

4.3 Numerical simulations

In this section, we present the results of several numerical examples and compare them with those of the main theorems presented in Sections 4.1 and 4.2. For numerical implementation, we used the fourth-order Runge-Kutta method with the time step depend on K,

4.3.1 Non-flocking configurations

First, we see that mono-cluster flocking does not emerge under the condition of $K < K_0$. Figure 4.1 shows the randomly chosen initial configurations. Throughout this subsection, we will use this configuration for all simulations with any K in order to exclude the effects of the initial data. We checked not only maximal velocity difference, but also the variance of velocities:

$$\mathcal{V}(t) = \Big(\sum_{\substack{1\le\alpha\le n \\ 1\le i\le N_\alpha}} \|\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_c(t)\|^2 \Big)^{1/2}.$$

This functions in Figure 4.2 show that the critical coupling strength is in the range of 1 to 10^2, where these values are calculated after a long time, at $t = 100K$. We can see that there is a large difference in the variance, between 1 to 10^2. Furthermore, this value is clearly between the following values:

$$K_0 = 0.0020, \ K^0 = 4855.$$

4.3.2 Emergence of multi-cluster flocking

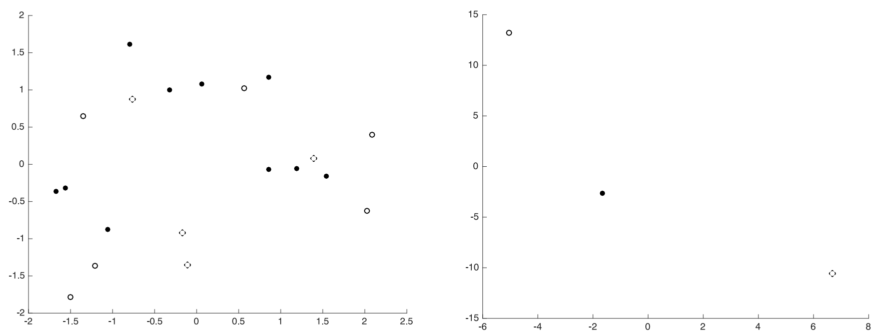
Here, we present simulation results for the conditions of Theorem 4.2.1. Local multi-cluster flocking should occur when $K_1 < K < K_2$. Figure 4.3 shows the precisely chosen initial configurations. Note that the initial data satisfy the condition of Theorem 4.2.1. Now, we let the velocities be different even in the same group in order to observe more than three clusters. To measure the local flocking, we will also use the variance of velocities. For each α, let

$$\mathcal{V}_\alpha(t) = \Big(\sum_{\substack{1\le\alpha\le R \\ 1\le i\le N_\alpha}} \|\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\alpha c}(t)\|^2 \Big)^{1/2}.$$

In this case, the function $\|\hat{\boldsymbol{v}}_\alpha(t)\|_{2,\infty}$ shows similar behavior since our initial configuration is close to multi-cluster flocking. Figure 4.4 shows that these factors are calculated after a long time, at $t = 100K$. We can see that mono-cluster flocking vanishes near 10 to 10^2, and multi-cluster flocking vanishes near 1. This large difference in variance occurs outside the range of the following values:

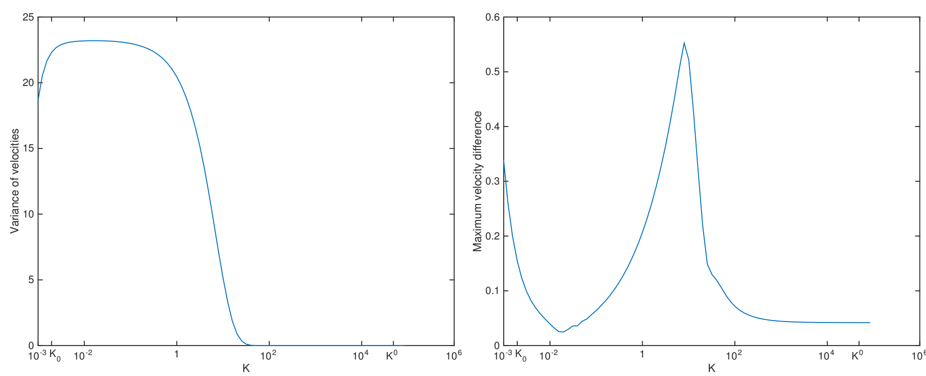
$$K_1 = 2.9152, \quad K_2 = 5.0670,$$

which confirms Theorem 4.2.1.



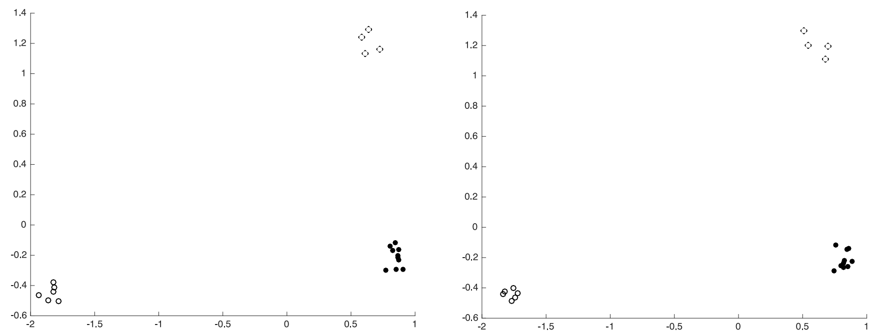
(a) Initial positions of particles (b) Initial velocities of particles

Figure 4.1: Initial configurations for the non-flocking phenomenon (markers for different groups)



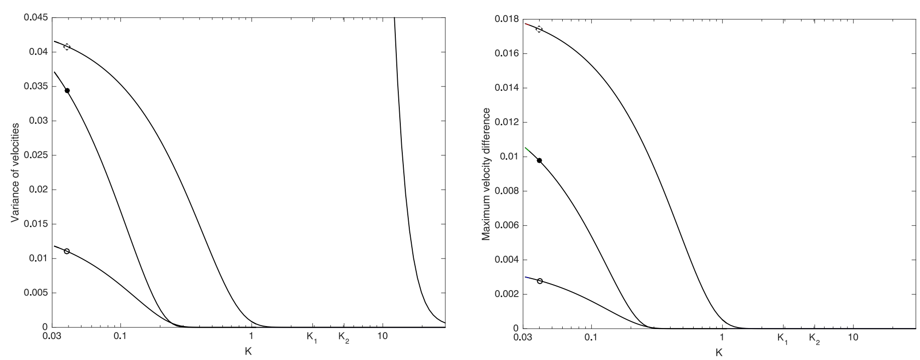
(a) the variance of velocities (b) the maximal difference of velocities

Figure 4.2: Flocking and non-flocking phenomena for different K at time $t = 100K$



(a) Initial positions of particles (b) Initial velocities of particles

Figure 4.3: Initial configurations for the multi-cluster flocking phenomenon (markers for different groups)



(a) the variance of velocities (b) the maximal difference of velocities

Figure 4.4: Local flocking and non-flocking phenomena for different K at time $t = 100K$ (markers for different groups, non-marked for the global variance)

Chapter 5

Existence of bi-cluster flocking with unit speed constraint

In this chapter, we study a unit-speed constrainted flocking model introduced in Section 2.4. The contents of this chapter correspond with the ones of Chapter 3. First, we suggest a set of configurations tend to bi-cluster flocking mimicking the situation of separating two particles. Second, we use differential inequalities and Lyapunov functionals to prove it. This chapter is based on the joint work in [19].

5.1 A generalized two-dimensional J-K model

In this section, we consider a generalized two-dimensional J-K model with a metric-dependent communication weight ψ, compared to the original J-K model where the interaction between agents is all-to-all. We show that the bi-cluster flocking phenomena occurs for an admissible class of initial configurations. To this end, we first reformulate the J-K system in (2.4.13) into an expanded system that is suitable for bi-cluster flocking analysis. Note that the bi-cluster configuration cannot be an equilibrium for the J-K model, but it can be achieved asymptotically. To measure the formation of bi-cluster flocking, we introduce new functionals that simplify the analysis. Imagine that the whole ensemble can be split into two flocking clusters so that each cluster moves with the same velocity in two different directions; thus, the metric distance between the two clusters approaches infinity as time goes to infinity. In contrast, the velocity diameter of each cluster tends to zero. After introducing several functionals, we derive a SDDI, and then, under an admissible class of initial configurations, we show that bi-cluster flocking can occur using the continuity argument.

5.1.1 A reformulation of the J-K model

Assume that the system is composed of two groups $\mathcal{G}_1$ and $\mathcal{G}_2$ with $|\mathcal{G}_1| = N_1$ and $|\mathcal{G}_2| = N_2$. We relabel the position and heading of each agent with $(\boldsymbol{x}_{1i}, \theta_{1i})$, $i = 1, 2, \cdots, N_1$ and $(\boldsymbol{x}_{2j}, \theta_{2j})$, $j = 1, 2, \cdots, N_2$. Then, the system can be reformulated as an augmented system:

$$
\begin{aligned}
\frac{d\boldsymbol{x}_{1i}}{dt} &= (\cos\theta_{1i}, \sin\theta_{1i}), \quad \frac{d\boldsymbol{x}_{2j}}{dt} = (\cos\theta_{2j}, \sin\theta_{2j}), \quad t > 0, \\
\frac{d\theta_{1i}}{dt} &= \frac{K}{N}\sum_{k=1}^{N_1} \psi(|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}|)\sin(\theta_{1k} - \theta_{1i}) \\
&\quad + \frac{K}{N}\sum_{k=1}^{N_2} \psi(|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}|)\sin(\theta_{2k} - \theta_{1i}), \\
\frac{d\theta_{2j}}{dt} &= \frac{K}{N}\sum_{k=1}^{N_2} \psi(|\boldsymbol{x}_{2k} - \boldsymbol{x}_{2j}|)\sin(\theta_{2k} - \theta_{2j}) \\
&\quad + \frac{K}{N}\sum_{k=1}^{N_1} \psi(|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}|)\sin(\theta_{1k} - \theta_{2j}),
\end{aligned}
\tag{5.1.1}
$$

where $i = 1, 2, \cdots, N_1$, $j = 1, 2, \cdots, N_2$, and $1 \le N_1 \le N_2$.

Next, we define new quantities to measure the local averages and local fluctuations of the state variables $(\boldsymbol{x}_{\alpha i}, \theta_{\alpha i})$ in each group:

$$
\begin{aligned}
\boldsymbol{x}_{1c} &:= \frac{1}{N_1}\sum_{k=1}^{N_1} \boldsymbol{x}_{1k}, \quad \boldsymbol{x}_{2c} := \frac{1}{N_2}\sum_{k=1}^{N_2} \boldsymbol{x}_{2k}, \\
\theta_{1c} &:= \frac{1}{N_1}\sum_{k=1}^{N_1} \theta_{1k}, \quad \theta_{2c} := \frac{1}{N_2}\sum_{k=1}^{N_2} \theta_{2k},
\end{aligned}
$$

$$\hat{\boldsymbol{x}}_{\alpha i} \;:=\; \boldsymbol{x}_{\alpha i} - \boldsymbol{x}_{\alpha c}, \quad \hat{\theta}_{\alpha i} = \theta_{\alpha i} - \theta_{\alpha c}, \quad \alpha = 1, 2.$$

One can expect that if the initial condition is close to the central position and central velocity of each group and the groups are sufficiently separated, then bi-cluster flocking occurs.

We introduce several functionals to manage crowding in each flocking group. For each flocking group, the spatial and angular fluctuations of group members are summed:

$$\begin{aligned}
\mathcal{V}_\alpha(\boldsymbol{x}) &:= \sqrt{\sum_{k=1}^{N_\alpha} |\hat{\boldsymbol{x}}_{\alpha k}|^2}, \quad \mathcal{V}_\alpha(\Theta) := \sqrt{\sum_{k=1}^{N_\alpha} \hat{\theta}_{\alpha k}^2}, \quad \alpha = 1, 2 \\
\mathcal{V}(\boldsymbol{x}) &:= \mathcal{V}_1(\boldsymbol{x}) + \mathcal{V}_2(\boldsymbol{x}), \quad \mathcal{V}(\Theta) := \mathcal{V}_1(\Theta) + \mathcal{V}_2(\Theta), \\
D_\alpha(\Theta) &:= \max_{i,j} |\theta_{\alpha i} - \theta_{\alpha j}|, \quad D(\Theta) := D_1(\Theta) + D_2(\Theta).
\end{aligned}$$

Then, it is easy to verify that

$$\frac{1}{\sqrt{N}}\mathcal{V}(\Theta) \le D(\Theta) \le \sqrt{2}\mathcal{V}(\Theta). \tag{5.1.2}$$

5.1.2 A class of admissible initial data

Suppose there exists $r_0 > 0$ such that

$$\min_{i,j}(x_{1i}^1(0) - x_{2j}^1(0)) =: r_0 > 0. \tag{5.1.3}$$

By the continuity of the solution, there exists $T > 0$ such that

$$|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)| \ge t + \frac{r_0}{2} \quad \text{for } t \in [0, T) \quad \text{and} \quad \forall i, j \in \{1, \cdots, N\}. \tag{5.1.4}$$

Fix the following set $\mathcal{T}$ and its supremum T^*:

$$\mathcal{T} := \{T \in (0, \infty] \;:\; (5.1.4) \text{ holds for } t \in [0, T)\}, \quad T^* := \sup \mathcal{T}.$$

Since ψ is nonincreasing, it follows that

$$\psi(|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)|) \le \psi(t + \frac{r_0}{2}) \quad \text{in } [0, T^*).$$

We are now ready to present a class of well-prepared initial data satisfying $\mathcal{C}_2$, which will lead to a bi-cluster flocking configuration.

$$
\begin{aligned}
&(\mathcal{C}_2 0): r_0 = \min_{i,j}(x^1_{1i}(0) - x^1_{2j}(0)) > 0,\\
&(\mathcal{C}_2 1):\ (i)\ \theta_{1i}(0) \in \Big(-\frac{\pi}{6}, \frac{\pi}{6}\Big), \quad \theta_{2j}(0) \in \Big(\frac{5\pi}{6}, \frac{7\pi}{6}\Big),\\
&\qquad\qquad \text{for all}\ \ i = 1, 2, \cdots, N_1, \quad j = 1, 2, \cdots, N_2,\\
&\qquad\quad (ii)\ \mathcal{V}(\Theta_0) < \min\left\{\frac{\pi}{24}, \frac{A\beta_0}{4\sqrt{2}}\right\},\\
&(\mathcal{C}_2 2):\ (i)\ K\int_{\frac{r_0}{2}}^{\infty}\psi(s)ds < \frac{\pi}{12}, \quad \mathcal{V}(\Theta_0) > \frac{2K}{\beta_0}\sqrt{\frac{2N_1N_2}{N}}\psi\Big(\frac{r_0}{2}\Big),\\
&\qquad\quad (ii)\ \frac{2K}{\beta_0}\sqrt{\frac{N_1N_2}{N}}\int_{\frac{r_0}{2}}^{\infty}\psi(s)ds < \frac{A}{2},
\end{aligned}
$$

where β_0 and A are positive constants defined by

$$
\beta_0 := \frac{KN_1}{2N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}_0)), \quad \psi(\sqrt{2}(\mathcal{V}(\boldsymbol{x}_0) + A)) := \frac{1}{2}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}_0)). \tag{5.1.5}
$$

The first condition $(\mathcal{C}_2 1)$ implies that the perturbations of the two groups are sufficiently small, each group is close to the flocking state, and they are initially separated from each other. The second condition $(\mathcal{C}_2 2)$ implies that the distance between the two groups is sufficiently large, which will decrease the interaction between them. It is easy to verify that the set of the initial data is not empty.

5.1.3 Time-evolution of functionals

In this subsection, we derive differential inequalities for the functionals $\mathcal{V}(\boldsymbol{x})$ and $\mathcal{V}(\Theta)$:

$$
\left|\frac{d\mathcal{V}(\boldsymbol{x})}{dt}\right| \le \sqrt{2}\mathcal{V}(\Theta), \qquad \frac{d}{dt}\mathcal{V}(\Theta) \le -\frac{\beta_0}{2}\mathcal{V}(\Theta) + K\sqrt{\frac{2N_1N_2}{N}}\psi(t + \frac{r_0}{2}), \quad t > 0. \tag{5.1.6}
$$

In the following two lemmas, we derive the inequalities in (5.1.6).

Lemma 5.1.1. *Suppose that* $(\boldsymbol{x}_{1i}(t), \theta_{1i}(t))$ *and* $(\boldsymbol{x}_{2j}(t), \theta_{2j}(t))$ *are solutions to system* (5.1.1). *Then*

$$\left|\frac{d\mathcal{V}_1(\boldsymbol{x})}{dt}\right| \le \sqrt{2}\mathcal{V}_1(\Theta), \quad \left|\frac{d\mathcal{V}_2(\boldsymbol{x})}{dt}\right| \le \sqrt{2}\mathcal{V}_2(\Theta), \quad \left|\frac{d\mathcal{V}(\boldsymbol{x})}{dt}\right| \le \sqrt{2}\mathcal{V}(\Theta), \quad t > 0.$$

Proof. It needs direct but long calculation from the equations for any i,

$$2\hat{\boldsymbol{x}}_{1i} \cdot \frac{d\hat{\boldsymbol{x}}_{1i}}{dt} = 2\hat{\boldsymbol{x}}_{1i} \cdot \Big(\cos\theta_{1i} - \frac{1}{N_1}\sum_{k=1}^{N_1} \cos\theta_{1k}, \sin\theta_{1i} - \frac{1}{N_1}\sum_{k=1}^{N_1} \sin\theta_{1k} \Big).$$

□

Lemma 5.1.2. *Suppose that conditions* $(\mathcal{C}_2 1)(ii)$ *and* $(\mathcal{C}_2 2)$ *in Section 5.1.2 hold. Then*

$$\begin{aligned} &(i)\ \frac{d}{dt}\mathcal{V}(\Theta) \le -\frac{\beta_0}{2}\mathcal{V}(\Theta) + K\sqrt{\frac{2N_1N_2}{N}}\psi(t+\frac{r_0}{2}), \quad t \in (0, T^*), \\ &(ii)\ \sup_{t\in[0,T^*)} \mathcal{V}(\Theta(t)) < 2\mathcal{V}(\Theta_0), \qquad \sup_{t\in[0,T^*)} \mathcal{V}(\boldsymbol{x}(t)) < \mathcal{V}(\boldsymbol{x}_0) + A, \end{aligned} \tag{5.1.7}$$

where $T^*, r_0,$ *and* A *are positive constants defined in* (5.1.3), (5.1.4), *and* (5.1.5), *respectively.*

Proof. First, recall from (5.1.4) that

$$T^* := \sup\Big\{ T \in (0,\infty] \ : \ \min_{i,j} |x_{1i}(t) - x_{2j}(t)| \ge t + \frac{r_0}{2} \quad \text{for } t \in [0,T) \Big\} > 0. \tag{5.1.8}$$

In the sequel, desired estimates in (5.1.7) hold in the time-interval $[0, T^*)$; specifically, consider the set $\mathcal{S}$:

$$\mathcal{S} := \{ T \in (0, T^*] \ : \ \text{estimates (i) and (ii) in (5.1.7) hold for } t \in [0, T) \}.$$

In the following two steps, we will show that $\mathcal{S}$ is nonempty, and its supremum is equal to T^*.

• Step A ($\mathcal{S}$ is not empty): Since $\beta_0 := \frac{KN_1}{2N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}_0))$, these are initially ture. by the continuity of $\mathcal{V}(\Theta)$ and $\mathcal{V}(\boldsymbol{x})$, $\mathcal{S}$ is not empty.

• Step B ($\sup \mathcal{S} = T^*$): By Step A, we know that the set $\mathcal{S}$ has a supremum.

$$S^* := \sup \mathcal{S} \in (0, T^*].$$

First, we must rule out the possibility of

$$S^* < T^*. \tag{5.1.9}$$

Assume that (5.1.9) holds. In this case,

$$\lim_{t \to S^*-} \mathcal{V}(x(t)) = \mathcal{V}(x_0) + A \quad \text{or} \quad \lim_{t \to S^*-} \mathcal{V}(\Theta(t)) = 2\mathcal{V}(\Theta_0).$$

⋄ (Case 1) Suppose

$$\lim_{t \to S^*-} \mathcal{V}(x(t)) = \mathcal{V}(x_0) + A. \tag{5.1.10}$$

Note that relations (5.1.2) and $(\mathcal{C}_2 1)(ii)$ yield

$$|\hat{\theta}_{1i} - \hat{\theta}_{1j}| \le D(\Theta(t)) \le \sqrt{2}\mathcal{V}(\Theta) \le 2\sqrt{2}\mathcal{V}(\Theta_0) < \frac{\pi}{6} \quad \text{in } [0, \tau).$$

This implies

$$\frac{\sin(\hat{\theta}_{1i} - \hat{\theta}_{1j})}{\hat{\theta}_{1i} - \hat{\theta}_{1j}} > \frac{\sin(2\sqrt{2}\mathcal{V}(\Theta_0))}{2\sqrt{2}\mathcal{V}(\Theta_0)} > \frac{1}{2} \quad \text{in } [0, \tau).$$

On the other hand, by using (5.1.1) and $\theta_{1i} = \hat{\theta}_{1i} + \theta_{1c}$, we obtain

$$\begin{aligned} \frac{d\hat{\theta}_{1i}}{dt} =& \frac{K}{N} \sum_{k=1}^{N_1} \psi(|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}|) \sin(\hat{\theta}_{1k} - \hat{\theta}_{1i}) \\ &+ \frac{K}{N} \sum_{k=1}^{N_2} \psi(|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}|) \sin(\theta_{2k} - \theta_{1i}) - \frac{d\theta_{1c}}{dt}. \end{aligned}$$

This yields

$$\begin{aligned} 2\sum_{i=1}^{N_1} \hat{\theta}_{1i} \frac{d\hat{\theta}_{1i}}{dt} =& \frac{2K}{N} \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \psi(|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}|) \sin(\hat{\theta}_{1k} - \hat{\theta}_{1i}) \hat{\theta}_{1i} \\ &+ \frac{2K}{N} \sum_{i=1}^{N_1} \sum_{k=1}^{N_2} \psi(|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}|) \sin(\theta_{2k} - \theta_{1i}) \hat{\theta}_{1i}. \end{aligned}$$

By definition of $\mathcal{V}_1(\Theta)$, it follows that for $t \in (0, \tau)$,

$$\begin{aligned} 2\mathcal{V}_1(\Theta)\frac{d\mathcal{V}_1(\Theta)}{dt} &= -\frac{K}{N}\sum_{i=1}^{N_1}\sum_{k=1}^{N_1}\psi(|\boldsymbol{x}_{1k}-\boldsymbol{x}_{1i}|)\sin(\hat{\theta}_{1k}-\hat{\theta}_{1i})(\hat{\theta}_{1k}-\hat{\theta}_{1i}) \\ &\quad + \frac{2K}{N}\sum_{i=1}^{N_1}\sum_{k=1}^{N_2}\psi(|\boldsymbol{x}_{2k}-\boldsymbol{x}_{1i}|)\sin(\theta_{2k}-\theta_{1i})\hat{\theta}_{1i} \\ &\le -\frac{KN_1\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}))}{N}\mathcal{V}_1^2(\Theta) + \frac{2KN_2\sqrt{N_1}}{N}\psi(t+\frac{r_0}{2})\mathcal{V}_1(\Theta), \end{aligned}$$

where the following inequality is applied;

$$\sin(\hat{\theta}_{1k}-\hat{\theta}_{1i})(\hat{\theta}_{1k}-\hat{\theta}_{1i}) > \frac{1}{2}(\hat{\theta}_{1k}-\hat{\theta}_{1i})^2 \quad \text{in } (0,\tau).$$

This implies a differential inequality for $\mathcal{V}_1(\Theta)$:

$$\frac{d\mathcal{V}_1(\Theta)}{dt} \le -\frac{KN_1}{2N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}))\mathcal{V}_1(\Theta) + \frac{KN_2\sqrt{N_1}}{N}\psi(t+\frac{r_0}{2}) \quad \text{in } (0,\tau). \tag{5.1.11}$$

Similarly, we can get the same equation for $\mathcal{V}_2$, hence for $N_1 \le N_2$,

$$\frac{d\mathcal{V}(\Theta)}{dt} \le -\frac{KN_1}{2N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}))\mathcal{V}(\Theta) + K\sqrt{\frac{2N_1N_2}{N}}\psi\Big(t+\frac{r_0}{2}\Big) \quad \text{in } (0,\tau). \tag{5.1.12}$$

Again, we use the assumption of β_0 to (5.1.12),

$$\frac{d\mathcal{V}(\Theta)}{dt} \le -\frac{\beta_0}{2}\mathcal{V}(\Theta) + K\sqrt{\frac{2N_1N_2}{N}}\psi\Big(t+\frac{r_0}{2}\Big) \quad \text{in } (0,\tau). \tag{5.1.13}$$

We integrate (5.1.13) using Gronwall's lemma to find

$$\mathcal{V}(\Theta(t)) \le \mathcal{V}(\Theta_0)e^{-\frac{\beta_0}{2}t} + K\sqrt{\frac{2N_1N_2}{N}}\int_0^t \psi\Big(s+\frac{r_0}{2}\Big)e^{\frac{\beta_0}{2}(s-t)}ds \quad \text{in } (0,\tau). \tag{5.1.14}$$

It follows from Lemma 5.1.1 and (5.1.14) that

$$\begin{aligned}
&|\mathcal{V}(\boldsymbol{x}(t)) - \mathcal{V}(\boldsymbol{x}_0)| \\
&\quad \le \int_0^t \Big|\frac{d}{ds}\mathcal{V}(\boldsymbol{x}(s))\Big| ds \le \sqrt{2}\int_0^t \mathcal{V}(\Theta(s))ds \\
&\quad \le \sqrt{2}\mathcal{V}(\Theta_0)\int_0^t e^{-\frac{\beta_0}{2}s}ds + \sqrt{2}K\sqrt{\frac{2N_1N_2}{N}}\int_0^t\int_0^s \psi\Big(\tilde{s}+\frac{r_0}{2}\Big)e^{\frac{\beta_0}{2}(\tilde{s}-s)}d\tilde{s}ds \\
&\quad \le \frac{2\sqrt{2}\mathcal{V}(\Theta_0)}{\beta_0} + \frac{2K}{\beta_0}\sqrt{\frac{N_1N_2}{N}}\int_{\frac{r_0}{2}}^\infty \psi(s)ds, \quad \text{in } (0,\tau).
\end{aligned} \tag{5.1.15}$$

Thus, (5.1.15) implies

$$\mathcal{V}(\boldsymbol{x}(t)) \le \mathcal{V}(\boldsymbol{x}_0) + \frac{2\sqrt{2}\mathcal{V}(\Theta_0)}{\beta_0} + \frac{2K}{\beta_0}\sqrt{\frac{N_1N_2}{N}}\int_{\frac{r_0}{2}}^\infty \psi(s)ds \quad t\in[0,\tau).$$

Then, it follows from $(\mathcal{C}_21)(ii)$ and $(\mathcal{C}_22)(ii)$ that the sum of the last two terms is strictly less than A, i.e.,

$$\mathcal{V}(\boldsymbol{x}(t)) < \mathcal{V}(\boldsymbol{x}_0) + A - \epsilon, \quad \text{for } t\in[0,\tau), \epsilon \ll 1. \tag{5.1.16}$$

Finally, combining (5.1.13) and (5.1.16) yields

$$\lim_{t\to S^*-}\mathcal{V}(x(t)) < \mathcal{V}(x_0) + A,$$

which contradicts (5.1.10).

◇ (Case 2): Now, suppose

$$\lim_{t\to S^*-}\mathcal{V}(\Theta(t)) = 2\mathcal{V}(\Theta_0). \tag{5.1.17}$$

We use the estimate

$$I(t) = \int_0^t \psi\left(s+\frac{r_0}{2}\right)e^{\frac{\beta_0}{2}(s-t)}ds \le \psi\Big(\frac{r_0}{2}\Big)\int_0^t e^{\frac{\beta_0}{2}(s-t)}ds \le \frac{2}{\beta_0}\psi\Big(\frac{r_0}{2}\Big),$$

assumption $(\mathcal{C}_22)$, and (5.1.14) to obtain

$$\begin{aligned}
\mathcal{V}(\Theta(t)) &\le \mathcal{V}(\Theta_0)e^{-\frac{\beta_0}{2}t} + \frac{2K}{\beta_0}\sqrt{\frac{2N_1N_2}{N}}\psi\Big(\frac{r_0}{2}\Big) \\
&< \mathcal{V}(\Theta_0)e^{-\frac{\beta_0}{2}t} + \mathcal{V}(\Theta_0), \quad \text{in } [0,S^*).
\end{aligned} \tag{5.1.18}$$

Letting $t \to S^*$ in (5.1.18), we have

$$\lim_{t\to S^*-} \mathcal{V}(\Theta(t)) \le \mathcal{V}(\Theta_0)(e^{-\frac{\beta_0}{2}S^*} + 1) < 2\mathcal{V}(\Theta_0),$$

which contradicts (5.1.17).
Therefore, combining two cases, $S^* = T^*$ and we get the desired result. $\square$

5.1.4 Emergence of bi-cluster flocking

In this subsection, we present our first main result on the bi-cluster flocking estimate of system (2.4.13) for well-prepared initial data satisfying assumptions $(\mathcal{C}_2)$ in Section 5.1.2. Note that once we show that

$$\begin{aligned}
&\sup_{0\le t<\infty} \mathcal{V}(\boldsymbol{x}(t)) < \mathcal{V}(\boldsymbol{x}_0) + A, \quad \lim_{t\to\infty} \mathcal{V}(\Theta(t)) = 0,\\
&\liminf_{0\le t<\infty} \min_{i,j} |\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)| = \infty,
\end{aligned}$$

we achieve the desired bi-cluster flocking in the sense of Definition 2.1.1.

Theorem 5.1.1. (Emergent of bi-cluster flocking) *Suppose that conditions $(\mathcal{C}_2 1)$ and $(\mathcal{C}_2 2)$ hold. Then $T^* = \infty$, and the following bi-flocking estimates hold: for $t \in [0, \infty)$,*

$$\begin{aligned}
&(i)\ \mathcal{V}(\boldsymbol{x}(t)) < \mathcal{V}(\boldsymbol{x}_0) + A, \quad |\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)| > t + \frac{r_0}{2},\\
&(ii)\ \mathcal{V}(\Theta(t)) \le C\Big[e^{-\frac{\beta_0}{4}t} + \psi\Big(\frac{t + r_0}{2}\Big)\Big].
\end{aligned}$$

Proof. • Step A: We will show that Lemma 5.1.2 holds with $T^* = \infty$.

In (5.1.8), assume that $T^* < \infty$. Then there exist i_0 and j_0 such that

$$|x_{1i_0}(T^*) - x_{2j_0}(T^*)| = T^* + \frac{r_0}{2}. \tag{5.1.19}$$

Since

$$\frac{d\theta_{1c}}{dt} = \frac{K}{N_1 N}\sum_{i=1}^{N_1}\sum_{k=1}^{N_2} \psi(|x_{2k} - x_{1i}|)\sin(\theta_{2k} - \theta_{1i}),$$

it follows that

$$|\theta_{1c}(t) - \theta_{1c}(0)| \le K \int_0^t \psi(s + \frac{r_0}{2}) ds \le K \int_{\frac{r_0}{2}}^{\infty} \psi(s) ds, \quad \text{in } [0, T^*). \tag{5.1.20}$$

Now, for each θ_{1i}, we use (5.1.20) to obtain

$$\begin{aligned} |\theta_{1i}(t)| &\le |\theta_{1i}(t) - \theta_{1c}(t)| + |\theta_{1c}(t) - \theta_{1c}(0)| + |\theta_{1c}(0)| \\ &\le \mathcal{V}(\Theta(t)) + |\theta_{1c}(0)| + K \int_{\frac{r_0}{2}}^{\infty} \psi(s) ds \\ &\le 2\mathcal{V}(\Theta_0) + |\theta_{1c}(0)| + K \int_{\frac{r_0}{2}}^{\infty} \psi(s) ds, \quad t \in [0, T^*). \end{aligned} \tag{5.1.21}$$

Hence, $(\mathcal{C}_2 1)$ and $(\mathcal{C}_2 2)(i)$ imply

$$\mathcal{V}(\Theta_0) < \frac{\pi}{24}, \quad |\theta_{1c}(0)| < \frac{\pi}{6}, \quad K \int_{\frac{r_0}{2}}^{\infty} \psi(s) ds < \frac{\pi}{12}. \tag{5.1.22}$$

Combining (5.1.21) and (5.1.22) yields

$$\theta_{1i}(t) \in (-\frac{\pi}{3}, \frac{\pi}{3}) \quad \text{in } [0, T^*). \tag{5.1.23}$$

Similarly, we have

$$\theta_{2j}(t) \in (\frac{2\pi}{3}, \frac{4\pi}{3}) \quad \text{in } [0, T^*). \tag{5.1.24}$$

Since $\dfrac{dx^1_{\alpha i}}{dt} = \cos\theta_{\alpha i}$, it follows from (5.1.23) and (5.1.24) that

$$\dot{x}^1_{1i} = \cos\theta_{1i}(t) > \frac{1}{2}, \qquad \dot{x}^1_{2j} = \cos\theta_{2j}(t) < -\frac{1}{2}, \quad t \in [0, T^*).$$

This yields

$$\begin{aligned} |\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)| &\ge |x^1_{1i}(t) - x^1_{2j}(t)| \\ &= \left| x^1_{1i}(0) - x^1_{2j}(0) + \int_0^t (\cos\theta_{1i}(s) - \cos\theta_{2j}(s)) ds \right| \\ &\ge t + |x^1_{1i}(0) - x^1_{2j}(0)| > t + \frac{r_0}{2}, \quad t \in [0, T^*). \end{aligned}$$

From continuity and letting $t \to T^*-$, we obtain

$$\lim_{t\to T^*-} |\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)| > T^* + \frac{r_0}{2},$$

which contradicts (5.1.19). Thus, Lemma 5.1.2 holds with $T^* = \infty$.

• Step B (Estimate of $\mathcal{V}(\Theta)$): Note that $\mathcal{V}(\Theta)$ satisfies

$$\frac{d}{dt}\mathcal{V}(\Theta) \le -\frac{\beta_0}{2}\mathcal{V}(\Theta) + K\sqrt{\frac{2N_1N_2}{N}}\psi(t+\frac{r_0}{2}), \quad t \in (0,\infty).$$

We now apply Lemma A.0.1 with

$$\alpha = \frac{\beta_0}{2} \quad \text{and} \quad f(t) = K\sqrt{\frac{2N_1N_2}{N}}\psi(t+\frac{r_0}{2})$$

to derive the desired estimate:

$$\mathcal{V}(\Theta(t)) \le \frac{2K}{\beta_0}\sqrt{\frac{2N_1N_2}{N}}\psi(\frac{t+r_0}{2}) + \mathcal{V}(\Theta_0)e^{-\frac{\beta_0 t}{2}} + K\sqrt{\frac{2N_1N_2}{N}}\psi(\frac{r_0}{2})e^{-\frac{\beta_0 t}{4}}.$$

□

Proposition 5.1.1. (Relaxation of local averaged velocities) *Suppose conditions* $(\mathcal{C}_2 1)$ *and* $(\mathcal{C}_2 2)$ *hold. Then there exist asymptotic velocities* θ_{1c}^∞ *and* θ_{2c}^∞ *such that*

$$|\theta_{1c}(t) - \theta_{1c}^\infty| + |\theta_{2c}(t) - \theta_{2c}^\infty| \le C\int_t^\infty \psi\Big(s+\frac{r_0}{2}\Big)ds \quad \text{as } t\to\infty,$$

where C *is a positive constant.*

Proof. Note that the local phase average θ_{1c} satisfies

$$\theta_{1c}(t) = \theta_{1c}(0) + \frac{K}{N_1N}\sum_{i=1}^{N_1}\sum_{k=1}^{N_2}\int_0^t \psi(|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}|)\sin(\theta_{2k}-\theta_{1i})ds. \quad (5.1.25)$$

We set the asymptotic state θ_{1c}^∞ as follows:

$$\theta_{1c}^\infty := \theta_{1c}(0) + \frac{K}{N_1N}\sum_{i=1}^{N_1}\sum_{k=1}^{N_2}\int_0^\infty \psi(|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}|)\sin(\theta_{2k}-\theta_{1i})ds. \quad (5.1.26)$$

It follows from (5.1.25) and (5.1.26) that

$$|\theta_{1c}(t) - \theta_{1c}^{\infty}| \le K \int_t^{\infty} \psi\Big(s + \frac{r_0}{2}\Big) ds. \tag{5.1.27}$$

Note that the right-hand side goes to zero due to integrability of ψ. We conclude the result since this is also true for θ_{2c}^{∞}. $\square$

Remark 5.1.1. *By combining results of Theorem 5.1.1 and Proposition 5.1.1,*

$$\begin{aligned}
&|\theta_{1i}(t) - \theta_1^{\infty}| + |\theta_{2j}(t) - \theta_2^{\infty}| \\
&\quad \le (|\theta_{1i}(t) - \theta_{1c}(t)| + |\theta_{2j}(t) - \theta_{2c}(t)|) + |\theta_{1c}(t) - \theta_1^{\infty}| + |\theta_{2c}(t) - \theta_2^{\infty}| \\
&\quad \le \mathcal{V}(\Theta(t)) + 2K \int_t^{\infty} \psi\Big(s + \frac{r_0}{2}\Big) ds \\
&\quad \le C\Big[e^{-\frac{\beta_0}{4}t} + \psi\Big(\frac{t + r_0}{2}\Big)\Big] + 2K \int_t^{\infty} \psi\Big(s + \frac{r_0}{2}\Big) ds.
\end{aligned}$$

In particular, for the C-S communication weight $\psi(s) = \frac{1}{(1+s)^{\beta}}$ with $\beta > 1$, we can find the explicit decay rate

$$|\theta_{1i}(t) - \theta_1^{\infty}| + |\theta_{2j}(t) - \theta_2^{\infty}| \le C(1+t)^{-(\beta-1)} \quad \text{as } t \to \infty.$$

5.2 The multi-dimensional C-S model with the unit speed constraint

In this section, we present a formation of bi-cluster flocking for the C-S model with the unit speed constraint, which was introduced in [24]:

$$\begin{aligned}
&\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \quad 1 \le i \le N, \ t > 0, \\
&\frac{d\boldsymbol{v}_i}{dt} = \frac{K}{N} \sum_{k=1}^{N} \psi(\|\boldsymbol{x}_k - \boldsymbol{x}_i\|) \Big(\boldsymbol{v}_k - \frac{\langle \boldsymbol{v}_i, \boldsymbol{v}_k \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle} \boldsymbol{v}_i\Big).
\end{aligned} \tag{5.2.1}$$

As noted in Theorem 2.4.2, when the initial configuration is close to a mono-cluster flocking configuration, the emergence of mono-cluster flocking can be guaranteed. Thus, if the sufficient conditions stated in Theorem 2.4.2 are violated, the number of clusters that emerge asymptotically cannot be determined. In the future, we will focus on the formation of bi-cluster flocking using an idea similar to those in the previous section.

5.2.1 A reformulation of the C-S model

We assume the system is composed of two subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$ with $|\mathcal{G}_1| = N_1$ and $|\mathcal{G}_2| = N_2$. We relabel the position and velocity of each agent with $(\boldsymbol{x}_{1i}, \boldsymbol{v}_{1i})$, $i = 1, 2, \cdots, N_1$ and $(\boldsymbol{x}_{2j}, \boldsymbol{v}_{2j})$, $j = 1, 2, \cdots, N_2$, respectively. In this setting, system (5.2.1) can be rewritten as

$$\begin{aligned}
&\dot{\boldsymbol{x}}_{1i} = \boldsymbol{v}_{1i}, \quad \dot{\boldsymbol{x}}_{2j} = \boldsymbol{v}_{2j}, \\
&\dot{\boldsymbol{v}}_{1i} = \frac{K}{N}\sum_{k=1}^{N_1} \psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}\|)\Big(\boldsymbol{v}_{1k} - \frac{\langle \boldsymbol{v}_{1i}, \boldsymbol{v}_{1k}\rangle}{\langle \boldsymbol{v}_{1i}, \boldsymbol{v}_{1i}\rangle}\boldsymbol{v}_{1i}\Big) \\
&\qquad + \frac{K}{N}\sum_{k=1}^{N_2} \psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)\Big(\boldsymbol{v}_{2k} - \frac{\langle \boldsymbol{v}_{1i}, \boldsymbol{v}_{2k}\rangle}{\langle \boldsymbol{v}_{1i}, \boldsymbol{v}_{1i}\rangle}\boldsymbol{v}_{1i}\Big), \\
&\dot{\boldsymbol{v}}_{2j} = \frac{K}{N}\sum_{k=1}^{N_2} \psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{2j}\|)\Big(\boldsymbol{v}_{2k} - \frac{\langle \boldsymbol{v}_{2j}, \boldsymbol{v}_{2k}\rangle}{\langle \boldsymbol{v}_{2j}, \boldsymbol{v}_{2j}\rangle}\boldsymbol{v}_{2j}\Big) \\
&\qquad + \frac{K}{N}\sum_{k=1}^{N_1} \psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}\|)\Big(\boldsymbol{v}_{1k} - \frac{\langle \boldsymbol{v}_{2j}, \boldsymbol{v}_{1k}\rangle}{\langle \boldsymbol{v}_{2j}, \boldsymbol{v}_{2j}\rangle}\boldsymbol{v}_{2j}\Big),
\end{aligned} \tag{5.2.2}$$

where $i = 1, 2, \cdots, N_1$ and $j = 1, 2, \cdots, N_2$. Without loss of generality, we assume

$$\|\boldsymbol{v}_i(t)\| = 1 \quad \text{and} \quad 1 \le N_1 \le N_2. \tag{5.2.3}$$

Next, we define the local averages and local fluctuations with respect to space and velocity to analyze (5.2.2):

$$\begin{aligned}
&\boldsymbol{x}_{1c} = \frac{1}{N_1}\sum_{k=1}^{N_1} \boldsymbol{x}_{1k}, \quad \boldsymbol{x}_{2c} = \frac{1}{N_2}\sum_{k=1}^{N_2} \boldsymbol{x}_{2k}, \\
&\boldsymbol{v}_{1c} = \frac{1}{N_1}\sum_{k=1}^{N_1} \boldsymbol{v}_{1k}, \quad \boldsymbol{v}_{2c} = \frac{1}{N_2}\sum_{k=1}^{N_2} \boldsymbol{v}_{2k}, \\
&\hat{\boldsymbol{x}}_{\alpha i} = \boldsymbol{x}_{\alpha i} - \boldsymbol{x}_{\alpha c}, \quad \hat{\boldsymbol{v}}_{\alpha i} = \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha c}, \quad \alpha = 1, 2.
\end{aligned} \tag{5.2.4}$$

Lemma 5.2.1. *Let* $(\boldsymbol{x}, \boldsymbol{v})$ *be a solution to* (5.2.1) *and* (5.2.3). *Then the local* *averages and fluctuations in* (5.2.4) *satisfy the following coupled system:*

$$\begin{aligned}
\dot{\hat{\boldsymbol{x}}}_{1i} =& \hat{\boldsymbol{v}}_{1i}, \quad \dot{\hat{\boldsymbol{x}}}_{2i} = \hat{\boldsymbol{v}}_{2i} \\
\dot{\hat{\boldsymbol{v}}}_{1i} =& \frac{K}{N}\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k}-\boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{1k}-\boldsymbol{v}_{1i}) \\
&+\frac{K}{N}\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k}-\boldsymbol{x}_{1i}\|)\langle \boldsymbol{v}_{1i}-\boldsymbol{v}_{1k},\boldsymbol{v}_{1i}\rangle \boldsymbol{v}_{1i} \\
&+\frac{K}{N}\sum_{k=1}^{N_2}\psi(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2k}-\langle \boldsymbol{v}_{1i},\boldsymbol{v}_{2k}\rangle \boldsymbol{v}_{1i})-\dot{\boldsymbol{v}}_{1c} \\
\dot{\hat{\boldsymbol{v}}}_{2i} =& \frac{K}{N}\sum_{k=1}^{N_2}\psi(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{2i}\|)(\boldsymbol{v}_{2k}-\boldsymbol{v}_{2i}) \\
&+\frac{K}{N}\sum_{k=1}^{N_2}\psi(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{2i}\|)\langle \boldsymbol{v}_{2i}-\boldsymbol{v}_{2k},\boldsymbol{v}_{2i}\rangle \boldsymbol{v}_{2i} \\
&+\frac{K}{N}\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k}-\boldsymbol{x}_{2i}\|)(\boldsymbol{v}_{1k}-\langle \boldsymbol{v}_{2i},\boldsymbol{v}_{1k}\rangle \boldsymbol{v}_{2i})-\dot{\boldsymbol{v}}_{2c}.
\end{aligned} \tag{5.2.5}$$

Proof. The proof directly follows from the equation. See Lemma 3.1.2. □

Next, we introduce ℓ_2-type functionals that measure the total fluctuations of each flocking group:

$$\begin{aligned}
\mathcal{V}_\alpha(\boldsymbol{x}) :=& \sqrt{\sum_{k=1}^{N_\alpha}\|\hat{\boldsymbol{x}}_{\alpha k}\|^2}, \quad \mathcal{V}_\alpha(\boldsymbol{v}) := \sqrt{\sum_{k=1}^{N_\alpha}\|\hat{\boldsymbol{v}}_{\alpha k}\|^2}, \quad \alpha = 1,2 \\
\mathcal{V}(\boldsymbol{x}) :=& \mathcal{V}_1(\boldsymbol{x})+\mathcal{V}_2(\boldsymbol{x}), \quad \mathcal{V}(\boldsymbol{v}) := \mathcal{V}_1(\boldsymbol{v})+\mathcal{V}_2(\boldsymbol{v}).
\end{aligned}$$

Lemma 5.2.2. *Suppose* $(\boldsymbol{x}_{1i}(t),\boldsymbol{v}_{1i}(t))$ *and* $(\boldsymbol{x}_{2j}(t),\boldsymbol{v}_{2j}(t))$ *for* $i=1,2,\cdots,N_1$ *and* $j=1,2,\cdots,N_2$ *are the solutions to the coupled system in* (5.2.5)*. Then* *the functionals* $\mathcal{V}(\boldsymbol{x})$ *and* $\mathcal{V}(\boldsymbol{v})$ *satisfy*

$$
\begin{aligned}
&(i)\ \left|\frac{d\mathcal{V}(\boldsymbol{x})}{dt}\right| \le\ \mathcal{V}(\boldsymbol{v}), \quad t>0,\\
&(ii)\ \frac{d\mathcal{V}(\boldsymbol{v})}{dt} \le -\frac{KN_1}{N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}))\mathcal{V}(\boldsymbol{v})\\
&\qquad\qquad + \frac{KN_2}{N}\psi(0)(\mathcal{V}(\boldsymbol{v}))^2 + 2K\sqrt{\frac{2N_1N_2}{N}}\psi_M,
\end{aligned}
$$

where $\psi_M := \max_{i,j}\psi(\|\boldsymbol{x}_{1i}-\boldsymbol{x}_{2j}\|)$.

Proof. These are basically from the equation (5.2.5). The proof is based on the one of Lemma 3.1.3, but we need a little more than that from the lack of symmetry. This difficulty can be figured out from the following facts:

$$
\begin{aligned}
&\langle \boldsymbol{v}_{1i}-\boldsymbol{v}_{1k}, \boldsymbol{v}_{1i}\rangle = \frac{1}{2}\|\boldsymbol{v}_{1k}-\boldsymbol{v}_{1i}\|^2, \quad \|\boldsymbol{x}_{1k}-\boldsymbol{x}_{1i}\| \le \sqrt{2}\mathcal{V}(\boldsymbol{x}),\\
&\|\boldsymbol{v}_{1i}\| = \|\boldsymbol{v}_{2j}\| = 1.
\end{aligned}
$$

□

5.2.2 A class of admissible initial data

In this subsection, we present a class of admissible initial data leading to asymptotic bi-cluster flocking. First, we define

$$
\begin{aligned}
&\beta_0 := \frac{KN_1}{N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}_0)), \quad \psi(\sqrt{2}(\mathcal{V}(x_0)+A)) = \frac{3}{4}\psi(\sqrt{2}\mathcal{V}(x_0)),\\
&\boldsymbol{e}_d := (0,0,\cdots,0,1), \quad r_0 := \min_{i,j}\langle \boldsymbol{x}_{1i}(0)-\boldsymbol{x}_{2j}(0), \boldsymbol{e}_d\rangle.
\end{aligned}
\tag{5.2.6}
$$

Below, we list the conditions for admissible initial data:

$$
\begin{aligned}
(\mathcal{C}_3 1): &(i)\ \langle \boldsymbol{v}_{1i}, \boldsymbol{e}_d \rangle \in (\frac{\sqrt{3}}{2}, 1], \quad \langle \boldsymbol{v}_{2j}, \boldsymbol{e}_d \rangle \in [-1, -\frac{\sqrt{3}}{2}), \qquad \text{for all } i, j, \\
&(ii)\ \mathcal{V}(\boldsymbol{v}_0) < \min\left\{ \frac{\sqrt{3}-1}{8}, \frac{A\beta_0}{4}, \frac{N\beta_0}{8N_2 K\psi(0)} \right\}, \\
, (\mathcal{C}_3 2): &(i)\ \mathcal{V}(\boldsymbol{v}_0) > \frac{4\sqrt{2}K}{\beta_0}\sqrt{\frac{N_1N_2}{N}}\psi(\frac{r_0}{2}), \\
&\frac{A}{2} > \frac{4\sqrt{2}K}{\beta_0}\sqrt{\frac{N_1N_2}{N}}\int_{\frac{r_0}{2}}^{\infty}\psi(s)ds, \\
&(ii)\ \frac{128K^3N_1N_2}{N^2\beta_0^2}\psi(0)\psi(\frac{r_0}{2})\Big[\frac{2}{\beta_0}\psi(\frac{r_0}{2}) + \int_{\frac{r_0}{2}}^{\infty}\psi(s)ds\Big] \\
&\quad + \frac{2K\psi(0)}{N\beta_0}\mathcal{V}^2(\boldsymbol{v}_0) + 2K\int_{\frac{r_0}{2}}^{\infty}\psi(s)ds < \frac{\sqrt{3}-1}{4}.
\end{aligned}
$$

Condition $(\mathcal{C}_3 1)$ implies the perturbations of the two groups are sufficiently small, each group is close to a flocking state, and they are separate from each other initially. Condition $(\mathcal{C}_3 2)$ implies the distance between the two groups is sufficiently large, which will decrease the interaction between them. It is easy to verify that the set of initial data is not empty.

For any i, j, we have $\|\boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0)\| \geq r_0$. By the continuity of the solution, there exists a $T > 0$ such that

$$\|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\| \geq t + \frac{r_0}{2} \quad \text{for } t \in [0, T) \text{ and all } i, j.$$

Let $T_2^* = \sup T$. Then

$$\psi(\|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\|) \leq \psi(t + \frac{r_0}{2}) \quad \text{in } [0, T_2^*), \quad \text{and for all } i, j.$$

5.2.3 Time-evolution of functionals

In this subsection, we study the time-evolution estimates of the functionals $\mathcal{V}(\boldsymbol{x})$ and $\mathcal{V}(\boldsymbol{v})$.

Lemma 5.2.3. *The following estimate holds:*

$$\int_0^t \psi(s+\frac{r_0}{2})e^{\frac{\beta_0(s-t)}{2}}ds \leq \frac{2}{\beta_0}\psi\Big(\frac{r_0}{2}\Big)e^{-\frac{\beta_0 t}{4}} + \frac{2}{\beta_0}\psi\Big(\frac{t+r_0}{2}\Big).$$

Proof. Direct integration similar to (A.0.1) in Appendix. □

Proposition 5.2.1. *Suppose that conditions* $(\mathcal{C}_3 1)(i)$ *and* $(\mathcal{C}_3 2)(i)(ii)$ *hold. Then there exists a positive constant* $T_2^* \in (0,\infty]$ *such that for all* $t \in [0, T_2^*)$,

$$\begin{aligned}&(i)\ \mathcal{V}(\boldsymbol{v}) < 2\mathcal{V}(\boldsymbol{v}_0), \quad \frac{d\mathcal{V}(\boldsymbol{v})}{dt} \leq -\frac{\beta_0}{2}\mathcal{V}(\boldsymbol{v}) + 2K\sqrt{\frac{2N_1N_2}{N}}\psi\Big(t+\frac{r_0}{2}\Big),\\ &(ii)\ \mathcal{V}(\boldsymbol{x}) < \mathcal{V}(\boldsymbol{x}_0) + A,\end{aligned} \tag{5.2.7}$$

where $T_2^*, r_0,$ *and* A *are positive constants defined in* (5.2.6).

Proof. We follow the same bootstrapping argument used to prove Lemma 5.1.2. Note that

$$T_2^* := \sup\Big\{T \in (0,\infty]\ :\ \min_{i,j}|x_{1i}(t) - x_{2j}(t)| \geq t + \frac{r_0}{2} \quad \text{for } t \in [0,T)\Big\} > 0.$$

Consider the set $\mathcal{S}_2$:

$$\mathcal{S}_2 := \{T \in (0, T_2^*]\ :\ \text{estimates (i) and (ii) in (5.2.7) hold for } t \in [0,T)\}.$$

In the following two steps, we will show that $\mathcal{S}_2$ is nonempty, and its supremum is equal to T_2^*.

⋄ Step A ($\mathcal{S}_2$ is not empty): A type of bootstrapping argument is used:

- Step A.1: Rough estimates of $\mathcal{V}(\boldsymbol{v})$ and $\mathcal{V}(\boldsymbol{x})$.

- Step A.2: Refined estimate of $\mathcal{V}(\boldsymbol{v})$ using the rough estimate of $\mathcal{V}(\boldsymbol{v})$ and the second Gronwall inequality in Lemma 5.2.2.

- Step A.3: Estimate of $\mathcal{V}(\boldsymbol{x})$ using the first Gronwall inequality in Lemma 5.2.2.

- Step A.4: Repeat the above process again to derive an optimal estimate.

• Step A.1 (rough estimates of $\mathcal{V}(\boldsymbol{x})$ and $\mathcal{V}(\boldsymbol{v})$): From the assumption $(\mathcal{C}_3 1)(ii)$ and the relation

$$\frac{KN_1}{N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}_0)) = \beta_0,$$

there exists $\tau \in (0, T_2^*)$ such that

$$\begin{aligned} &\frac{KN_2}{N}\psi(0)\mathcal{V}(\boldsymbol{v}(t)) < \frac{2KN_2}{N}\psi(0)\mathcal{V}(\boldsymbol{v}_0) < \frac{1}{4}\beta_0 \iff \mathcal{V}(\boldsymbol{v}(t)) < 2\mathcal{V}(\boldsymbol{v}_0), \\ &\frac{KN_1}{N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}(t))) > \frac{3\beta_0}{4} \iff \mathcal{V}(x(t)) < \mathcal{V}(x_0) + A, \quad t \in [0, \tau). \end{aligned} \tag{5.2.8}$$

Thus, $\tau \in \mathcal{S}_2$, i.e., the set $\mathcal{S}_2$ is not empty.

• Step A.2 (Refined estimates of $\mathcal{V}(\boldsymbol{x})$ and $\mathcal{V}(\boldsymbol{v})$): Note that it follows from Lemma 5.2.2(ii) that

$$\frac{d\mathcal{V}(\boldsymbol{v})}{dt} \le -\frac{KN_1}{N}\psi(\sqrt{2}\mathcal{V}(\boldsymbol{x}))\mathcal{V}(\boldsymbol{v}) + \frac{KN_2}{N}\psi(0)(\mathcal{V}(\boldsymbol{v}))^2 + 2K\sqrt{\frac{2N_1N_2}{N}}\psi_M. \tag{5.2.9}$$

The a priori estimates in (5.2.8) and (5.2.9) are used to obtain

$$\frac{d\mathcal{V}(\boldsymbol{v})}{dt} \le -\frac{\beta_0}{2}\mathcal{V}(\boldsymbol{v}) + 2K\sqrt{\frac{2N_1N_2}{N}}\psi\Big(t + \frac{r_0}{2}\Big) \quad \text{in } (0, \tau).$$

Gronwall's inequality implies a refined estimate for $\mathcal{V}(\boldsymbol{v})$:

$$\mathcal{V}(\boldsymbol{v}(t)) \le \mathcal{V}(\boldsymbol{v}_0)e^{-\frac{\beta_0}{2}t} + 2K\sqrt{\frac{2N_1N_2}{N}}\int_0^t \psi\Big(s + \frac{r_0}{2}\Big)e^{\frac{\beta_0}{2}(s-t)}ds \quad \text{in } [0, \tau). \tag{5.2.10}$$

• Step A.3 (Refined estimates of $\mathcal{V}(\boldsymbol{x})$): Using Lemma 5.2.2, (5.2.10), and Fubini's theorem, we obtain

$$\mathcal{V}(\boldsymbol{x}(t)) \le \mathcal{V}(\boldsymbol{x}_0) + \frac{2\mathcal{V}(\boldsymbol{v}_0)}{\beta_0} + \frac{4\sqrt{2}K}{\beta_0}\sqrt{\frac{N_1N_2}{N}}\int_{\frac{r_0}{2}}^{\infty}\psi(s)ds \quad \text{in } [0, \tau],$$

where the same estimate in (5.1.15) in Lemma 5.1.2 is used.

• Step A.4 ($\sup \mathcal{S}_2 = T_2^*$): By Step A.1, we know that the set $\mathcal{S}_2$ has a supremum, so we set

$$S_2^* := \sup \mathcal{S}_2 \in (0, T_2^*].$$

Now we must rule out the possibility that

$$S_2^* < T_2^* \tag{5.2.11}$$

to obtain the desired estimate. Assume that (5.2.11) holds. It follows that

$$\lim_{t \to S_2^* -} \mathcal{V}(\boldsymbol{x}(t)) = \mathcal{V}(\boldsymbol{x}_0) + A \quad \text{or} \quad \lim_{t \to S_2^* -} \mathcal{V}(\boldsymbol{v}(t)) = 2\mathcal{V}(\boldsymbol{v}_0).$$

The conditions $(\mathcal{C}_3 1)(ii)$ and $(\mathcal{C}_3 2)(i)$ provide us

$$\mathcal{V}(\boldsymbol{x}(t)) < \mathcal{V}(\boldsymbol{x}_0) + A - \epsilon, \quad t \in [0, \tau), \quad \epsilon \ll 1.$$

On the other hand, if

$$\lim_{t \to S_2^* -} \mathcal{V}(\boldsymbol{v}(t)) = 2\mathcal{V}(\boldsymbol{v}_0), \tag{5.2.12}$$

the assumption $(\mathcal{C}_3 2)(i)$ and (5.2.10) imply

$$\begin{aligned} \mathcal{V}(\boldsymbol{v}(t)) &\leq \mathcal{V}(\boldsymbol{v}_0) e^{-\frac{\beta_0}{2} t} + \frac{4\sqrt{2}K}{\beta_0} \sqrt{\frac{N_1 N_2}{N}} \psi\Big(\frac{r_0}{2}\Big) \\ &< \mathcal{V}(\boldsymbol{v}_0) e^{-\frac{\beta_0}{2} t} + \mathcal{V}(\boldsymbol{v}_0), \quad \text{in } [0, S_2^*). \end{aligned} \tag{5.2.13}$$

Letting $t \to S^* -$ in (5.2.13), we have

$$\lim_{t \to S^* -} \mathcal{V}(\boldsymbol{v}(t)) \leq \mathcal{V}(\boldsymbol{v}_0)(e^{-\frac{\beta_0}{2} S^*} + 1) < 2\mathcal{V}(\boldsymbol{v}_0),$$

which contradicts (5.2.12). Therefore, we obtain our desired result. □

5.2.4 Emergence of bi-cluster flocking

In this subsection, we prove bi-cluster flocking for system (5.2.1) using Lemma 5.2.1 with $T_2^* = \infty$.

Theorem 5.2.1. (Emergence of bi-cluster flocking) *Suppose conditions* $(\mathcal{C}_3 1)$ *and* $(\mathcal{C}_3 2)$ *hold. Then* $T_2^* = \infty$, *and the following bi-flocking estimates hold: for* $t \in [0, \infty)$,

$$(i)\ \mathcal{V}(\boldsymbol{x}(t)) < \mathcal{V}(\boldsymbol{x}_0) + A, \quad |\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)| > t + \frac{r_0}{2},$$
$$(ii)\ \mathcal{V}(\boldsymbol{v}(t)) \leq C\Big[e^{-\frac{\beta_0}{4}t} + \psi\Big(\frac{t + r_0}{2}\Big)\Big].$$

Proof. We split the estimates into two steps.

- Step A: Lemma 5.2.1 holds with $T_2^* = \infty$.

Suppose to the contrary. Then there exist i_0 and j_0 such that

$$\|\boldsymbol{x}_{1i_0}(T_2^*) - \boldsymbol{x}_{2j_0}(T_2^*)\| = T_2^* + \frac{r_0}{2}. \tag{5.2.14}$$

On the other hand, note that

$$\begin{aligned}\dot{\boldsymbol{v}}_{1c} &= \frac{K}{2NN_1}\sum_{i=1}^{N_1}\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}\|)\|\boldsymbol{v}_{1k} - \boldsymbol{v}_{1i}\|^2\boldsymbol{v}_{1i} \\ &\quad + \frac{K}{NN_1}\sum_{i=1}^{N_1}\sum_{k=1}^{N_2}\psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2k} - \langle \boldsymbol{v}_{1i}, \boldsymbol{v}_{2k}\rangle \boldsymbol{v}_{1i}),\end{aligned} \tag{5.2.15}$$

where the following relation is used:

$$\begin{aligned}&\frac{K}{NN_1}\sum_{i=1}^{N_1}\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{1k} - \langle \boldsymbol{v}_{1i}, \boldsymbol{v}_{1k}\rangle \boldsymbol{v}_{1i}) \\ &\qquad = \frac{K}{2NN_1}\sum_{i=1}^{N_1}\sum_{k=1}^{N_1}\psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}\|)\|\boldsymbol{v}_{1k} - \boldsymbol{v}_{1i}\|^2\boldsymbol{v}_{1i}.\end{aligned}$$

Next, we integrate (5.2.15) and use the relations

$$
\begin{aligned}
&\Big\| \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \psi(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}\|) \|\boldsymbol{v}_{1k} - \boldsymbol{v}_{1i}\|^2 \boldsymbol{v}_{1i} \Big\| \\
&\qquad \le \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \psi(0) \|\boldsymbol{v}_{1k} - \boldsymbol{v}_{1i}\|^2 = \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \psi(0) \|\boldsymbol{v}_{1k} - \boldsymbol{v}_{1c} + \boldsymbol{v}_{1c} - \boldsymbol{v}_{1i}\|^2 \\
&\qquad = 2N_1 \psi(0) \sum_{i=1}^{N_1} \|\boldsymbol{v}_{1c} - \boldsymbol{v}_{1i}\|^2 = 2N_1 \psi(0) \mathcal{V}_1^2(\boldsymbol{v}), \\
&\Big\| \sum_{i=1}^{N_1} \sum_{k=1}^{N_2} \psi(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2k} - \langle \boldsymbol{v}_{2k}, \boldsymbol{v}_{1i} \rangle \boldsymbol{v}_{1i}) \Big\| \le 2N_1 N_2 \psi\Big(t + \frac{r_0}{2}\Big)
\end{aligned} \tag{5.2.16}
$$

to obtain

$$
\|\boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}(0)\| \le \frac{K\psi(0)}{N} \int_0^t \mathcal{V}^2(\boldsymbol{v}(s)) ds + 2K \int_0^t \psi\Big(s + \frac{r_0}{2}\Big) ds, \quad t \in [0, T_2^*). \tag{5.2.17}
$$

We now use (5.2.10) to show that

$$
\mathcal{V}^2(\boldsymbol{v}(t)) \le 2\Big[\mathcal{V}^2(\boldsymbol{v}_0) e^{-\beta_0 t} + \frac{8K^2 N_1 N_2}{N} \Big(\int_0^t \psi(s + \frac{r_0}{2}) e^{\frac{\beta_0(s-t)}{2}} ds \Big)^2 \Big]. \tag{5.2.18}
$$

It follows from (5.2.18) and Lemma 5.2.3 that for $t \in [0, T_2^*)$,

$$
\mathcal{V}^2(\boldsymbol{v}(t)) \le 2\mathcal{V}^2(\boldsymbol{v}_0) e^{-\beta_0 t} + \frac{128 K^2 N_1 N_2}{N \beta_0^2} \Big[\psi^2\Big(\frac{r_0}{2}\Big) e^{-\frac{\beta_0 t}{2}} + \psi^2\Big(\frac{t + r_0}{2}\Big) \Big]. \tag{5.2.19}
$$

In (5.2.17), (5.2.19) is used to obtain

$$
\begin{aligned}
&\|\boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}(0)\| \\
&\quad \le \frac{K\psi(0)}{N} \int_0^t \mathcal{V}^2(\boldsymbol{v}(s)) ds + 2K \int_0^t \psi\Big(s + \frac{r_0}{2}\Big) ds \\
&\quad \le \frac{2K\psi(0)}{N\beta_0} \mathcal{V}^2(\boldsymbol{v}_0) + 2K \int_0^t \psi(s + \frac{r_0}{2}) ds \\
&\qquad + \frac{128 K^3 \psi(0) N_1 N_2}{N^2 \beta_0^2} \Big[\frac{2}{\beta_0} \psi^2(\frac{r_0}{2}) + \psi(\frac{r_0}{2}) \int_0^t \psi(\frac{s + r_0}{2}) ds \Big],
\end{aligned}
$$

for the time $t \in [0, T_2^*)$. On the other hand, condition $(\mathcal{C}_3 2)(ii)$ implies

$$\|\boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}(0)\| < \frac{\sqrt{3}-1}{4}, \quad \text{in } [0, T_2^*).$$

Using $(\mathcal{C}_3 1)$, we have

$$\begin{aligned}
\langle \boldsymbol{v}_{1i}(t), \boldsymbol{e}_d \rangle &= \langle \boldsymbol{v}_{1c}(0), \boldsymbol{e}_d \rangle + \langle \boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}(0), \boldsymbol{e}_d \rangle + \langle \boldsymbol{v}_{1i}(t) - \boldsymbol{v}_{1c}(t), \boldsymbol{e}_d \rangle \\
&> \frac{\sqrt{3}}{2} - \|\boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}(0)\| - 2\mathcal{V}(\boldsymbol{v}_0) > \frac{1}{2}, \quad \text{in } [0, T_2^*).
\end{aligned}$$

Similarly, we obtain

$$\langle \boldsymbol{v}_{2j}(t), \boldsymbol{e}_d \rangle < -\frac{1}{2} \quad \text{in } [0, T_2^*).$$

Thus,

$$\begin{aligned}
\|\boldsymbol{x}_{1i}(t) - \boldsymbol{x}_{2j}(t)\| &\geq \left\| \int_0^t \langle \boldsymbol{v}_{1i}(s) - \boldsymbol{v}_{2j}(s), \boldsymbol{e}_d \rangle ds + \langle \boldsymbol{x}_{1i}(0) - \boldsymbol{x}_{2j}(0), \boldsymbol{e}_d \rangle \right\| \\
&> t + \frac{r_0}{2} \quad \text{in } [0, T_2^*],
\end{aligned}$$

which contradicts (5.2.14). Therefore,

$$T_2^* = \infty.$$

• Step B (Estimate of $\mathcal{V}(\boldsymbol{v})$): Applying Gronwall's inequality to (5.2.7) and Lemma 5.2.3 yields

$$\begin{aligned}
\mathcal{V}(\boldsymbol{v}(t)) &\leq \mathcal{V}(\boldsymbol{v}_0)e^{-\frac{\beta_0}{2}t} + 2K\sqrt{\frac{2N_1N_2}{N}} \int_0^t \psi\Big(s + \frac{r_0}{2}\Big) e^{\frac{\beta_0}{2}(s-t)} ds \\
&\leq \mathcal{V}(\boldsymbol{v}_0)e^{-\frac{\beta_0}{2}t} + \frac{4K}{\beta_0}\sqrt{\frac{2N_1N_2}{N}} \Big(\psi(\frac{r_0}{2})e^{-\frac{\beta_0 t}{4}} + \psi(\frac{t+r_0}{2})\Big) \\
&\leq C\Big[e^{-\frac{\beta_0 t}{4}} + \psi(\frac{t+r_0}{2})\Big]
\end{aligned}$$

□

Remark 5.2.1. *It is easy to verify that the estimates given in Theorem 5.2.1 imply asymptotic bi-cluster flocking in the sense of Definition 2.1.1.*

In the following proposition, we also show that there are asymptotic velocities such that bi-clusters evolve to them.

Proposition 5.2.2. *Suppose conditions* $(\mathcal{C}_3 1)$ *and* $(\mathcal{C}_3 2)$ *hold. Then there exist constant asymptotic states* $\boldsymbol{v}_{1c}^\infty$ *and* $\boldsymbol{v}_{2c}^\infty$ *such that*

$$\|\boldsymbol{v}_{1c}(t)-\boldsymbol{v}_{1c}^\infty\|+\|\boldsymbol{v}_{2c}(t)-\boldsymbol{v}_{2c}^\infty\| \le C\Big[e^{-\frac{\beta_0 t}{2}}+\int_t^\infty \psi\Big(s+\frac{r_0}{2}\Big)ds\Big] \quad \text{as } t\to\infty,$$

where C *is a positive constant.*

Proof is basically same as in Theorem 3.2.2, from the equation

$$\begin{aligned}\boldsymbol{v}_{1c}(t) = \boldsymbol{v}_{1c}(0) &+ \frac{K}{2NN_1}\sum_{i=1}^{N_1}\sum_{k=1}^{N_1}\int_0^t \psi(\|\boldsymbol{x}_{1k}-\boldsymbol{x}_{1i}\|)\|\boldsymbol{v}_{1k}-\boldsymbol{v}_{1i}\|^2\boldsymbol{v}_{1i}ds\\ &+\frac{K}{NN_1}\sum_{i=1}^{N_1}\sum_{k=1}^{N_2}\int_0^t \psi(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{1i}\|)(\boldsymbol{v}_{2k}-\langle \boldsymbol{v}_{1i},\boldsymbol{v}_{2k}\rangle\boldsymbol{v}_{1i})ds.\end{aligned}$$

5.3 Numerical simulations

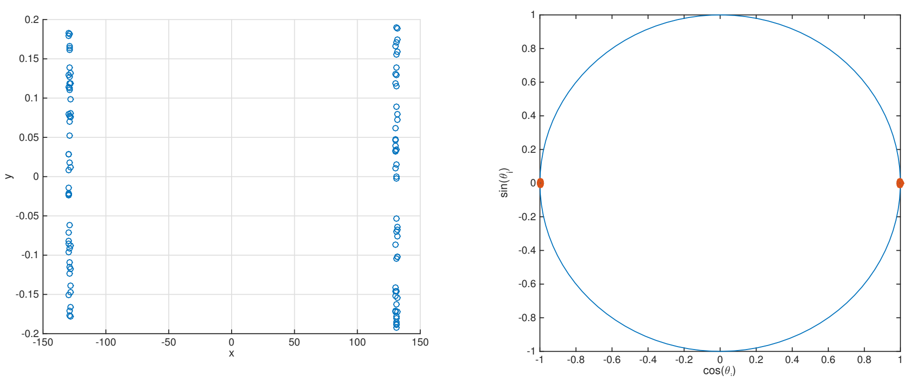
In this section, we provide numerical simulations on the two dimensional model, the generalized J-K model. In order to compare it to Theorem 5.1.1, the initial data were chosen to satisfy the conditions the conditions in Section 5.1.3 (see Figure 5.1). We used the fourth-order Runge-Kutta method. The common parameters used in the simulations were as follows,

$$\psi(s)=\frac{1}{(1+s^2)^{\frac{1}{2}}},\quad N_1=N_2=50,\quad \Delta t=0.01.$$

Particles in each group were randomly distributed around designated points. Their velocities indicated almost the opposite direction with respect to that of another group. We performed numerical integrations of (2.4.13) with different coupling strengths, $K=0.5,1,2,4,8$, to assess their effects.

In Figure 5.2, we consider the time-variations of local fluctuations. Notice that the local variation of heading angles tends to zero exponentially fast, and the local variation of spatial configurations also relaxes to some positive constant exponentially fast. In both figures, as the coupling strength doubles, the time it takes to achieve a relaxed state reduces to almost half of that of the preceding one.

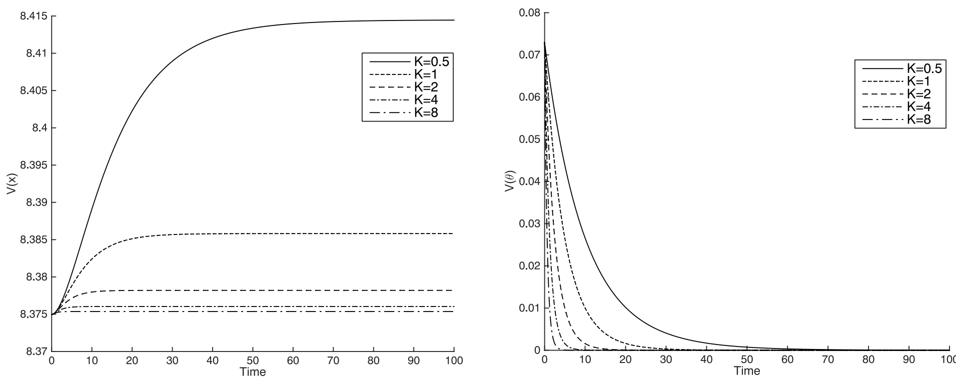
In Figure 5.3, we plot the intergroup distance and phase difference. Observe that the intergroup distance tends to infinity almost linearly, and regardless of the coupling strength, the distance grows at the same rate; hence, the graph appears to be one line. The phase difference converges to some constant exponentially fast, and the convergence time and coupling strength are inversely proportional. Note that although the time taken to reach a certain constant varies for different coupling strengths, they all go to the same constant regardless of the coupling strengths.



(a) Initial position

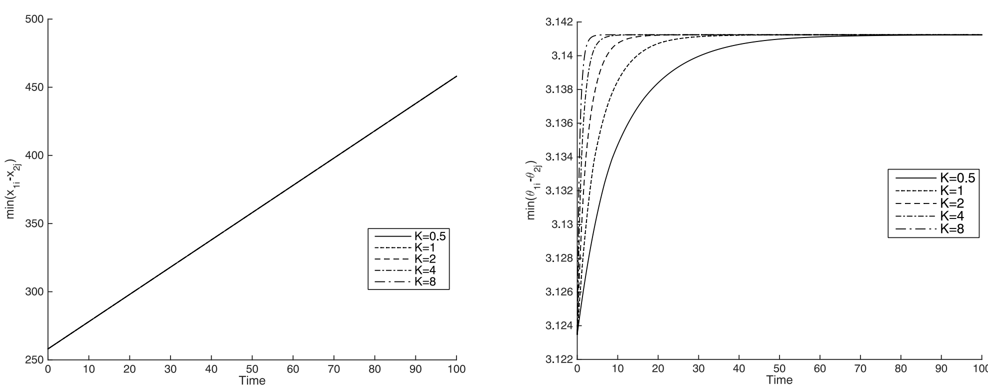
(b) Initial velocity

Figure 5.1: Initial configuration: The generalized J-K model



(a) Dynamics of $\mathcal{V}(\boldsymbol{x})$ (b) Dynamics of $\mathcal{V}(\Theta)$

Figure 5.2: The evidence of local flocking



(a) Dynamics of $\min_{i,j}(|x_{1i} - x_{2j}|)$ (b) Dynamics of $\min_{i,j}(|\theta_{1i} - \theta_{2j}|)$

Figure 5.3: The evidence of separation

Chapter 6

Multi-cluster flocking with unit speed constraint

In this chapter, we analyze the unit speed constrained model with the arguments of Chapter 4. In Chapter 5, we considered a set of initial configurations that is close to the separating two-particle model. We continue to analyze the same model with other situations, totally separating conditions and multi-cluster separating conditions, which appeared in Chapter 4. In the same way, we can study on the critical coupling strength of C-S model with unit speed constraint. This chapter is based on the joint work in [43].

6.1 A necessary condition for mono-cluster flocking

In this section, we provide a framework for the non-existence of mono-cluster flocking and state a *necessary condition* for the emergence of a mono-cluster flocking. As in the arguments of Chapter 4, we study the condition that every particle with different velocities does not flock each other.

Before we start the analysis, we recall the flocking estimates on the mono-cluster formation for (2.4.13). We set

$$D(\boldsymbol{v}) := \max_{i,j} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|.$$

Theorem 6.1.1. [20] *Suppose that the coupling strength and initial configuration* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *satisfy*

$$K > 0,\ \|\boldsymbol{v}_i^0\| = 1, \quad \min_{\forall\ i\neq j}\langle \boldsymbol{v}_i^0, \boldsymbol{v}_j^0\rangle > 0, \quad 0 < D(\boldsymbol{v}_0) < \frac{K\mathcal{A}(\boldsymbol{v}_0)}{2}\int_{D(\boldsymbol{x}_0)}^{\infty}\psi(s)ds,$$

Then, for any solution $\big(\boldsymbol{x}(t), \boldsymbol{v}(t)\big)$ *to system* (2.4.13)*, there exists a positive constant* d_x^∞ *such that*

$$\sup_{0\le t<\infty} D\big(\boldsymbol{x}(t)\big) \le d_x^\infty, \quad t \ge 0, \quad D\big(\boldsymbol{v}(t)\big) \le D(\boldsymbol{v}_0)e^{-KC_0\psi(d_x^\infty t)}.$$

Remark 6.1.1. *Note that Theorem 6.1.1 yields a sufficient condition for a mono-cluster flocking. For a small coupling strength* $K \ll 1$*, bi-cluster and multi-cluster flockings can emerge from some initial configurations. It has been shown that local flocking, in particular bi-cluster flocking, can emerge from some well-prepared configurations close to bi-cluster configurations in Chapter 4 and 5.*

6.1.1 A framework and main result

In this subsection, we will introduce a framework for the non-existence of mono-cluster flocking. Let $\mathcal{G} := \{(\boldsymbol{x}_{i0}, \boldsymbol{v}_{i0})\}_{i=1}^N$ be an initial non-flocking configuration of the ensemble of C-S particles. Then, we set subensembles $\mathcal{G}_1, \cdots, \mathcal{G}_n$ of the total ensemble $\mathcal{G}$ according to initial velocity: for $\alpha = 1, \cdots, n$,

$$(\boldsymbol{x}_{\alpha i}, \boldsymbol{v}_{\alpha i}), (\boldsymbol{x}_{\alpha j}, \boldsymbol{v}_{\alpha j}) \in \mathcal{G}_\alpha \quad \Longleftrightarrow \quad \boldsymbol{v}_{\alpha i0} = \boldsymbol{v}_{\alpha j}^0, \quad \text{for all } i, j \le |\mathcal{G}_\alpha| =: N_\alpha.$$

Since we assume the initial configuration is not in the mono-cluster flocking state, we have $n \ge 2$, and the original system (2.4.13) can be rewritten as:

$$\begin{aligned}
&\dot{\boldsymbol{x}}_{\alpha i} = \boldsymbol{v}_{\alpha i}, \quad t > 0, \quad i = 1, 2, \cdots, N_\alpha,\\
&\dot{\boldsymbol{v}}_{\alpha i} = \frac{K}{N}\sum_{k=1}^{N_\alpha}\psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|)\left(\boldsymbol{v}_{\alpha k} - \frac{\langle \boldsymbol{v}_{\alpha k}, \boldsymbol{v}_{\alpha i}\rangle}{\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i}\rangle}\boldsymbol{v}_{\alpha i}\right)\\
&\qquad + \frac{K}{N}\sum_{\beta\neq\alpha}\sum_{k=1}^{N_\beta}\psi(\|\boldsymbol{x}_{\beta k} - \boldsymbol{x}_{\alpha i}\|)\left(\boldsymbol{v}_{\beta k}(t) - \frac{\langle \boldsymbol{v}_{\beta k}, \boldsymbol{v}_{\alpha i}\rangle}{\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i}\rangle}\boldsymbol{v}_{\alpha i}\right),\\
&(\boldsymbol{x}_{\alpha i}(0), \boldsymbol{v}_{\alpha i}(0)) = (\boldsymbol{x}_{\alpha i0}, \boldsymbol{v}_{\alpha i0}), \quad \|\boldsymbol{v}_{\alpha i0}\| = 1.
\end{aligned} \tag{6.1.1}$$

Here we assume the short-range communication weight such that

$$\psi(s) = \frac{1}{(1+s^2)^{\frac{\beta}{2}}}, \quad \beta \geq 1.$$

For conveniences, we introduce local averages:

$$\boldsymbol{x}_{\alpha c} := \frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha} \boldsymbol{x}_{\alpha i}, \qquad \boldsymbol{v}_{\alpha c} := \frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha} \boldsymbol{v}_{\alpha i}. \tag{6.1.2}$$

Now, we describe the geometry of initial separation between sub-ensembles. For a given initial configuration $(\boldsymbol{x}_0, \boldsymbol{v}_0)$, we set

$$\begin{aligned} &\theta_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \min_{\beta \neq \alpha} \arccos(\boldsymbol{v}_{\beta c}(0) \cdot \boldsymbol{v}_{\alpha c}(0)), \quad D(\boldsymbol{x}_0) := \max_{\beta\neq\alpha, i, k} \|\boldsymbol{x}_{\alpha i 0} - \boldsymbol{x}_{\beta k 0}\|, \\ &T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \max_{\beta\neq\alpha,i,k} \left\{0, -\frac{\big(\boldsymbol{x}_{\alpha i 0} - \boldsymbol{x}_{\beta k 0}\big)\cdot \boldsymbol{v}_{\alpha c}(0)}{\lambda_0}\right\}, \quad \lambda_0 := \cos\frac{\theta_0}{4} - \cos\frac{3\theta_0}{8}. \end{aligned} \tag{6.1.3}$$

For notational simplicity, we suppress $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ dependence in T_0, K_0 in the following:

$$\theta_0 := \theta_0(\boldsymbol{x}_0, \boldsymbol{v}_0), \qquad T_0 := T_0(\boldsymbol{x}_0, \boldsymbol{v}_0).$$

Remark 6.1.2. *We can easily see that* $\theta_0 \in (0, \pi]$.

We next introduce a coupling strength $K_0(\boldsymbol{x}_0, \boldsymbol{v}_0)$ depending on the geometry of the initial configuration $(\boldsymbol{x}_0, \boldsymbol{v}_0)$.

• If initial configuration satisfies

$$\min_{\beta\neq\alpha,i,k} \big(\boldsymbol{x}_{\alpha i 0} - \boldsymbol{x}_{\beta k 0}\big)\cdot \boldsymbol{v}_{\alpha c}(0) < 0,$$

then, we set

$$\begin{aligned} K_0(\boldsymbol{x}_0, \boldsymbol{v}_0) &:= \min\left\{\frac{1-\cos\frac{\theta_0}{8}}{2T_0}, \ \frac{\cos\frac{\theta_0}{8} - \cos\frac{\theta_0}{4}}{D(\boldsymbol{x}_0) + 2T_0}, \ \frac{\lambda_0(\cos\frac{\theta_0}{8} - \cos\frac{\theta_0}{4})}{(1-\gamma_N)\int_0^\infty \psi(s)ds}\right\}, \\ \gamma_N &:= \frac{\min N_\beta}{N}. \end{aligned}$$

• If initial configuration satisfies

$$\min_{\beta\neq\alpha,i,k}\big(\boldsymbol{x}_{\alpha i0}-\boldsymbol{x}_{\beta k0}\big)\cdot\boldsymbol{v}_{\alpha c}(0)\geq 0,$$

then, we set

$$K_0(\boldsymbol{x}_0,\boldsymbol{v}_0):=\frac{\bar{\lambda}_0(1-\cos\frac{\theta_0}{8})}{(1-\gamma_N)\int_0^\infty\psi(s)ds},\qquad \bar{\lambda}_0=\cos\frac{\theta_0}{8}-\cos\frac{7\theta_0}{8}.$$

Now we are ready to state our main result as follows.

Theorem 6.1.2. *Let* $(\boldsymbol{x},\boldsymbol{v})$ *be a global solution to* (2.4.13) *with initial data satisfying*

$$\max_{i\neq j}\|\boldsymbol{v}_{i0}-\boldsymbol{v}_{j0}\|>0.$$

If $K<K_0(\boldsymbol{x}_0,\boldsymbol{v}_0)$*, then we have*

$$\min_{\alpha\neq\beta,i,k}\sup_{0\leq t<\infty}\|\boldsymbol{x}_{\alpha i}(t)-\boldsymbol{x}_{\beta k}(t)\|=\infty,\qquad \min_{\alpha\neq\beta,i,k}\liminf_{t\to\infty}\|\boldsymbol{v}_{\alpha i}(t)-\boldsymbol{v}_{\beta k}(t)\|>0.$$

i.e., mono-cluster flocking does not occur asymptotically. Moreover, each groups are separating.

6.1.2 Dynamics of local averages and fluctuations

In this subsection, we provide estimates on the local averages and fluctuations defined in (6.1.2). Now, we introduce a useful function which is crucial for the study of the Cucker-Smale model with unit speed:

$$v_\alpha^m(t):=\min_{1\leq i\leq N_\alpha}\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{e},\quad t\geq 0, \tag{6.1.4}$$

where $\boldsymbol{e}$ represents a unit constant vector, which will be replaced later by a fixed vector $\boldsymbol{e}_\alpha(T_0)$ depending on initial data. Then, we have the following proposition with respect to function $\boldsymbol{v}_\alpha^m(t)$ defined in (6.1.4) as follows.

Proposition 6.1.1. *Let* $\big(\boldsymbol{x}_{\alpha i},\boldsymbol{v}_{\alpha i}\big)$, $\alpha=1,\cdots,n$ *be a solution to system (6.1.1). Then, for any* α *and* $\boldsymbol{e}$*, we have*

$$\dot{v}_\alpha^m(t)\geq -K(1-\gamma_N)\psi_M(t),\ t\in[0,T),\ \psi_M(t):=\max_{\beta\neq\alpha,i,k}\psi(\|\boldsymbol{x}_{\beta k}(t)-\boldsymbol{x}_{\alpha i}(t)\|),$$

where T *is the time satisfying* $\langle\boldsymbol{v}_{\alpha i}(t),\boldsymbol{e}\rangle\geq 0$ *for all* $t\in[0,T)$.

Proof. Each $\boldsymbol{v}_{\alpha i}(t)$ (resp each $\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{e}$) is a real analytic function with values in $\mathbb{R}^d$ (resp in $\mathbb{R}$) on $t\in[0,+\infty)$. Thus, $v_\alpha^m(t)$ is piecewise analytic, hence there exists time steps $0=t_{\alpha 0}<t_{\alpha 1}<t_{\alpha 2}<\cdots$ such that for each $k\in\{0,1,2,\cdots\}$, there exists $i_k\in\{1,\cdots,N_\alpha\}$ satisfying $v_\alpha^m(t)=\boldsymbol{v}_{\alpha i_k}(t)\cdot\boldsymbol{e}$ for all $t\in[t_{\alpha k},t_{\alpha(k+1)}]$. For shorthand, we assume that

$$\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{e}=v_\alpha^m(t),\quad t\in[t_{\alpha k},t_{\alpha(k+1)}].$$

Note that as $\boldsymbol{v}_{\beta k}-\langle\boldsymbol{v}_{\alpha i},\boldsymbol{v}_{\beta k}\rangle\boldsymbol{v}_{\alpha i}$ is the component of $\boldsymbol{v}_{\beta k}$ that is orthogonal to $\boldsymbol{v}_{\alpha i}$, then

$$\|\boldsymbol{v}_{\beta k}-\langle\boldsymbol{v}_{\alpha i},\boldsymbol{v}_{\beta k}\rangle\boldsymbol{v}_{\alpha i}\|\le 1,$$

and

$$\langle\boldsymbol{v}_{\alpha k},\boldsymbol{e}\rangle-\langle\boldsymbol{v}_{\alpha k},\boldsymbol{v}_{\alpha i}\rangle\langle\boldsymbol{v}_{\alpha i},\boldsymbol{e}\rangle\ge\langle\boldsymbol{v}_{\alpha k},\boldsymbol{e}\rangle-\langle\boldsymbol{v}_{\alpha i},\boldsymbol{e}\rangle\ge 0.$$

Thus, we have

$$\begin{aligned}
&\dot{\boldsymbol{v}}_{\alpha i}(t)\cdot\boldsymbol{e}\\
&\quad=\frac{K}{N}\sum_{k=1}^{N_\alpha}\psi(\|\boldsymbol{x}_{\alpha k}(t)-\boldsymbol{x}_{\alpha i}(t)\|)\big(\boldsymbol{v}_{\alpha k}(t)-\langle\boldsymbol{v}_{\alpha i}(t),\boldsymbol{v}_{\alpha k}(t)\rangle\boldsymbol{v}_{\alpha i}(t)\big)\cdot\boldsymbol{e}\\
&\qquad+\frac{K}{N}\sum_{\beta\neq\alpha}\sum_{k=1}^{N_\beta}\psi(\|\boldsymbol{x}_{\beta k}(t)-\boldsymbol{x}_{\alpha i}(t)\|)\big(\boldsymbol{v}_{\beta k}(t)-\langle\boldsymbol{v}_{\alpha i}(t),\boldsymbol{v}_{\beta k}(t)\rangle\boldsymbol{v}_{\alpha i}(t)\big)\cdot\boldsymbol{e}\\
&\quad\ge-\frac{K}{N}(N-N_\alpha)\psi_M(t)\|\boldsymbol{v}_{\beta k}-\langle\boldsymbol{v}_{\alpha i},\boldsymbol{v}_{\beta k}\rangle\boldsymbol{v}_{\alpha i}\|\\
&\quad\ge-\frac{K(N-N_\alpha)}{N}\psi_M(t)\ge-K(1-\gamma_N)\psi_M(t),\quad t\in[0,T).
\end{aligned}$$

□

6.1.3 Non-existence of mono-cluster flocking

In this subsection, we will provide the proof of Theorem 6.1.2. We first briefly outline our strategy as follows. The idea of our main results can be split into three stages. For a given initial configuration $(\boldsymbol{x}_0,\boldsymbol{v}_0)$,

- Initial stage (from mixed configuration to segregated configuration): there exists a $T_0 \geq 0$ such that, for any i, k, and $\beta \neq \alpha$,

$$\big(\boldsymbol{x}_{\alpha i}(T_0) - \boldsymbol{x}_{\beta k}(T_0)\big) \cdot \boldsymbol{v}_{\alpha c}(T_0) \geq 0.$$

- Intermediate stage (maintaining segregated configuration): there exists $T_0^* > T_0$ which has the desired properties for all $t \in [T_0, T_0^*)$,

$$\min_{\alpha\neq\beta,i,k} \Big\{ \big(\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\beta k}(t)\big) \cdot \boldsymbol{e}_\alpha(T_0) \Big\} > \lambda_0, \quad \|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| > \lambda_0(t - T_0),$$

where $\boldsymbol{e}_\alpha(T_0)$ is the unit vector in the direction of $\boldsymbol{v}_{\alpha c}(T_0)$.

- Final stage (emergence of non-mono cluster configuration): finally we show that

$$T_0^* = \infty$$

and obtain the non-existence of mono-cluster flocking.

Emergence of segregated configurations

In this subsection, we will show that the configuration at time T_0 is well segregated:

$$\varDelta_{\alpha i,\beta k}(T_0) := \big(\boldsymbol{x}_{\alpha i}(T_0) - \boldsymbol{x}_{\beta k}(T_0)\big) \cdot \boldsymbol{v}_{\alpha c}(T_0) \geq 0. \tag{6.1.5}$$

Recall that

$$T_0 := \max_{\beta\neq\alpha,i,k} \Big\{ - \frac{\varDelta_{\alpha i,\beta k}(0)}{\lambda_0}, 0 \Big\}.$$

In the sequel, we assume without loss of generality that

$$\varDelta_{\alpha i,\beta k}(0) < 0 \quad \text{so that} \quad T_0 > 0.$$

Otherwise, $T_0 = 0$ and the desired estimate (6.1.5) holds trivially, and all the lemmas from Lemma 6.1.1 to Lemma 6.1.4 can be proved with better estimates. We stated this argument in the proof of Theorem 6.1.2, at the end of this section. As in the definition of $\boldsymbol{e}_\alpha(T_0)$, we set

$$\boldsymbol{e}_\alpha(t) = \frac{\boldsymbol{v}_{\alpha c}(t)}{\|\boldsymbol{v}_{\alpha c}(t)\|}.$$

Lemma 6.1.1. *Let* $(\boldsymbol{x}_{\alpha i}, \boldsymbol{v}_{\alpha i})$ *be a global solution to* (6.1.1) *with non-flocking initial data* $(\boldsymbol{x}_{\alpha i0}, \boldsymbol{v}_{\alpha i0})$*. If the coupling strength* K *satisfies*

$$0 < K < \frac{1 - \cos\frac{\theta_0}{8}}{2T_0},$$

then the following estimates hold: for $t \in [0, T_0]$ *and* $\beta \neq \alpha$*,*

$$\begin{aligned}
&(i)\ \boldsymbol{v}_{\alpha i}(t) \cdot \boldsymbol{v}_{\alpha c}(t) > \cos\frac{\theta_0}{8}, \quad \boldsymbol{v}_{\beta k}(t) \cdot \boldsymbol{v}_{\alpha c}(t) < \cos\frac{7\theta_0}{8},\\
&(ii)\ \boldsymbol{v}_{\alpha i}(t) \cdot \boldsymbol{v}_{\beta k}(t) < \cos\frac{7\theta_0}{8}, \quad \boldsymbol{e}_{\alpha}(t) \cdot \boldsymbol{e}_{\beta}(t) < \cos\frac{5\theta_0}{8}.
\end{aligned}$$

Proof. (i) Note that

$$\|\boldsymbol{v}_{\alpha k}(t) - \langle \boldsymbol{v}_{\alpha i}(t), \boldsymbol{v}_{\alpha k}(t)\rangle \boldsymbol{v}_{\alpha i}(t)\| \leq 1.$$

Now, we use system (6.1.1), the assumption of ψ in (1.0.2), and the above relation to get for any $\alpha \in \{1, \cdots, n\}$,

$$\|\dot{\boldsymbol{v}}_{\alpha i}(t)\| \leq \frac{KN_{\alpha}}{N} + \frac{K(N - N_{\alpha})}{N} = K. \tag{6.1.6}$$

Similarly, by direct calculation, we have

$$\|\dot{\boldsymbol{v}}_{\alpha c}(t)\| \leq K. \tag{6.1.7}$$

Thus, we combine estimates (6.1.6) and (6.1.7) to obtain

$$\left|\frac{d\big(\boldsymbol{v}_{\alpha i}(t) \cdot \boldsymbol{v}_{\alpha c}(t)\big)}{dt}\right| \leq 2K. \tag{6.1.8}$$

Then, we use estimate (6.1.8) and the assumption of K to get

$$\boldsymbol{v}_{\alpha i}(t) \cdot \boldsymbol{v}_{\alpha c}(t) \geq \boldsymbol{v}_{\alpha i}(0) \cdot \boldsymbol{v}_{\alpha c}(0) - 2KT_0 = 1 - 2KT_0 > \cos\frac{\theta_0}{8}, \quad t \in [0, T_0].$$

For the second estimate, we use the estimate of $\boldsymbol{v}_{\alpha c}(t)$ in (i) for all $\alpha \in \{1, \cdots, n\}$ to obtain that for any $\beta \neq \alpha$

$$\left|\frac{d\big(\boldsymbol{v}_{\beta k}(t) \cdot \boldsymbol{v}_{\alpha c}(t)\big)}{dt}\right| \leq |\dot{\boldsymbol{v}}_{\beta k}(t) \cdot \boldsymbol{v}_{\alpha c}(t)| + |\boldsymbol{v}_{\beta k}(t) \cdot \dot{\boldsymbol{v}}_{\alpha c}(t)| \leq 2K.$$

Thus, we obtain the following from the assumption of K.

$$\begin{aligned}\boldsymbol{v}_{\beta k}(t)\cdot\boldsymbol{v}_{\alpha c}(t) &\le \boldsymbol{v}_{\beta k}(0)\cdot\boldsymbol{v}_{\alpha c}(0)+2KT_0\\ &\le \cos\theta_0+2KT_0 \ \le \cos\theta_0+1-\cos\frac{\theta_0}{8}\\ &\le \cos\frac{7\theta_0}{8},\quad t\in[0,T_0].\end{aligned}$$

Here the last inequality is from properties of cosine functions. For $\theta_0\in(0,\pi]$,

$$\cos\frac{\theta_0}{8}+\cos\frac{7\theta_0}{8}=2\cos(\frac{\theta_0}{2})\cos(\frac{3\theta_0}{8})\ge 2\cos(\frac{\theta_0}{2})\cos(\frac{\theta_0}{2})=1+\cos\theta_0.$$

(ii) By using a similar analysis as in the second estimate of (i), we can derive the first estimate in (ii). For the last inequality, we use (i) and the definition of $\boldsymbol{e}_\alpha(t)$ to see that for any $\alpha\in\{1,\cdots,n\}$,

$$\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{e}_\alpha(t)\ge\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{v}_{\alpha c}(t)>\cos\frac{\theta_0}{8},\quad t\in[0,T_0] \tag{6.1.9}$$

Now we combine relation (6.1.9) and the previous estimates to get

$$\begin{aligned}&\arccos\big(\boldsymbol{e}_\alpha(t)\cdot\boldsymbol{e}_\beta(t)\big)\\ &\quad\ge\arccos\big(\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{v}_{\beta k}(t)\big)-\arccos\big(\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{e}_\alpha(t)\big)-\arccos\big(\boldsymbol{v}_{\beta k}(t)\cdot\boldsymbol{e}_\beta(t)\big)\\ &\quad>\frac{7\theta_0}{8}-\frac{\theta_0}{8}-\frac{\theta_0}{8}=\frac{5\theta_0}{8},\quad t\in[0,T_0].\end{aligned}$$

Hence, we obtain

$$\boldsymbol{e}_\alpha(t)\cdot\boldsymbol{e}_\beta(t)<\cos\frac{5\theta_0}{8},\quad t\in[0,T_0].$$

□

Lemma 6.1.2. *Let* $(\boldsymbol{x}_{\alpha i},\boldsymbol{v}_{\alpha i})$ *be a global solution to* (6.1.1) *with non-flocking initial data* $(\boldsymbol{x}_{\alpha i0},\boldsymbol{v}_{\alpha i0})$. *If the coupling strength* K *satisfies*

$$0<K<\min\Big\{\frac{1-\cos\frac{\theta_0}{8}}{2T_0},\ \frac{\cos\frac{\theta_0}{8}-\cos\frac{\theta_0}{4}}{D(\boldsymbol{x}_0)+2T_0}\Big\},$$

then, we have

$$\min_{\beta\neq\alpha,i,k}\varDelta_{\alpha i,\beta k}(T_0)>0.$$

Proof. For the desired estimate, we claim:

$$\min_{\beta\neq\alpha,i,k} \frac{d}{dt}\mathit{\Delta}_{\alpha i,\beta k}(t) > \lambda_0, \quad t \in [0, T_0]. \tag{6.1.10}$$

Proof of claim (6.1.10): For all $t \in [0, T_0]$, $\alpha \neq \beta$ and i, k,

$$\begin{aligned}
\big\|\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\beta k}(t)\big\| &= \Big\|\big(\boldsymbol{x}_{\alpha i0} - \boldsymbol{x}_{\beta k0}\big) + \int_0^t \big(\boldsymbol{v}_{\alpha i}(s) - \boldsymbol{v}_{\beta k}(s)\big)ds\Big\| \\
&\leq \big\|\boldsymbol{x}_{\beta k0} - \boldsymbol{x}_{\alpha i0}\big\| + 2T_0 \\
&\leq D(\boldsymbol{x}_0) + 2T_0,
\end{aligned}$$

By Lemma 6.1.1 and the assumption of K, we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathit{\Delta}_{\alpha i,\beta k}(t) \\
&\quad = \big(\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\beta k}(t)\big)\cdot\boldsymbol{v}_{\alpha c}(t) + \big(\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\beta k}(t)\big)\cdot\dot{\boldsymbol{v}}_{\alpha c}(t) \\
&\quad = \boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{v}_{\alpha c}(t) - \boldsymbol{v}_{\beta k}(t)\cdot\boldsymbol{v}_{\alpha c}(t) + \big(\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\beta k}(t)\big)\cdot\dot{\boldsymbol{v}}_{\alpha c}(t) \\
&\quad > \cos\frac{\theta_0}{8} - \cos\frac{7\theta_0}{8} - \big(D(\boldsymbol{x}_0) + 2T_0\big)K \\
&\quad > \cos\frac{\theta_0}{4} - \cos\frac{7\theta_0}{8} \qquad > \lambda_0, \quad t \in [0, T_0].
\end{aligned}$$

Now the claim (6.1.10) is proved. We integrate relation (6.1.10) to obtain

$$\mathit{\Delta}_{\alpha i,\beta k}(t) > \mathit{\Delta}_{\alpha i,\beta k}(0) + \lambda_0 t, \quad t \in (0, T_0].$$

Then, the defining relation of T_0 in (6.1.3) implies

$$\mathit{\Delta}_{\alpha i,\beta k}(T_0) > \mathit{\Delta}_{\alpha i,\beta k}(0) + \lambda_0 T_0 \geq 0.$$

We now take an minimum over α, β, i and k to obtain the desired result. $\square$

Proof of Theorem 6.1.2

In this subsection, we provide the proof of Theorem 6.1.2. Recall that we defined a normal vector in the direction of $\boldsymbol{v}_{\alpha c}(T_0)$:

$$\boldsymbol{e}_\alpha(T_0) := \frac{\boldsymbol{v}_{\alpha c}(T_0)}{\|\boldsymbol{v}_{\alpha c}(T_0)\|}.$$

Note that it is a well-defined since $\boldsymbol{v}_{\alpha c}(T_0)$ cannnot be zero from previous lemmas. We define

$$T_0^* := \sup \left\{ T \in [T_0, \infty) \;\middle|\; \min_{\alpha,i} \big(\boldsymbol{v}_{\alpha i}(t) \cdot \boldsymbol{e}_\alpha(T_0)\big) > \cos\frac{\theta_0}{4}, \text{ for all } t \in [T_0, T] \right\}. \tag{6.1.11}$$

Lemma 6.1.3. *Let $(\boldsymbol{x}, \boldsymbol{v})$ be a global solution to* (6.1.1) *with non-flocking initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$. If the coupling strength K satisfies*

$$0 < K < \frac{1 - \cos\frac{\theta_0}{8}}{2T_0}.$$

Then we have, for $t \in [T_0, T_0^*)$,

$$\begin{aligned}
&(i) \quad \max_{\beta k} \big(\boldsymbol{v}_{\beta,k}(t) \cdot \boldsymbol{e}_\alpha(T_0)\big) < \cos\frac{3\theta_0}{8},\\
&(ii) \quad \min_{\alpha\neq\beta,i,k} \Big\{ \big(\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\beta k}(t)\big) \cdot \boldsymbol{e}_\alpha(T_0) \Big\} > \lambda_0,
\end{aligned}$$

where $\lambda_0 := \cos\frac{\theta_0}{4} - \cos\frac{3\theta_0}{8}$.

Proof. (i) We use Lemma 6.1.1 to get that for $t \in [T_0, T_0^*)$

$$\begin{aligned}
&\arccos\big(\boldsymbol{v}_{\beta k}(t) \cdot \boldsymbol{e}_\alpha(T_0)\big)\\
&\quad \geq \arccos\big(\boldsymbol{e}_\alpha(T_0) \cdot \boldsymbol{e}_\beta(T_0)\big) - \arccos\big(\boldsymbol{v}_{\beta k}(t) \cdot \boldsymbol{e}_\beta(T_0)\big) > \frac{5\theta_0}{8} - \frac{\theta_0}{4} = \frac{3\theta_0}{8}.
\end{aligned}$$

Hence, we obtain

$$\boldsymbol{v}_{\beta k}(t) \cdot \boldsymbol{e}_\alpha(T_0) < \cos\frac{3\theta_0}{8}, \quad t \in [T_0, T_0^*).$$

(ii) By the definition of T_0^* and estimate (i), assertion (ii) holds trivially. $\square$

Lemma 6.1.4. *Let $(\boldsymbol{x}, \boldsymbol{v})$ be a global solution to* (2.4.13) *with non-flocking initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$. If the coupling strength K satisfies*

$$0 < K < \min\left\{ \frac{1 - \cos\frac{\theta_0}{8}}{2T_0}, \frac{\cos\frac{\theta_0}{8} - \cos\frac{\theta_0}{4}}{D(\boldsymbol{x}_0) + 2T_0} \right\}.$$

Then, we have

$$T_0^* > T_0 \quad \textit{and} \quad \psi_M(t) < \psi\Big(\lambda_0(t - T_0)\Big) \textit{ for } t \in (T_0, T_0^*).$$

Proof. (i) It follows from Lemma 6.1.1 that we have

$$\boldsymbol{v}_{\alpha i}(T_0) \cdot \boldsymbol{e}_\alpha(T_0) > \cos\frac{\theta_0}{4}.$$

Hence, we have $T_0^* > T_0$.

(ii) We use Lemma 6.1.2 and Lemma 6.1.3 to obtain

$$\begin{aligned}\|\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\beta k}(t)\| &\geq \big(\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\beta k}(t)\big) \cdot \boldsymbol{e}_\alpha(T_0) \\ &> \int_{T_0}^t \big(\boldsymbol{v}_{\alpha i}(s) - \boldsymbol{v}_{\beta k}(s)\big) \cdot \boldsymbol{e}_\alpha(T_0) ds \\ &> \lambda_0(t - T_0), \quad t \in (T_0, T_0^*).\end{aligned}$$

Thus, by the non-increasing property of $\psi(t)$, we get the conclusion. □

We are now ready to provide the proof of Theorem 6.1.2 as follows.

The proof of Theorem 6.1.2. Let $(\boldsymbol{x}, \boldsymbol{v})$ be a global solution to (2.4.13) with non-flocking initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$. If the coupling strength K satisfies

$$K < K_0.$$

Then, we claim: for $t \in (T_0, \infty)$,

$$\min_{\beta \neq \alpha, i, k} \Big(\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\beta k}(t)\Big) \cdot \boldsymbol{e}_\alpha(T_0) > \lambda_0, \quad \|\boldsymbol{x}_{\alpha i}(t) - \boldsymbol{x}_{\beta k}(t)\| > \lambda_0(t - T_0).$$

For the proof of the above claim, we consider two cases:

$$\text{Either} \quad T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) > 0, \quad \text{or} \quad T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) = 0.$$

• Case A: Suppose that we have $T_0(\boldsymbol{x}_0, \boldsymbol{v}_0) > 0$. Then, it follows from the arguments in Lemma 6.1.4 that $T_0^* > T_0$.
Suppose that

$$T_0^* < \infty.$$

Then, by definition in (6.1.11), there exist α, i such that

$$\boldsymbol{v}_{\alpha i}(T_0^*) \cdot \boldsymbol{e}_\alpha(T_0) = \cos\frac{\theta_0}{4}. \tag{6.1.12}$$

On the other hand, we use Proposition 6.1.1, Lemma 6.1.1, Lemma 6.1.4 and the assumption of K to obtain

$$\begin{aligned}\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{e}_\alpha(T_0) &\ge \boldsymbol{v}_\alpha^m(t) \ge \boldsymbol{v}_\alpha^m(T_0) - \frac{K(N-N_\alpha)}{N}\int_{T_0}^t \psi_M(s)ds \\ &\ge \cos\frac{\theta_0}{8} - \frac{K(1-\gamma_N)}{\lambda_0}\int_0^\infty \psi(s)ds > \cos\frac{\theta_0}{4}, \quad t\in[T_0,T_0^*].\end{aligned}$$

In particular, we have

$$\boldsymbol{v}_{\alpha i}(T_0^*)\cdot\boldsymbol{e}_\alpha(T_0) > \cos\frac{\theta_0}{4}$$

This contradicts inequality (6.1.12). Thus, we have $T_0^*=\infty$. Therefore, the conclusion (ii) of Lemma 6.1.3 implies the conclusion of Theorem 6.1.2.

• Case B: Suppose that we have

$$T_0(\boldsymbol{x}_0,\boldsymbol{v}_0)=0.$$

In this case, recall that

$$K_0 = \frac{1-\cos\frac{\theta_0}{8}}{(1-\gamma_N)\int_0^\infty \psi(s)ds}, \quad \bar{\lambda}_0 = \cos\frac{\theta_0}{8} - \cos\frac{7\theta_0}{8}.$$

Then, for $K<K_0$, we use the similar arguments in Case A. The conclusion of Lemma 6.1.2 is just (6.1.5) and same for Lemma 6.1.4. Lemma 6.1.1 can be proved without smallness of K and we can improve the result into $\boldsymbol{e}_\alpha(t)\cdot\boldsymbol{e}_\beta(t) < \cos\frac{7\theta_0}{8}$. From the same reason, the conclusion of Lemma 6.1.3 became $\min\limits_{\alpha\ne\beta,i,k}\left\{\big(\boldsymbol{v}_{\alpha i}(t)-\boldsymbol{v}_{\beta k}(t)\big)\cdot\boldsymbol{e}_\alpha(T_0)\right\} > \bar{\lambda}_0$. Hence we obtain

$$\|\boldsymbol{v}_{\alpha i}(t)-\boldsymbol{v}_{\beta k}(t)\| > \bar{\lambda}_0, \qquad \|\boldsymbol{x}_{\alpha i}(t)-\boldsymbol{x}_{\beta k}(t)\| > \bar{\lambda}_0 t, \quad \text{for all} \ \ t>0.$$

Finally, it follows from Case A and Case B that we complete the proof of Theorem 6.1.2. □

6.2 Emergence of multi-cluster flocking

In this section, we present an emergence of multi-cluster flocking to the Cucker-Smale model (2.4.13). In Section 6.1, we divided the particles into n sub-ensembles $\mathcal{G}_1, \cdots, \mathcal{G}_n$ according to their initial velocities, and showed that for a small coupling strength $K < K_0$, any two different particles in different groups do not flock. Thus, it is natural to ask whether two different particles in the same group will flock or not in a small coupling regime. In the sequel, we will concentrate this question by allowing the initial velocities of different particles in the same group to be slightly different.

Consider the Cucker-Smale flocking system with n sub-ensembles $\mathcal{G}_\alpha,\ \alpha = 1, 2, \cdots, n$:

$$\begin{aligned}
&\dot{\boldsymbol{x}}_{\alpha i} = \boldsymbol{v}_{\alpha i}, \quad t \geq 0, \quad \alpha = 1, 2, \cdots, n, \quad i = 1, \cdots, N_\alpha, \\
&\dot{\boldsymbol{v}}_{\alpha i} = \frac{K}{N} \sum_{k=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|) \left(\boldsymbol{v}_{\alpha k} - \frac{\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha k} \rangle}{\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i} \rangle} \boldsymbol{v}_{\alpha i} \right) \\
&\qquad + \frac{K}{N} \sum_{\beta \neq \alpha} \sum_{k=1}^{N_\beta} \psi(\|\boldsymbol{x}_{\beta k} - \boldsymbol{x}_{\alpha i}\|) \left(\boldsymbol{v}_{\beta k} - \frac{\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\beta k} \rangle}{\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i} \rangle} \boldsymbol{v}_{\alpha i} \right).
\end{aligned} \tag{6.2.13}$$

6.2.1 A framework and main result

As in Section 6.1, we define some parameters θ_0, δ_0 and r_0 related to the separations of each sub-ensembles:

$$\begin{aligned}
&\theta_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \min_{\beta \neq \alpha} \arccos \big(\boldsymbol{v}_{\beta c}(0) \cdot \boldsymbol{v}_{\alpha c}(0) \big), \\
&\delta_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \max_{\alpha, i} \arccos \big(\boldsymbol{v}_{\alpha i 0} \cdot \boldsymbol{v}_{\alpha c}(0) \big) \\
&r_0(\boldsymbol{x}_0, \boldsymbol{v}_0) := \min_{\alpha \neq \beta, i, k} \big(\boldsymbol{x}_{\alpha i 0} - \boldsymbol{x}_{\beta k 0} \big) \cdot \frac{\boldsymbol{v}_{\alpha c}(0)}{\|\boldsymbol{v}_{\alpha c}(0)\|}, \\
&\Lambda_0 := \cos\Big(\frac{\theta_0}{3} + \frac{\delta_0}{3} \Big) - \cos\Big(\frac{2\theta_0}{3} - \frac{\delta_0}{3} \Big).
\end{aligned}$$

Now, we introduce the local fluctuations and l_2- type functionals that measure the total fluctuations of each group:

$$\hat{\boldsymbol{x}}_{\alpha i} := \boldsymbol{x}_{\alpha i} - \boldsymbol{x}_{\alpha c}, \quad \hat{\boldsymbol{v}}_{\alpha i} := \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha c}, \quad \mathcal{X}_\alpha := \Big(\sum_{i=1}^{N_\alpha} \hat{\boldsymbol{x}}_{\alpha i}^2 \Big)^{\frac{1}{2}}, \quad \mathcal{V}_\alpha := \Big(\sum_{i=1}^{N_\alpha} \hat{\boldsymbol{v}}_{\alpha i}^2 \Big)^{\frac{1}{2}}.$$

We next state our framework $(\mathcal{C}_3)$ for a multi-cluster flocking as follows.

- $(\mathcal{C}_3 1)$ (Initial configuration): Initial configuration is well-separated and initial fluctuations are sufficiently small in the sense that

$$r_0 \geq 0, \quad \delta_0 \in \Big[0, \frac{1}{2}\theta_0\Big), \qquad \mathcal{V}_\alpha^0 \leq \frac{1}{4}\psi(\sqrt{2}\mathcal{X}_\alpha^0).$$

- $(\mathcal{C}_3 2)$ (Coupling strength): The coupling strength takes an intermediate value and the initial distance is large such that

$$\begin{aligned} &(i)\ \psi\big(\sqrt{2}(\mathcal{X}_\alpha^0 + A)\big) \geq \frac{3}{4}\psi(\sqrt{2}\mathcal{X}_\alpha^0), \\ &(ii)\ K < K_1 := \min\Big\{ \frac{\Lambda_0\big(\cos\delta_0 - \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3})\big)}{(1-\gamma_N)\int_0^\infty \psi(s)ds}, \\ &\qquad\qquad \frac{\Lambda_0 \psi(\sqrt{2}\mathcal{X}_\alpha^0)}{4\sqrt{N}(1-\gamma_N)\int_0^\infty \psi(s)ds} \Big\}, \end{aligned}$$

  where A and β_α are positive constants defined by the following relations

$$A := \frac{4\mathcal{V}_\alpha^0}{\beta_\alpha} + \frac{4K\sqrt{N}(1-\gamma_N)}{N\beta_\alpha\Lambda_0}\int_{r_0}^{+\infty}\psi(s)ds, \quad \beta_\alpha := KN_\alpha\psi(\sqrt{2}\mathcal{X}_\alpha^0). \tag{6.2.14}$$

Remark 6.2.1. *(i) By the assumption $(\mathcal{C}_3 1)$, we know that $\Lambda_0 > 0$,*
(ii) For fixed $\mathcal{X}_\alpha^0, \theta_0, \delta_0$, if $\mathcal{V}_\alpha^0 \ll 1$ and $r_0 \gg 1$, we can always choose such K satisfying $(\mathcal{C}_3 2)$.

Theorem 6.2.1. *Suppose that the framework* $(\mathcal{C}_3)$ *holds, and let* $(\boldsymbol{x}_{\alpha i}, \boldsymbol{v}_{\alpha i})$ *be a solution to system (6.2.13) with initial configuration* $(\boldsymbol{x}_{\alpha i0}, \boldsymbol{v}_{\alpha i0})$. *Then, we have the following estimates:*

$$
\begin{aligned}
&(i) \quad \min_{\beta\neq\alpha,i,k} \|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| > r_0 + \Lambda_0 t, \quad t \in (0,\infty),\\
&(ii) \quad \mathcal{X}_\alpha(t) < \mathcal{X}_\alpha^0 + A,\\
&(iii) \quad \mathcal{V}_\alpha(t) \le C_\alpha \max\Big\{ e^{-\frac{K\psi(\sqrt{2}\mathcal{X}_\alpha^0)t}{4}}, \psi\Big(r_0 + \frac{\Lambda_0 t}{2}\Big)\Big\}, \ \textit{for some } C_\alpha > 0,
\end{aligned}
$$

i.e., the multi-cluster flocking emerges.

Remark 6.2.2. *In Section 6.1, it follows from the classification that we assume* $\boldsymbol{v}_{\alpha i0} = \boldsymbol{v}_{\alpha c}(0)$ *for any* $\alpha \in \{1, \cdots, n\}$. *Thus the initial assumption of* δ_0 *in the above theorem is satisfied naturally.*

6.2.2 Dynamics of local averages and fluctuations

In this subsection, we study the time-evolution of local averages and fluctuations. We use the same definition as in (6.1.2).

Lemma 6.2.1. *Let* $\big(\boldsymbol{x}_{\alpha i}, \boldsymbol{v}_{\alpha i}\big)$ *be a solution to system* (6.2.13). *Then, local averages and fluctuations satisfy*

$$
\begin{cases}
\dot{\boldsymbol{x}}_{\alpha c} = \boldsymbol{v}_{\alpha c}, \quad t \ge 0, \quad \alpha = 1, 2, \cdots, n,\\
\dot{\boldsymbol{v}}_{\alpha c} = \dfrac{K}{2NN_\alpha} \displaystyle\sum_{k=1}^{N_\alpha}\sum_{i=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|)\|\hat{\boldsymbol{v}}_{\alpha i} - \hat{\boldsymbol{v}}_{\alpha k}\|^2 \boldsymbol{v}_{\alpha i}\\
\quad + \dfrac{K}{NN_\alpha} \displaystyle\sum_{\beta\neq\alpha}\sum_{k=1}^{N_\beta}\sum_{i=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\beta k} - \boldsymbol{x}_{\alpha i}\|)\big(\boldsymbol{v}_{\beta k} - \langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\beta k}\rangle \boldsymbol{v}_{\alpha i}\big).
\end{cases}
\tag{6.2.15}
$$

and

$$
\begin{cases}
\dot{\hat{\boldsymbol{x}}}_{\alpha i}(t) = \hat{\boldsymbol{v}}_{\alpha i}(t), \quad t \ge 0, \quad \alpha = 1, \cdots, n, \quad i = 1, 2, \cdots, N_\alpha,\\
\dot{\hat{\boldsymbol{v}}}_{\alpha i}(t) = -\dot{\boldsymbol{v}}_{\alpha c} + \dfrac{K}{N} \displaystyle\sum_{k=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|)\big(\boldsymbol{v}_{\alpha k} - \boldsymbol{v}_{\alpha i}\big)\\
\qquad + \dfrac{K}{N} \displaystyle\sum_{k=1}^{N_\alpha} \psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|)\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha k}\rangle \boldsymbol{v}_{\alpha i}\\
\qquad + \dfrac{K}{N} \displaystyle\sum_{\beta\neq\alpha}\sum_{k=1}^{N_\beta} \psi(\|\boldsymbol{x}_{\beta k} - \boldsymbol{x}_{\alpha i}\|)\big(\boldsymbol{v}_{\beta k} - \langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\beta k}\rangle \boldsymbol{v}_{\alpha i}\big).
\end{cases}
\tag{6.2.16}
$$

The proof is basically same as Lemma 3.1.2. For the lack of symmetry, note that

$$\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha k}\rangle = \langle \boldsymbol{v}_{\alpha k}, \boldsymbol{v}_{\alpha k}\rangle - \langle \boldsymbol{v}_{\alpha k}, \boldsymbol{v}_{\alpha i}\rangle,$$

hence we have

$$\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha k}\rangle = \frac{1}{2}\|\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha k}\|^2 = \frac{1}{2}\|\hat{\boldsymbol{v}}_{\alpha i} - \hat{\boldsymbol{v}}_{\alpha k}\|^2. \tag{6.2.17}$$

In the following proposition, we derive estimates on the time-derivatives of $\mathcal{X}_\alpha$ and $\mathcal{V}_\alpha$.

Proposition 6.2.1. *Let* $\big(\boldsymbol{x}_{\alpha i}, \boldsymbol{v}_{\alpha i}\big), \alpha = 1, \cdots, n$ *be a solution to system (6.1.1). Then we have, for any* α*,*

$$(i)\ \frac{d\mathcal{X}_\alpha}{dt} \le \mathcal{V}_\alpha, \quad a.e.\ t \in [0, \infty),$$
$$(ii)\ \frac{d\mathcal{V}_\alpha}{dt} \le -\frac{KN_\alpha}{N}\big(\psi(\sqrt{2}\mathcal{X}_\alpha) - \mathcal{V}_\alpha\big)\mathcal{V}_\alpha + K\sqrt{N}(1-\gamma_N)\psi_M,$$

where $\psi_M := \max\limits_{\beta\neq\alpha, i, k} \psi(\|\boldsymbol{x}_{\beta k} - \boldsymbol{x}_{\alpha i}\|)$.

Proof. The outline of the proof is same as that of Lemma 3.1.3. For the part (ii), we multiply the second equation of (6.2.16) by $2\hat{\boldsymbol{v}}_{\alpha i}(t)$ and sum the resulting relation over $i = 1, \cdots, N_\alpha$. Then

$$\begin{aligned}\frac{d\mathcal{V}_\alpha^2}{dt} &= -2\sum_{i=1}^{N_\alpha}\langle \hat{\boldsymbol{v}}_{\alpha i}, \dot{\boldsymbol{v}}_{\alpha c}\rangle \\ &\quad + \frac{2K}{N}\sum_{k=1}^{N_\alpha}\sum_{i=1}^{N_\alpha}\psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|)\langle \hat{\boldsymbol{v}}_{\alpha i}, \hat{\boldsymbol{v}}_{\alpha k} - \hat{\boldsymbol{v}}_{\alpha i}\rangle \\ &\quad + \frac{2K}{N}\sum_{k=1}^{N_\alpha}\sum_{i=1}^{N_\alpha}\psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|)\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha k}\rangle\langle \hat{\boldsymbol{v}}_{\alpha i}, \boldsymbol{v}_{\alpha i}\rangle \\ &\quad + \frac{2K}{N}\sum_{\beta\neq\alpha}\sum_{k=1}^{N_\beta}\sum_{i=1}^{N_\alpha}\psi(\|\boldsymbol{x}_{\beta k} - \boldsymbol{x}_{\alpha i}\|)\langle \hat{\boldsymbol{v}}_{\alpha i}, \boldsymbol{v}_{\beta k} - \langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\beta k}\rangle \boldsymbol{v}_{\alpha i}\rangle \\ &=: \mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23} + \mathcal{I}_{24}.\end{aligned} \tag{6.2.18}$$

• (Estimate on $\mathcal{I}_{21}$): It is easy to see that

$$\mathcal{I}_{21} = 0.$$

• (Estimate on $\mathcal{I}_{22}$): We exchange $i \longleftrightarrow k$ to get

$$\mathcal{I}_{22} = -\frac{K}{N}\sum_{k=1}^{N_\alpha}\sum_{i=1}^{N_\alpha}\psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|)\|\hat{\boldsymbol{v}}_{\alpha i} - \hat{\boldsymbol{v}}_{\alpha k}\|^2 \le -\frac{2N_\alpha K}{N}\psi(\sqrt{2}\mathcal{X}_\alpha)\mathcal{V}_\alpha^2.$$

• (Estimate on $\mathcal{I}_{23}$): We use relation $\langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha k}\rangle = \frac{1}{2}\|\hat{\boldsymbol{v}}_{\alpha i} - \hat{\boldsymbol{v}}_{\alpha k}\|^2$ derived in (6.2.17) to get that

$$\mathcal{I}_{23} = -\frac{K}{N}\sum_{k=1}^{N_\alpha}\sum_{i=1}^{N_\alpha}\psi(\|\boldsymbol{x}_{\alpha k} - \boldsymbol{x}_{\alpha i}\|)\|\hat{\boldsymbol{v}}_{\alpha i} - \hat{\boldsymbol{v}}_{\alpha k}\|^2\langle \hat{\boldsymbol{v}}_{\alpha i}, \boldsymbol{v}_{\alpha i}\rangle.$$

Thus, we use the upper bound of ψ and $\boldsymbol{v}_{\alpha i}$ to obtain

$$\begin{aligned}
|\mathcal{I}_{23}| &\le \frac{K}{N}\sum_{k=1}^{N_\alpha}\sum_{i=1}^{N_\alpha}\|\hat{\boldsymbol{v}}_{\alpha i} - \hat{\boldsymbol{v}}_{\alpha k}\|^2\|\hat{\boldsymbol{v}}_{\alpha i}\| \\
&= \frac{K}{N}\sum_{k=1}^{N_\alpha}\sum_{i=1}^{N_\alpha}(\|\hat{\boldsymbol{v}}_{\alpha i}\|^2 + \|\hat{\boldsymbol{v}}_{\alpha k}\|^2)\|\hat{\boldsymbol{v}}_{\alpha i}\| \\
&\le \frac{K}{N}(N_\alpha\mathcal{V}_\alpha^3 + \sqrt{N_\alpha}\mathcal{V}_\alpha^3) \le \frac{2N_\alpha K}{N}\mathcal{V}_\alpha^3,
\end{aligned}$$

where in the second equality we have used that

$$-2\sum_{k=1}^{N_\alpha}\sum_{i=1}^{N_\alpha}\langle \hat{\boldsymbol{v}}_{\alpha i}, \hat{\boldsymbol{v}}_{\alpha k}\rangle\|\hat{\boldsymbol{v}}_{\alpha i}\| = -2\left\langle \sum_{k=1}^{N_\alpha}\hat{\boldsymbol{v}}_{\alpha k}, \sum_{i=1}^{N_\alpha}\hat{\boldsymbol{v}}_{\alpha i}\|\hat{\boldsymbol{v}}_{\alpha i}\|\right\rangle = 0.$$

• (Estimate on $\mathcal{I}_{24}$): Note that

$$\|\boldsymbol{v}_{\beta k} - \langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\beta k}\rangle\boldsymbol{v}_{\alpha i}\| \le 1.$$

Hence

$$|\mathcal{I}_{24}| \le \frac{2K\psi_M}{N}\sum_{\beta\ne\alpha}\sum_{k=1}^{N_\beta}\sum_{i=1}^{N_\alpha}\|\hat{\boldsymbol{v}}_{\alpha i}\|\|\boldsymbol{v}_{\beta k} - \langle \boldsymbol{v}_{\alpha i}, \boldsymbol{v}_{\beta k}\rangle\boldsymbol{v}_{\alpha i}\| \le \frac{2\sqrt{N_\alpha}(N - N_\alpha)K}{N}\psi_M\mathcal{V}_\alpha.$$

In (6.2.18), we combine all estimates of $\mathcal{I}_{1i},\ i = 1, \cdots, 4$ to obtain

$$\begin{aligned}\frac{d\mathcal{V}_\alpha^2}{dt} &\le -\frac{2N_\alpha K}{N}\psi(\sqrt{2}\mathcal{X}_\alpha)\mathcal{V}_\alpha^2 + \frac{2N_\alpha K}{N}\mathcal{V}_\alpha^3 + \frac{2\sqrt{N_\alpha}(N-N_\alpha)K}{N}\psi_M\mathcal{V}_\alpha\\ &\le -\frac{2KN_\alpha}{N}\big(\psi(\sqrt{2}\mathcal{X}_\alpha) - \mathcal{V}_\alpha\big)\mathcal{V}_\alpha^2 + 2K\sqrt{N}(1-\gamma_N)\psi_M\mathcal{V}_\alpha.\end{aligned}$$

We now divide the above relation by $2\mathcal{V}_\alpha$ to obtain the desired estimate. □

6.2.3 Proof on multi-cluster flocking

In this subsection, we prove Theorem 6.2.1, the emergence of multi-cluster flocking configurations for the Cucker-Smale dynamics.

Definition 6.2.1. *Define*

$$T_1^* := \sup\left\{T > 0 \ \middle|\ \min_{\alpha,i}\big(\boldsymbol{v}_{\alpha i}(t)\cdot\boldsymbol{e}_\alpha(0)\big) > \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}), \quad t \in [0, T]\right\}, \tag{6.2.19}$$

where $\boldsymbol{e}_\alpha(0)$ is the unit vector in the direction of $\boldsymbol{v}_{\alpha c}(0)$ as before.

$$\boldsymbol{e}_\alpha(0) := \frac{\boldsymbol{v}_{\alpha c}(0)}{\|\boldsymbol{v}_{\alpha c}(0)\|}.$$

Lemma 6.2.2. *Let $\big(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t)\big),\ \alpha = 1, \cdots, n$ be the solution to system (6.2.13) with initial data satisfying $(\mathcal{C}_3 1)$. Then, we have for $t \in [0, T_1^*)$,*

$$\begin{aligned}&(i) \quad \max_{\beta,k}\big(\boldsymbol{v}_{\beta,k}(t)\cdot\boldsymbol{e}_\alpha(T_0)\big) < \cos(\frac{2\theta_0}{3} - \frac{\delta_0}{3}),\\ &(ii) \quad \min_{\beta\neq\alpha,i,k}\big(\boldsymbol{v}_{\alpha i}(t) - \boldsymbol{v}_{\beta k}(t)\big)\cdot\boldsymbol{e}_\alpha(0) > \Lambda_0,\end{aligned}$$

where Λ_0 is a constant depending on θ_0 and δ_0:

$$\Lambda_0 := \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}) - \cos(\frac{2\theta_0}{3} - \frac{\delta_0}{3}).$$

Proof. (i) For any $\beta \neq \alpha$ and $1 \le k \le N_\beta$, we can get that

$$\begin{aligned}\arccos\big(\boldsymbol{v}_{\beta k}(t)\cdot\boldsymbol{e}_\alpha(0)\big) &\ge \arccos\big(\boldsymbol{e}_\alpha(0)\cdot\boldsymbol{e}_\beta(0)\big) - \arccos\big(\boldsymbol{v}_{\beta k}(t)\cdot\boldsymbol{e}_\beta(0)\big)\\ &> \theta_0 - \frac{1}{3}(\theta_0 + \delta_0) = \frac{1}{3}(2\theta_0 - \delta_0), \quad t \in [0, T_1^*).\end{aligned}$$

Thus, we have

$$\max_{\beta,k}\big(\boldsymbol{v}_{\beta,k}(t)\cdot\boldsymbol{e}_\alpha(T_0)\big)<\cos(\frac{2\theta_0}{3}-\frac{\delta_0}{3}),\quad t\in[0,T_1^*).$$

(ii) By the definition of T_1^* and estimate (i), assertion (ii) holds trivially. $\square$

Lemma 6.2.3. *Let* $\big(\boldsymbol{x}_{\alpha i}(t),\boldsymbol{v}_{\alpha i}(t)\big)$, $\alpha=1,\cdots,n$ *be the solution to system (6.2.13) with initial data satisfying* $(\mathcal{C}_3 1)$*. Then, we have*

$$T_1^*>0\quad \textit{and}\quad \psi_M(t)<\psi(r_0+\Lambda_0 t)\ \textit{for}\ t\in(0,T_1^*).$$

Proof. (i) By the assumptions $(\mathcal{C}_3 1)$ on initial data, we have

$$\arccos\big(\boldsymbol{v}_{\alpha i}(0)\cdot\boldsymbol{e}_\alpha(0)\big)\le\delta_0<\frac{1}{3}(\theta_0+\delta_0).$$

Thus, by the continuity, we can conclude $T_1^*>0$.

(ii) By the initial assumptions and Lemma 6.2.2, for any $\beta\neq\alpha$, $1\le i\le N_\alpha$ and $1\le k\le N_\beta$,

$$\begin{aligned}\|\boldsymbol{x}_{\alpha i}(t)-\boldsymbol{x}_{\beta k}(t)\| &\ge \big(\boldsymbol{x}_{\alpha i}(t)-\boldsymbol{x}_{\beta k}(t)\big)\cdot\boldsymbol{e}_\alpha(0)\\ &=\big(\boldsymbol{x}_{\alpha i}(0)-\boldsymbol{x}_{\beta k}(0)\big)\cdot\boldsymbol{e}_\alpha(0)+\int_0^t\big(\boldsymbol{v}_{\alpha i}(s)-\boldsymbol{v}_{\beta k}(s)\big)\cdot\boldsymbol{e}_\alpha(0)\\ &>r_0+\Lambda_0 t,\quad t\in(0,T_1^*).\end{aligned}$$

Thus, by the non-increasing property of $\psi(t)$, we have the result. $\square$

We are now ready to prove Theorem 6.2.1.

The proof of Theorem 6.2.1. Suppose that the framework $(\mathcal{C}_3)$ holds, and let $(\boldsymbol{x}_{\alpha i}(t),\boldsymbol{v}_{\alpha i}(t))$ be a solution to system (6.2.13) with initial configuration $(\boldsymbol{x}_{\alpha i0},\boldsymbol{v}_{\alpha i0})$. Then, we claim $T_1^*=+\infty$ and

$(i)\quad \min_{\beta\neq\alpha,i,k}\|\boldsymbol{x}_{\alpha i}(t)-\boldsymbol{x}_{\beta j}(t)\|>\Lambda_0 t+r_0,$

$(ii)\quad \mathcal{X}_\alpha(t)<\mathcal{X}_\alpha^0+A,\quad \text{where } A=\dfrac{4\mathcal{V}_0}{\beta_\alpha}+\dfrac{4K\sqrt{N}(1-\gamma_N)}{N\beta_\alpha\Lambda_0}\displaystyle\int_{r_0}^{+\infty}\psi(s)ds,$

with $\beta_\alpha = KN_\alpha \psi(\sqrt{2}\mathcal{X}_\alpha^0)$,

$$(iii)\quad \mathcal{V}_\alpha(t) \le C_\alpha \max\Big\{e^{-\frac{K\psi(\sqrt{2}\mathcal{X}_\alpha^0)t}{4}}, \psi\Big(r_0 + \frac{\Lambda_0 t}{2}\Big)\Big\},\ \text{for some } C_\alpha > 0.$$

• (Estimate of estimate (i)): It follows from Lemma 6.2.3 that we have $T_1^* > 0$. Now suppose that $T_1^* < +\infty$.
Then by definition in (6.2.19), there exists $\alpha \in \{1, \cdots, n\}$ and $1 \le i_0 \le N_\alpha$ such that

$$\boldsymbol{v}_{\alpha i_0}(T_1^*) \cdot \boldsymbol{e}_\alpha(0) = \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}). \tag{6.2.20}$$

On the other hand, we use Proposition 6.1.1 and Lemma 6.2.3 to have that for any $\alpha \in \{1, \cdots, n\}$

$$\dot{v}_\alpha^m(t) \ge -K(1-\gamma_N)\psi(r_0 + \Lambda_0 t), \quad t \in [0, T_1^*]. \tag{6.2.21}$$

Thus, we use relation (6.2.21) and the assumption of K to obtain

$$\begin{aligned} v_\alpha^m(T_1^*) &\ge v_\alpha^m(0) - K(1-\gamma_N)\int_0^{T_1^*} \psi(r_0 + \lambda_0 t)dt \\ &\ge \cos\delta_0 - \frac{K(1-\gamma_N)}{\Lambda_0}\int_0^\infty \psi(s)ds \quad > \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}). \end{aligned}$$

Then, we have

$$\boldsymbol{v}_{\alpha i}(T_1^*) \cdot \boldsymbol{e}_\alpha(0) \ge v_\alpha^m(T_1^*) > \cos(\frac{\theta_0}{3} + \frac{\delta_0}{3}).$$

This contradicts to relation (6.2.20). Thus we have $T_1^* = +\infty$.
Now we apply the arguments of Lemma 6.2.3 to derive the estimate:

$$\|\boldsymbol{x}_{\beta k}(t) - \boldsymbol{x}_{\alpha i}(t)\| > r_0 + \Lambda_0 t, \quad t > 0.$$

• (Estimate of estimate (ii)): We claim that

$$\mathcal{V}_\alpha(t) < \mathcal{V}_\alpha^0 + \frac{K\sqrt{N}(1-\gamma_N)}{\Lambda_0}\int_{r_0}^\infty \psi(s)ds, \quad \mathcal{X}_\alpha(t) < \mathcal{X}_\alpha^0 + A, \tag{6.2.22}$$

where A and β_α are defined in (6.2.14).

To prove the claim (6.2.22), we set

$$\hat{T}^* := \sup\{T \geq 0 \mid \text{Claim (6.2.22) holds,} \quad \text{for all } t \in [0, T]\}.$$

We only need to prove that $\hat{T}^* = +\infty$. Clearly $\hat{T}^* > 0$.

Suppose that $\hat{T}^* < +\infty$. By definition of $\hat{T}^*$, we have

$$\mathcal{V}_\alpha(\hat{T}^*) = \mathcal{V}_\alpha^0 + \frac{K\sqrt{N}(1-\gamma_N)}{\Lambda_0}\int_{r_0}^\infty \psi(s)ds \quad \text{or} \quad \mathcal{X}_\alpha(\hat{T}^*) = \mathcal{X}_\alpha^0 + A. \tag{6.2.23}$$

On the other hand, if we use assumption ($\mathcal{C}_3$1) and ($\mathcal{C}_3$2), then for $t \in [0, \hat{T}^*]$,

$$\begin{aligned}
&\psi(\sqrt{2}\mathcal{X}_\alpha) - \mathcal{V}_\alpha \\
&\quad \geq \psi(\sqrt{2}(\mathcal{X}_\alpha^0 + A)) - \mathcal{V}_\alpha^0 + \frac{K\sqrt{N}(1-\gamma_N)}{\Lambda_0}\int_{r_0}^\infty \psi(s)ds \\
&\quad > \frac{3}{4}\psi(\sqrt{2}\mathcal{X}_\alpha^0) - \frac{1}{4}\psi(\sqrt{2}\mathcal{X}_\alpha^0) - \frac{1}{4}\psi(\sqrt{2}\mathcal{X}_\alpha^0) \quad = \frac{1}{4}\psi(\sqrt{2}\mathcal{X}_\alpha^0).
\end{aligned}$$

Thus, by Proposition 6.2.1 and Lemma 6.2.3, we have

$$\begin{aligned}
\frac{d\mathcal{V}_\alpha(t)}{dt} &\leq -\frac{KN_\alpha}{N}\big(\psi(\sqrt{2}\mathcal{X}_\alpha) - \mathcal{V}_\alpha\big)\mathcal{V}_\alpha(t) + K\sqrt{N}(1-\gamma_N)\psi(\Lambda_0 t + r_0) \\
&< -\frac{\beta_\alpha}{4}\mathcal{V}_\alpha(t) + K\sqrt{N}(1-\gamma_N)\psi(\Lambda_0 t + r_0), \quad t \in [0, \hat{T}^*],
\end{aligned} \tag{6.2.24}$$

where $\beta_\alpha = KN_\alpha\psi(\sqrt{2}\mathcal{X}_\alpha^0)$. We Integrate (6.2.24) directly and apply Gronwall's inequality to obtain

$$\mathcal{V}_\alpha(t) < \mathcal{V}_\alpha^0 e^{-\frac{\beta_\alpha}{4}t} + K\sqrt{N}(1-\gamma_N)\int_0^t \psi(\Lambda_0 s + r_0)e^{-\frac{\beta_\alpha}{4}(t-s)}ds, \quad t \in (0, \hat{T}^*]. \tag{6.2.25}$$

In particular, we obtain

$$\mathcal{V}_\alpha(\hat{T}^*) < \mathcal{V}_\alpha^0 + \frac{K\sqrt{N}(1-\gamma_N)}{\Lambda_0}\int_{r_0}^\infty \psi(s)ds. \tag{6.2.26}$$

It follows from the inequality (6.2.25) that we have

$$
\begin{aligned}
|\mathcal{X}_\alpha(t) - \mathcal{X}_\alpha^0| &\le \int_0^t |\frac{d}{ds}\mathcal{X}_\alpha(s)|ds \le \int_0^t \mathcal{V}_\alpha(s)ds \\
&< \mathcal{V}_\alpha^0 \int_0^t e^{-\frac{\beta_\alpha}{4}\tau} d\tau + K\sqrt{N}(1-\gamma_N)\int_0^t \int_0^s \psi(\Lambda_0\tau + r_0)e^{-\frac{\beta_\alpha}{4}(s-\tau)}d\tau ds \\
&\le \frac{4\mathcal{V}_\alpha^0}{\beta_\alpha} + \frac{4K\sqrt{N}(1-\gamma_N)}{\beta_\alpha\Lambda_0}\int_{r_0}^{+\infty}\psi(s)ds, \quad t \in (0, \hat{T}^*].
\end{aligned}
$$

Thus, we get

$$
\mathcal{X}_\alpha(\hat{T}^*) < \mathcal{X}_\alpha^0 + \frac{4\mathcal{V}_0}{\beta_\alpha} + \frac{4K\sqrt{N}(1-\gamma_N)}{N\beta_\alpha\Lambda_0}\int_{r_0}^{+\infty}\psi(s)ds. \tag{6.2.27}
$$

The inequalities (6.2.26) and (6.2.27) contradict the assertion (6.2.23). Thus we obtain $\hat{T}^* = +\infty$.

(iii) For all $t \in (0, +\infty)$, we use assertions (6.2.25) to get that

$$
\mathcal{V}_\alpha(t) < \mathcal{V}_\alpha^0 e^{-\frac{\beta_\alpha}{4}t} + \frac{4K\sqrt{N}(1-\gamma_N)\psi(\Lambda_0 t + r_0)}{\beta_\alpha}\big(\psi(r_0)e^{-\frac{\beta_\alpha}{4}t} + \psi(\frac{\Lambda_0}{2}t + r_0)\big).
$$

Thus we have $\mathcal{V}_\alpha(t) \to 0$ as $t \to +\infty$. □

6.3 Numerical simulations

In this section, we present several numerical examples and compare them with analytical results in the previous sections, in particular Theorem 6.1.2 and Theorem 6.2.1. For numerical integrations, we use the fourth-order Runge-Kutta method and well prepared initial configurations and parameter values in the model (2.4.13) as follows:

$$
\Delta t = 0.01, \quad d = 2, \quad \psi(s) = \frac{1}{(1+s^2)}, \qquad \text{for } t \in [0, 2000]
$$

in order to get clear visualizations and computations.

6.3.1 Non-existence of mono-cluster flocking

Recall that Theorem 6.1.2 deals with initial conditions leading to the complete separation of each ensemble of particles. We start with some ensembles of particles whose initial velocities are same in each ensemble. We choose the following parameters of initial data:

$$N = 10, \quad N_1 = 3, \quad N_2 = 4, \quad N_3 = 3.$$

Figure 6.1(a) represents initial spatial configuration. The initial positions are chosen randomly based on the fixed central positions of groups, whereas initial velocities are chosen to collide with other groups. In this situation, the relative parameters employed in the simulation are

$$\theta_0 = 1.5708, \quad T_0 = 122.18, \quad K_0 = 0.0147, \quad K = 0.9 \times K_0$$

and they satisfy the sufficient condition in Theorem 6.1.2. On the other hand, Figure 6.1(b) illustrates the conclusion of Lemma 6.1.4. It shows that each group is separating at least after the time T_0.

Figure 6.2 and 6.3 denote the temporal evolutions of $D(\boldsymbol{x}(t))$ and $\theta(\boldsymbol{x}(t),\boldsymbol{v}(t))$, which measure the diameters of positions and velocities of the whole ensemble, respectively. Note that their velocities do not change much initially since K is quite small. This is the basic idea of proofs on Lemma 6.1.1, Lemma 6.1.2, Lemma 6.1.3, and Lemma 6.1.4.

6.3.2 Total separation of particles

The initial positions and velocities are chosen uniformly in $[-1,1]^2$ and S^1, respectively. This randomness makes T_0 and K_0 get extremely big and small values, respectively.

$$K_0 = 1.1658 \times 10^{-8}, \quad T_0 = 18884.$$

Figure 6.4(a) shows a realization of initial data. Figure 6.4(b) shows that total separation condition is satisfied near $t = 30$ for $K = 0.9 \times K_0$. The notable point is, however, that Theorem 6.1.2 is generally satisfied with respect to initial data if we give small enough K. On the other hand, the result of Theorem 6.2.1 cannot be applied to general initial data, since the existence of local flocking itself is not guaranteed for general initial configuration.

6.3.3 Emergence of multi-cluster flockings

Now we focus on Theorem 6.2.1. The results are similar to the subsection 6.3.3, but the difference comes from the fluctuation of initial velocities. Since Theorem 6.2.1 has restriction on $\mathcal{X}^0$ and $\mathcal{V}^0$, the L^2-norm of fluctuation, its result depend on the number of particles N. Hence we set small number of particles as follows, for the convenience of visualization.

$$N = 10, \quad N_1 = 3, \quad N_2 = 4, \quad N_3 = 3.$$

Initial configuration is also chosen in a similar way as in subsection 6.3.1. The reference positions and velocities of each group are fixed and we give small random fluctuation for each particle. In contrast to the previous setting, we set separating initial conditions to see the behavior after separation.

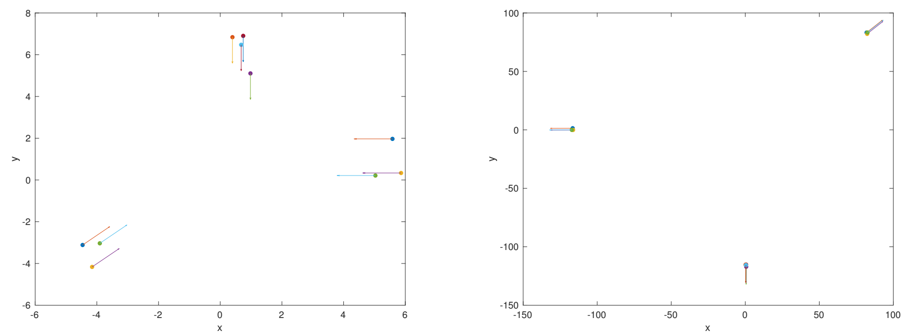
Figure 6.5(a) shows the initial spatial distribution which is separating. Relative positions are scaled larger to guarantee the conditions of Theorem 6.2.1. This configuration has the following parameters,

$$\begin{aligned}
&\delta_0 = \frac{\pi}{10000}, \quad \theta_0 = 1.5708, \\
&\mathcal{X}^0 = 0.1579, \quad \mathcal{V}^0 = 1.6203 \times 10^{-4}, \quad A = 0.2548,
\end{aligned}$$

where the coupling strengths are chosen as follows.

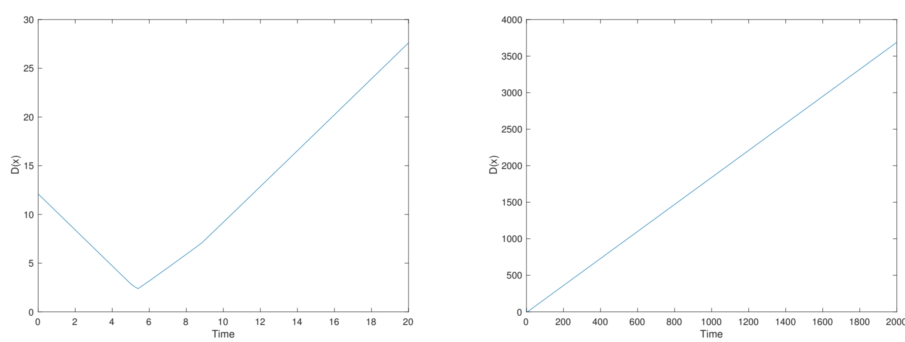
$$K_1 = 0.0251, \quad K = 0.9 \times K_1 < K_1.$$

Hence all the restrictions for Theorem 6.2.1 are satisfied. Here δ_0 is chosen to be quite small value in order to satisfy the condition $(\mathcal{C}_3 2)$.
In Figure 6.5(b), we can observe the minimal difference of velocity angle is nearly constant, which implies K is so small that the interaction between each groups are little. On the other hand, K is large enough to make local flocking of each group, as we can see Figure 6.6. Smaller values break Theorem 6.2.1, for example, we tested $K = 0.09K_1$ makes $\mathcal{X} > \mathcal{X}^0 + A$ and $K = 0.009K_1$ makes more clusters than three.



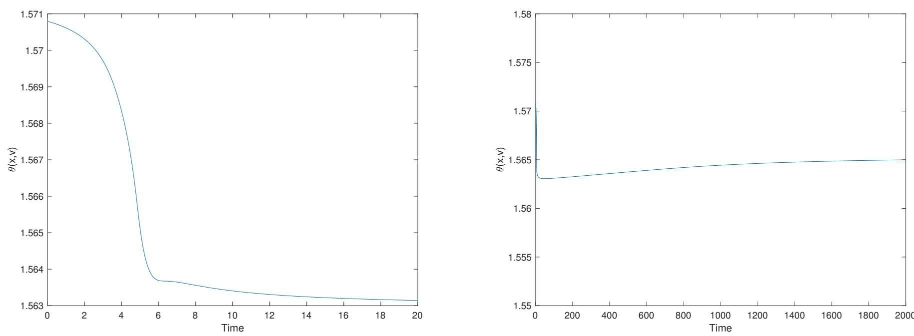
(a) Initial spatial configuration $\boldsymbol{x}_0$ (b) Spatial distribution at time T_0

Figure 6.1: Position-velocity configurations



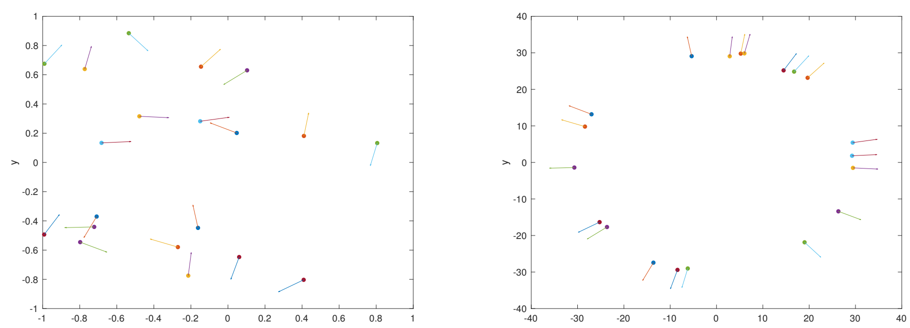
(a) $D(\boldsymbol{x}(t))$ for time $t \in (0, 20)$ (b) $D(\boldsymbol{x}(t))$ for time $t \in (0, 2000)$

Figure 6.2: Diameter of positions $D(\boldsymbol{x})$ along time axis



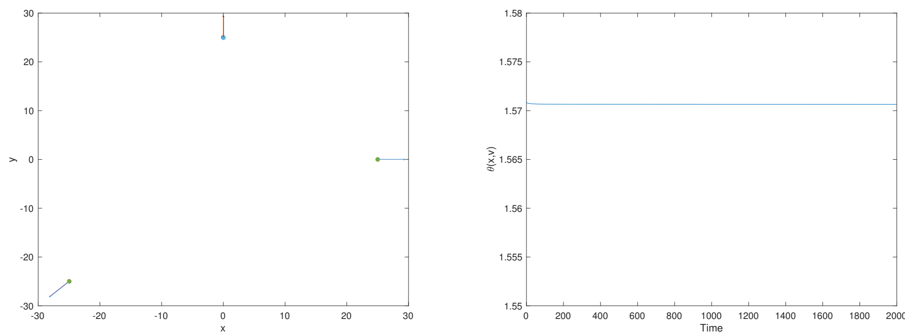
(a) $\theta(\boldsymbol{x}(t), \boldsymbol{v}(t))$ for time $t \in (0, 20)$ (b) $\theta(\boldsymbol{x}(t), \boldsymbol{v}(t))$ for time $t \in (0, 2000)$

Figure 6.3: Temporal evolution of $\theta(\boldsymbol{x}(t), \boldsymbol{v}(t))$



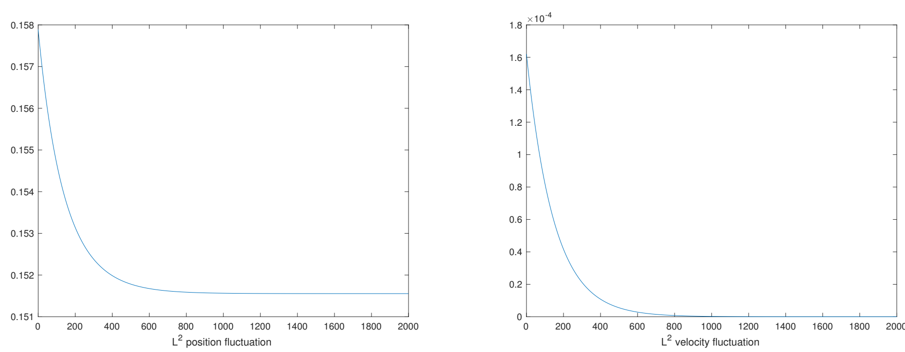
(a) Initial position-velocity configuration (b) Position-velocity configuration at $t = 30$

Figure 6.4: Randomly chosen Position-velocity distribution



(a) Initial spatial distribution $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ (b) $\theta(\boldsymbol{x}(t), \boldsymbol{v}(t))$ for time $t \in (0, 2000)$

Figure 6.5: Initial configuration and its evidence of non-flocking



(a) Temporal evolution of position fluctuation $\mathcal{X}(t)$ (b) Temporal evolution of Position fluctuation $\mathcal{V}(t)$

Figure 6.6: Emergence of local flocking

Chapter 7

Local flocking scenarios with two ensemble coupling network

In this chapter, we are addressing the following situation. Suppose that two homogeneous ensembles of C-S particles are interacting in the whole space. Then, what will happen asymptotically after they begin to interact? Do they form a single ensemble moving together? Or do they diverge as a separate flocking ensemble after the initial mixing? We can think of several possible scenarios from this situation. Here, the main interest is to suggest three frameworks, which sufficiently lead to possible scenarios. Each one represents global flocking, bi-cluster flocking, and partial flocking, respectively. This work is a continuation of the contents in Chapter 3 and 4, and the arguments will be applied to Chapter 8. This chapter is based on the joint work in [45].

To fix the idea, the notation $\mathcal{G}_1 = \{(\boldsymbol{x}_{1i}, \boldsymbol{v}_{1i})\}_{i=1}^{N_1}$ and $\mathcal{G}_2 = \{(\boldsymbol{x}_{2j}, \boldsymbol{v}_{2j})\}_{j=1}^{N_2}$ will be used as two homogeneous C-S ensembles. In contrast with Chapter 3, the interaction between two subsystems $\mathcal{G}_1$ and $\mathcal{G}_2$ might have different coupling network. Here the adjective "homogeneous" means that each C-S particle in the same subsystem has the same mass, so that each particle is indistinguishable. Let $\big(\boldsymbol{x}_{\alpha i}(t), \boldsymbol{v}_{\alpha i}(t)\big) \in \mathbb{R}^{2d}$ be the phase-space coordinate of the ith Cucker-Smale flocking agent in group $\mathcal{G}_\alpha$. Consider the interacting

Cucker-Smale flocking system:

$$
\begin{aligned}
&\dot{\boldsymbol{x}}_{1i} = \boldsymbol{v}_{1i}, \quad \dot{\boldsymbol{x}}_{2j} = \boldsymbol{v}_{2j}, \quad t > 0, \quad i = 1, 2, \cdots, N_1, \quad j = 1, 2, \cdots, N_2, \\
&\dot{\boldsymbol{v}}_{1i} = \frac{K_1}{N_1} \sum_{k=1}^{N_1} \psi_1(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{1i}\|)\big(\boldsymbol{v}_{1k} - \boldsymbol{v}_{1i}\big) \\
&\qquad\quad + \frac{K_d}{N_2} \sum_{k=1}^{N_2} \psi_d(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)\big(\boldsymbol{v}_{2k} - \boldsymbol{v}_{1i}\big), \\
&\dot{\boldsymbol{v}}_{2j} = \frac{K_2}{N_2} \sum_{k=1}^{N_2} \psi_2(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{2j}\|)\big(\boldsymbol{v}_{2k} - \boldsymbol{v}_{2j}\big) \\
&\qquad\quad + \frac{K_d}{N_1} \sum_{k=1}^{N_1} \psi_d(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}\|)\big(\boldsymbol{v}_{1k} - \boldsymbol{v}_{2j}\big),
\end{aligned}
\tag{7.0.1}
$$

where K_1, K_2, and K_d are nonnegative intra-system and inter-system coupling strengths, and the communication weight $\psi_\alpha : \mathbb{R}_+ \to \mathbb{R}$ is Lipschitz continuous and satisfies the following conditions:

$$
\begin{aligned}
&0 < \psi_\alpha(s) \le \psi_\alpha(0) = 1 < +\infty, \quad \psi_\alpha(s) \in L^1(\mathbb{R}_+), \quad \alpha = 1, 2, d, \\
&(\psi_\alpha(s_2) - \psi_\alpha(s_1))(s_2 - s_1) \le 0, \quad s_1, s_2 \in \mathbb{R}_+.
\end{aligned}
\tag{7.0.2}
$$

Note that if we turn off inter-system coupling strength $K_d = 0$, then system (7.0.1) becomes the collection of two C-S models. The well-posedness of system (7.0.1) - (7.0.2) is obvious owing to the standard Cauchy-Lipschitz theory of ordinary differential equations.

The main story of this chapter is threefold. First, we present a sufficient framework for a mono-cluster flocking to the combined system (7.0.1)-(7.0.2). It turns out that the key factor for the emergence of mono-cluster flocking is basically dependent on the inter-system coupling strength K_d. For a large inter-system coupling strength K_d, the combined system leads to mono-cluster flocking for any nonnegative intra-system coupling strengths K_1 and K_2 (Theorem 7.1.1) for some admissible class of initial configurations. Second, we deal with a sufficient framework for the bi-cluster flocking of subsystems $\mathcal{G}_1$ and $\mathcal{G}_2$. In this case, the inter-system coupling strength should be small, but the intra-system coupling strength should be large. We quantify this plausible guess by providing explicit lower and upper bounds for K_α and K_d in terms of initial configuration only. Third, we present a sufficient framework for a partial flocking. More precisely, we present the conditions for local flocking of subsystem $\mathcal{G}_1$, where $\mathcal{G}_2$ does not flock (Theorem 7.2.2).

7.1 Emergence of mono-cluster flocking

In this section, we study a sufficient condition for the mono-cluster flocking in the interaction of two homogeneous C-S ensembles. We will see that the inter-ensemble coupling strength K_d will play a key role in the mono-cluster flocking estimates as long as the intra-ensemble coupling strengths K_1 and K_2 are nonnegative.

7.1.1 Estimates on moments and functionals

Before we start the analysis, we need longer and more complex calculations than before. In this subsection, we focus on the temporal evolution of the normalized first and second velocity moments. For this, we set

$$\begin{aligned}
\boldsymbol{v}_{1c} &:= \frac{1}{N_1}\sum_{i=1}^{N_1}\boldsymbol{v}_{1i}, \quad \boldsymbol{v}_{2c} := \frac{1}{N_2}\sum_{i=1}^{N_2}\boldsymbol{v}_{2i}, \\
m_{2,1} &:= \frac{1}{N_1}\sum_{i=1}^{N_1}\|\boldsymbol{v}_{1i}\|^2, \quad m_{2,2} := \frac{1}{N_2}\sum_{i=1}^{N_2}\|\boldsymbol{v}_{2i}\|^2, \\
M_1 &:= \boldsymbol{v}_{1c} + \boldsymbol{v}_{2c}, \quad M_2 := m_{2,1} + m_{2,2}.
\end{aligned}$$

As we did before, we need to construct differential inequalities for these moments. To start with, the first and second velocity moments satiesfy the following equations.

Lemma 7.1.1. *Let $\big(\boldsymbol{x}, \boldsymbol{v}\big)$ be a global solution of the coupled system (7.0.1). Then, we have*

$$(i)\ \frac{d\boldsymbol{v}_{1c}}{dt} = \frac{K_d}{N_1N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{1i}\|)\big(\boldsymbol{v}_{2k}-\boldsymbol{v}_{1i}\big).$$

$$
\begin{aligned}
(ii)\ \frac{d\boldsymbol{v}_{2c}}{dt} &= -\frac{K_d}{N_1N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{1i}-\boldsymbol{x}_{2k}\|)\big(\boldsymbol{v}_{2k}-\boldsymbol{v}_{1i}\big).\\
(iii)\ \frac{dm_{2,1}}{dt} &= -\frac{K_1}{N_1^2}\sum_{k=1}^{N_1}\sum_{i=1}^{N_1}\psi_1(\|\boldsymbol{x}_{1k}-\boldsymbol{x}_{1i}\|)\big\|\boldsymbol{v}_{1k}-\boldsymbol{v}_{1i}\big\|^2\\
&\quad+\frac{2K_d}{N_1N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{1i}\|)\langle\boldsymbol{v}_{1i},\boldsymbol{v}_{2k}-\boldsymbol{v}_{1i}\rangle.\\
(iv)\ \frac{dm_{2,2}}{dt} &= -\frac{K_2}{N_2^2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_2}\psi_2(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{2i}\|)\big\|\boldsymbol{v}_{2k}-\boldsymbol{v}_{2i}\big\|^2\\
&\quad-\frac{2K_d}{N_1N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{1i}-\boldsymbol{x}_{2k}\|)\langle\boldsymbol{v}_{2k},\boldsymbol{v}_{2k}-\boldsymbol{v}_{1i}\rangle.
\end{aligned}
$$

We omit the proof since it is similar to Lemma 3.1.2.

Remark 7.1.1. *If we define the total momentum P and energy E,*

$$
\begin{aligned}
P &:= \sum_{i=1}^{N_1}\boldsymbol{v}_{1i}+\sum_{i=1}^{N_2}\boldsymbol{v}_{2i}=N_1\boldsymbol{v}_{1c}+N_2\boldsymbol{v}_{2c},\\
E &:= \frac{1}{2}\Big(\sum_{i=1}^{N_1}\|\boldsymbol{v}_{1i}\|^2+\sum_{i=1}^{N_2}\|\boldsymbol{v}_{2i}\|^2\Big)=\frac{1}{2}\Big(N_1\mathcal{V}_1^2+N_2\mathcal{V}_2^2\Big),
\end{aligned}
$$

then they satisfy the following estimates:

$$
\begin{aligned}
\dot{P} &= N_1\dot{\boldsymbol{v}}_{1c}+N_2\dot{\boldsymbol{v}}_{2c}\\
&= K_d\Big(\frac{1}{N_2}-\frac{1}{N_1}\Big)\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{1i}\|)\big(\boldsymbol{v}_{2k}-\boldsymbol{v}_{1i}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{E} &= \frac{1}{2}N_1\dot{m}_{2,1}+\frac{1}{2}N_2\dot{m}_{2,2}\\
&= -\frac{K_1}{2N_1}\sum_{k=1}^{N_1}\sum_{i=1}^{N_1}\psi_1(\|\boldsymbol{x}_{1k}-\boldsymbol{x}_{1i}\|)\big\|\boldsymbol{v}_{1k}-\boldsymbol{v}_{1i}\big\|^2\\
&\quad-\frac{K_2}{2N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_2}\psi_2(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{2i}\|)\big\|\boldsymbol{v}_{2k}-\boldsymbol{v}_{2i}\big\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{K_d}{N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{2k}-\boldsymbol{x}_{1i}\|)\langle \boldsymbol{v}_{1i}, \boldsymbol{v}_{2k}-\boldsymbol{v}_{1i}\rangle \\
&- \quad \frac{K_d}{N_1}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{1i}-\boldsymbol{x}_{2k}\|)\langle \boldsymbol{v}_{2k}, \boldsymbol{v}_{2k}-\boldsymbol{v}_{1i}\rangle.
\end{aligned}
$$

As a direct corollary of Lemma 7.1.1, we have the estimates for M_1 and M_2 as follows.

Corollary 7.1.1. *Let $(\boldsymbol{x}_1, \boldsymbol{v}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{v}_2)$ be a global solution of the coupled system (7.0.1). Then, we have*

$$
M_1(t) = M_1(0), \qquad M_2(t) \le M_2(0), \quad t \ge 0.
$$

From the analysis of moments, the first momentum is conserved and the propagation velocities cannot go to infinity. Next, we introduce nonlinear functionals measuring the formation of mono-cluster flocking for system (7.0.1), and derive a system of dissipative differential inequalities (SDDI). In order to study the global flocking, we introduce the global averages and fluctuations around them:

$$
\begin{aligned}
&\boldsymbol{x}_c := \frac{1}{2}\Big(\frac{1}{N_1}\sum_{i=1}^{N_1}\boldsymbol{x}_{1i} + \frac{1}{N_2}\sum_{j=1}^{N_2}\boldsymbol{x}_{2j}\Big), \quad \boldsymbol{v}_c := \frac{1}{2}\Big(\frac{1}{N_1}\sum_{i=1}^{N_1}\boldsymbol{v}_{1i} + \frac{1}{N_2}\sum_{j=1}^{N_2}\boldsymbol{v}_{2j}\Big), \\
&\hat{\boldsymbol{x}}_{\alpha i} := \boldsymbol{x}_{\alpha i} - \boldsymbol{x}_c, \quad \hat{\boldsymbol{v}}_{\alpha i} := \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_c, \quad \alpha = 1, 2.
\end{aligned}
$$

Then $(\boldsymbol{x}_c, \boldsymbol{v}_c)$ and $(\hat{\boldsymbol{x}}_\alpha, \hat{\boldsymbol{v}}_\alpha)$ satisfy

$$
\begin{aligned}
&\dot{\boldsymbol{x}}_c = \boldsymbol{v}_c, \quad \dot{\boldsymbol{v}}_c = 0, \quad \dot{\hat{\boldsymbol{x}}}_{1i} = \hat{\boldsymbol{v}}_{1i}, \quad \dot{\hat{\boldsymbol{x}}}_{2j} = \hat{\boldsymbol{v}}_{2j}, \quad t > 0, \\
&\dot{\hat{\boldsymbol{v}}}_{1i} = \frac{K_1}{N_1}\sum_{k=1}^{N_1}\psi_1(\|\hat{\boldsymbol{x}}_{1k}-\hat{\boldsymbol{x}}_{1i}\|)\big(\hat{\boldsymbol{v}}_{1k}-\hat{\boldsymbol{v}}_{1i}\big) \\
&\qquad + \frac{K_d}{N_2}\sum_{k=1}^{N_2}\psi_d(\|\hat{\boldsymbol{x}}_{2k}-\hat{\boldsymbol{x}}_{1i}\|)\big(\hat{\boldsymbol{v}}_{2k}-\hat{\boldsymbol{v}}_{1i}\big), \\
&\dot{\hat{\boldsymbol{v}}}_{2j} = \frac{K_2}{N_2}\sum_{k=1}^{N_2}\psi_2(\|\hat{\boldsymbol{x}}_{2k}-\hat{\boldsymbol{x}}_{2j}\|)\big(\hat{\boldsymbol{v}}_{2k}-\hat{\boldsymbol{v}}_{2j}\big) \\
&\qquad + \frac{K_d}{N_1}\sum_{k=1}^{N_1}\psi_d(\|\hat{\boldsymbol{x}}_{1k}-\hat{\boldsymbol{x}}_{2j}\|)\big(\hat{\boldsymbol{v}}_{1k}-\hat{\boldsymbol{v}}_{2j}\big).
\end{aligned}
\tag{7.1.1}
$$

Note that the dynamics of $(\boldsymbol{x}_c, \boldsymbol{v}_c)$ and $(\hat{\boldsymbol{x}}_\alpha, \hat{\boldsymbol{v}}_\alpha)$ are coupled except for $N_1 = N_2$. We now define Lyapunov functionals $\mathcal{X}$ and $\mathcal{V}$ as the weighted l^2-norms:

$$\begin{aligned}
\mathcal{X} &:= \Big(\frac{1}{N_1}\sum_{i=1}^{N_1}\|\hat{\boldsymbol{x}}_{1i}\|^2 + \frac{1}{N_2}\sum_{j=1}^{N_2}\|\hat{\boldsymbol{x}}_{2j}\|^2\Big)^{\frac{1}{2}},\\
\mathcal{V} &:= \Big(\frac{1}{N_1}\sum_{i=1}^{N_1}\|\hat{\boldsymbol{v}}_{1i}\|^2 + \frac{1}{N_2}\sum_{j=1}^{N_2}\|\hat{\boldsymbol{v}}_{2j}\|^2\Big)^{\frac{1}{2}}.
\end{aligned} \tag{7.1.2}$$

Note that $\mathcal{X}$ and $\mathcal{V}$ measure the deviations from the global averages, and it is easy to see that the functional $\mathcal{X}$ and $\mathcal{V}$ are Lipschitz continuous in t, so it is differentiable for almost all $t \in (0, \infty)$. Before we proceed to the flocking estimate, we recall the definition of mono-cluster flocking as follows.

Definition 7.1.1. *Let* $(\boldsymbol{x}, \boldsymbol{v})$ *be a global solution to* (7.0.1)*. We call the subsystems* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *exhibit a time-asymptotic mono-cluster flocking if* $\mathcal{X}$ *and* $\mathcal{V}$ *satisfy*

$$\sup_{0\le t<\infty} \mathcal{X}(t) < \infty, \qquad \lim_{t\to\infty} \mathcal{V}(t) = 0.$$

Note that, in the following sections, we let $\boldsymbol{x} := (\boldsymbol{x}_1, \boldsymbol{x}_2)$, $\boldsymbol{v} := (\boldsymbol{v}_1, \boldsymbol{v}_2)$, and $N_1 + N_2 := N$.

Proposition 7.1.1. *Let* $(\boldsymbol{x}, \boldsymbol{v})$ *be a global solution to* (7.0.1) *with*

$$K_\alpha \ge 0, \qquad \alpha = 1, 2, \qquad K_d > 0.$$

Then, the Lyapunov functionals defined in (7.1.2) *satisfy*

$$\Big|\frac{d\mathcal{X}}{dt}\Big| \le \mathcal{V}, \qquad \frac{d\mathcal{V}}{dt} \le -K_d\psi_d\big(\sqrt{2N}\mathcal{X}\big)\mathcal{V}, \quad a.e.\ t \in (0, \infty).$$

Proof. The first inequality is from the direct calculaltion. For the second inequality, we multiply $(7.1.1)_2$ by $2\hat{\boldsymbol{v}}_{1i}$ and $(7.1.1)_3$ by $2\hat{\boldsymbol{v}}_{2j}$, and sum the results together. Using similar calculations to the proof of Lemma 7.1.1 (ii),

$$\begin{aligned}
\frac{d\mathcal{V}^2}{dt} &= \frac{2}{N_1}\sum_{i=1}^{N_1}\langle \hat{\boldsymbol{v}}_{1i}, \dot{\hat{\boldsymbol{v}}}_{1i}\rangle + \frac{2}{N_2}\sum_{j=1}^{N_2}\langle \hat{\boldsymbol{v}}_{2j}, \dot{\hat{\boldsymbol{v}}}_{2j}\rangle \\
&\le -\frac{2K_d}{N_1N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\hat{\boldsymbol{x}}_{2k}-\hat{\boldsymbol{x}}_{1i}\|)\|\hat{\boldsymbol{v}}_{2k}-\hat{\boldsymbol{v}}_{1i}\|^2 \\
&\le -\frac{2K_d}{N_1}\psi_d\big(\sqrt{2N}\mathcal{X}\big)\sum_{i=1}^{N_1}\|\hat{\boldsymbol{v}}_{1i}\|^2 - \frac{2K_d}{N_2}\psi_d\big(\sqrt{2N}\mathcal{X}\big)\sum_{k=1}^{N_2}\|\hat{\boldsymbol{v}}_{2k}\|^2 \\
&\quad + 4K_d\psi_d\big(\sqrt{2N}\mathcal{X}\big)\Big(\frac{1}{N_1}\sum_{i=1}^{N_1}\hat{\boldsymbol{v}}_{1i}\Big)\cdot\Big(\frac{1}{N_2}\sum_{k=1}^{N_2}\hat{\boldsymbol{v}}_{2k}\Big),
\end{aligned} \tag{7.1.3}$$

where we used

$$K_\alpha \ge 0, \quad \alpha = 1,2, \quad \|\hat{\boldsymbol{x}}_{2k}-\hat{\boldsymbol{x}}_{1i}\|^2 \le 2N\mathcal{X}^2.$$

On the other hand, note that

$$\frac{1}{N_1}\sum_{i=1}^{N_1}\hat{\boldsymbol{v}}_{1i} + \frac{1}{N_2}\sum_{k=1}^{N_2}\hat{\boldsymbol{v}}_{2k} = \Big(\frac{1}{N_1}\sum_{i=1}^{N_1}\boldsymbol{v}_{1i} + \frac{1}{N_2}\sum_{k=1}^{N_2}\boldsymbol{v}_{2k}\Big) - 2\boldsymbol{v}_c = 0.$$

Then, we substitute the relation $\frac{1}{N_2}\sum_{k=1}^{N_2}\hat{\boldsymbol{v}}_{2k} = -\frac{1}{N_1}\sum_{i=1}^{N_1}\hat{\boldsymbol{v}}_{1i}$ into (7.1.3),

$$\begin{aligned}
\frac{d\mathcal{V}^2}{dt} &\le -\frac{2K_d}{N_1}\psi\big(\sqrt{2N}\mathcal{X}\big)\sum_{i=1}^{N_1}\|\hat{\boldsymbol{v}}_{1i}\|^2 - \frac{2K_d}{N_2}\psi_d\big(\sqrt{2N}\mathcal{X}\big)\sum_{k=1}^{N_2}\|\hat{\boldsymbol{v}}_{2k}\|^2 \\
&= -2K_d\psi_d\big(\sqrt{2N}\mathcal{X}\big)\mathcal{V}^2.
\end{aligned}$$

This yields the desired differential inequality for $\mathcal{V}$. □

7.1.2 Proof on the mono-cluster flocking pheonomena

In this subsection, we provide the proof of the emergence of mono-cluster flocking using the SDDI in Proposition 7.1.1. We now present our first main result.

Theorem 7.1.1. *Suppose that initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *are given and the intra- and inter-ensemble coupling strengths* K_α *and* K_d *satisfy the following conditions:*

$$K_\alpha \geq 0, \quad \alpha = 1, 2, \qquad K_d > \frac{\mathcal{V}_0}{\int_{\mathcal{X}_0}^{\infty} \psi_d(\sqrt{2N}x)dx}. \tag{7.1.4}$$

Then, for global solution $(\boldsymbol{x}, \boldsymbol{v})$ *to* (7.0.1), *there exists a positive constant* x_{1M} *such that*

$$\sup_{0 \leq t < \infty} \mathcal{X}(t) \leq x_{1M}, \qquad \mathcal{V}(t) \leq \mathcal{V}_0 e^{-K_d \psi_d\left(\sqrt{2N} x_{1M}\right) t}, \quad \text{as } t \to \infty.$$

Proof. • Step A (Existence of x_{1M}): It follows from Proposition 7.1.1 that we have

$$\left|\frac{d\mathcal{X}}{dt}\right| \leq \mathcal{V}, \qquad \frac{d\mathcal{V}}{dt} \leq -K_d \psi_d\big(\sqrt{2N}\mathcal{X}\big)\mathcal{V}, \quad \text{a.e. } t \in (0, \infty). \tag{7.1.5}$$

We now define a Lyapunov functional $\mathcal{L}_0$ following [47]:

$$\mathcal{L}_0(t) := \mathcal{V}(t) + K_d \int_0^{\mathcal{X}(t)} \psi_d(\sqrt{2N}x)dx, \quad t \in (0, \infty). \tag{7.1.6}$$

Then, we use (7.1.5) and (7.1.6) to obtain

$$\frac{d\mathcal{L}_0}{dt} = \frac{d\mathcal{V}}{dt} + K_d \psi_d(\sqrt{2N}\mathcal{X})\frac{d\mathcal{X}}{dt} \leq -K_d \psi_d(\mathcal{X})\Big(\mathcal{V} - \frac{d\mathcal{X}}{dt}\Big) \leq 0.$$

This yields

$$\mathcal{L}_0(t) \leq \mathcal{L}_0(0), \quad t \in (0, \infty),$$

or equivalently

$$\mathcal{V}(t) + K_d \int_{\mathcal{X}_0}^{\mathcal{X}(t)} \psi_d(\sqrt{2N}\xi)d\xi \leq \mathcal{V}_0, \quad t \in (0, \infty).$$

In particular, this yields

$$K_d \int_{\mathcal{X}_0}^{\mathcal{X}(t)} \psi_d(\sqrt{2N}\xi)d\xi \leq \mathcal{V}_0, \quad t \in (0, \infty). \tag{7.1.7}$$

We set

$$\mathcal{F}(\beta) := K_d \int_{\mathcal{X}_0}^{\beta} \psi_d(\sqrt{2N}\xi)d\xi, \quad \beta \geq 0.$$

Then, $\mathcal{F}(\beta)$ is a continuous and increasing function of β, and by assumption (7.1.4), we have

$$0 = \mathcal{F}(\mathcal{X}_0) < \mathcal{V}_0 < \lim_{\beta\to\infty} \mathcal{F}(\beta).$$

Hence, by the intermediate value theorem, we can choose the largest value of x_{1M} such that

$$K_d \int_{\mathcal{X}_0}^{x_{1M}} \psi_d(\sqrt{2N}\xi)d\xi = \mathcal{V}_0.$$

Then, we claim

$$\sup_{0\le t<\infty} \mathcal{X}(t) \le x_{1M}. \tag{7.1.8}$$

Proof of claim (7.1.8): Suppose not, i.e., there exists $t_* \in (0,\infty)$ such that

$$\mathcal{X}(t_*) > x_{1M}.$$

Then, for such $\mathcal{X}(t_*)$, we have

$$K_d \int_{\mathcal{X}_0}^{\mathcal{X}(t_*)} \psi_d(\sqrt{2N}\xi)d\xi > K_d \int_{\mathcal{X}_0}^{x_{1M}} \psi_d(\sqrt{2N}\xi)d\xi = \mathcal{V}_0,$$

which is contradictory to (7.1.7).

• Step B (Exponential decay of $\mathcal{V}$): We use (7.1.8) and the non-increasing property of ψ_d to obtain

$$\frac{d\mathcal{V}(t)}{dt} \le -K_d\psi_d\big(\sqrt{2N}\mathcal{X}\big)\mathcal{V} \le -K_d\psi_d\big(\sqrt{2N}x_{1M}\big)\mathcal{V}(t), \quad \text{a.e. } t\in(0,\infty).$$

This yields the desired result. □

Remark 7.1.2. *1. Note that in* (7.0.2)*, we assume that the communication weights ψ_α is assumed to be Lipschitz continuous to guarantee the global well-posedness of the coupled system* (7.0.1)*. However, in the proofs of Theorem 7.1.1, Theorem 7.2.1 and Theorem 7.2.2, we only need ψ_α to be integrable; while in Corollary 7.2.1 and Corollary 7.2.2, we need the boundedness of ψ_α to guarantee the existence of the finite time T_0. Thus, in principle our flocking estimates can be done for the coupled particle system* (7.0.1) *and its kinetic* *counterpart with singular communication weights [12, 47, 69] in a priori settings. However, we leave this issue for future work.*

2. The condition (7.1.4) *on the lower bound for K_d implies that, as $\mathcal{V}_0$ increases or $\mathcal{X}_0$ increases, the lower bound for K_d increases. This is what we can expect to happen.*

3. Consider the system with a bi-partite interaction, i.e., there is no intra-ensemble interaction, i.e., $K_1 = K_2 = 0$: for $i = 1, 2, \cdots, N_1, \quad j = 1, 2, \cdots, N_2$,

$$\begin{aligned}
\dot{\boldsymbol{x}}_{1i} &= \boldsymbol{v}_{1i}, \quad \dot{\boldsymbol{x}}_{2j} = \boldsymbol{v}_{2j}, \quad t > 0, \\
\dot{\boldsymbol{v}}_{1i} &= \frac{K_d}{N_2} \sum_{k=1}^{N_2} \psi_d(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)\big(\boldsymbol{v}_{2k} - \boldsymbol{v}_{1i}\big), \\
\dot{\boldsymbol{v}}_{2j} &= \frac{K_d}{N_1} \sum_{k=1}^{N_1} \psi_d(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}\|)\big(\boldsymbol{v}_{1k} - \boldsymbol{v}_{2j}\big).
\end{aligned}$$

Then, the result of Theorem 7.1.1 yields that, as long as the inter-ensemble coupling strength K_d is sufficiently large, we still have mono-cluster flocking for the initial configuration. This is a rather counterintuitive result.

In the following two sections, we study the formation of bi-cluster and multi-cluster flocking.

7.2 Emergence of the local flocking phenomena

In this section, we study the dynamics of system (7.0.1) in a small inter-coupling regime $K_d \ll 1$. In this regime, we present sufficient conditions where each sub-ensemble $\mathcal{G}_1$ and $\mathcal{G}_2$ flock by themselves, but there is no mono-cluster flocking. Note that, for a large inter-ensemble coupling regime $K_d > \dfrac{\mathcal{V}_0}{\int_{\mathcal{X}_0}^{\infty} \psi_d(\sqrt{2N}x)dx}$, we have a mono-cluster flocking wherein two sub-ensembles flock together independent of the detailed geometry of the initial configurations.

7.2.1 Description of bi-cluster flocking

In this subsection, we briefly discuss our main results on the formation of bi-cluster flocking. Since we have bi-cluster flocking asymptotics in mind, we introduce local ensemble averages and local fluctuations around them: for $\alpha = 1, 2$, we set

$$\begin{aligned}
&\boldsymbol{x}_{\alpha c} := \frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha}\boldsymbol{x}_{\alpha i}, \quad \boldsymbol{v}_{\alpha c} := \frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha}\boldsymbol{v}_{\alpha i}, \quad \hat{\boldsymbol{x}}_{\alpha i} := \boldsymbol{x}_{\alpha i} - \boldsymbol{x}_{\alpha c},\\
&\hat{\boldsymbol{v}}_{\alpha i} := \boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha c}, \quad \mathcal{X}_\alpha := \Big(\frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha}\|\hat{\boldsymbol{x}}_{\alpha i}\|^2\Big)^{\frac{1}{2}}, \quad \mathcal{V}_\alpha := \Big(\frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha}\|\hat{\boldsymbol{v}}_{\alpha i}\|^2\Big)^{\frac{1}{2}},\\
&\|\hat{\boldsymbol{x}}_\alpha\|_\infty := \sup_{1\le i\le N_\alpha}\|\hat{\boldsymbol{x}}_{\alpha i}\|, \quad \|\hat{\boldsymbol{v}}_\alpha\|_\infty := \sup_{1\le i\le N_\alpha}\|\hat{\boldsymbol{v}}_{\alpha i}\|.
\end{aligned}$$

Here we use the same notation for the local fluctuations as for the global fluctuations in Section 7.1 for notational simplicity. Then, it is easy to see that

$$\sum_{i=1}^{N_\alpha}\hat{\boldsymbol{x}}_{\alpha i} = 0, \qquad \sum_{i=1}^{N_\alpha}\hat{\boldsymbol{v}}_{\alpha i} = 0, \quad \alpha = 1, 2.$$

And then $(\boldsymbol{x}_{\alpha c}, \boldsymbol{v}_{\alpha c})$ and $(\hat{\boldsymbol{x}}_\alpha, \hat{\boldsymbol{v}}_\alpha)$ satisfy

$$
\begin{aligned}
&\dot{\boldsymbol{x}}_{1c} = \boldsymbol{v}_{1c}, \quad \dot{\boldsymbol{x}}_{2c} = \boldsymbol{v}_{2c}, \quad \dot{\hat{\boldsymbol{x}}}_{1i} = \hat{\boldsymbol{v}}_{1i}, \quad \dot{\hat{\boldsymbol{x}}}_{2j} = \hat{\boldsymbol{v}}_{2j}, \quad t > 0,\\
&\dot{\boldsymbol{v}}_{1c} = \frac{K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)\big(\boldsymbol{v}_{2k} - \boldsymbol{v}_{1i}\big),\\
&\dot{\boldsymbol{v}}_{2c} = \frac{K_d}{N_1 N_2} \sum_{k=1}^{N_1} \sum_{j=1}^{N_2} \psi_d(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}\|)\big(\boldsymbol{v}_{1k} - \boldsymbol{v}_{2j}\big),\\
&\dot{\hat{\boldsymbol{v}}}_{1i} = -\dot{\boldsymbol{v}}_{1c} + \frac{K_1}{N_1} \sum_{k=1}^{N_1} \psi_1(\|\hat{\boldsymbol{x}}_{1k} - \hat{\boldsymbol{x}}_{1i}\|)\big(\hat{\boldsymbol{v}}_{1k} - \hat{\boldsymbol{v}}_{1i}\big)\\
&\qquad + \frac{K_d}{N_2} \sum_{k=1}^{N_2} \psi_d(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)\big(\boldsymbol{v}_{2k} - \boldsymbol{v}_{1i}\big),\\
&\dot{\hat{\boldsymbol{v}}}_{2j} = -\dot{\boldsymbol{v}}_{2c} + \frac{K_2}{N_2} \sum_{k=1}^{N_2} \psi_2(\|\hat{\boldsymbol{x}}_{2k} - \hat{\boldsymbol{x}}_{2j}\|)\big(\hat{\boldsymbol{v}}_{2k} - \hat{\boldsymbol{v}}_{2j}\big)\\
&\qquad + \frac{K_d}{N_1} \sum_{k=1}^{N_1} \psi_d(\|\boldsymbol{x}_{1k} - \boldsymbol{x}_{2j}\|)\big(\boldsymbol{v}_{1k} - \boldsymbol{v}_{2j}\big).
\end{aligned}
\tag{7.2.1}
$$

Definition 7.2.1. *Let $(\boldsymbol{x}, \boldsymbol{v})$ be a global solution to the coupled system* (7.0.1).

1. *The subsystem $\mathcal{G}_i$ exhibits a time-asymptotic flocking if and only if*

$$\sup_{0 \le t < \infty} \mathcal{X}_i(t) < \infty, \qquad \lim_{t \to \infty} \mathcal{V}_i(t) = 0, \quad i = 1, 2.$$

2. *The whole system $(\mathcal{G}_1, \mathcal{G}_2)$ exhibits a time-asymptotic bi-cluster flocking if and only if both subsystems $\mathcal{G}_1$ and $\mathcal{G}_2$ exhibit a time-asymptotic flocking, but the whole system does not exhibit a time-asymptotic mono-cluster flocking.*

3. *The whole system $(\mathcal{G}_1, \mathcal{G}_2)$ exhibits a time-asymptotic partial flocking if and only if only one of $\mathcal{G}_1$ and $\mathcal{G}_2$ exhibits a time-asymptotic flocking, but the other does not.*

Our main results on the emergence of bi-cluster flocking can be summarized as follows.

Theorem 7.2.1. *Suppose that the following framework* $(\mathcal{C}_4)$ *holds for the initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *to system* (7.0.1).

- $(\mathcal{C}_4 1)$*: (Restriction on initial configurations)*

$$
\begin{aligned}
&\lambda_0 := \frac{1}{2}\|\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0)\| > 0,\\
&\max_{1\le i\le N_1}\|\boldsymbol{v}_{1i}(0) - \boldsymbol{v}_{1c}(0)\| \le \frac{1}{4}\lambda_0, \quad \max_{1\le k\le N_2}\|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2c}(0)\| \le \frac{1}{4}\lambda_0,\\
&\min_{\substack{1\le i\le N_1\\ 1\le k\le N_2}}\big\{(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{1i}(0))\cdot(\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0))\big\} \ge 0.
\end{aligned}
$$

- $(\mathcal{C}_4 2)$*: (Restriction on coupling strengths): for* $\alpha = 1, 2$,

$$
\begin{aligned}
&K_\alpha > \frac{\mathcal{V}_\alpha(0) + \frac{K_d\sqrt{2NM_2(0)}}{\lambda_0}\int_0^\infty \psi_d(x)dx}{\int_{\mathcal{X}_\alpha(0)}^\infty \psi_\alpha(\sqrt{2N_\alpha}x)dx},\\
&0 \le K_d < \frac{\lambda_0^2}{12\sqrt{2NM_2(0)}\int_0^\infty \psi_d(x)dx}.
\end{aligned}
$$

Then, the whole system $(\mathcal{G}_1, \mathcal{G}_2)$ *exhibits a time-asymptotic bi-cluster flocking. More precisely, for the solution* $(\boldsymbol{x}, \boldsymbol{v})$ *to system* (7.0.1) *with initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$*, there exist positive constants* x_α^∞ *and* C_α*,* $\alpha = 1, 2$ *that depend only on the initial data and* ψ *such that*

$$
\begin{aligned}
&\sup_{0\le t<\infty}\mathcal{X}_\alpha(t) \le x_\alpha^\infty, \quad \mathcal{V}_\alpha(t) \le C_\alpha \max\Big\{e^{-\frac{K_\alpha\psi_\alpha(\sqrt{2N_\alpha}x_\alpha^\infty)t}{2}}, \psi_d\Big(\frac{\lambda_0}{2}t\Big)\Big\},\\
&\inf_{0\le t<\infty, i,k}\|\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{1i}(t)\| \ge \lambda_0, \quad \min_{i,k}\|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{1i}(t)\| \ge \lambda_0 t, \quad t \in [0, \infty).
\end{aligned}
$$

Remark 7.2.1. *1. The last geometric condition*

$$
\min_{\substack{1\le i\le N_1\\ 1\le k\le N_2}}\big\{(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{1i}(0))\cdot(\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0))\big\} \ge 0
$$

means that the particles in different groups depart each other initially. Actually, this geometric condition is not that crucial for the validity of Theorem 7.2.1 as can be seen in Corollary 7.2.1. This condition will be attained in a finite time for proper coupling strengths, even if we begin with initial data *that do not satisfy this condition.*

2. The smallness condition on K_d is needed to prevent mono-cluster flocking, whereas the largeness condition on K_α is needed to enable flocking of each subsystem.

As we did in Chapter 4 and 6, we can get rid of the condition

$$\min_{\substack{1\le i\le N_1\\1\le k\le N_2}} \big\{(\boldsymbol{x}_{2k}(0)-\boldsymbol{x}_{1i}(0))\cdot(\boldsymbol{v}_{2c}(0)-\boldsymbol{v}_{1c}(0))\big\}\ge 0.$$

In order to guarantee the separation, we use the following time stamp.

$$T_0 := \frac{1}{\lambda_0^2}\max_{\substack{1\le i\le N_1\\1\le k\le N_2}}\Big|(\boldsymbol{x}_{2k}(0)-\boldsymbol{x}_{1i}(0))\cdot(\boldsymbol{v}_{2c}(0)-\boldsymbol{v}_{1c}(0))\Big|.$$

Corollary 7.2.1. *Suppose that the following framework $(\mathcal{C}_5)$ holds for the initial data $(\boldsymbol{x}_0,\boldsymbol{v}_0)$ to system* (7.0.1).

- $(\mathcal{C}_5 1)$*: (Restriction on initial configurations)*

$$\begin{aligned}&\lambda_0 := \frac{1}{2}\|\boldsymbol{v}_{2c}(0)-\boldsymbol{v}_{1c}(0)\|>0,\\ &\max_{1\le i\le N_1}\|\boldsymbol{v}_{1i}(0)-\boldsymbol{v}_{1c}(0)\|\le\frac{1}{4}\lambda_0,\quad \max_{1\le k\le N_2}\|\boldsymbol{v}_{2k}(0)-\boldsymbol{v}_{2c}(0)\|\le\frac{1}{4}\lambda_0.\end{aligned}$$

- $(\mathcal{C}_5 2)$*: (Restriction on coupling strengths): for $\alpha=1,2$,*

$$\begin{aligned}0\le K_d<\min\Bigg\{&\frac{\lambda_0}{16\sqrt{2NM_2(0)}T_0},\frac{\lambda_0^2}{24\sqrt{2NM_2(0)}\int_0^\infty\psi_d(x)dx},\\ &\frac{\lambda_0^2}{2\big(D(\boldsymbol{x}_1(0),\boldsymbol{x}_2(0))+\sqrt{2NM_2(0)}T_0\big)\sqrt{2NM_2(0)}}\Bigg\},\\ K_\alpha>&\frac{P_\alpha(0)+\frac{K_d\sqrt{2NM_2(0)}}{\lambda_0}\int_0^\infty\psi_d(x)dx}{\int_{R_\alpha(0)}^\infty\psi_\alpha(\sqrt{2N_\alpha}x)dx},\quad \alpha=1,2.\end{aligned}$$

where we have used some quantities that only depend on the initial data:

$$\begin{aligned}&D(\boldsymbol{x}_1(0),\boldsymbol{x}_2(0)) := \max_{i,k}\|\boldsymbol{x}_{2k}(0)-\boldsymbol{x}_{1i}(0)\|,\\ &P_\alpha(0) := \mathcal{V}_\alpha(0)+\frac{\lambda_0}{16},\quad \alpha=1,2,\quad R_\alpha(0) := \mathcal{X}_\alpha(0)+P_\alpha(0)T_0.\end{aligned}\tag{7.2.2}$$

Then, the whole system $(\mathcal{G}_1, \mathcal{G}_2)$ *exhibits a time-asymptotic bi-cluster flocking. More precisely, for the solution* $(\boldsymbol{x}, \boldsymbol{v})$ *to system* (7.0.1) *with initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$*, there exist* x_α^∞ *and* C_α*,* $\alpha = 1, 2$*, that only depend on the initial data and* ψ *such that*

$$\sup_{T_0 \le t < \infty} \mathcal{X}_\alpha(t) \le x_\alpha^\infty, \quad \mathcal{V}_\alpha(t) \le C_\alpha \max\Big\{ e^{-\frac{K_\alpha \psi_d(\sqrt{2N_\alpha} x_\alpha^\infty)(t-T_0)}{2}}, \psi_d\Big(\frac{\lambda_0(t-T_0)}{4}\Big)\Big\},$$
$$\inf_{i,k,0 \le t < \infty} \|\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{1i}(t)\| \ge \frac{\lambda_0}{2}, \quad \min_{i,k} \|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{1i}(t)\| \ge \frac{\lambda_0}{2} t, , \quad t \in [T_0, \infty).$$

Remark 7.2.2. *The technique using the separated time* T_0 *is described in Theorem 4.1.1 in Chapter 4 and Theorem 6.1.2 in Chapter 6. There is a detailed explanation in [45], hence we omit in this Chapter.*

7.2.2 Emergence of bi-cluster flocking

In this subsection, we present a proof of Theorem 7.2.1 on the formation of bi-cluster flockings resulting from the interaction of two C-S ensembles in the low inter-coupling regime $K_d \ll 1$.

Proposition 7.2.1. *Suppose that the coupling strengths satisfy*

$$K_\alpha \ge 0, \quad \alpha = 1, 2, \qquad K_d \ge 0,$$

and let $(\boldsymbol{x}, \boldsymbol{v})$ *be a global solution to* (7.0.1)*. Then, for* $\alpha = 1, 2$*, we have*

$$(i) \ \Big\| \frac{d\boldsymbol{v}_{\alpha c}}{dt} \Big\| \le K_d \sqrt{2NM_2(0)}\psi_{dM}, \ a.e. \ t \in (0, \infty),$$
$$(ii) \ \frac{d\mathcal{X}_\alpha}{dt} \le \mathcal{V}_\alpha, \quad \frac{d\mathcal{V}_\alpha}{dt} \le -K_\alpha \psi_\alpha(\sqrt{2N_\alpha}\mathcal{X}_\alpha)\mathcal{V}_\alpha + K_d\sqrt{2NM_2(0)}\psi_{dM},$$

where ψ_{dM} *is the time-dependent maximal communication weight between distinct ensembles:*

$$\psi_{dM}(t) := \max_{\substack{1 \le i \le N_1 \\ 1 \le k \le N_2}} \psi_d(\|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{1i}(t)\|) \ge 0.$$

Proof. Since the estimates for subsystem $\mathcal{G}_2$ are the same as for subsystem $\mathcal{G}_1$, we only treat estimates for $\alpha = 1$.

(i) We use (7.2.1) and the following inequality

$$\|\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{1i}(t)\| \leq \sqrt{2NM_2(t)} \leq \sqrt{2NM_2(0)}$$

to conclude the result,

$$\begin{aligned}\left\|\frac{d\boldsymbol{v}_{1c}}{dt}\right\| &= \frac{K_d}{N_1N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)\|\boldsymbol{v}_{2k} - \boldsymbol{v}_{1i}\| \\ &\leq K_d\sqrt{2NM_2(0)}\psi_{dM}(t).\end{aligned}$$

(ii) Since the first inequality can be proved similarly with Proposition 7.1.1, here we only prove the second one. We multiply $(7.2.1)_4$ by $2\hat{\boldsymbol{v}}_{1i}$ and sum the results with respect to i to obtain

$$\begin{aligned}\frac{d\mathcal{V}_1^2}{dt} &= \frac{2}{N_1}\sum_{i=1}^{N_1}\langle\hat{\boldsymbol{v}}_{1i}, \dot{\hat{\boldsymbol{v}}}_{1i}\rangle \\ &= -\frac{2}{N_1}\Big\langle\sum_{i=1}^{N_1}\hat{\boldsymbol{v}}_{1i}, \dot{\boldsymbol{v}}_{1c}\Big\rangle - \frac{K_1}{N_1^2}\sum_{k=1}^{N_1}\sum_{i=1}^{N_1}\psi_1(\|\hat{\boldsymbol{x}}_{1k} - \hat{\boldsymbol{x}}_{1i}\|)\big\|\hat{\boldsymbol{v}}_{1k} - \hat{\boldsymbol{v}}_{1i}\big\|^2 \\ &\quad + \frac{2K_d}{N_1N_2}\sum_{k=1}^{N_2}\sum_{i=1}^{N_1}\psi_d(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|)\langle\hat{\boldsymbol{v}}_{1i}, \boldsymbol{v}_{2k} - \boldsymbol{v}_{1i}\rangle \\ &\leq -\frac{K_1}{N_1^2}\psi_1(\sqrt{2N_1}\mathcal{X}_1)\sum_{k=1}^{N_1}\sum_{i=1}^{N_1}\big\|\hat{\boldsymbol{v}}_{1k} - \hat{\boldsymbol{v}}_{1i}\big\|^2 + 2K_d\sqrt{2NM_2(0)}\psi_{dM}\mathcal{V}_1.\end{aligned} \tag{7.2.3}$$

On the other hand, note that

$$\sum_{i=1}^{N_1}\sum_{k=1}^{N_1}\|\hat{\boldsymbol{v}}_{1k} - \hat{\boldsymbol{v}}_{1i}\|^2 = 2N_1\sum_{k=1}^{N_1}\|\hat{\boldsymbol{v}}_{1i}\|^2 = 2N_1^2\mathcal{V}_1^2. \tag{7.2.4}$$

Putting (7.2.4) into (7.2.3) leads to the desired inequality,

$$\frac{d\mathcal{V}_1}{dt} \leq -K_1\psi_1(\sqrt{2N_1}\mathcal{X}_1)\mathcal{V}_1 + K_d\sqrt{2NM_2(0)}\psi_{dM}.$$

The inequality for $\mathcal{V}_2$ can be proved in the same way. $\square$

Proposition 7.2.2. *Suppose that the coupling strengths satisfy*

$$K_\alpha \geq 0, \quad \alpha = 1, 2, \qquad K_d \geq 0,$$

and let $(\boldsymbol{x}, \boldsymbol{v})$ *be a global solution to* (7.0.1). *Then, for* $\alpha = 1, 2$, *we have*

$$\|\hat{\boldsymbol{v}}_\alpha(t_2)\|_\infty - \|\hat{\boldsymbol{v}}_\alpha(t_1)\|_\infty \leq 2K_d\sqrt{2NM_2(0)} \int_{t_1}^{t_2} \psi_{dM}(t)dt,$$

for any time step $0 \leq t_1 \leq t_2 < \infty$.

Proof. We only prove $\alpha = 1$ since the case $\alpha = 2$ can be treated in the same way. We set

$$\mathcal{F}(t) = 2K_d\sqrt{2NM_2(0)}\psi_{dM}(t).$$

We claim that for any $t_1 \in [0, \infty)$, there exists $\Delta t > 0$ such that

$$\|\hat{\boldsymbol{v}}_1(t_2)\|_\infty - \|\hat{\boldsymbol{v}}_1(t_1)\|_\infty \leq \int_{t_1}^{t_2} \mathcal{F}(t)dt, \quad \forall t_2 \in (t_1, t_1 + \Delta t]. \tag{7.2.5}$$

Proof of claim (7.2.5). Now we take an arbitrary $t_1 \in [0, \infty)$. Set

$$I_{t_1} := \{1 \leq j \leq N_1 \;\Big|\; \|\hat{\boldsymbol{v}}_{1j}(t_1)\| = \|\hat{\boldsymbol{v}}_1(t_1)\|_\infty\}.$$

For any $j \in I_{t_1}$, we have

$$\begin{aligned}\frac{d\|\hat{\boldsymbol{v}}_{1j}(t)\|^2}{dt}\Big|_{t=t_1} = \Big\{ &\frac{2K_1}{N_1}\sum_{k=1}^{N_1} \psi_1(\|\hat{\boldsymbol{x}}_{1k}(t) - \hat{\boldsymbol{x}}_{1j}(t)\|)\langle \hat{\boldsymbol{v}}_{1j}(t), \hat{\boldsymbol{v}}_{1k}(t) - \hat{\boldsymbol{v}}_{1j}(t)\rangle \\ &+ \frac{2K_d}{N_2}\sum_{k=1}^{N_2} \psi_d(\|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{1j}(t)\|)\langle \hat{\boldsymbol{v}}_{1j}(t), \boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{1j}(t) \\ &- 2\hat{\boldsymbol{v}}_{1j} \cdot \dot{\boldsymbol{v}}_{1c}\rangle\Big\}\Big|_{t=t_1}.\end{aligned}$$

We use $\langle \hat{\boldsymbol{v}}_{1j}(t_1), \hat{\boldsymbol{v}}_{1k}(t_1) - \hat{\boldsymbol{v}}_{1j}(t_1)\rangle \leq 0$ and the estimate of $\|\dot{\boldsymbol{v}}_{\alpha c}\|$ in Proposition 7.2.1 to find

$$\frac{d\|\hat{\boldsymbol{v}}_{1j}(t)\|}{dt}\Big|_{t=t_1} < \mathcal{F}(t_1).$$

From the continuity, there exists $\Delta t > 0$ such that for any $t \in [t_1, t_1 + \Delta t]$, any $j \in I_{t_1}$ and any $i \in \{1, \cdots, N\} \backslash I_{t_1}$, it holds that

$$\frac{d\|\hat{\boldsymbol{v}}_{1j}(t)\|}{dt} < \mathcal{F}(t) \quad \text{and} \quad \|\hat{\boldsymbol{v}}_{1i}(t)\| < \|\hat{\boldsymbol{v}}_1(t)\|_\infty.$$

Thus, for any $t_2 \in [t_1, t_1 + \Delta t]$, there exists $j \in I_{t_1}$ such that $\|\hat{\boldsymbol{v}}_{1j}(t_2)\| = \|\hat{\boldsymbol{v}}_1(t_2)\|_\infty$. Hence, we have

$$\begin{aligned} &\|\hat{\boldsymbol{v}}_1(t_2)\|_\infty - \|\hat{\boldsymbol{v}}_1(t_1)\|_\infty = \|\hat{\boldsymbol{v}}_{1j}(t_2)\| - \|\hat{\boldsymbol{v}}_{1j}(t_1)\| \\ &\qquad = \int_{t_1}^{t_2} \frac{d\|\hat{\boldsymbol{v}}_{1j}(t)\|}{dt} < \int_{t_1}^{t_2} \mathcal{F}(t)dt, \quad \text{for all } t_2 \in (t_1, t_1 + \Delta t]. \end{aligned}$$

Thus, the claim (7.2.5) holds. Now we set

$$T^\natural := \sup\{t \in (t_1, \infty) \,\Big|\, \|\hat{\boldsymbol{v}}_1(s)\|_\infty - \|\hat{\boldsymbol{v}}_1(t_1)\|_\infty \le \int_{t_1}^{s} \mathcal{F}(\tau)d\tau,\ \forall s \in [t_1, t]\}.$$

Now we claim

$$T^\natural = \infty.$$

Otherwise, we assume $T^\natural < \infty$. Then

$$\|\hat{\boldsymbol{v}}_1(T^\natural)\|_\infty - \|\hat{\boldsymbol{v}}_1(t_1)\|_\infty = \int_{t_1}^{T^\natural} \mathcal{F}(t)dt.$$

We use claim (7.2.5) to know there exists $\Delta t > 0$ such that

$$\|\hat{\boldsymbol{v}}_1(s)\|_\infty - \|\hat{\boldsymbol{v}}_1(T^\natural)\|_\infty \le \int_{T^\natural}^{s} \mathcal{F}(t)dt, \quad \text{for all } s \in (T^\natural, T^\natural + \Delta t].$$

Thus, we can have the following for $s \in [t_1, T^\natural + \Delta t]$.

$$\|\hat{\boldsymbol{v}}_1(s)\|_\infty - \|\hat{\boldsymbol{v}}_1(t_1)\|_\infty \le (\int_{T^\natural}^{s} + \int_{t_1}^{T^\natural})\mathcal{F}(t)dt = \int_{t_1}^{s} \mathcal{F}(t)dt.$$

This contradicts the definition of $T^\natural$. Therefore, the conclusion follows. $\square$

In the following two subsections, we proceed to prove Theorem 7.2.1.

- Step A (Local-in-time estimate): We will show that each sub-ensemble $\mathcal{G}_\alpha$ satisfies the flocking estimates for some finite time T and $t \in [0, T)$:

$$\sup_{0 \le t < T} \mathcal{X}_\alpha(t) < x_\alpha^\infty, \quad \mathcal{V}_\alpha(t) \le C_\alpha \max\Big\{ e^{-\frac{K_\alpha \psi_\alpha(\sqrt{2N_\alpha} x_\alpha^\infty) t}{2}}, \psi_d\Big(\frac{\lambda_0 t}{2}\Big)\Big\}$$
$$\inf_{0 \le t < T, i, k} \|\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{1i}(t)\| \ge \lambda_0, \quad \min_{i,k} \|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{1i}(t)\| \ge \lambda_0 t.$$

- Step B (Continuation to the whole time interval): We will show that time T in Step A can be chosen to be infinity.

Step A : Local-in-time flocking estimates

To find the time interval where all desired flocking estimates hold, we set

$$\begin{aligned} \boldsymbol{e}_{1,2}^0 &:= \frac{\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0)}{\|\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0)\|}, \quad T_1^* := \sup \mathcal{T}_1, \\ \mathcal{T}_1 &:= \Big\{ T \in [0, +\infty) \ \Big| \ \min_{i,k} \{(\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{1i}(t)) \cdot e_{1,2}^0\} > \lambda_0, \ \forall\ t \in [0, T) \Big\}. \end{aligned} \tag{7.2.6}$$

We first show that T_1^* exists and is positive.

Lemma 7.2.1. *Let* $(\boldsymbol{x}, \boldsymbol{v})$ *be a global solution to* (7.0.1) *with initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *satisfying* $(\mathcal{C}_4 1)$ *in Theorem 7.2.1. Then, we have*

$$T_1^* > 0 \quad \textit{and} \quad \psi_{dM}(t) \le \psi_d(\lambda_0 t), \quad \textit{for all} \ \ t \in [0, T_1^*).$$

Proof. (i) We first show that $T^* > 0$.

$$\begin{aligned} &(\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1i}(0)) \cdot \boldsymbol{e}_{1,2}^0 \\ &\quad = \frac{(\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0))}{\|\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0)\|} \cdot \Big(\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0) - \hat{\boldsymbol{v}}_{1i}(0) + \hat{\boldsymbol{v}}_{2k}(0) \Big) \\ &\quad \ge \|\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0)\| - \|\hat{\boldsymbol{v}}_1(0)\|_\infty - \|\hat{\boldsymbol{v}}_2(0)\|_\infty \\ &\quad \ge \frac{3\lambda_0}{2} > \lambda_0. \end{aligned}$$

We now take a minimum over i and k to obtain

$$\min_{i,k} \{(\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1i}(0)) \cdot \boldsymbol{e}_{1,2}^0\} > \lambda_0.$$

Then, by the continuity, there exists $\delta > 0$ such that

$$\min_{i,k}\{(\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{1i}(t)) \cdot \boldsymbol{e}^0_{1,2}\} > \lambda_0, \quad t \in [0, \delta), \quad \text{i.e.,} \quad \delta \in \mathcal{T}_1.$$

Hence $T_1^* \geq \delta > 0$.

(ii) For $t \in [0, T_1^*)$, we have

$$\begin{aligned}\|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{1i}(t)\| &\geq \big(\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{1i}(t)\big) \cdot \boldsymbol{e}^0_{1,2} \\ &= \big(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{1i}(0)\big) \cdot \boldsymbol{e}^0_{1,2} \\ &\quad + \int_0^t \big(\boldsymbol{v}_{2k}(s) - \boldsymbol{v}_{1i}(s)\big) \cdot \boldsymbol{e}^0_{1,2} ds \quad \geq \lambda_0 t.\end{aligned}$$

Thus, by the non-increasing property of ψ_d, we have

$$\psi_{dM}(t) \leq \psi_d(\lambda_0 t), \quad \text{for all } t \in [0, T_1^*).$$

$\square$

Lemma 7.2.2. (Flocking estimate in $[0, T_1^*)$) *Suppose that the initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *satisfy* $(\mathcal{C}_4 1)$ *and the coupling strengths satisfy*

$$K_\alpha > \frac{\mathcal{V}_\alpha(0) + \frac{K_d\sqrt{2NM_2(0)}}{\lambda_0}\int_0^\infty \psi_d(x)dx}{\int_{\mathcal{X}_\alpha(0)}^\infty \psi_\alpha(\sqrt{2N_\alpha}x)dx}, \quad \alpha = 1, 2, \quad K_d \geq 0.$$

Then, for the solution $(\boldsymbol{x}, \boldsymbol{v})$ *to system* (7.0.1) *with initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$*, there exist positive constants* x_α^∞ *and* C_α *independent of time* t *such that*

$$\begin{aligned}&\sup_{0 \leq t < T_1^*} \mathcal{X}_\alpha(t) < x_\alpha^\infty, \\ &\mathcal{V}_\alpha(t) \leq C_\alpha \max\left\{e^{-\frac{K_\alpha\psi_\alpha(\sqrt{2N_\alpha}x_\alpha^\infty)t}{2}}, \psi_d\left(\frac{\lambda_0 t}{2}\right)\right\}, \ t \in [0, T_1^*).\end{aligned}$$

Proof. (i) (Existence of an upper bound x_α^∞): We fix $\alpha \in \{1, 2\}$ and define a Lyapunov functional $\mathcal{L}_{1\alpha}$:

$$\mathcal{L}_{1\alpha}(t) := \mathcal{V}_\alpha(t) + K_\alpha \int_0^{\mathcal{X}_\alpha(t)} \psi_\alpha(\sqrt{2N_\alpha}x)dx.$$

It follows from Proposition 7.2.1 and Lemma 7.2.1 that

$$\begin{aligned}\frac{d\mathcal{L}_{1\alpha}(t)}{dt} &= \frac{d}{dt}\mathcal{V}_\alpha(t) + K_\alpha\psi_\alpha(\sqrt{2N_\alpha}\mathcal{X}_\alpha(t))\frac{d}{dt}\mathcal{X}_\alpha(t) \\ &\le -K_\alpha\psi_\alpha(\sqrt{2N_\alpha}\mathcal{X}_\alpha(t))\Big(\mathcal{V}_\alpha(t) - \frac{d}{dt}\mathcal{X}_\alpha(t)\Big) + K_d\sqrt{2NM_2(0)}\psi_{dM} \\ &\le K_d\sqrt{2NM_2(0)}\psi_d(\lambda_0 t), \quad t \in [0, T_1^*).\end{aligned}$$

We integrate the aforementioned relation to obtain

$$\mathcal{V}_\alpha(t) + K_\alpha\int_{\mathcal{X}_\alpha(0)}^{\mathcal{X}_\alpha(t)}\psi_\alpha(\sqrt{2N_\alpha}x)dx \le \mathcal{V}_\alpha(0) + \frac{K_d\sqrt{2NM_2(0)}}{\lambda_0}\int_0^\infty \psi_d(x)dx.$$

In particular, this yields

$$K_\alpha\int_{\mathcal{X}_\alpha(0)}^{\mathcal{X}_\alpha(t)}\psi_\alpha(\sqrt{2N_\alpha}x)dx \le \mathcal{V}_\alpha(0) + \frac{K_d\sqrt{2NM_2(0)}}{\lambda_0}\int_0^\infty \psi_d(x)dx,\ t \in [0, T_1^*). \tag{7.2.7}$$

On the other hand, the assumption on K_α implies

$$\mathcal{V}_\alpha(0) + \frac{K_d\sqrt{2NM_2(0)}}{\lambda_0}\int_0^\infty \psi_d(x)dx < K_\alpha\int_{\mathcal{X}_\alpha(0)}^\infty \psi_\alpha(\sqrt{2N_\alpha}x)dx. \tag{7.2.8}$$

We use (7.2.7) and (7.2.8) to see the existence of a solution to the following equation with variable x_α^∞:

$$K_\alpha\int_{\mathcal{X}_\alpha(0)}^{x_\alpha^\infty}\psi_\alpha(\sqrt{2N_\alpha}x)dx = \mathcal{V}_\alpha(0) + \frac{K_d\sqrt{2NM_2(0)}}{\lambda_0}\int_0^\infty \psi_d(x)dx.$$

We set x_α^∞ to be its largest possible positive value. Then, by the same argument employed in Theorem 7.1.1, we have

$$\mathcal{X}_\alpha(t) \le x_\alpha^\infty, \quad \text{for all } \ t \in [0, T_1^*).$$

(ii) (Decay estimate of $\|\hat{\boldsymbol{v}}_\alpha(t)\|_\infty$): By the estimate (ii) in Proposition 7.2.1, we have

$$\frac{d\mathcal{V}_\alpha}{dt} \le -K_\alpha\psi_\alpha(\sqrt{2N_\alpha}x_\alpha^\infty)\mathcal{V}_\alpha + K_d\sqrt{2NM_2(0)}\psi_d(\lambda_0 t).$$

We now apply Lemma A.0.1 in Appendix with

$$a := K_\alpha \psi_\alpha(\sqrt{2N_\alpha} x_\alpha^\infty), \quad f := K_d \sqrt{2NM_2(0)} \psi_d(\lambda_0 t),$$

to find the desired flocking estimate:

$$\begin{aligned} \mathcal{V}_\alpha(t) \ &\le \ \mathcal{V}_\alpha(0) e^{-K_\alpha \psi_\alpha(\sqrt{2N_\alpha} x_\alpha^\infty) t} \\ &\quad + \frac{K_d \sqrt{2NM_2(0)}}{K_\alpha \psi(\sqrt{2N_\alpha} x_\alpha^\infty)} \Big[e^{-\frac{K_\alpha \psi_d(\sqrt{2N_\alpha} x_\alpha^\infty) t}{2}} + \psi_d\Big(\frac{\lambda_0 t}{2}\Big)\Big]. \end{aligned}$$

$\square$

As a final step, we are now ready to complete the proof of Theorem 7.2.1 by showing that $T_1^* = \infty$.

Step B: Continuation to the whole time interval

Suppose that initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ and coupling strengths satisfy the framework $(\mathcal{F}_A)$. We claim

$$T_1^* = \infty. \tag{7.2.9}$$

Proof of claim (7.2.9): Suppose not, i.e., $0 < T_1^* < \infty$. Then, it follows from the definition of T^* in (7.2.6) that

$$(\boldsymbol{v}_{2k}(T_1^*) - \boldsymbol{v}_{1i}(T_1^*)) \cdot \boldsymbol{e}_{1,2}^0 = \lambda_0. \tag{7.2.10}$$

On the other hand, we use Proposition 7.2.1 and Proposition 7.2.2 to obtain: for $\alpha = 1, 2$,

$$\begin{aligned} &\|\boldsymbol{v}_{\alpha c}(T_1^*) - \boldsymbol{v}_{\alpha c}(0)\| \le \frac{K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x) dx, \\ &\|\hat{\boldsymbol{v}}_\alpha(T_1^*)\|_\infty \le \|\hat{\boldsymbol{v}}_\alpha(0)\|_\infty + \frac{2K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x) dx. \end{aligned} \tag{7.2.11}$$

We use assumptions on the initial data K_d and (7.2.11) to derive the following relation:

$$\begin{aligned}
&(\boldsymbol{v}_{2k}(T_1^*) - \boldsymbol{v}_{1i}(T_1^*)) \cdot \boldsymbol{e}_{1,2}^0 \\
&\qquad \geq \|\boldsymbol{v}_{2c}(0) - \boldsymbol{v}_{1c}(0)\| - \|\boldsymbol{v}_{1c}(T_1^*) - \boldsymbol{v}_{1c}(0)\| - \|\boldsymbol{v}_{2c}(T_1^*) - \boldsymbol{v}_{2c}(0)\| \\
&\qquad\quad - \|\hat{\boldsymbol{v}}_1(T_1^*)\|_\infty - \|\hat{\boldsymbol{v}}_2(T_1^*)\|_\infty \\
&\qquad \geq 2\lambda_0 - \|\hat{\boldsymbol{v}}_1(0)\|_\infty - \|\hat{\boldsymbol{v}}_2(0)\|_\infty - \frac{6K_d\sqrt{2NM_2(0)}}{\lambda_0}\int_0^\infty \psi_d(s)ds \\
&\qquad > \lambda_0.
\end{aligned}$$

This contradicts relation (7.2.10). Thus $T_1^* = \infty$.

7.2.3 Description of partial flocking

In this subsection, we briefly discuss the main results for the emergence of partial flocking (see Definition 7.2.1) for some class of initial configurations under the following situation:

$$K_1 \gg 1, \quad K_2 \ll 1, \quad K_d \ll 1.$$

In this case, subsystem $\mathcal{G}_1$ flocks, but the other subsystem, $\mathcal{G}_2$, does not. More precisely, our result is as follows.

Theorem 7.2.2. *Suppose that the following framework* $(\mathcal{C}_6)$ *holds for the initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *to system* (7.0.1)*.*

- $(\mathcal{C}_6 1)$*: (Restriction on initial configurations)*

$$\begin{aligned}
&\max_{1\leq i\leq N_1} \|\boldsymbol{v}_{1i}(0) - \boldsymbol{v}_{1c}(0)\| < \frac{1}{4}\min_{1\leq k\leq N_2} \|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0)\|, \\
&\boldsymbol{v}_{2k}(0) \neq \boldsymbol{v}_{2i}(0), \textit{for } i \neq k, \\
&\min_{\substack{1\leq i\leq N_1\\1\leq k\leq N_2}} \{(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{1i}(0)) \cdot (\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0))\} \geq 0 \quad \textit{and} \\
&\min_{1\leq i\neq k\leq N_2} \{(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{2i}(0)) \cdot (\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0))\} \geq 0.
\end{aligned}$$

- $(\mathcal{C}_6 2)$: *(Restriction on coupling strengths):*

$$
\begin{aligned}
0 \le K_d &< \frac{\Lambda_0 \min\{\mu_0, \frac{\Lambda_0}{4}\}}{4\sqrt{2NM_2(0)}\int_0^\infty \psi_d(x)dx},\\
0 \le K_2 &< \frac{\mu_0 \min\{\mu_0, \frac{\Lambda_0}{2}\}}{4\sqrt{2NM_2(0)}\int_0^\infty \psi_2(x)dx},\\
K_1 &> \frac{\mathcal{V}_1(0) + \frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx}{\int_{\mathcal{X}_1(0)}^\infty \psi_1(\sqrt{2N_1}x)dx},
\end{aligned}
$$

where positive constants Λ_0 and μ_0 are given by the following relations:

$$
\Lambda_0 := \frac{1}{2}\min_{1\le k\le N_2}\|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0)\|, \quad \mu_0 := \frac{1}{2}\min_{1\le i\ne k\le N_2}\|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0)\|.
$$

Then, the subsystem $\mathcal{G}_1$ and $\mathcal{G}_2$ exhibit a time-asymptotic partial flocking. More precisely, for the solution $(\boldsymbol{x}, \boldsymbol{v})$ to system (7.0.1) with initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$, there exist $\bar{x}_1^\infty$ and C_1 that only depend on the initial data and ψ_1 such that for any $t \ge 0$,

$$
\begin{aligned}
&\sup_{0\le t<\infty}\mathcal{X}_1(t) < \bar{x}_1^\infty, \qquad \mathcal{V}_1(t) \le C_1 \max\Big\{e^{-\frac{K_1\psi_1(\sqrt{2N_1}\bar{x}_1^\infty)t}{2}}, \psi_d\Big(\frac{\Lambda_0 t}{2}\Big)\Big\},\\
&\inf_{\substack{0\le t<\infty\\ 1\le i\ne k\le N_2}}\|\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{2i}(t)\| \ge \mu_0, \qquad \min_{1\le i\ne k\le N_2}\|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{2i}(t)\| \ge \mu_0 t.
\end{aligned}
$$

As a corollary of Theorem 7.2.2, we get rid of the assumptions

$$
\begin{aligned}
&\min_{\substack{1\le i\le N_1\\ 1\le k\le N_2}}\{(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{1i}(0))\cdot(\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0))\} \ge 0, \text{ and}\\
&\min_{1\le i\ne k\le N_2}\{(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{2i}(0))\cdot(\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0))\} \ge 0.
\end{aligned}
$$

For this, we define

$$
\begin{aligned}
T_1 &:= \frac{1}{\Lambda_0^2}\max_{\substack{1\le i\le N_1\\ 1\le k\le N_2}}|(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{1i}(0))\cdot(\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0))|,\\
T_2 &:= \frac{1}{\mu_0^2}\max_{1\le i\ne k\le N_2}|(\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{2i}(0))\cdot(\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0))|,\\
T_0 &:= \max\{T_1, T_2\} \ge 0.
\end{aligned}
$$

Corollary 7.2.2. *Suppose that the following framework* $(\mathcal{C}_7)$ *holds for the initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *to system* (7.0.1).

- $(\mathcal{C}_7 1)$*: (Restriction on initial configurations)*

$$\begin{aligned}&\max_{1\le i\le N_1} \|\boldsymbol{v}_{1i}(0)-\boldsymbol{v}_{1c}(0)\| < \frac{1}{4}\min_{1\le k\le N_2}\|\boldsymbol{v}_{2k}(0)-\boldsymbol{v}_{1c}(0)\|,\\ &\boldsymbol{v}_{2k}(0)\neq \boldsymbol{v}_{2i}(0), \textit{for } i\neq k.\end{aligned}$$

- $(\mathcal{C}_7 2)$*: (Restriction on coupling strengths):*

$$\begin{aligned}
0 \le K_d &< \min\Big\{\frac{\min\{\Lambda_0,\mu_0\}}{16\sqrt{2NM_2(0)}T_0}, \frac{\Lambda_0\min\{\mu_0,\frac{\Lambda_0}{4}\}}{8\sqrt{2NM_2(0)}\int_0^\infty \psi_d(x)dx},\\
&\qquad \frac{\min\{\Lambda_0^2,\mu_0^2\}}{4\big(D(\boldsymbol{x}_1(0),\boldsymbol{x}_2(0))+\sqrt{2NM_2(0)}T_0\big)\sqrt{2NM_2(0)}}\}\Big\},\\
0 \le K_2 &< \min\Big\{\frac{\min\{\Lambda_0,\mu_0\}}{16\sqrt{2NM_2(0)}T_0}, \frac{\mu_0\min\{\mu_0,\frac{\Lambda_0}{2}\}}{8\sqrt{2NM_2(0)}\int_0^\infty \psi_2(x)dx},\\
&\qquad \frac{\min\{\Lambda_0^2,\mu_0^2\}}{4\big(D(\boldsymbol{x}_1(0),\boldsymbol{x}_2(0))+\sqrt{2NM_2(0)}T_0\big)\sqrt{2NM_2(0)}},\Big\},\\
K_1 &> \frac{P_1(0)+\frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx}{\int_{R_1(0)}^\infty \psi_1(\sqrt{2N_1}x)dx},
\end{aligned}$$

where $P_1(0)$ *and* $R_1(0)$ *are defined in* (7.2.2).

Then, the subsystems $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *exhibit a time-asymptotic partial flocking. More precisely, for the solution* $(\boldsymbol{x},\boldsymbol{v})$ *to system* (7.0.1) *with initial data* $(\boldsymbol{x}_0,\boldsymbol{v}_0)$*, the following estimates hold: there exist* $\overline{x}_1^\infty > 0$ *and* T_0 *such that*

$$\begin{aligned}
&\sup_{0\le t<\infty}\mathcal{X}_1(t) < \bar{x}_1^\infty, \quad \mathcal{V}_1(t)\le C_2\max\Big\{e^{-\frac{K_1\psi_1(\sqrt{2N_1}\bar{x}_1^\infty)(t-T_0)}{2}}, \psi_d\Big(\frac{\Lambda_0(t-T_0)}{4}\Big)\Big\},\\
&\inf_{\substack{T_0\le t<\infty\\ 1\le i\neq k\le N_2}}\|\boldsymbol{v}_{2k}(t)-\boldsymbol{v}_{2i}(t)\| \ge \frac{\mu_0}{2},\\
&\min_{1\le i\neq k\le N_2}\|\boldsymbol{x}_{2k}(t)-\boldsymbol{x}_{2i}(t)\| > \frac{\mu_0}{2}(t-T_0),\ t\in[T_0,\infty).
\end{aligned}$$

As we mentioned in Remark 7.2.2, we only present the proof of Theorem 7.2.2 in the following section.

7.2.4 Emergence of partial flocking

We first remind the system of differential inequalities on our functionals.

Proposition 7.2.3. *Suppose that the coupling strengths satisfy*

$$K_\alpha \geq 0, \quad \alpha = 1, 2, \qquad K_d \geq 0,$$

and let $(\boldsymbol{x}, \boldsymbol{v})$ *be a global solution to* (7.0.1). *Then, for* $\alpha = 1, 2$, *we have*

$$(i) \ \left\| \frac{d\boldsymbol{v}_{\alpha c}}{dt} \right\| < K_d \sqrt{2NM_2(0)}\psi_{dM}, \ \ a.e. \ t \in (0, \infty),$$

$$(ii) \ \left\| \frac{d\boldsymbol{v}_{2k}(t)}{dt} \right\| \leq K_2 \sqrt{2NM_2(0)}\psi_{2M}(t) + K_d \sqrt{2NM_2(0)}\psi_{dM}(t),$$

$$(iii) \ \frac{d\mathcal{X}_1}{dt} \leq \mathcal{V}_1, \quad \frac{d\mathcal{V}_1}{dt} \leq -K_1 \psi_1(\sqrt{2N_1}\mathcal{X}_1)\mathcal{V}_1 + K_d \sqrt{2NM_2(0)}\psi_{dM},$$

$$(iv) \ \|\hat{\boldsymbol{v}}_\alpha(t_2)\|_\infty - \|\hat{\boldsymbol{v}}_\alpha(t_1)\|_\infty \leq 2K_d \sqrt{2NM_2(0)} \int_{t_1}^{t_2} \psi_{dM}(t)dt,$$

for any $0 \leq t_1 \leq t_2 < \infty$.

where ψ_{dM} *and* ψ_{2M} *are given by the following relations:*

$$\psi_{dM} := \max_{\substack{1 \leq i \leq N_1 \\ 1 \leq k \leq N_2}} \psi_d(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{1i}\|), \quad \psi_{2M} := \max_{1 \leq i \neq k \leq N_2} \psi_2(\|\boldsymbol{x}_{2k} - \boldsymbol{x}_{2i}\|). \tag{7.2.12}$$

Proof. It is an analogue of the proof of Proposition 7.2.1 and Proposition 7.2.2. □

In the following two subsections, we prove Theorem 7.2.2 as follows.

- Step A (Local-in-time estimate): We will show that, for some finite time T, subsystem $\mathcal{G}_1$ satisfies the flocking estimate, but subsystem $\mathcal{G}_2$ does not:

$$\sup_{0 \leq t < T} \mathcal{X}_1(t) < \overline{x}_1^\infty, \quad \mathcal{V}_1(t) \leq C_1 \max\left\{ e^{-\frac{K_1 \psi_1(\sqrt{2N_1}\overline{x}_1^\infty)t}{2}}, \psi_d\Big(\frac{\Lambda_0 t}{2}\Big) \right\},$$
$$\min_{i \neq k} \|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{2i}(t)\| \geq Ct, \quad \text{for some } C > 0, \text{ for all } t \in [0, T).$$

- Step B (Continuation to the whole time interval): We will show that the time T in Step A can be chosen to be infinity.

Step A : Local-in-time flocking estimates

In this part, we show that the flocking estimates hold at least locally in time. For this, we set

$$\begin{aligned}
&\boldsymbol{e}^0_{2k,1} := \frac{\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0)}{\|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0)\|}, \qquad \boldsymbol{e}^0_{2k,2i} := \frac{\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0)}{\|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0)\|},\\
&\quad T_0^* := \sup\Big\{T \in [0,\infty) \ \Big| \ \min_{i,k}\{(\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{1i}(t)) \cdot \boldsymbol{e}^0_{2k,1}\} > \Lambda_0, \forall t \in [0,T)\Big\},\\
&\quad \hat{T}_0^* := \sup\Big\{T \in [0,T_0^*) \ \Big| \ \min_{i\neq k}\{(\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{2i}(t)) \cdot \boldsymbol{e}^0_{2k,2i}\} > \mu_0, \forall t \in [0,T)\Big\},
\end{aligned} \tag{7.2.13}$$

Lemma 7.2.3. *Suppose that initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ satisfy the following relations:*

$$\begin{aligned}
&\max_{1\le i\le N_1} \|\boldsymbol{v}_{1i}(0) - \boldsymbol{v}_{1c}(0)\| < \frac{1}{4}\min_{1\le k\le N_2} \|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0)\|,\\
&\boldsymbol{v}_{2k}(0) \neq \boldsymbol{v}_{2i}(0), \quad \textit{for } i \neq k.
\end{aligned}$$

Then, we have

$$T_0^* > 0 \quad \textit{and} \quad \hat{T}_0^* > 0.$$

Proof. We use assumptions to see

$$\begin{aligned}
(\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1i}(0)) \cdot \boldsymbol{e}^0_{2k,1} &= (\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0) - \hat{\boldsymbol{v}}_{1i}(0)) \cdot \boldsymbol{e}^0_{2k,1}\\
&\ge \|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0)\| - \|\hat{\boldsymbol{v}}_1(0)\|_\infty \ge \frac{3\Lambda_0}{2} > \Lambda_0,\\
(\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0)) \cdot \boldsymbol{e}^0_{2k,2i} &= \|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0)\| \ge 2\mu_0 > \mu_0.
\end{aligned}$$

Then, by the continuity argument, we have $T_0^* > 0$ and $\hat{T}_0^* > 0$. □

Lemma 7.2.4. *Suppose that the initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ satisfy $(\mathcal{C}_6 1)$. Let ψ_{2M} and ψ_{dM} be the functions defined in* (7.2.12)*. Then, they satisfy*

$$\psi_{2M}(t) \le \psi_2(\mu_0 t), \qquad \psi_{dM}(t) \le \psi_d(\Lambda_0 t), \quad t \in [0, \hat{T}_0^*).$$

Proof. For $t \in [0, \hat{T}_0^*]$ and $i \neq k$, we have

$$\begin{aligned}
\|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{2i}(t)\| &\ge (\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{2i}(t)) \cdot \boldsymbol{e}^0_{2k,2i}\\
&= (\boldsymbol{x}_{2k}(0) - \boldsymbol{x}_{2i}(0)) \cdot \boldsymbol{e}^0_{2k,2i} + \int_0^t (\boldsymbol{v}_{2k}(s) - \boldsymbol{v}_{2i}(s)) \cdot \boldsymbol{e}^0_{2k,2i} ds\\
&\ge \mu_0 t.
\end{aligned}$$

Similarly, we have

$$\|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{1i}(t)\| \geq \Lambda_0 t.$$

Thus, by the non-increasing property of ψ_2 and ψ_d, we have the desired estimates. $\square$

Lemma 7.2.5. *Suppose that the initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ satisfy $(\mathcal{C}_6 1)$, then the following estimate holds.*

$$\|\boldsymbol{v}_2(t)-\boldsymbol{v}_2(0)\|_\infty \leq \frac{K_2\sqrt{2NM_2(0)}}{\mu_0}\int_0^\infty \psi_2(x)dx+\frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx,$$

for $t \in [0, \hat{T}_0^*)$, where $\|\boldsymbol{v}_2(t) - \boldsymbol{v}_2(0)\|_\infty := \max\limits_{1\leq i\leq N_2} \|\boldsymbol{v}_{2i}(t) - \boldsymbol{v}_{2i}(0)\|$.

Proof. It follows from Proposition 7.2.3 that, for $i \in \{1, \cdots, N_2\}$,

$$\begin{aligned}
\|\boldsymbol{v}_{2i}(t) - \boldsymbol{v}_{2i}(0)\| &\leq K_2\sqrt{2NM_2(0)}\int_0^t \psi_2(\mu_0 x)dx \\
&\quad + K_d\sqrt{2NM_2(0)}\int_0^t \psi_d(\Lambda_0 x)dx \\
&\leq \frac{K_2\sqrt{2NM_2(0)}}{\mu_0}\int_0^\infty \psi_2(x)dx \\
&\quad + \frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx.
\end{aligned}$$

$\square$

Lemma 7.2.6. *Suppose that the initial data $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ satisfy $(\mathcal{C}_6 1)$ and the coupling strengths K_2 and K_d satisfy*

$$\begin{aligned}
&0 \leq K_d < \frac{\Lambda_0\mu_0}{4\sqrt{2NM_2(0)}\int_0^\infty \psi_d(x)dx}, \qquad \textit{and} \\
&0 \leq K_2 < \frac{\mu_0^2}{4\sqrt{2NM_2(0)}\int_0^\infty \psi_2(x)dx}.
\end{aligned}$$

Then, we have $\hat{T}_0^* = T_0^*$.

Proof. It follows from Lemma 7.2.3 that we have $\hat{T}_0^* > 0$. We now assume that

$$\hat{T}_0^* < T_0^*.$$

Then, the definition of $\hat{T}_0^*$ implies

$$(\boldsymbol{v}_{2k}(\hat{T}_0^*) - \boldsymbol{v}_{2i}(\hat{T}_0^*)) \cdot \boldsymbol{e}^0_{2k,2i} = \mu_0. \tag{7.2.14}$$

On the other hand, by the initial assumption and Lemma 7.2.5, for $t \in [0, \hat{T}_0^*]$ and $i \neq k$

$$\begin{aligned}
&(\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{2i}(t)) \cdot \boldsymbol{e}^0_{2k,2i} \\
&\quad \geq \|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{2i}(0)\| - \|\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{2k}(0)\| - \|\boldsymbol{v}_{2i}(t) - \boldsymbol{v}_{2i}(0)\| \\
&\quad \geq 2\mu_0 - \left(\frac{2K_2}{\mu_0}\int_0^\infty \psi_2(x)dx + \frac{2K_d}{\Lambda_0}\int_0^\infty \psi_d(x)dx\right)\sqrt{2NM_2(0)} \\
&\quad > 2\mu_0 - \mu_0 = \mu_0.
\end{aligned}$$

The last inequality is from the assumptions of K_2 and K_d. This gives a contradiction to (7.2.14). Hence we obtain $\hat{T}_0^* = T_0^*$. $\square$

Lemma 7.2.7. *Keep the assumption of Lemma 7.2.6, we have*

$$\begin{aligned}
&(i)\ \|\boldsymbol{v}_{1c}(t) - \boldsymbol{v}_{1c}(0)\| \leq \frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx, t \in [0, T_0^*). \\
&(ii)\ \|\hat{\boldsymbol{v}}_1(t)\|_\infty \leq \|\hat{\boldsymbol{v}}_1(0)\|_\infty + \frac{2K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx, t \in [0, T_0^*).
\end{aligned}$$

Proof. The estimates follow directly from the Proposition 7.2.3. $\square$

Lemma 7.2.8. (Local-in-time flocking estimate) *Suppose that the initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *satisfy* $(\mathcal{C}_6 1)$ *and the coupling strengths satisfy*

$$\begin{aligned}
&0 \leq K_d < \frac{\mu_0\Lambda_0}{4\sqrt{2NM_2(0)}\int_0^\infty \psi_d(x)dx}, \\
&0 \leq K_2 < \frac{\mu_0^2}{4\sqrt{2NM_2(0)}\int_0^\infty \psi_2(x)dx}, \\
&K_1 > \frac{\mathcal{V}_1(0) + \frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx}{\int_{\mathcal{X}_1(0)}^\infty \psi_1(\sqrt{2N_1}x)dx}.
\end{aligned}$$

Then, for the solution $(\boldsymbol{x}, \boldsymbol{v})$ *to system* (7.0.1) *with initial data* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$*, there exist positive constants* $\bar{x}_1^\infty$ *and* C_1 *independent of time* t *such that*

$$\begin{aligned}
&\sup_{0\le t<\infty} \mathcal{X}_1(t) < \bar{x}_1^\infty, \quad \mathcal{V}_1(t) \le C_1 \max\Big\{ e^{-\frac{K_1\psi_1(\sqrt{2N_1}\bar{x}_1^\infty)t}{2}}, \psi_d\Big(\frac{\Lambda_0 t}{2}\Big)\Big\},\\
&\inf_{\substack{0\le t<\infty\\ 1\le i\neq k\le N_2}} \|\boldsymbol{v}_{2k}(t) - \boldsymbol{v}_{2i}(t)\| \ge \mu_0,\\
&\min_{1\le i\neq k\le N_2} \|\boldsymbol{x}_{2k}(t) - \boldsymbol{x}_{2i}(t)\| \ge \mu_0 t, \quad t\in[0, T_0^*].
\end{aligned}$$

Proof. (i) (Existence of $\bar{x}_1^\infty$): Define a Lyapunov functional $\mathcal{L}_2$:

$$\mathcal{L}_2(t) := \mathcal{V}_1(t) + K_1\int_0^{\mathcal{X}_1(t)} \psi_1(\sqrt{2N_1}x)dx.$$

It follows from Proposition 7.2.3 and Lemma 7.2.4 that we have

$$\begin{aligned}
\frac{d\mathcal{L}_2(t)}{dt} &= \frac{d}{dt}\mathcal{V}_1(t) + K_1\psi_1(\sqrt{2N_1}\mathcal{X}_1(t))\frac{d}{dt}\mathcal{X}_1(t)\\
&\le -K_1\psi_1(\sqrt{2N_1}\mathcal{X}_1(t))\Big(\mathcal{V}_1(t) - \frac{d}{dt}\mathcal{X}_1(t)\Big) + K_d\sqrt{2NM_2(0)}\psi_{dM}\\
&\le K_d\sqrt{2NM_2(0)}\psi_d\Big(\Lambda_0 t\Big), \quad t\in[0,T_0^*).
\end{aligned}$$

We integrate the aforementioned relation to obtain

$$\mathcal{V}_1(t) + K_1\int_{\mathcal{X}_1(0)}^{\mathcal{X}_1(t)} \psi_1(\sqrt{2N_1}x)dx \le \mathcal{V}_1(0) + \frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx.$$

In particular, this yields

$$K_1\int_{\mathcal{X}_1(0)}^{\mathcal{X}_1(t)} \psi_1(\sqrt{2N_1}x)dx \le \mathcal{V}_1(0) + \frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx, \quad t\in[0,T_0^*). \tag{7.2.15}$$

On the other hand, the condition on K_1 implies

$$\mathcal{V}_1(0) + \frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx < K_1\int_{\mathcal{X}_1(0)}^\infty \psi_1(\sqrt{2N_1}x)dx.$$

We set $\bar{x}_1^\infty$ to be a positive number satisfying the following relation:

$$K_1\int_{\mathcal{X}_1(0)}^{\bar{x}_1^\infty} \psi_1(\sqrt{2N_1}x)dx = \mathcal{V}_1(0) + \frac{K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx. \tag{7.2.16}$$

Then, by using (7.2.15) and (7.2.16), we have

$$\sup_{0\le t<T_0^*} \mathcal{X}_1(t) < \bar{x}_1^\infty, \quad t \in [0, T_0^*].$$

(ii) (Decay estimate of $\mathcal{V}_1$): It follows from Proposition 7.2.3 and the result of (i) that we have

$$\frac{d\mathcal{V}_1}{dt} \le -K_1\psi_1(\sqrt{2N_1}\bar{x}_1^\infty)\mathcal{V}_1 + K_d\sqrt{2NM_2(0)}\psi_{dM}.$$

This yields the desired decay estimate of $\mathcal{V}_1$.
The two remaining estimates are direct results of Lemma 7.2.4 and Lemma 7.2.6.

□

Step B: Continuation to the whole time interval

In this part, we complete the proof of Theorem 7.2.2. Suppose that the initial data and coupling strength satisfy the framework $(\mathcal{C}_6)$ in Theorem 7.2.2. Then, it follows from Lemma 7.2.3 that we have

$$T_0^* > 0.$$

Suppose that $T_0^* < \infty$. Then, by the definition of T_0^* in (7.2.13), we have

$$(\boldsymbol{v}_{2k}(T_0^*) - \boldsymbol{v}_{1i}(T_0^*)) \cdot \boldsymbol{e}^0_{2k,1} = \Lambda_0. \tag{7.2.17}$$

It follows from Lemma 7.2.4-Lemma 7.2.7 that we have

$$\begin{aligned}
&(\boldsymbol{v}_{2k}(T_0^*) - \boldsymbol{v}_{1i}(T_0^*)) \cdot \boldsymbol{e}^0_{2k,1} \\
&\quad \ge \|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0)\| \\
&\qquad - \|\hat{\boldsymbol{v}}_{1i}(T_0^*)\| - \|\boldsymbol{v}_{2k}(T_0^*) - \boldsymbol{v}_{2k}(0)\| - \|\boldsymbol{v}_{1c}(T_0^*) - \boldsymbol{v}_{1c}(0)\| \\
&\quad \ge \|\boldsymbol{v}_{2k}(0) - \boldsymbol{v}_{1c}(0)\| - \|\hat{\boldsymbol{v}}_1(0)\|_\infty \\
&\qquad - \frac{K_2\sqrt{2NM_2(0)}}{\mu_0}\int_0^\infty \psi_2(x)dx - \frac{4K_d\sqrt{2NM_2(0)}}{\Lambda_0}\int_0^\infty \psi_d(x)dx \\
&\quad > \Lambda_0,
\end{aligned}$$

where we have used the initial assumption and the assumptions of K_2, K_d to get the last inequality. This gives a contradiction to (7.2.17). Thus $T_0^* = \infty$. Now we apply Lemma 7.2.8 with $T_0^* = \infty$ to get the desired estimates and complete the proof of Theorem 7.2.2.

Chapter 8

Bi-cluster flocking on hydrodynamic Cucker-Smale model

The purpose of this chapter is to provide two frameworks leading to mono-cluster and bi-cluster flockings in terms of communication weight, coupling strengths, and regularity and size of initial configurations. We did those things on particle-based N-body dynamics in the previous chapters. Now we treat hydrodynamic model (2.5.29) mentioned in Section 2.5. This chapter is based on the joint work in [44].

Before proceeding, a few comments on notation are in order. Here we use the same notation for vectors and scalars, which excludes bold, double, or arrow notations. For any nonnegative integer k, $H^k := H^k(\Omega)$ denotes the kth-order Sobolev space on Ω, and for simplicity, we omit Ω-dependence in norm whenever there is no confusion, i.e., $\|u\|_{H^k} := \|u\|_{H^k(\Omega)}$. On the other hand, $\mathcal{C}^k(I; E)$ is the space of k times continuously differentiable functions from an interval $I \subset \mathbb{R}$ into a Banach space E. The derivative character ∇^k denotes any spatial partial derivative ∂^α with a multi-index α with $|\alpha| = k$.

In this chapter, we are interested in a coupled system of hydrodynamic models (2.5.26) for $(\rho_i, u_i),\ i = 1, 2$, on each domain $\Omega_1(t)$ and $\Omega_2(t)$, which is separated. The equations of motion are stated in Section 2.5;

$$
\begin{aligned}
&\partial_t \rho_1 + \nabla \cdot (\rho_1 u_1) = 0, \quad \partial_t \rho_2 + \nabla \cdot (\rho_2 u_2) = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+,\\
&\rho_1 \partial_t u_1 + \rho_1 u_1 \cdot \nabla u_1\\
&\qquad = \kappa_{11} \int_{\Omega_1(t)} \rho_1(x)\rho_1(y)\psi(|y-x|)(u_1(y)-u_1(x))dy\\
&\qquad + \kappa_{12} \int_{\Omega_2(t)} \rho_1(x)\rho_2(y)\psi(|y-x|)(u_2(y)-u_1(x))dy,\\
&\rho_2 \partial_t u_2 + \rho_2 u_2 \cdot \nabla u_2\\
&\qquad = \kappa_{22} \int_{\Omega_2(t)} \rho_2(x)\rho_2(y)\psi(|y-x|)(u_2(y)-u_2(x))dy\\
&\qquad + \kappa_{21} \int_{\Omega_1(t)} \rho_2(x)\rho_1(y)\psi(|y-x|)(u_1(y)-u_2(x))dy,
\end{aligned}
\tag{8.0.1}
$$

subject to initial data

$$
(\rho_i, u_i)(x,0) = (\rho_{i0}, u_{i0}), \quad x \in \mathbb{R}^d,\ i = 1,2. \tag{8.0.2}
$$

Here, we consider the fluid regions $\Omega_1(t)$ and $\Omega_2(t)$ are the connected compact supports of the densities ρ_1 and ρ_2 at time t, respectively, and κ_{ii} and κ_{ij} are intra- and inter-coupling strengths, which are assumed to be nonnegative.

We assume ρ_1 and ρ_2 are strictly positive on each regions $\Omega_1(t)$ and $\Omega_2(t)$, respectively, so that there are jumps in the mass densities near the boundary of domain. This allow us to avoid difficulties of vacuum. Moreover, initial data are sufficiently regular to satisfy

$$
(\rho_{i0}, u_{i0}) \in H^s(\Omega_i) \times H^{s+1}(\Omega_i),\ s > 1 + \frac{d}{2},
$$

so that we have a classical solution

$$
(\rho_i, u_i) \in H^s(\Omega_i) \times H^{s+1}(\Omega_i),
$$

which is $\mathcal{C}^1$ by the Sobolev embedding theorem.

This chapter consists of two main result. First, we present a framework leading to mono-cluster flocking on (2.5.29). The framework is formulated in terms of the initial configuration, coupling strengths, and communication weight. The intercoupling strengths κ_{12} and κ_{21} are assumed to be the same, whereas intracoupling strengths are sufficiently large so that two ensembles of macroscopic C-S fluids are combined into one single cluster.

Our second result addresses the case in which bi-cluster flocking is guaranteed. To guarantee bi-cluster flocking, the intracoupling strengths should be sufficiently large, but the intercoupling strengths should be sufficiently small so that two ensemble configurations are sustained for all time. As noticed in Chapter 3 and 4, for a technical reason, we require that the initial configurations be well prepared in the sense that they are close to a bi-cluster flocking configuration; otherwise, in general, we may have multi-cluster flockings. These restrictions are encoded in our second framework for bi-cluster flocking in Section 3.2.

8.1 Lagrangian formulation and variables

In order to construct differential inequalities on Lyapunov functionals, we discuss a Lagrangian formulation for the coupled system (8.0.1) and give several basic a priori estimates for the propagation of velocity moments corresponding to mass, momentum, and energy. In this way, we can avoid free boundary problems while we need positive minimum density of particles.

8.1.1 Lagrangian formulation

As noticed in [41] for the hydrodynamic C-S model (2.5.26) in the vacuum regime, it is convenient to reformulate the system (8.0.1) in terms of Lagrangian variables so that system (8.0.1) becomes an integro-differential system in Lagrangian coordinates, where the computational domain is fixed as the initial domain. Now, we briefly discuss the Lagrangian formulation of (8.0.1), loosely following the presentation in [21, 41]. First, we introduce the triplet (η_i, q_i, v_i) for Lagrangian variables associated with macroscopic observables (ρ_i, u_i), consisting of the forward particle path $\eta_i = \eta_i(x, t)$, Lagrangian mass $q_i = q_i(x, t)$, and velocity densities $v_i = v_i(x, t)$. We set $\Omega_1 := \Omega_1(0)$

and $\Omega_2 := \Omega_2(0)$ for notational convenience. Then, for a fixed $x \in \Omega_i$,

$$\begin{cases} \dfrac{d\eta_i(x,t)}{dt} = u_i(\eta_i(x,t),t), \quad t > 0, \quad i = 1,2, \\ \eta_i(x,0) = x \end{cases} \tag{8.1.1}$$

and

$$q_i(x,t) := \rho_i(\eta_i(x,t),t), \quad v_i(x,t) := u_i(\eta_i(x,t),t). \tag{8.1.2}$$

In the following lemma, we study the evolution of the Lagrangian triplet (η_i, q_i, v_i).

Lemma 8.1.1. *Let (ρ_i, u_i) be a sufficiently smooth solution to system* (8.0.1) *and* (8.0.2)*. Then, the Lagrangian variables (η_i, q_i, v_i) defined by* (8.1.1) *and* (8.1.2) *satisfy the following relations:*

$$\begin{aligned}
&\eta_i(x,t) = x + \int_0^t v_i(x,\tau)d\tau, \\
&q_i(x,t) = \rho_{i0}(x) det(\nabla \eta_i(x,t))^{-1}, \quad x \in \Omega_i, \ i = 1,2, \\
&\partial_t v_1 = \kappa_{11} \int_{\Omega_1} q_1(y,0)\psi(|\eta_1(y) - \eta_1(x)|)(v_1(y) - v_1(x))dy \\
&\qquad + \kappa_{12} \int_{\Omega_2} q_2(y,0)\psi(|\eta_2(y) - \eta_1(x)|)(v_2(y) - v_1(x))dy, \\
&\partial_t v_2 = \kappa_{22} \int_{\Omega_2} q_2(y,0)\psi(|\eta_2(y) - \eta_2(x)|)(v_2(y) - v_2(x))dy \\
&\qquad + \kappa_{21} \int_{\Omega_1} q_1(y,0)\psi(|\eta_1(y) - \eta_2(x)|)(v_1(y) - v_2(x))dy, \\
&v_i(x,0) = u_{i0}.
\end{aligned} \tag{8.1.3}$$

Proof. In the following, we only sketch the derivation of the relations for q_1 and v_1; the estimates for q_2 and v_2 can be done similarly (see [41] for corresponding estimates for (2.5.26)).

• (Estimate for q_1): We evaluate the continuity equation for ρ_1 on the Lagrangian path $(\eta_i(x,t),t) \in \mathbb{R}^d \times \mathbb{R}_+$:

$$\begin{aligned}
0 &= \partial_t \rho_1 + \nabla \cdot (\rho_1 u_1)\Big|_{(\eta_1(x,t),t)} \\
&= (\partial_t \rho_1 + u_1 \cdot \nabla \rho_1)\Big|_{(\eta_1(x,t),t)} + \rho_1 \nabla \cdot u_1\Big|_{(\eta_1(x,t),t)}.
\end{aligned} \tag{8.1.4}$$

It follows from the defining relations (8.1.2) that

$$\begin{aligned}
&(\partial_t \rho_1 + u_1 \cdot \nabla \rho_1)\Big|_{(\eta_1(x,t),t)} = \partial_t q_1(x,t) \quad \text{and} \\
&\nabla \cdot u_1\Big|_{(\eta_1(x,t),t)} = \sum_{i,j=1}^{d} \partial_{x_j} v_{1i} \partial_{x_i} (\eta_1^{-1})_j = \sum_{i,j=1}^{d} (\nabla \eta_1)^{-1}_{j,i} \partial_{x_j} v_{1i}.
\end{aligned} \tag{8.1.5}$$

Then, we combine (8.1.4) and (8.1.5) to obtain

$$\partial_t q_1 + q_1 \sum_{i,j=1}^{d} (\nabla \eta)^{-1}_{j,i} \partial_{x_j} v_{1i} = 0, \quad x \in \Omega_1, \ t > 0. \tag{8.1.6}$$

We also note that

$$\partial_t \mathrm{det}(\nabla \eta_1(x,t)) = \mathrm{det}(\nabla \eta_1(x,t)) \sum_{i,j=1}^{n} (\nabla \eta_1)^{-1}_{j,i} \partial_{x_j} v_{1i}. \tag{8.1.7}$$

We now combine (8.1.6) and (8.1.7) to obtain the relation for q_1:

$$\partial_t \Big(q_1(x,t) \mathrm{det}(\nabla \eta_1(x,t) \Big) = 0,$$

and integrate the above relation in t and use $\nabla \eta_1(x,0) = I$ to find

$$\begin{aligned}
&q_1(x,t) \mathrm{det}(\nabla \eta_1(x,t)) = \rho_{10}(x) \mathrm{det}(\nabla \eta_1(x,0) = \rho_{10}(x), \\
&\text{i.e.,} \quad q_1(x,t) = \rho_{10}(x) \mathrm{det}(\nabla \eta_1(x,t))^{-1}.
\end{aligned} \tag{8.1.8}$$

• (Estimate for v_1): Note that the left hand side of the momentum equation in (8.0.1) along the Lagrangian path can be rewritten as

$$\partial_t(\rho_1 u_1) + \nabla \cdot (\rho_1 u_1 \otimes u_1)\Big|_{(\eta_1(x,t),t)} = \rho_1(\partial_t u_1 + u_1 \cdot \nabla u_1)\Big|_{(\eta_1(x,t),t)} = q_1(x,t) \partial_t v_1(x,t).$$

Hence, the momentum equation becomes

$$\begin{aligned}
q_1 \partial_t v_1 = \kappa_{11} q_1 &\int_{\Omega_1} q_1(y,t) \mathrm{det}(\nabla \eta_1(y,t)) \psi(|\eta_1(x,t), \eta_1(y,t)|)(v_1(y) - v_1(x)) dy \\
+ \kappa_{12} q_1 &\int_{\Omega_2} q_2(y,t) \mathrm{det}(\nabla \eta_2(y,t)) \psi(|\eta_1(x,t), \eta_2(y,t)|)(v_2(y) - v_1(x)) dy.
\end{aligned} \tag{8.1.9}$$

Then, we now substitute the relation (8.1.8) into (8.1.9) to get the desired estimate. □

Remark 8.1.1. *When ρ_i is strictly positive on $Int(\Omega_i)$ (the interior of Ω_i),*

$$\inf_{x\in Int(\Omega)} (\rho_{i0}(x)) > 0, \quad i = 1, 2,$$

the following two statements are equivalent:
(1) There are smooth solutions (u_i, ρ_i) of system (8.0.1) with (8.0.2) in $\Omega_i(t)$.
(2) There are smooth solutions (v_i, q_i, η_i) of system (8.1.3) in $\Omega_i \times \mathbb{R}_+$.

8.1.2 Macroscopic quantities

In this subsection, we study the dynamics of macroscopic quantities corresponding to total mass, momentum, and energy. For $t \geq 0$, we set

$$\begin{aligned}
\text{mass:} \quad M_{i0}(t) &:= \int_{\Omega_1} q_i(x,0)dx, \quad i = 1, 2,\\
\text{momentum:} \quad M_1(t) &:= \int_{\Omega_1} q_1(x,0)v_1(x,t)dx + \int_{\Omega_2} q_2(x,0)v_2(x,t)dx,\\
\text{energy:} \quad M_2(t) &:= \int_{\Omega_1} q_1(x,0)|v_1(x,t)|^2dx + \int_{\Omega_2} q_2(x,0)|v_2(x,t)|^2dx.
\end{aligned}$$

It follows from Remark 8.1.1 that we can easily check that the definitions of the macroscopic quantities are equivalent to the settings above in Lagrangian coordinates. In particular, it is easy to see that the total mass is conserved along the flow (8.0.1):

$$M_{10}(t) + M_{20}(t) = M_{10}(0) + M_{20}(0), \quad t \geq 0.$$

In the following lemma, for the simplicity of presentation, we suppress t-dependence in η_i and v_i:

$$\eta_i(x) := \eta_i(x,t), \qquad v_i(x) := v_i(x,t), \quad i = 1, 2,$$

and we study the temporal evolution of M_1 and M_2.

Lemma 8.1.2. *Let (q_i, v_i), $i = 1, 2$, be the classical solution to system (8.1.3) decaying at $|x| = \infty$ sufficiently fast. Then, we have the following a priori* *estimates for $t > 0$:*

(i) $$\frac{d}{dt}\int_{\Omega_1} q_1(x,0)v_1(x)dx = \kappa_{12}\iint_{\Omega_1\times\Omega_2} q_1(x,0)q_2(y,0)\psi(|\eta_2(y)-\eta_1(x)|)(v_2(y)-v_1(x))dydx.$$

(ii) $$\frac{d}{dt}\int_{\Omega_2} q_2(x,0)v_2(x)dx = \kappa_{21}\iint_{\Omega_1\times\Omega_2} q_1(y,0)q_2(x,0)\psi(|\eta_1(y)-\eta_2(x)|)(v_1(y)-v_2(x))dydx.$$

(iii) $$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_1} q_1(x,0)|v_1(x)|^2dx &= -\frac{\kappa_{11}}{2}\iint_{\Omega_1\times\Omega_1} q_1(x,0)q_1(y,0)\psi(|\eta_1(y)-\eta_1(x)|)|v_1(y)-v_1(x)|^2dydx\\ &+ \kappa_{12}\iint_{\Omega_1\times\Omega_2} q_1(x,0)q_2(y,0)\psi(|\eta_2(y)-\eta_1(x)|)\langle v_1(x), v_2(y)-v_1(x)\rangle dy.\end{aligned}$$

(iv) $$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_2} q_2(x,0)|v_2(x)|^2dx &= -\frac{\kappa_{22}}{2}\iint_{\Omega_2\times\Omega_2} q_2(x,0)q_2(y,0)\psi(|\eta_2(y)-\eta_2(x)|)|v_2(y)-v_2(x)|^2dydx\\ &+ \kappa_{21}\iint_{\Omega_1\times\Omega_2} q_1(y,0)q_2(x,0)\psi(|\eta_1(y)-\eta_2(x)|)\langle v_2(x), v_1(y)-v_2(x)\rangle dy.\end{aligned}$$

Proof. (i) and (ii): We multiply $(8.1.3)_2$ by $q_1(x,0)$ and integrate the resulting relation with respect to x in Ω_1 and then use the symmetric trick $x \longleftrightarrow y$ to obtain

$$\frac{d}{dt}\int_{\Omega_1} q_1(x,0)v_1(x)dx = \kappa_{12}\iint_{\Omega_1\times\Omega_2} q_1(x,0)q_2(y,0)\psi(|\eta_2(y)-\eta_1(x)|)(v_2(y)-v_1(x))dydx.$$

Similarly, we have the same statement for $q_2(x,0)v_2(x)$.

(iii) and (iv): We multiply $(8.1.3)_2$ by $q_1(x,0)v_1(x)$ and integrate the resulting relation with respect to x in Ω_1 to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_1} q_1(x,0)|v_1(x)|^2 dx \\
&\quad = \kappa_{11}\iint_{\Omega_1\times\Omega_1} q_1(x,0)q_1(y,0)\psi(|\eta_1(y)-\eta_1(x)|)\langle v_1(x), v_1(y)-v_1(x)\rangle dydx \\
&\quad + \kappa_{12}\iint_{\Omega_1\times\Omega_2} q_1(x,0)q_2(y,0)\psi(|\eta_2(y)-\eta_1(x)|)\langle v_1(x), v_2(y)-v_1(x)\rangle dydx \\
&\quad = -\frac{\kappa_{11}}{2}\iint_{\Omega_1\times\Omega_1} q_1(x,0)q_1(y,0)\psi(|\eta_1(y)-\eta_1(x)|)|v_1(y)-v_1(x)|^2 dydx \\
&\quad + \kappa_{12}\iint_{\Omega_1\times\Omega_2} q_1(x,0)q_2(y,0)\psi(|\eta_2(y)-\eta_1(x)|)\langle v_1(x), v_2(y)-v_1(x)\rangle dydx.
\end{aligned}$$

□

As a direct corollary of Lemma 8.1.2, we have conservation of total momentum and dissipation of total energy.

Corollary 8.1.1. *Suppose that the intercoupling strengths are symmetric:*

$$\kappa_{12} = \kappa_{21}, \tag{8.1.10}$$

and let (q_i, v_i) be a smooth solution to system (8.1.3). *Then, we have the following: for $t > 0$,*

(i) $\dfrac{dM_1}{dt} = 0.$

(ii)
$$\begin{aligned}
\frac{dM_2}{dt} &= -\kappa_{11}\iint_{\Omega_1\times\Omega_1} q_1(x,0)q_1(y,0)\psi(\eta_1(y),\eta_1(x))|v_1(y)-v_1(x)|^2 dydx \\
&- \kappa_{22}\iint_{\Omega_2\times\Omega_2} q_2(x,0)q_2(y,0)\psi(\eta_2(y),\eta_2(x))|v_2(y)-v_2(x)|^2 dydx \\
&- 2\kappa_{12}\iint_{\Omega_1\times\Omega_2} q_1(x,0)q_2(y,0)\psi(\eta_2(y),\eta_1(x))|v_2(y)-v_1(x)|^2 dydx.
\end{aligned}$$

Proof. The estimates follow from the symmetric trick $x \longleftrightarrow y$ and the symmetry relation (8.1.10). □

Remark 8.1.2. *Note that M_2 is nonincreasing along the flow* (8.1.3)*:*

$$M_2(t) \le M_2(0), \quad t \ge 0.$$

8.2 Description of frameworks and main results

In this section, we present two frameworks employed in the emergence of mono-cluster and bi-cluster flocking in later sections, and we discuss main results under the proposed frameworks. Before we present the frameworks, we introduce local and global averages for velocity and density as follows. For $i = 1, 2$,

$$\begin{aligned} v_{ic}(t) &:= \frac{1}{M_{i0}} \int_{\Omega_i} q_i(x,0) v_i(x,t)dx, \ \ M_{i0} := \int_{\Omega_i} q_i(x,0)dx, \\ \eta_{ic}(t) &:= \frac{1}{M_{i0}} \int_{\Omega_i} q_i(x,0) \eta_i(x,t)dx, \\ v_c(t) &:= \frac{1}{2}\Big(v_{1c} + v_{2c}\Big), \quad \eta_c(t) := \frac{1}{2}\Big(\eta_{1c} + \eta_{2c}\Big). \end{aligned} \tag{8.2.1}$$

Before we define concepts of mono and bi-cluster flockings, we introduce several functionals measuring the local and global fluctuations around the local and global averages defined in (8.2.1): for $i = 1, 2$,

$$\begin{aligned} \mathcal{X}_i(t) &:= \|\eta_i(t) - \eta_{ic}(t)\|_{L^\infty}, \quad \mathcal{V}_i(t) := \|v_i(t) - v_{ic}(t)\|_{L^\infty}, \\ \mathcal{X}(t) &:= \max\Big\{\|\eta_1(\cdot,t) - \eta_c(t)\|_{L^\infty}, \|\eta_2(\cdot,t) - \eta_c(t)\|_{L^\infty}\Big\}, \\ \mathcal{V}(t) &:= \max\Big\{\|v_1(\cdot,t) - v_c(t)\|_{L^\infty}, \|v_2(\cdot,t) - v_c(t)\|_{L^\infty}\Big\}, \\ \mathcal{X}_d(t) &:= \min_{x\in\Omega_1, y\in\Omega_2} |x_1(x,t) - x_2(y,t)|, \\ \mathcal{V}_d(t) &:= \min_{x\in\Omega_1, y\in\Omega_2} |v_1(x,t) - v_2(y,t)|. \end{aligned} \tag{8.2.2}$$

where the norms of η_i and v_i are taken on the domain Ω_i.

Note that $(\mathcal{V}_i, \mathcal{X})$ and $(\mathcal{X}, \mathcal{V})$ measure the velocity and spatial fluctuations around the local averages and global averages, respectively. Of course, for a single ensemble, these functionals coincide.

We next recall the definitions of the mono-cluster and bi-cluster flockings in terms of the functionals defined in (8.2.2).

Definition 8.2.1. *Let $Z = \{(\eta_i, q_i, v_i)\}_{i=1}^2$ be a classical global solution to the Lagrangian system* (8.1.3)*.*

1. *The Lagrangian configuration Z exhibits an asymptotic "mono-cluster flocking" if the functionals $\mathcal{X}$ and $\mathcal{V}$ satisfy*

$$\sup_{0\le t<\infty} \mathcal{X}(t) < \infty, \quad \lim_{t\to\infty} \mathcal{V}(t) = 0.$$

2. *The Lagrangian configuration Z exhibits an asymptotic "bi-cluster flocking" if the functionals $\mathcal{X}_i$, $\mathcal{V}_i$, and $\mathcal{V}_d$ satisfy*

$$\sup_{0\le t<\infty} \mathcal{X}_i(t) < \infty, \quad \lim_{t\to\infty} \mathcal{V}_i(t) = 0, \quad 1 \le i \le 2, \qquad \inf_{0\le t<\infty} \mathcal{V}_d(t) > 0.$$

In the following two subsection, we discuss two frameworks for the emergence of mono-clusters and bi-clusters.

8.2.1 Description of mono-cluster flocking

In this subsection, we list the framework $(\mathcal{C}_8)$ in terms of the initial data, coupling strengths, and communication weight.

- $(\mathcal{C}_8 1)$: Initial supports of ρ_{i0} are compact, disjoint and with smooth boundary:

$$\mathcal{L}^d(\mathrm{spt}(\rho_{i0})) < \infty, \quad \rho_{i0}(x) > 0, \quad x \in \mathrm{Int}(\Omega_i),\ i = 1, 2, \quad \mathcal{X}(0) > 0,$$

  where $\mathcal{L}^d$ is a d-dimensional Lebesgue measure in $\mathbb{R}^d$.

- $(\mathcal{C}_8 2)$: Initial data are sufficiently regular:

$$(q_{i0}, v_{i0}) \in H^s(\Omega_i) \times H^{s+1}(\Omega_i), \quad i = 1, 2, \quad s > 1 + \frac{d}{2}.$$

- $(\mathcal{C}_8 3)$: The intercoupling strengths are symmetric and bounded below:

$$\kappa_{12} = \kappa_{21}, \quad \min_{1\le i,j\le 2} \kappa_{ij} \|\rho_{j0}\|_{L^1} > \frac{\mathcal{V}_0}{\int_{\mathcal{X}_0}^\infty \psi(2x)dx}.$$

- ($\mathcal{C}_8 4$): The communication weight ψ takes the form given in [26]:

$$\psi(r) = \frac{1}{(1+r^2)^{\frac{\beta}{2}}}.$$

Note that the non-vacuum condition in ($\mathcal{C}_8 1$) is required to exclude the blow-up of the smooth solutions in a finite-time (see [40] for the single C-S ensemble), and the conditions ($\mathcal{C}_8 1$) and ($\mathcal{C}_8 2$) guarantees that the initial data are $\mathcal{C}^1$, so we can expect a $\mathcal{C}^1$ solution globally in time. Finally, the third condition ($\mathcal{C}_8 3$) is needed to enforce the ensembles to make one ensemble.

For the global existence of smooth solutions to (8.1.3), we next introduce the solution space $\mathcal{Q}_k(T)$: For $T \in (0, \infty]$, we set

$$\begin{aligned} \mathcal{Q}_k(T) \;\; := \;\; & \Big\{(q_i, v_i) : q_i \in \mathcal{C}^0\big([0,T); H^k\big) \bigcap \mathcal{C}^1\big([0,T); H^{k-1}\big), \\ & v_i \in \mathcal{C}^0\big([0,T); H^{k+1}\big) \bigcap \mathcal{C}^1\big([0,T); H^k\big)\Big\}. \end{aligned}$$

We next present our first main result under the framework ($\mathcal{C}_8$),

Theorem 8.2.1. *Suppose that the framework* ($\mathcal{C}_8$) *holds. Then, there exists a positive constant* ε_0 *depending only on* ρ_{i0} *such that if* $\|v_{i0}\|_{H^{s+1}(\Omega_i)} < \varepsilon_0,\ i = 1, 2,$ *then the Cauchy problem* (8.0.1) *has a unique classical solution* $(\rho_i, u_i),\ i = 1, 2,$ *satisfying regularity and flocking estimates:*

$$\begin{aligned} &(i)\ (q, v) \in \mathcal{Q}_s(\infty), \quad \eta \in \mathcal{C}^0\big([0,\infty); H^{s+1}\big). \\ &(ii)\ \sup_{0 \le t < \infty} \mathcal{X}(t) < x_\infty, \quad \mathcal{V}(t) \le \mathcal{V}_0 e^{-C_1 \psi(2x_\infty)t}, \quad t \ge 0, \end{aligned}$$

where $C_1 := \min\limits_{1 \le i \le 2} \|\rho_{i0}\|_{L^1(\Omega_i)} \times \min\limits_{1 \le i,j \le 2} \kappa_{ij}$.

8.2.2 Description of bi-cluster flocking

In this subsection, we study a sufficient framework ($\mathcal{C}_9$) leading to bi-cluster flocking in terms of the initial data, coupling strengths, and communication weight. Unlike the framework ($\mathcal{C}_8$), our initial configurations should be well prepared in the sense that they is close to a bi-cluster configuration initially and that intercoupling strengths are sufficiently small, whereas intracoupling strengths should be sufficiently large to keep each sub-ensemble coherent. These heuristic arguments are formalized in the following framework.

- $(\mathcal{C}_9 1)$: Initial supports of ρ_{i0} are compact, disjoint and with smooth boundary: for $i = 1, 2$,

$$\begin{aligned}
&\mathcal{L}^d(\mathrm{spt}(\rho_{i0})) < \infty, \quad \rho_{i0}(x) > 0, \quad x \in \mathrm{Int}(\Omega_i), \quad \|\rho_{i0}\|_{L^1(\Omega_i)} > 0, \\
&\mathcal{X}(0) > 0, \quad \lambda_0 := \frac{1}{2}|v_{2c}(0) - v_{1c}(0)| > 0, \quad \mathcal{V}_{i0} := \mathcal{V}_i(0) < \frac{\lambda_0}{2}, \\
&r_0 := \min\left[(\eta_2(x,0) - \eta_1(y,0)) \cdot \frac{(v_{2c}(0) - v_{1c}(0))}{|v_{2c}(0) - v_{1c}(0)|}\right] > 0,
\end{aligned}$$

  where $\mathcal{L}^d$ is a d-dimensional Lebesgue measure in $\mathbb{R}^d$.

- $(\mathcal{C}_9 2)$: Initial data are sufficiently regular:

$$(q_{i0}, v_{i0}) \in H^s(\Omega_i) \times H^{s+1}(\Omega_i), \quad i = 1, 2, \quad s > 1 + \frac{d}{2}.$$

- $(\mathcal{C}_9 3)$: The intercoupling strengths are symmetric and bounded above, whereas intracoupling strength is sufficiently large:

$$\begin{aligned}
&\kappa_{ii} > \frac{\mathcal{V}_{i0} + \frac{2C\kappa_{12}\sqrt{2M_2(0)}}{\lambda_0}\int_{r_0}^{\infty}\psi(s)ds}{\int_{\mathcal{X}_{i0}}^{+\infty}\psi(2s)ds}, \quad i = 1, 2, \\
&0 < \kappa_{21} = \kappa_{12} < \frac{\lambda_0}{12C\sqrt{2M_2(0)}\int_{r_0}^{\infty}\psi(s)ds},
\end{aligned}$$

  where $C := \Pi_{i=1}^2\|\rho_{i0}\|_{L^1} + \max\limits_{1\le i\le 2}\{\|\rho_{i0}\|_{L^2}\|\rho_{i0}\|_{L^1}^{1/2}\}$ is a positive constant depending only on ρ_{i0}.

- $(\mathcal{C}_9 4)$: The communication weight ψ takes the form given in [26]:

$$\psi(r) = \frac{1}{(1+r^2)^{\frac{\beta}{2}}}, \quad \beta > 1.$$

Before we state our second main result, we comment on the framework $(\mathcal{C}_9)$. As can be seen form the first condition $(\mathcal{C}_9 1)$, the initial data need to be well-prepared, and initial separation r_0 between two sub-ensembles is assumed to be positive in the sense that initial configurations are in the trend of bi-cluster flocking. The regularity ($\mathcal{C}_9 2$) of initial configuration is needed to guarantee the existence of smooth $\mathcal{C}^1$-solutions. The condition ($\mathcal{C}_9 3$) is required to guaranteed to keep two sub-ensembles in the evolution process. For later use, we set

$$\mathcal{R}_0 := \max\{\lambda_0^{-\beta}, (1+r_0^2)^{-\beta/2}\}, \tag{8.2.3}$$

Then, our second main result is on the emergence of bi-cluster flockiong.

Theorem 8.2.2. *Suppose that the framework ($\mathcal{C}_9$) holds. Then, there exists a positive constant ε_0 depending only on ρ_{i0} such that if*

$$\max_{1\le i\le 2} \|\nabla_x v_{i0}\|_{H^s} + \kappa_{12}\mathcal{R}_0 < \varepsilon_0,$$

then the Cauchy problem (8.0.1) has a unique classical solution (ρ_i, u_i), $i = 1, 2$ given by the following:

$$\begin{aligned}
&(i)\ (q, v) \in \mathcal{Q}_s(\infty), \quad \eta \in \mathcal{C}^0\big([0,\infty); H^{s+1}\big).\\
&(ii)\ \mathcal{V}_d(t) > \lambda_0, \quad \mathcal{V}_i(t) \le \tilde{C}_{ii} \max\Big\{e^{-\frac{\kappa_{ii}\psi(2\bar{x}_\infty)t}{2}}, \psi\Big(\frac{\lambda_0 t}{4}\Big)\Big\},\\
&\quad \mathcal{X}_i(t) < \infty,\ t \ge 0, \quad i = 1, 2,
\end{aligned}$$

where $\bar{x}_\infty$ is a constant determined by initial conditions and coupling strength.

Remark 8.2.1. *By the standard Sobolev embedding theorem, the solutions $(q_i, v_i) \in \mathcal{Q}_s(\infty)$, $s > 1 + \frac{d}{2}$ in Theorem 8.2.1 and Theorem 8.2.2 are $\mathcal{C}^1$, that is, $(q_i, v_i) \in \mathcal{C}^1(\Omega_i \times [0,\infty))$. In addition, we need $\kappa_{12}\mathcal{R}_0$ to control the global existence. It is reasonable since both $\kappa_{12} = 0$ and $\mathcal{R}_0 = 0$ imply steady state. Controlling this factor, we can make the system close to a bi-cluster flocking situation.*

In the following section, we study the emergent property of system (8.1.3).

8.3 Dynamics of the coupled C-S system

In this section, we present a priori estimates for mono-cluster and bi-cluster flockings under the frameworks ($\mathcal{C}_8$) and ($\mathcal{C}_9$). For the flocking estimate, we will employ the Lyapunov functional approach in [47]. For this, we first derive a system of dissipative differential inequalities for the functionals introduced in (8.2.2), and then we construct explicit Lyapunov-type functionals leading to the uniform bounds and zero convergence for $(\mathcal{X}_i, \mathcal{V}_i)$ and $(\mathcal{X}, \mathcal{V})$, respectively.

8.3.1 Dynamics of mono-cluster flocking

In this subsection, we present a mono-cluster flocking estimate under the framework of $(\mathcal{C}_8)$ by deriving the system of dissipative differential inequalities for $\mathcal{X}$ and $\mathcal{V}$ introduced in (8.2.2).

Lemma 8.3.1. *Let $T_* \in (0, \infty]$ be a positive number, and let (η_i, q_i, v_i) be a classical solution to system* (8.1.3) *in* $[0, T_*)$ *and* $(\mathcal{X}, \mathcal{V})$ *be functionals defined in* (8.2.2)*. Then, we have the following estimates:*

$$
\begin{aligned}
&(i)\ \frac{d\mathcal{X}(t)}{dt} \le \mathcal{V}(t), \quad a.e.\ \ t \in (0, T_*),\\
&(ii)\ \frac{d\mathcal{V}(t)}{dt} \le -\Big(\min_{i,j} \kappa_{ij} \|\rho_{j0}\|_{L^1}\Big) \psi(2\mathcal{X}(t))\mathcal{V}(t).
\end{aligned}
$$

Proof. (i) (Temporal variation of $\mathcal{X}$): Since the functional $\mathcal{X}$ is Lipschitz continuous, it is differentiable at almost all $t \in [0, T_*)$. Then, without loss of generality, we can pick $t \in [0, T_*)$ such that $\mathcal{X}(t)$ are differentiable at t, and we can choose x^* with the following property:

$$
\mathcal{X}(t) = |\eta_i(x_i^*, t) - \eta_c(t)|, \quad \text{for some } i \in \{1, 2\}.
$$

Now we need to compare $\frac{d\mathcal{X}^2}{dt}$ and $\frac{d}{dt}|\eta_i(x^*) - \eta_c|^2$. For small value h,

$$
\begin{aligned}
&\frac{1}{h}\left(\mathcal{X}^2(t) - \mathcal{X}^2(t-h)\right)\\
&\quad = \frac{1}{h}\left(|\eta_i(x_i^*, t) - \eta_c(t)|^2 - \max_{x \in \Omega_1} |\eta_i(x_i, t-h) - \eta_c(t-h)|^2\right)\\
&\quad \le \frac{1}{h}\left(|\eta_i(x_i^*, t) - \eta_c(t)|^2 - |\eta_i(x_i^*, t-h) - \eta_c(t-h)|^2\right).
\end{aligned}
$$

By passing to the limit $h \to 0$ in the above relation, we obtain

$$
\frac{d\mathcal{X}^2(t)}{dt} \le \frac{d}{dt}|\eta_i(x^*) - \eta_c|^2, \quad \text{a.e. } \ t \in (0, T_*).
$$

Then we have

$$
\begin{aligned}
\frac{1}{2}\frac{d\mathcal{X}^2}{dt} &\le \left|\frac{1}{2}\frac{d}{dt}|\eta_i(x^*) - \eta_c|^2\right| = |\langle \eta_i(x^*) - \eta_c, v_i(x^*) - v_c\rangle| \\
&\le |\eta_i(x^*) - \eta_c| \cdot \mathcal{V}(t) \le \mathcal{X}(t)\mathcal{V}(t).
\end{aligned}
$$

This implies the result.

(ii) (Temporal variation of $\mathcal{V}$): For a proper $t \in [0, T_*)$, we assume that $\mathcal{V}$ is differentiable at t and, without loss of generality, assume that $\mathcal{V}$ satisfies

$$
\mathcal{V}(t) = |v_1(x_1^*, t) - v_c(t)|, \quad \text{for some } x_1^* \in \Omega_1. \tag{8.3.1}
$$

In particular, this implies

$$
\begin{aligned}
|v_1(x_1^*, t) - v_c(t)| &= \max_{x \in \Omega_1} |v_1(x, t) - v_c(t)|, \\
|v_1(x_1^*, t) - v_c(t)| &\ge \max_{x \in \Omega_2} |v_2(x, t) - v_c(t)|.
\end{aligned} \tag{8.3.2}
$$

From this condition, we have

$$
\begin{aligned}
&\langle v_1(x_1^*) - v_c, v_1(y) - v_1(x_1^*)\rangle \\
&\qquad = \langle v_1(x_1^*) - v_c, v_1(y) - v_c\rangle - |v_1(x_1^*) - v_c|^2 \le 0, \\
&\langle v_1(x_1^*) - v_c, v_2(y) - v_1(x_1^*)\rangle \le 0, \text{ similarly.}
\end{aligned} \tag{8.3.3}
$$

Now we use (8.1.3) with (8.3.1),

$$
\begin{aligned}
\frac{1}{2}\frac{d\mathcal{V}^2}{dt} &\le \frac{1}{2}\frac{d}{dt}|v_1(x_1^*) - v_c|^2 \\
&= \kappa_{11}\int_{\Omega_1} q_1(y,0)\psi(|\eta_1(y) - \eta_1(x_1^*)|)\langle v_1(x_1^*) - v_c, v_1(y) - v_1(x_1^*)\rangle dy \\
&+ \kappa_{12}\int_{\Omega_2} q_2(y,0)\psi(|\eta_2(y) - \eta_1(x_1^*)|)\langle v_1(x_1^*) - v_c, v_2(y) - v_1(x_1^*)\rangle dy.
\end{aligned} \tag{8.3.4}
$$

Then, we continue the estimate in (8.3.4) using (8.3.3) to obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d\mathcal{V}^2}{dt} &\le -\kappa_{11}\psi_m^{11}(t)\big|v_1(x_1^*)-v_c\big|^2\|\rho_{10}\|_{L^1} \\
&\quad + \kappa_{11}\psi_m^{11}(t)\int_{\Omega_1} q_1(y,0)\langle v_1(x_1^*)-v_c, v_1(y)-v_c\rangle dy \\
&\quad - \kappa_{12}\psi_m^{12}(t)\big|v_1(x_1^*)-v_c\big|^2\|\rho_{20}\|_{L^1} \\
&\quad + \kappa_{12}\psi_m^{12}(t)\int_{\Omega_2} q_2(y,0)\langle v_1(x_1^*)-v_c, v_2(y)-v_c\rangle dy \\
&=: \mathcal{I}_{41}+\mathcal{I}_{42}+\mathcal{I}_{43}+\mathcal{I}_{44},
\end{aligned}
\tag{8.3.5}
$$

where we suppressed the t dependence in v_i and η_i for notational simplicity, and ψ_m^{ij} are defined as follows:

$$
\psi_m^{11}(t) := \min_{x,y\in\Omega_1}\psi(|\eta_1(y)-\eta_1(x)|) \quad \text{and} \quad \psi_m^{12}(t) := \min_{x\in\Omega_1,y\in\Omega_2}\psi(|\eta_2(y)-\eta_1(x)|).
$$

Note that the terms $\mathcal{I}_{41}$ and $\mathcal{I}_{43}$ are nonpositive, so we only need to estimate the remaining terms $\mathcal{I}_{42}$ and $\mathcal{I}_{44}$. From the definition of v_c,

$$
\begin{aligned}
\mathcal{I}_{42} &= \kappa_{11}\|\rho_{10}\|_{L^1(\Omega_1)}\psi_m^{11}(t)\langle v_1(x_1^*)-v_c, v_{1c}-v_c\rangle, \\
\mathcal{I}_{44} &= \kappa_{12}\|\rho_{20}\|_{L^1(\Omega_2)}\psi_m^{12}(t)\langle v_1(x_1^*)-v_c, v_{2c}-v_c\rangle \\
&= -\kappa_{12}\|\rho_{20}\|_{L^1(\Omega_1)}\psi_m^{12}(t)\langle v_1(x_1^*)-v_c, v_{1c}-v_c\rangle,
\end{aligned}
$$

hence $\mathcal{I}_{42}$ and $\mathcal{I}_{44}$ have opposite signs. We have two cases;

$$
\text{either} \quad \mathcal{I}_{42}\le 0,\ \mathcal{I}_{44}\ge 0 \quad \text{or} \quad \mathcal{I}_{42}\ge 0,\ \mathcal{I}_{44}\le 0.
$$

• Subcase A.1 ($\mathcal{I}_{42}\le 0$ and $\mathcal{I}_{44}\ge 0$): In this case, we have

$$
\mathcal{I}_{41}+\mathcal{I}_{42} \le -\kappa_{11}\psi_m^{11}(t)\big|v_1(x_1^*)-v_c\big|^2\int_{\Omega_1} q_1(y,0)dy \tag{8.3.6}
$$

and

$$
\begin{aligned}
\mathcal{I}_{43}+\mathcal{I}_{44} &= \kappa_{12}\psi_m^{12}(t)\|\rho_{20}\|_{L^1}\Big(\langle v_1(x_1^*)-v_c, v_{2c}-v_c\rangle - \big|v_1(x_1^*)-v_c\big|^2\Big) \\
&\le \kappa_{12}\psi_m^{12}(t)\|\rho_{20}\|_{L^1}|v_1(x_1^*)-v_c|\cdot\Big(\max_{x\in\Omega_1}|v_2-v_c| - \big|v_1(x_1^*)-v_c\big|\Big) \\
&\le 0,
\end{aligned}
\tag{8.3.7}
$$

where we used the maximality of x_1^* in (8.3.1).

In (8.3.4), we now combine all estimates (8.3.6) and (8.3.7) to obtain

$$\frac{1}{2}\frac{d}{dt}|v_1(x_1^*,t)-v_c(t)|^2 \le -\kappa_{11}\|\rho_{10}\|_{L^1}\psi_m^{11}(t)\big|v_1(x_1^*)-v_c\big|^2. \tag{8.3.8}$$

• Subcase A.2 ($\mathcal{I}_{42} \ge 0$ and $\mathcal{I}_{44} \le 0$): Similar to Subcase A.1, we use the relation

$$\mathcal{I}_{41}+\mathcal{I}_{42} \le 0$$

to obtain

$$\frac{1}{2}\frac{d}{dt}|v_1(x_1^*,t)-v_c(t)|^2 \le -\kappa_{12}\|\rho_{20}\|_{L^1}\psi_m^{12}(t)\big|v_1(x_1^*)-v_c\big|^2. \tag{8.3.9}$$

Note that both $\psi_m^{11}(t)$ and $\psi_m^{12}(t)$ satisfy

$$\psi_m^{11}(t) \ge \psi(2\mathcal{X}(t)), \quad \psi_m^{12}(t) \ge \psi(2\mathcal{X}(t)). \tag{8.3.10}$$

Finally, in (8.3.5) we combine relations (8.3.8), (8.3.9), and (8.3.10) to conclude that

$$\frac{d\mathcal{V}}{dt} \le -\min\{\kappa_{11}\|\rho_{10}\|_{L^1},\ \kappa_{12}\|\rho_{20}\|_{L^1}\}\psi(2\mathcal{X}(t))\mathcal{V}(t). \tag{8.3.11}$$

• Case B: For a proper $t \in [0, T_*)$, suppose that the maximum is obtained in Ω_2; i.e., there exists $x_2^* \in \Omega_2$ such that

$$|v_2(x_2^*,t)-v_c(t)| := \Big\{\max_{x\in\Omega_1}|v_1(x,t)-v_c(t)|,\ \max_{x\in\Omega_2}|v_2(x,t)-v_c(t)|\Big\}.$$

We can get

$$\frac{d\mathcal{V}(t)}{dt} \le -\min\{\kappa_{22}\|\rho_{20}\|_{L^1},\kappa_{21}\|\rho_{10}\|_{L^1}\}\psi(2\mathcal{X}(t))\mathcal{V}(t). \tag{8.3.12}$$

Thus, we use (8.3.11), (8.3.12), and assumption ($\mathcal{C}_8 3$) in Section 3 to conclude that

$$\frac{d\mathcal{V}(t)}{dt} \le -\Big(\min_{i,j}\kappa_{ij}\|\rho_{j0}\|_{L^1}\Big)\psi(2\mathcal{X}(t))\mathcal{V}(t), \quad t \in (0, T_*).$$

□

Proposition 8.3.1. *Let $T_* \in (0, \infty]$ be a positive number, and suppose that the framework $(\mathcal{C}_8)$ holds. Let (η_i, q_i, v_i) be a classical solution to system (8.1.3) in $[0, T_*)$ and let $(\mathcal{X}, \mathcal{V})$ be functionals defined in (8.2.2). Then, there exists a positive constant $x_\infty \in (0, \infty)$ such that*

$$\sup_{0 \le t < T_*} \mathcal{X}(t) \le x_\infty, \quad \mathcal{V}(t) \le \mathcal{V}_0 \exp\Big[- \min_{i,j} \kappa_{ij} \|\rho_{j0}\|_{L^1} \psi(2x_\infty) t \Big], \quad t \in (0, T_*);$$

i.e., mono-cluster flocking in the sense of Definition 8.2.1 occurs asymptotically.

Proof. Define a Lyapunov functional

$$\mathcal{L}(t) = \mathcal{V}(t) + \Big(\min_{i,j} \kappa_{ij} \|\rho_{j0}\|_{L^1} \Big) \int_0^{\mathcal{X}(t)} \psi(2x) dx.$$

Then, we use Lemma 8.3.1 to obtain the nonincreasing property of $\mathcal{L}(t)$. The assumption $(\mathcal{C}_8 3)$ leads to the existence of x_∞ through the Lyapunov arguments in Theorem 3.2.1 of Chapter 3. For the second inequality, we use the upper bound of $\mathcal{X}$ and the nonincreasing property of $\psi(t)$ to get

$$\frac{d\mathcal{V}}{dt} \le -\Big(\min_{i,j} \kappa_{ij} \|\rho_{j0}\|_{L^1} \Big) \psi(2x_\infty) \mathcal{V}(t). \tag{8.3.13}$$

Then we integrate (8.3.13) with respect to t over $\mathbb{R}^+$ to obtain

$$\mathcal{V}(t) \le \mathcal{V}_0 \exp\Big[- \min_{i,j} \kappa_{ij} \|\rho_{j0}\|_{L^1} \psi(2x_\infty) t \Big], \quad t \in [0, T_*).$$

□

8.3.2 Bi-cluster flocking

In this subsection, we present the emergence of bi-cluster flocking for some well-prepared configurations $(\mathcal{C}_9)$. In the previous subsection, we studied the emergence of mono-cluster flocking in the large-coupling-strength regime:

$$\kappa_{12} = \kappa_{21}, \qquad \min_{i,j} \kappa_{ij} \|\rho_{j0}\|_{L^1} > \frac{\mathcal{V}_0}{\int_{\mathcal{X}_0}^\infty \psi(2x) dx}.$$

However, when the above condition is violated, then there are many scenarios possible; mono-cluster, bi-cluster, and multi-cluster flockings are possible depending on the geometry of initial configurations as noted for the single ensemble of C-S particles [18, 19, 45]. In this subsection, we focus on the formation of bi-cluster flocking within the framework $(\mathcal{C}_9)$. Our strategy for this can be summarized as follows:

- Step A: We derive temporal variations of the functionals $\mathcal{X}_i$ and $\mathcal{V}_i$.
- Step B: We show that the functional $\mathcal{V}_d$ measuring the velocity differences between two subensembles is bounded below by some positive constant.
- Step C: We show that the bi-flocking estimate holds in a local-in-time interval, and then we further show that these local-in-time estimates can be prolonged to the whole time interval by a continuation argument.

We next present a priori estimates for the emergence of bi-cluster flocking for system (8.1.3).

Lemma 8.3.2. *Let T_* be a positive number and let (η_i, q_i, v_i) be a classical solution to system* (8.1.3) *in* $[0, T_*)$ *and let $(\mathcal{X}_i, \mathcal{V}_i)$ be functionals defined in* (8.2.2)*. Suppose that the intercoupling strengths satisfy*

$$\kappa_{12} = \kappa_{21}.$$

Then, we have the following estimates: for $i = 1, 2$,

$$\frac{d\mathcal{X}_i}{dt} \leq \mathcal{V}_i, \qquad \frac{d\mathcal{V}_i}{dt} \leq -\kappa_{ii}\psi(2\mathcal{X}_i)\mathcal{V}_i + 2C_2\kappa_{12}\sqrt{2M_2(0)}\psi_M, \quad t \in (0, T_*),$$

where C_2 and ψ_M is a positive constant depending only on ρ_{i0} and a nonnegative-valued function:

$$\begin{aligned} C_2 &:= \|\rho_{10}\|_{L^1}\|\rho_{20}\|_{L^1} + \max\{\|\rho_{i0}\|_{L^2}\|\rho_{i0}\|_{L^1}^{1/2}\}, \\ \psi_M(t) &:= \max_{x\in\Omega_1, y\in\Omega_2} \psi(\eta_2(y) - \eta_1(x)). \end{aligned}$$

Proof. We basically follow arguments similar to those in Lemma 8.3.1 to derive the differential inequalities for $\mathcal{X}_i$ and $\mathcal{V}_i$. Since the estimates for $\mathcal{X}_i$ are the same as in Lemma 8.3.1, we only focus on the estimates for $\mathcal{V}_i$. Without loss of generality, we can take proper $t \in [0, T_*)$ such that $x_1^{**}(t) \in \Omega_1$ satisfying the relation

$$|v_1(x_1^{**}, t) - v_{1c}(t)| := \max_{x\in\Omega_1} |v_1(x,t) - v_{1c}(t)|.$$

We use (8.2.2), Lemma 8.1.2, Remark 8.1.2, and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&\|\rho_{10}\|_{L^1}\Big|\frac{dv_{1c}(t)}{dt}\Big| \\
&\quad = \kappa_{12}\Big|\iint_{\Omega_1\times\Omega_2} q_1(x,0)q_2(y,0)\psi(|\eta_2(y)-\eta_1(x)|)(v_2(y)-v_1(x))dydx\Big| \\
&\quad \le \kappa_{12}\psi_M(t)\iint_{\Omega_1\times\Omega_2}\Big[(\sqrt{q_1(x,0)q_2(y,0)})(\sqrt{q_1(x,0)}\sqrt{q_2(y,0)}|v_2(y)|) \\
&\qquad\qquad + (\sqrt{q_1(x,0)q_2(y,0)})(\sqrt{q_1(x,0)}|v_1(x)|)\sqrt{q_2(y,0)}\Big]dydx \\
&\quad \le \kappa_{12}\psi_M(t)\sqrt{M_2(0)}\Big(\|\rho_{10}\|_{L^1}\|\rho_{20}\|_{L^1}^{\frac{1}{2}} + \|\rho_{10}\|_{L^1}^{\frac{1}{2}}\|\rho_{20}\|_{L^1}\Big).
\end{aligned} \tag{8.3.14}$$

This yields

$$\left|\frac{dv_{1c}(t)}{dt}\right| \le \kappa_{12}\sqrt{M_2(0)}\Big(\|\rho_{20}\|_{L^1}^{\frac{1}{2}} + \|\rho_{10}\|_{L^1}^{-\frac{1}{2}}\|\rho_{20}\|_{L^1}\Big)\psi_M(t).$$

Now we use system (8.1.3) to find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_1(x_1^{**}(t),t) - v_{1c}(t)|^2 \\
&\quad = \kappa_{11}\int_{\Omega_1} q_1(y,0)\psi(|\eta_1(y)-\eta_1(x_1^{**})|)\langle v_1(x_1^{**}) - v_{1c}, v_1(y) - v_1(x_1^{**})\rangle dy \\
&\quad - \kappa_{12}\int_{\Omega_2} q_2(y,0)\psi(|\eta_2(y)-\eta_1(x_1^{**}))\langle v_1(x_1^{**}) - v_{1c}, v_2(y) - v_1(x_1^{**})\rangle dy \\
&\quad - \langle v_1(x_1^{**}) - v_{1c}, \dot{v}_{1c}\rangle \\
&\quad =: \mathcal{I}_{51} + \mathcal{I}_{52} + \mathcal{I}_{53}.
\end{aligned}$$

We next estimate $\mathcal{I}_{51}$, $\mathcal{I}_{52}$, and $\mathcal{I}_{53}$ one by one.

• Case A (Estimate of $\mathcal{I}_{51}$): By the choice of $x_1^{**}(t)$, we know that

$$\mathcal{I}_{51} \leq -\kappa_{11}\psi(2\mathcal{X}_1)\mathcal{V}_1^2.$$

• Case B (Estimate of $\mathcal{I}_{52}$): We use Corollary 8.1.1 to obtain

$$\begin{aligned}
\mathcal{I}_{52} &\leq \kappa_{12}\psi_M(t)|v_1(x_1^{**}) - v_{1c}| \int_{\Omega_2} q_2(y,0)|v_2(y) - v_1(x_1^{**})|dy \\
&\leq \kappa_{12}\psi_M(t)|v_1(x_1^{**}) - v_{1c}| \Big(\int_{\Omega_2} q_2^2(y,0)dy \Big)^{\frac{1}{2}} \Big(\int_{\Omega_2} q_2(y,0)|v_2(y) - v_1(x_1^{**})|^2 \Big)^{\frac{1}{2}} \\
&\leq C\kappa_{12}\sqrt{2M_2(0)}\psi_M(t)|v_1(x_1^{**}) - v_{1c}|,
\end{aligned}$$

where $C = \|\rho_{20}\|_{L^2}\|\rho_{20}\|_{L^1}^{1/2}$ is a positive constant depending only on ρ_{i0}.

• Case C (Estimate of $\mathcal{I}_{53}$): We use the estimate (8.3.14) to obtain

$$\mathcal{I}_{53} \leq C\kappa_{12}\sqrt{2M_2(0)}\psi_M(t)|v_1(x_{10}^{**}) - v_{1c}|,$$

where C is a positive constant depending only on ρ_{i0}. Hence we use the estimates of $\mathcal{I}_{51}$, $\mathcal{I}_{52}$, and $\mathcal{I}_{53}$ to conclude that

$$\begin{aligned}
\frac{d\mathcal{V}_1}{dt} &\leq \frac{1}{2}\frac{d}{dt}|v_1(x_1^{**}(t),t) - v_{1c}(t)|^2 \\
&\leq -\kappa_{11}\psi(2\mathcal{X}_1)\mathcal{V}_1 + 2C\kappa_{12}\sqrt{2M_2(0)}\psi_M,
\end{aligned}$$

where $C := \|\rho_{10}\|_{L^1}\|\rho_{20}\|_{L^1} + \|\rho_{20}\|_{L^2}\|\rho_{20}\|_{L^1}^{1/2}$ depending only on ρ_{i0}.

Similarly, we have

$$\frac{d\mathcal{V}_2}{dt} \leq -\kappa_{22}\psi(2\mathcal{X}_2)\mathcal{V}_2 + 2C\kappa_{12}\sqrt{2M_2(0)}\psi_M.$$

$\square$

Lemma 8.3.3. *Let T_* be a positive number, and suppose that the framework $(\mathcal{C}_9)$ holds, and let (η_i, q_i, v_i) be a classical solution to system* (8.1.3) *in* $[0, T_*)$ *and $\mathcal{V}_d(t)$ be the functional defined in* (8.2.2)*. Then, we have*

$$\psi_M(t) \leq \psi(r_0 + \lambda_0 t) \quad \textit{and} \quad \mathcal{V}_d(t) > \lambda_0, \quad t \in [0, T_*),$$

where λ_0 is given in framework $(\mathcal{C}_9 1)$ in Section 3.2.

Proof. (i) We first set

$$e := \frac{v_{2c}(0) - v_{1c}(0)}{|v_{2c}(0) - v_{1c}(0)|},$$

and we define

$$T_1 := \sup \Big\{ T \in (0, T_*] \Big| \min_{x \in \Omega_1, y \in \Omega_2} \{ \big(v_2(y,t) - v_1(x,t) \big) \cdot e \} > \lambda_0 \text{ holds for } t \in [0, T) \Big\}.$$

We next show that $T_1 = T_*$ by showing $T_1 > 0$ and $T_1 = T_*$.

• ($T_1 > 0$): We use the assumption ($\mathcal{C}_9 1$) to obtain

$$\big(v_2(y,0) - v_1(x,0) \big) \cdot e \geq |v_{2c}(0) - v_{1c}(0)| - \mathcal{V}_{10} - \mathcal{V}_{20} \ \ \geq \frac{3\lambda_0}{2} > \lambda_0.$$

Then, we use the continuity of $\min_{x \in \Omega_1, y \in \Omega_2} \{ \big(v_2(y,t) - v_1(x,t) \big) \cdot e \}$ to show that $T_1 > 0$.

• ($T_1 = T_*$): Now we assume $T_1 < T_*$. By definition of T_1, we have for some x and y

$$\big(v_2(y, T_1) - v_1(x, T_1) \big) \cdot e = \lambda_0. \tag{8.3.15}$$

By the assumption ($\mathcal{C}_9 1$), we know that for $t \in [0, T_1)$

$$\begin{aligned} |\eta_2(y,t) - \eta_1(x,t)| &\geq (\eta_2(y,t) - \eta_1(x,t)) \cdot e \\ &= (\eta_2(y,0) - \eta_1(x,0)) \cdot e + \int_0^t (v_2(y,s) - v_1(x,s)) \cdot e ds \\ &\geq r_0 + \lambda_0 t. \end{aligned}$$

Thus, by the nonincreasing property of $\psi(t)$, we have

$$\psi_M(t) \leq \psi(r_0 + \lambda_0 t), \quad t \in [0, T_*).$$

However, it follows from Lemma 8.3.2 that we have

$$\begin{aligned} \frac{d\mathcal{V}_i(t)}{dt} &\leq -\kappa_{ii} \psi(2\mathcal{X}_i) \mathcal{V}_i + 2C\kappa_{12} \sqrt{2M_2(0)} \psi_M(t) \\ &\leq 2C\kappa_{12} \sqrt{2M_2(0)} \psi(r_0 + \lambda_0 t), \quad i = 1, 2, \end{aligned} \tag{8.3.16}$$

where C is a positive constant given by

$$C := \|\rho_{10}\|_{L^1}\|\rho_{20}\|_{L^1} + \max_{1\le i\le 2}\{\|\rho_{i0}\|_{L^2}\|\rho_{i0}\|_{L^1}^{1/2}\}.$$

We integrate the above relation (8.3.16) directly over $[0, T_1)$ to get

$$\mathcal{V}_i(T_1) \le \mathcal{V}_{i0} + \frac{2C\kappa_{12}\sqrt{2M_2(0)}}{\lambda_0}\int_{r_0}^{\infty}\psi(s)ds, \quad i = 1, 2. \tag{8.3.17}$$

Moreover, we integrate relation (8.3.14) over $[0, T_1)$ to get

$$|v_{1c}(T_1) - v_{1c}(0)| \le \frac{C\kappa_{12}\sqrt{2M_2(0)}}{\lambda_0}\int_{r_0}^{\infty}\psi(s)ds. \tag{8.3.18}$$

Similarly, we can also have

$$|v_{2c}(T_1) - v_{2c}(0)| \le \frac{C\kappa_{12}\sqrt{2M_2(0)}}{\lambda_0}\int_{r_0}^{\infty}\psi(s)ds. \tag{8.3.19}$$

Now we combine relations (8.3.17), (8.3.18), and (8.3.19) and the assumption $(\mathcal{C}_9 1)$ and $(\mathcal{C}_9 3)$ to obtain

$$\begin{aligned}
&\big(v_2(x,t) - v_1(x,t)\big)\cdot e \\
&\quad\ge |v_{2c}(0) - v_{1c}(0)| - \mathcal{V}_{10} - \mathcal{V}_{20} - \frac{6C\kappa_{12}\sqrt{2M_2(0)}}{\lambda_0}\int_{r_0}^{\infty}\psi(s)ds \\
&\quad\ge \frac{3\lambda_0}{2} - \frac{6C\kappa_{12}\sqrt{2M_2(0)}}{\lambda_0}\int_{r_0}^{\infty}\psi(s)ds \\
&\quad> \lambda_0.
\end{aligned}$$

This contradict relation (8.3.15). Thus we have $T_1 = T_*$. □

We are now ready to present a priori bi-cluster flocking estimate in the following proposition.

Proposition 8.3.2. *Let $T_* \in (0, \infty]$ be a positive number, and suppose that the framework $(\mathcal{C}_9)$ holds. Let (η_i, q_i, v_i) be a classical solution to system (8.1.3) in $[0, T_*)$ and $(\mathcal{X}_i, \mathcal{V}_i)$ be functionals defined in (8.2.2). Then, the following assertions hold.*

1. *There exist positive constants* $\bar{x}_\infty \in (0,\infty)$ *and* $\bar{C}_1 = \bar{C}_1(\rho_{i0}, \mathcal{V}_{i0}, \kappa_{ij}, M_2(0), \bar{x}_\infty)$ *such that*

$$\sup_{0\le t<T} \mathcal{X}_i(t) \le \bar{x}_\infty, \qquad \mathcal{V}_i(t) \le \bar{C}_1\Big[e^{-\frac{1}{2}\kappa_{ii}\psi(2\bar{x}_\infty)t} + \psi\Big(r_0 + \frac{\lambda_0 t}{2}\Big)\Big], \quad t \in (0, T_*);$$

i.e., bi-cluster flocking in the sense of Definition 8.2.1 occurs asymptotically.

2. *If we assume* $\kappa_{12}\mathcal{R}_0 \ll 1$*, then we have*

$$\mathcal{V}_i(t) \le \Big(\mathcal{V}_1(0) + \frac{\kappa_{12}}{\kappa_{ii}}\mathcal{R}_0\Big)\frac{\mathcal{O}(1)}{(1+t)^\beta}.$$

Proof. (i) We define

$$\mathcal{L}_i(t) := \mathcal{V}_i(t) + \kappa_{ii}\int_0^{\mathcal{X}_i(t)} \psi(2s)ds.$$

Then, it follows from Lemma 8.3.2 and Lemma 8.3.3 that we have

$$\frac{d\mathcal{L}_i}{dt} \le 2C\kappa_{12}\sqrt{2M_2(0)}\psi(r_0+\lambda_0 t). \tag{8.3.20}$$

Integrating relation (8.3.20) directly yields

$$\mathcal{V}_i(t) + \kappa_{ii}\int_{\mathcal{X}_{i0}}^{\mathcal{X}_i(t)} \psi(2s)ds \le \mathcal{V}_{i0} + \frac{2C\kappa_{12}\sqrt{2M_2(0)}}{\lambda_0}\int_{r_0}^\infty \psi(s)ds.$$

We use the assumption of κ_{ii} in ($\mathcal{C}_9 3$) and similar reasoning as in [47] to obtain that there exist $\bar{x}_\infty$ such that

$$\max_{i=1,2}\sup_{0\le t<\infty} \mathcal{X}_i(t) \le \bar{x}_\infty.$$

(ii) We use the upper bound of $\mathcal{X}_i$ in (i) and the nonincreasing property of $\psi(t)$ to get

$$\psi(t) \ge \psi(2\bar{x}_\infty), \quad i = 1, 2,$$

and we use Lemma 8.3.2 to obtain

$$\frac{d\mathcal{V}_i(t)}{dt} \le -\kappa_{ii}\psi(2\bar{x}_\infty)\mathcal{V}_i(t) + 2C\kappa_{12}\sqrt{2M_2(0)}\psi(r_0+\lambda_0 t). \tag{8.3.21}$$

Using a Gronwall-type inequality, we integrate relation (8.3.21) with respect to t, getting

$$\begin{aligned}\mathcal{V}_i(t) \le &\left[\mathcal{V}_i(0) + \frac{2C\kappa_{12}\sqrt{2M_2(0)}\psi(r_0)}{\kappa_{ii}\psi(2\bar{x}_\infty)}\right] e^{-\frac{\kappa_{ii}\psi(2\bar{x}_\infty)t}{2}} \\ &+ \frac{2C\kappa_{12}\sqrt{2M_2(0)}\psi(r_0+\lambda_0 t/2)}{\kappa_{ii}\psi(2\bar{x}_\infty)}.\end{aligned}$$

Moreover, assume $(\mathcal{C}_9 3)$ and $\kappa_{12}\mathcal{R}_0 \ll 1$, where $\mathcal{R}_0$ is a positive constant defined in (8.2.3). Then the second coefficient term of the exponential decay is bounded by $\kappa_{12}\mathcal{R}_0$:

$$\frac{2C\kappa_{12}\sqrt{2M_2(0)}\psi(r_0)}{\kappa_{ii}\psi(2\bar{x}_\infty)} \le \mathcal{O}(1)\frac{\kappa_{12}}{\kappa_{ii}}\mathcal{R}_0, \qquad \psi(r_0+\lambda_0 t/2) \le \frac{\mathcal{O}(1)\mathcal{R}_0}{(1+t)^\beta},$$

where $\mathcal{O}(1)$ only depends on ρ_0, $M_2(0)$, and ψ. □

Remark 8.3.1. *Note that our coefficients in the above estimates are independent of the largest existence time T_*. Thus once we prove the global existence of a classical solution, we can immediately obtain the emergence of flocking as stated in the main theorems 8.2.1 and 8.2.2. The factor $(\kappa_{12}\mathcal{R}_0)$ plays a key role in bi-flocking estimates.*

8.4 Global existence of classical solutions

In this section, we study the global existence of classical solutions to the coupled system (8.0.1). In the previous section, we have shown that mono-cluster and bi-cluster flockings occur in the time interval where sufficiently smooth solutions are guaranteed. As a result of Remark 8.3.1, we need to show the global existence of a classical solution. Note that, for the mono-cluster case, the decay of velocity variations is always exponential; in contrast, the emergence of bi-cluster flocking can be algebraically slow for the Cucker-Smale communication weights (see Chapter 3, 4 for details). In the following, we only focus on the global existence of the bi-cluster flocking framework since the analysis for mono-cluster flocking is pretty much similar to the bi-cluster case. In the following two subsections, we will study the local existence and a priori estimates, but, because the former case is rather standard, we briefly sketch it and focus mostly on the a priori estimates.

8.4.1 Local existence of smooth solutions

In this subsection, we briefly sketch the local existence of classical solutions to system (8.1.3) by using the classical method in [61].

Proposition 8.4.1. *Suppose the framework* $(\mathcal{C}_9)$ *holds. For any positive constants* ε_1 *and* $s > 1 + \frac{d}{2}$, *let* ε_0 *be an arbitrary positive constant less than* ε_1. *Then there exist positive constants* T_* *such that if the initial data* u_{i0} *satisfy*

$$\|u_{i0}\|_{H^{s+1}} < \varepsilon_0,$$

then the system (8.1.3) *has a unique local solution* v_i *given by*

$$v_i(x,t) \in \mathcal{C}([0,T_*); H^{s+1}) \cap \mathcal{C}^1([0,T_*); H^s), \qquad \sup_{0 \le t < T_*} \|v_i(x,t)\|_{H^{s+1}} < \varepsilon_1.$$

Proof. The basic idea is to consider a sequence of functions $v_1^n(x,t)$ and $v_2^n(x)$ generated by the following iteration scheme: For $n = 0$, we set

$$v_i^0(x,t) = v_{i0}(x), \quad x \in \mathbb{R}^d, \ i = 1, 2,$$

and for $n \ge 1$, (v_1^{n+1}, v_2^{n+1}) is defined to be the solution of the following system:

$$\begin{aligned}
\partial_t v_1^{n+1} &= \kappa_{11} \int_{\Omega_1} q_1(y,0)(v_1^n(y) - v_1^n(x))\psi(\eta_1^{n+1}(y) - \eta_1^{n+1}(x))dy \\
&\quad + \kappa_{12} \int_{\Omega_2} q_2(y,0)(v_2^n(y) - v_1^n(x))\psi(\eta_2^{n+1}(y) - \eta_1^{n+1}(x))dy, \\
\partial_t v_2^{n+1} &= \kappa_{22} \int_{\Omega_2} q_2(y,0)(v_2^n(y) - v_2^n(x))\psi(\eta_2^{n+1}(y) - \eta_2^{n+1}(x))dy \\
&\quad + \kappa_{21} \int_{\Omega_1} q_1(y,0)(v_1^n(y) - v_2^n(x))\psi(\eta_1^{n+1}(y) - \eta_2^{n+1}(x))dy,
\end{aligned}$$

subject to fixed initial data

$$v_i^n(x,0) = v_i(x,0).$$

Here the Lagrangian path η^{n+1} is denoted by the relation

$$\eta_i^{n+1}(x,t) = x + \int_0^t v_i^n(x,\tau)d\tau.$$

Then with a classical process we can construct a limit function v_i from the sequence and this limit function is the solution with the desired property. The details can be found in [61] □

8.4.2 A priori estimates

In this subsection, we will construct an a priori estimate of the local classical solution by the continuity criterion. In the following two lemmatas, we present a priori H^k estimates for v_1. In the following, ∇ denotes the spatial gradient ∇_x.

Lemma 8.4.1. (Lower order estimates) *Suppose the framework* $(\mathcal{C}_9)$ *holds. For any positive constant* $T \in (0,\infty]$, *let* σ_1 *be positive constants satisfying the relation*

$$\beta > 1 + \sigma_1,$$

and let (η_i, q_i, v_i) *be a classical solution to system* (8.1.3) *in* $[0,T)$. *Then, there exists a positive number* ε_0 *such that if*

$$\|\nabla_x v_{i0}\|_{H^1} + \kappa_{12}\mathcal{R}_0 \le \varepsilon_0,$$

and moreover if $|\nabla_x \eta_i(x,t)|$ *is bounded, then there exists a positive constant* C_1 *such that*

$$\|v_i(t)\|_{L^2} \le \|v_{10}\|_{L^2} + \frac{C_1}{(1+t)^{\beta-1}}, \quad \|\nabla_x v_i(t)\|_{H^1} \le \frac{C_1}{(1+t)^{(\beta-\sigma_1)}}, \quad t \in [0,T),$$

where r_0 *and* λ_0 *are positive constant defined in* $(\mathcal{C}_9 1)$.

Proof. • (Zeroth-order estimate): It follows from Lemma 8.1.2 that we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_1}|v_1(x)|^2dx \\
&\quad= \kappa_{11}\iint_{\Omega_1\times\Omega_1} q_1(y,0)\psi(\eta_1(y)-\eta_1(x))(v_1(y)-v_1(x))\cdot v_1(x)dydx \\
&\quad+ \kappa_{12}\iint_{\Omega_2\times\Omega_1} q_2(y,0)\psi(\eta_2(y)-\eta_1(x))(v_2(y)-v_1(x))\cdot v_1(x)dydx \\
&\quad\le 2\kappa_{11}\iint_{\Omega_1\times\Omega_1} q_1(y,0)\mathcal{V}_1(t)|v_1(x)|dydx \\
&\quad+ \kappa_{12}\iint_{\Omega_2\times\Omega_1} q_2(y,0)|v_2(y)|\psi(\min(|\eta_2(y)-\eta_1(x)|))|v_1(x)|dydx \\
&\quad\le \frac{\mathcal{O}(1)(\kappa_{11}\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0)}{(1+t)^\beta}\int_{\Omega_1} q_1(y,0)dy\int_{\Omega_1}|v_1(x)|dx \\
&\quad+ \frac{\mathcal{O}(1)\kappa_{12}}{(1+(\lambda_0 t+r_0)^2)^{\beta/2}}\iint_{\Omega_2\times\Omega_1} q_2(y,0)|v_2(y)||v_1(x)|dydx \\
&\quad\le \frac{\mathcal{O}(1)}{(1+t)^\beta}\cdot(\kappa_{11}\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0)\|v_1\|_{L^2},
\end{aligned}
$$

where we use flocking estimate in Proposition 8.3.2 to control $\mathcal{V}_1(t)$. Here $\mathcal{O}(1)$ is a positive constant only depending on $\psi(2\bar{x}_\infty)$. Then we have

$$
\|v_1(t)\|_{L^2} \le \|v_{10}\|_{L^2} + \mathcal{O}(1)\frac{\kappa_{11}\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0}{(1+t)^{\beta-1}}.
$$

According to Remark 8.3.1, we can conclude that, when $\varepsilon_0 \ll 1$, $\psi(2\bar{x}_\infty)$ has a uniform lower bound and so does the constant $\mathcal{O}(1)$. Thus we have for ε_0 sufficiently small

$$
\|v_1(t)\|_{L^2} \le \|v_{10}\|_{L^2} + \frac{\mathcal{O}(1)}{(1+t)^{\beta-1}}.
$$

• (First-order estimates): By direct calculation, we have

$$
\begin{aligned}
\frac{d}{dt}\frac{|\nabla_x v_1|^2}{2} &= \kappa_{11}\int_{\Omega_1} q_1(y,0)(v_1(y)-v_1(x))\nabla_x(\psi(\eta_1(y)-\eta_1(x)))\cdot\nabla_x v_1 dy \\
&\quad - \kappa_{11}\int_{\Omega_1} q_1(y,0)\nabla_x v_1(\psi(\eta_1(y)-\eta_1(x)))\cdot\nabla_x v_1 dy \\
&\quad + \kappa_{12}\int_{\Omega_2} q_2(y,0)(v_2(y)-v_1(x))\nabla_x(\psi(\eta_2(y)-\eta_1(x)))\cdot\nabla_x v_1 dy \\
&\quad - \kappa_{12}\int_{\Omega_2} q_2(y,0)\nabla_x v_1(\psi(\eta_2(y)-\eta_1(x)))\cdot\nabla_x v_1 dy \\
&\le -\kappa_{11}\int_{\Omega_1} q_1(y,0)\psi(\mathcal{X}_1)dy|\nabla_x v_1|^2 \\
&\quad + \kappa_{11}\int_{\Omega_1} q_1(y,0)\mathcal{V}_1(t)|\psi'(\eta_1(y)-\eta_1(x))||\partial_x\eta_1(x)|dy|\nabla_x v_1| \\
&\quad + \kappa_{12}\int_{\Omega_2} q_2(y,0)|v_2(y)-v_1(x)||\psi'(\eta_2(y)-\eta_1(x))||\partial_x\eta_1(x)|dy|\nabla_x v_1| \\
&=: \mathcal{I}_{51}+\mathcal{I}_{52}+\mathcal{I}_{53}.
\end{aligned}
\tag{8.4.1}
$$

Below, we estimate the terms $\mathcal{I}_{5i}$ one by one.

◇ (Estimate on $\mathcal{I}_{51}$): We use the uniform lower bound of $\psi(\eta_1(y)-\eta_1(x))$ to obtain

$$\mathcal{I}_{51} \le -\kappa_{11}\psi(2\max(\mathcal{X}_1))\|\rho_{10}\|_{L^1}|\nabla_x v_1|^2.$$

◇ (Estimate on $\mathcal{I}_{52}$): We consider the term $|\nabla_x\eta|$. From the assumption that it is bounded, we can treat that the value is smaller than proper C_1.
By the way, If we have an ansatz for higher order norms, we can use the Sobolev inequality as follows. This argument will be used in the higher order estimates.
By direct calculation, we have

$$\nabla_x\eta_i(x,t) = I + \int_0^t \nabla_x v_i(x,\tau)d\tau.$$

According to the assumption of T_1^*, we can apply the Sobolev inequality to obtain

$$\|\nabla_x v_i\|_{L^\infty} \le \mathcal{O}(1)\|\nabla_x v_i\|_{H^s} \le \mathcal{O}(1)\frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^k \sigma_i)}},$$

when we have a natural ansatz for higher order,

$$\|\nabla_x v_i\|_{H^s} \le \frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^k \sigma_i)}},$$

for small σ_i. Thus, we have

$$|\nabla_x \eta_i(x,t)| \le \mathcal{O}(1)\Big(1+\int_0^{+\infty}|\nabla_x v_i(\tau)|d\tau\Big) \le \mathcal{O}(1)C_1.$$

From this, the estimate of $\mathcal{I}_{52}$ is

$$\mathcal{I}_{52} \le \mathcal{O}(1)C_1\kappa_{11}\|\rho_{10}\|_{L^1}\mathcal{V}_1(t)|\nabla_x v_1|.$$

$\diamond$ (Estimate on $\mathcal{I}_{52}$): For $\mathcal{I}_{53}$, we have

$$\begin{aligned}
&|\psi'(\eta_2(y)-\eta_1(x))| \le \mathcal{O}(1)|\psi(\eta_2(y)-\eta_1(x))| \quad \text{and} \\
&|v_{ic}| = \Big|\frac{\int_{\Omega_i} q_i v_i dx}{\int_{\Omega_i} q_i dx}\Big| \le \mathcal{O}(1)M_2(0).
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathcal{I}_{53} &\le \kappa_{12}\int_{\Omega_2} q_2(y,0)|v_2(y)-v_1(x)||\psi'(\eta_2(y)-\eta_1(x))||\nabla_x\eta_1(x)|dy|\nabla_x v_1| \\
&\le \kappa_{12}\int_{\Omega_2} q_2(y,0)\Big(|v_2(y)-v_{2c}|+|v_{1c}-v_1(x)|+|v_{1c}-v_{2c}|\Big) \\
&\quad \times |\psi'(\eta_2(y)-\eta_1(x))||\nabla_x\eta_1(x)|dy|\nabla_x v_1| \\
&\le \kappa_{12}\int_{\Omega_2} q_2(y,0)dy\Big(\mathcal{V}_1(t)+\mathcal{V}_2(t)+\mathcal{O}(1)M_2(0)\Big) \\
&\quad \times \frac{\mathcal{O}(1)}{(1+(\lambda_0 t+r_0)^2)^{\beta/2}}|\nabla_x v_1|C_1 \\
&\le \mathcal{O}(1)\|\rho_{20}\|_{L^1}\frac{\kappa_{12}\mathcal{R}_0}{(1+t)^\beta}\Big(\mathcal{V}_1(t)+\mathcal{V}_2(t)+\mathcal{O}(1)M_2(0)\Big)|\nabla_x v_1|C_1.
\end{aligned} \tag{8.4.2}$$

We combine all estimates (8.4.1) and (8.4.2) to obtain

$$\begin{aligned}\frac{d}{dt}|\nabla_x v_1| \le & -\kappa_{11}\psi(2\max(\mathcal{X}_1))\|\rho_{10}\|_{L^1(\Omega_1)}|\nabla_x v_1| \\ & + \mathcal{O}(1)C_1\kappa_{11}\|\rho_{10}\|_{L^1}\mathcal{V}_1(t)|\nabla_x v_1| \\ & + \mathcal{O}(1)C_1\|\rho_{20}\|_{L^1}\frac{\kappa_{12}\mathcal{R}_0}{(1+t)^\beta}(\mathcal{V}_1(t)+\mathcal{V}_2(t)+\mathcal{O}(1)M_2(0))|\nabla_x v_1|.\end{aligned}$$

Thus, applying Proposition 8.3.2, we have that $|\nabla_x v_1|$ would have the following algebraic decay rate:

$$\begin{aligned}|\nabla_x v_1| \le & \Big[|\nabla_x v_1(x,0)| + (\kappa_{11}\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0)\|\rho_{10}\|_{L^1(\Omega_1)}C_1 \\ & + \kappa_{12}\mathcal{R}_0\left(\mathcal{V}_2(0)+\mathcal{V}_1(0)+(\frac{\kappa_{12}}{\kappa_{11}}+\frac{\kappa_{12}}{\kappa_{22}})\mathcal{R}_0+M_2(0)\right)\|\rho_{20}\|_{L^1}C_1\Big]\frac{\mathcal{O}(1)}{(1+t)^\beta},\end{aligned}$$

where we use the algebraic decay part $|\nabla_x v_1(x,0)|\frac{\mathcal{O}(1)}{(1+t)^\beta}$ to absorb the exponential decay part. Since ρ_{i0} have uniformly compact supports Ω_i, we have

$$\|\nabla_x v_1\|_{L^2} \le \mathcal{O}(1)\|\nabla_x v_1\|_{L^\infty}.$$

According to Remark 8.3.1, we can conclude that, when $\epsilon_0 \ll 1$, ψ_{iL} has a uniform lower bound, so does the constant $\mathcal{O}(1)$. Thus we have, for ϵ_0 sufficiently small,

$$\|\nabla_x v_j\|_{L^2} \le \frac{C_1}{(1+t)^{(\beta-\sigma_1)}}, \quad \|\nabla_x v_j\|_{L^\infty} \le \frac{C_1}{(1+t)^{(\beta-\sigma_1)}}, \quad t\in[0,T_1^*).$$

□

As we mentioned in the proof, we don't need to assume the boundedness of $|\nabla_x \eta_i(x,t)|$ in the next lemma since we can use the highest order ansatz.

Lemma 8.4.2. *(Higher order estimates) Under the same assumptions as in Lemma 8.4.1, there exists a positive number ε_0 such that if*

$$\|\nabla_x v_{10}\|_{H^s} + \kappa_{12}\mathcal{R}_0 < \varepsilon_0,$$

then we have, for $t\in[0,T)$,

$$\|v_1(t)\|_{L^2} \le \|v_{10}\|_{L^2} + \frac{C_1}{(1+t)^{\beta-1}}, \quad \|\nabla_x v_1(t)\|_{H^s} \le \sum_{k=1}^{s+1}\frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^k\sigma_i)}}.$$

Proof. For higher order estimates, we first notice that if we choose $\varepsilon_0 < (s+1)\frac{C_1}{1+C_1}$, then in the initial layer, we can apply the continuity of the classical solution to obtain

$$\|\nabla_x v_1(t)\|_{H^{s+1}} < \sum_{k=1}^{s+1} \frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^k \sigma_i)}}, \quad t \in [0, T_1],$$

where $0 < T_1 \ll T$. Then we can set

$$T_1^* := \sup\Big\{\ t\ \in (0,T] \mid \|\nabla_x v_1(t)\|_{H^{s+1}} < \sum_{k=1}^{s+1} \frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^k \sigma_i)}}\Big\}.$$

We will show that $T_1^* = T$. If we suppose not, then we have

$$\|\nabla_x v_1(T_1^*)\|_{H^{s+1}} = \sum_{k=1}^{s+1} \frac{C_1}{(1+T_1^*)^{(\beta-\sum_{i=1}^k \sigma_i)}}. \tag{8.4.3}$$

Now we consider the estimates in the time interval $[0, T_1^*]$. Since we only have a Sobolev norm of order up to $(s+1)$, we cannot expect a classical derivative of v to exist for any higher order, thus we need to use an L^2 estimate:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla_x^k v_1\|_{L^2}^2 \\
&\quad = -\kappa_{11} \iint_{\Omega_1\times\Omega_1} q_1(y,0)\psi(\eta_1(y)-\eta_1(x))|\nabla_x^k v_1(x)|^2 dxdy \\
&\quad + \kappa_{11} \iint_{\Omega_1\times\Omega_1} q_1(y,0)\nabla_x^k(\psi(\eta_1(y)-\eta_1(x)))(v_1(y)-v_1(x))\nabla_x^k v_1(x) dxdy \\
&\quad - \kappa_{11} \sum_{1\le l\le k-1} \binom{k-1}{l} \\
&\qquad \times \iint_{\Omega_1\times\Omega_1} q_1(y,0)\nabla_x^l(\psi(\eta_1(y)-\eta_1(x)))(\nabla_x^{k-l} v_1(x))\nabla_x^k v_1(x) dxdy
\end{aligned}$$

$$
\begin{aligned}
&- \kappa_{12} \iint_{\Omega_2\times\Omega_1} q_2(y,0)\psi(\eta_2(y)-\eta_1(x))|\nabla_x^k v_1(x)|^2 dxdy \\
&+ \kappa_{12} \iint_{\Omega_2\times\Omega_1} q_2(y,0)\nabla_x^k(\psi(\eta_2(y)-\eta_1(x)))(v_2(y)-v_1(x))\nabla_x^k v_1(x) dxdy \\
&- \kappa_{12} \sum_{1\le l\le k-1} \binom{k-1}{l} \\
&\quad \times \iint_{\Omega_2\times\Omega_1} q_2(y,0)\nabla_x^l(\psi(\eta_2(y)-\eta_1(x)))(\nabla_x^{k-l} v_1(x))\nabla_x^k v_1(x) dxdy \\
&=: \mathcal{I}_{61}+\mathcal{I}_{62}+\mathcal{I}_{63}+\mathcal{I}_{64}+\mathcal{I}_{65}+\mathcal{I}_{66}.
\end{aligned} \tag{8.4.4}
$$

In the following, we estimate the terms $\mathcal{I}_{6i}$ one by one.

• (Estimate of $\mathcal{I}_{61}$): For $\mathcal{I}_{61}$, we directly have

$$\mathcal{I}_{61} \le -\kappa_{11}\psi(\max(\mathcal{X}_1))\|\rho_{10}\|_{L^1}\|\nabla_x^k v_1(x)\|_{L^2}^2.$$

• (Estimate of $\mathcal{I}_{62}$): For $\mathcal{I}_{62}$, as in the first-order estimate, we first note that

$$
\begin{aligned}
\nabla_x \eta_i(x,t) &= I + \int_0^t \nabla_x v_i(x,\tau)d\tau, \\
\nabla_x^l \eta_i(x,t) &= \int_0^t \nabla_x^l v_i(x,\tau)d\tau, \quad l\ge 2.
\end{aligned}
$$

Since we consider the time interval $[0,T_1^*]$, we get

$$\|\nabla_x^l \eta_i(t)\|_{L^2} \le \mathcal{O}(1)\int_0^t \|\nabla_x^l v_i(\tau)\|_{L^2}d\tau \le \mathcal{O}(1)C_1, \quad 1\le l\le s+1.$$

Thus, we have

$$\|\nabla_x \eta_i\|_{H^s} \le \mathcal{O}(1)C_1, \quad i=1,2.$$

We apply the Holder inequality and obtain

$$\mathcal{I}_{62} \le \kappa_{11}\mathcal{V}_1 \int_{\Omega_1} q_1(y,0)\Big(\int_{\Omega_1} |\nabla_x^k(\psi(\eta_1(y)-\eta_1(x)))|^2 dx\Big)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\times \Big(\int_{\Omega_1} |\nabla_x^k v_1(x)|^2 dx\Big)^{\frac{1}{2}} dy \\
&\leq \kappa_{11} \mathcal{V}_1 \|\rho_{10}\|_{L^1} \|\nabla_x \eta_1\|_{H^s}^a \|\nabla_x^k v_1\|_{L^2} \\
&\leq \mathcal{O}(1) \kappa_{11} \mathcal{V}_1 \|\rho_{10}\|_{L^1} C_1^a \|\nabla_x^k v_1\|_{L^2},
\end{aligned}
$$

where a is a constant dependent on k.

• (Estimate of $\mathcal{I}_{63}$): For $\mathcal{I}_{63}$, according to the commutator estimate, we have

$$
\begin{aligned}
\mathcal{I}_{63} \leq\ & \kappa_{11} \binom{k-1}{l} \|q_1\|_{L^\infty} \\
& \times \Big(\iint_{\Omega_1 \times \Omega_1} (\nabla_x^l (\psi(\eta_1(y) - \eta_1(x))) \nabla_x^{k-l} v_1(x))^2 dx dy \Big)^{\frac{1}{2}} \|\nabla_x^k v_1(x)\|_{L^2} \\
\leq\ & \mathcal{O}(1) \kappa_{11} \|q_1\|_{L^\infty} \Big[\|\nabla_x^{k-1} (\psi(\eta_1(y) - \eta_1(x)))\|_{L^2} \|\nabla_x v_1\|_{L^\infty} \\
& + \|\nabla_x (\psi(\eta_1(y) - \eta_1(x)))\|_{L^\infty} \|\nabla_x^{k-1} v\|_{L^2} \Big] \|\nabla_x^k v_1\|_{L^2} \\
\leq\ & \mathcal{O}(1) \kappa_{11} \|q_1\|_{L^\infty} \\
& \times \Big(\|\nabla_x \eta_1\|_{H^s}^a \|\nabla_x v_1\|_{L^\infty} + \|\nabla_x \eta_1\|_{L^\infty} \|\nabla_x^{k-1} v\|_{L^2} \Big) \|\nabla_x^k v_1\|_{L^2},
\end{aligned}
$$

where, as in the estimate of $\mathcal{I}_{62}$, a is a constant dependent on k. Now using the same formula, we get

$$
\mathcal{I}_{63} \leq \mathcal{O}(1) \kappa_{11} \|\rho_{10}\|_{L^1(\Omega_1)} (\|\nabla_x v_1\|_{L^\infty} + \|\nabla_x^{k-1} v_1\|_{L^2}) C_1^a \|\nabla_x^k v_1(x)\|_{L^2}.
$$

• (Estimate of $\mathcal{I}_{64}$): It is obvious to see that

$$
\mathcal{I}_{64} \leq 0.
$$

• (Estimate of $\mathcal{I}_{65}$): For $\mathcal{I}_{65}$, we apply Proposition 8.3.2 and Lemma 8.3.3 to obtain

$$
\begin{aligned}
\mathcal{I}_{65} \quad \leq \quad & \kappa_{12} \iint_{\Omega_2 \times \Omega_1} q_2(y,0) \Big(|v_2(y) - v_{2c}| + |v_{1c} - v_1(x)| + |v_{1c} - v_{2c}| \Big) \\
\times \quad & |\nabla_x^k \psi(\eta_2(y) - \eta_1(x))| |\nabla_x^k v_1(x)| dy dx \\
\leq \quad & \kappa_{12} (\mathcal{V}_2 + \mathcal{V}_1 + |v_{1c} - v_{2c}|) \int_{\Omega_2} q_2(y,0)
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad \Big(\int_{\Omega_1} |\nabla_x^k \psi(\eta_2(y)-\eta_1(x))|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega_1} |\nabla_x^k v_1(x)|^2 dx\Big)^{\frac{1}{2}} dy \\
&\leq \quad \mathcal{O}(1)\kappa_{12}\Big(\mathcal{V}_1(t)+\mathcal{V}_2(t)+M_2(0)\Big) \\
&\times \quad \psi(\min|\eta_1(x)-\eta_2(y)|)\|\rho_{20}\|_{L^1(\Omega_2)}\|\nabla_x \eta\|_{H^s}^a \|\nabla_x^k v_1(x)\|_{L^2} \\
&\leq \quad \mathcal{O}(1)\frac{C_1^a \kappa_{12}\mathcal{R}_0}{(1+t)^\beta}\Big(\mathcal{V}_1(t)+\mathcal{V}_2(t)+M_2(0)\Big)\|\rho_{20}\|_{L^1}\|\nabla_x^k v_1\|_{L^2}.
\end{aligned}
$$

- (Estimate of $\mathcal{I}_{66}$): We apply Lemma 8.3.3 to obtain

$$
\begin{aligned}
\mathcal{I}_{66} &\leq \mathcal{O}(1)\kappa_{12}\|q_2\|_{L^\infty}\psi(\min|\eta_1(x)-\eta_2(y)|) \\
&\quad \times (\|\nabla_x v_1\|_{L^\infty} + \|\nabla_x^{k-1} v_1\|_{L^2})\|\nabla_x^k v_1\|_{L^2} \\
&\leq \mathcal{O}(1)\kappa_{12}\mathcal{R}_0 \frac{\|\rho_{20}\|_{L^\infty}}{(1+t)^\beta}\Big(\|\nabla_x v_1\|_{L^\infty} + \|\nabla_x^{k-1} v_1\|_{L^2}\Big)\|\nabla_x^k v_1\|_{L^2}.
\end{aligned}
$$

In (8.4.4), combining the estimates of each part, we obtain

$$
\begin{aligned}
\frac{d}{dt}\|\nabla_x^k v_1\|_{L^2} \quad &\leq \quad -\kappa_{11}\psi(\max(\mathcal{X}_1))\|\rho_{10}\|_{L^1(\Omega_1)}\|\nabla_x^k v_1\|_{L^2} \\
&+ \quad \mathcal{O}(1)\kappa_{11}\mathcal{V}_1\|\rho_{10}\|_{L^1} C_1^a \\
&+ \quad \mathcal{O}(1)\kappa_{11}\|\rho_{10}\|_{L^1}(\|\nabla_x v_1\|_{L^\infty} + \|\nabla_x^{k-1} v_1\|_{L^2})C_1^a \\
&+ \quad \mathcal{O}(1)\frac{\kappa_{12}\mathcal{R}_0}{(1+t)^\beta}(\mathcal{V}_2(t)+\mathcal{V}_1(t)+M_2(0))\|\rho_{20}\|_{L^1} C_1^a \\
&+ \quad \mathcal{O}(1)\frac{\kappa_{12}\mathcal{R}_0}{(1+t)^\beta}(\|\nabla_x v_1\|_{L^\infty} + \|\nabla_x^{k-1} v_1\|_{L^2})\|\rho_{20}\|_{L^\infty}.
\end{aligned}
$$

We already have the estimate for $\nabla_x v_1$ as follows:

$$
\|\nabla_x v_j\|_{L^2} \leq \frac{C_1}{(1+t)^{(\beta-\sigma_1)}}, \quad \|\nabla_x v_j\|_{L^\infty} \leq \frac{C_1}{(1+t)^{(\beta-\sigma_1)}}.
$$

From now on, we will use induction on k. Suppose that the following inequality holds for $1 \leq m < k$:

$$
\|\nabla_x^m v_j\|_{L^2} \leq \frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^{m-1}\sigma_1)}}, \quad t \in [0, T_1^*).
$$

Then, $\nabla_x^k v_1$ satisfies

$$
\frac{d}{dt}\|\nabla_x^k v_1\|_{L^2} \quad \leq \quad -\kappa_{11}\psi(\max(\mathcal{X}_1))\|\rho_{10}\|_{L^1(\Omega_1)}\|\nabla_x^k v_1(x)\|_{L^2}
$$

$$
\begin{aligned}
&+ \quad \mathcal{O}(1)\frac{\kappa_{11}(\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0)}{(1+t)^{\beta}}\|\rho_{10}\|_{L^1}C_1^a \\
&+ \quad \mathcal{O}(1)\frac{\kappa_{11}}{(1+t)^{(\beta-\sum_{i=1}^{k-1}\sigma_i)}}\|\rho_{10}\|_{L^1(\Omega_1)}C_1^{(a+1)} \\
&+ \quad \mathcal{O}(1)\frac{\kappa_{12}\mathcal{R}_0}{(1+t)^{\beta}}(\mathcal{V}_2(t)+\mathcal{V}_1(t)+M_2(0))\|\rho_{20}\|_{L^1(\Omega_2)}C_1^a \\
&+ \quad \mathcal{O}(1)\frac{\kappa_{12}\mathcal{R}_0}{(1+t)^{\beta}}\|\rho_{20}\|_{L^\infty}C_1.
\end{aligned}
$$

Applying Proposition 8.3.2 gives us

$$
\begin{aligned}
\|\nabla_x^k v_1\|_{L^2} \le &\Big(\sum_{i=0}^{k}\|\nabla_x^i v_1(0)\|_{L^2}+\mathcal{O}(1)(\kappa_{11}\mathcal{V}_1(0)+\kappa_{12}\mathcal{R}_0)\Big) \\
&\times \frac{\max\{C_1^b | 1\le b\le a+1\}}{(1+t)^{\beta-\sum_{i=1}^{k-1}\sigma_i}}.
\end{aligned}
$$

Now the decay rate of $\|\nabla_x^k v_1\|_{L^2}$ is of higher order of t than that provided in the time interval $[0,T_1^*]$. However, we can find a proper ϵ_0 of

$$
\|\nabla_x v_1(0)\|_{H^s}+\kappa_{12}\mathcal{R}_0<\epsilon_0\ll 1
$$

such that the coefficient of the inequality is small enough. Note that

$$
\mathcal{V}_1(0)\le \mathcal{O}(1)|\nabla_x v_1(x,0)|\le \mathcal{O}(1)\|\nabla_x v_1(0)\|_{H^s},
$$

where the constant $\mathcal{O}(1)$ is independent of the choice of ϵ_0, if it is small enough. Therefore, we can choose sufficiently small ϵ_0, satisfying

$$
\|\nabla_x^k v_1\|_{L^2}\le \frac{C_1}{2(1+t)^{(\beta-\sum_{i=1}^{k}\sigma_i)}}.
$$

Therefore, we can close the induction on k. We now combine the estimates of each order to obtain

$$
\|\nabla_x v_1(t)\|_{H^s}<\sum_{k=1}^{s}\frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^{k}\sigma_i)}}+\frac{C_1}{2(1+t)^{(\beta-\sum_{i=1}^{s+1}\sigma_i)}},\quad t\in[0,T_1^*).
$$

According to the continuity of the classical solution, we have

$$
\|\nabla_x v_1(T_1^*)\|_{H^s}<\sum_{k=1}^{s+1}\frac{C_1}{(1+T_1^*)^{(\beta-\sum_{i=1}^{k}\sigma_i)}},
$$

which is a contradiction to (8.4.3). Thus, we can conclude that

$$\|\nabla_x v_1(t)\|_{H^s} \le \sum_{k=1}^{s+1} \frac{C_1}{(1+t)^{(\beta-\sum_{i=1}^k \sigma_i)}}, \quad t \in [0,T].$$

□

8.4.3 Proof on the global-time flocking phenomena

Now we are ready to prove Theorem 8.2.1 and Theorem 8.2.2.

Proof. • (Existence): Note that C_1 is independent of T, so we set

$$\varepsilon_1 > \|v_{10}\|_{L^2} + (s+2)C_1.$$

Then for initial data $u_{i0} \in H^{s+1}(\Omega_1)$ satisfying $\|u_0\|_{H^{s+1}} \le \varepsilon_0 < \varepsilon_1$, where ε_0 is given by Lemma 8.4.2, we define

$$T_e^* := \sup\{T \ge 0 : \sup_{0\le t\le T} \|v(t)\|_{H^{s+1}} < \varepsilon_1\}.$$

Note that Proposition 8.3.1 guarantees the local existence with initial condition $\|u_0\|_{H^{s+1}} < \varepsilon_1$. Therefore, we know that $T_e^* > 0$. Next we claim

$$T_e^* = \infty.$$

If we suppose not, i.e., $T_e^* < \infty$, then, by definition, we have

$$\sup_{0\le t\le T_e^*} \|v(t)\|_{H^{s+1}} = \varepsilon_1. \tag{8.4.5}$$

However, from Lemma 8.4.2, we obtain

$$\sup_{0\le t\le T_e^*} \|v(t)\|_{H^{s+1}} < \varepsilon_1.$$

This contradicts (8.4.5). Thus $T_e^* = \infty$. This complete the proof of global-in-time existence of smooth solutions.

• (Asymptotic behavior): We already have the global existence of classical solutions. Then Proposition 8.3.1 is completely justified if $(\mathcal{C}_8)$ holds, and similarly Proposition 8.3.2 is completely justified if $(\mathcal{C}_9)$ hold. This completes the proof. □

Chapter 9

Conclusion and future works

The Cucker-Smale model is a second-order particle-level system which describes flocking phenomena using attractive interactions on velocities. Since this model concerns the states at infinite time, it is important to suggest emergent motions analytically from initial data. From Chapter 3 to Chapter 7, we studied local flocking phenomena in Cucker-Smale model and its unit-speed model, based on the two particle system. In each chapter, we suggested a set of initial data which tend to flocking configurations asymptotically by considering differential inequalities of Lyapunov functionals. In order to prove flocking estimates, bootstrap argument was used on time to force particles follow specially chosen scenarios. This process is powerful enough to analyze the behavior of non-local system through the whole time line though it has a limitation that the initial data should be close to the multi-cluster formation. We used the same idea for the hydrodynamic model in Chapter 8. Using Lagrangian formulation, we avoid complex problems arise from the hydrodynamic settings including boundary problem and vacuum speed. By tracking the particle behavior from the bulk motion, we proved the existence of smooth solutions as well as the flocking phenomena.

It is worth to mention the properties of local flocking, which described in Section 3.2.2 and also in the end of other chapters. The difficulties of local flocking starts with the fact that local flocking configuration is not an equilibritum state. It is an ω-limit set on unbounded spacial area. Hence it signifies the algebraic convergence proved in Section 3.2.2 and disturb easy theories

for stability. Note that the conditions for flocking configurations need to set clusters before we analyze flocking groups. Compare to the synchronization models [79, 54, 59], flocking models [61, 72, 73, 74] do not have good variables representing the level of arrangement, called, order parameter. This is one more difficulty to distinguish infinite-time behaviors in flocking model.
Therefore, we can summarize some related problems as the followings.

1. The collision evading model. Flocking models are studied in various area, and Cucker-Smale model can be directly used for public opinion system or SNS spreading problem. However, for the physical robot control problem, it is important to evade collisions, which is impossible in Cucker-Smale model since it only has attracting force. The repulsion term makes hard to analyze flocking estimates.

2. The communication network effect on flocking. If the communication weight ψ contains negative value, then even the global flocking problem is not known. The modification of ψ is important for network related problems and also for collision evading problems. For example, if ψ is unbounded so that it is not locally integrable near 0, then two particles cannot collide each other. If ψ has bounded support, then local flocking occurs easily while global flocking seems to be hard. As we can see in two particle model, the change of communication weight distorts differential equations for Lyapunov functionals.

3. The number of cluster problem. From the initial data, it is practically important to know how many flocking clusters occur. This problem for generic initial data need the necessary and sufficient condition for multi-cluster flocking. This problem is difficult since the system has nonlocal integral interaction.

Appendix A

Gronwall type inequalities

Lemma A.0.1. *(i) Let* $y : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ *be a differentiable function satisfying*

$$y' \le -\alpha y + f, \quad t > 0, \qquad y(0) = y_0,$$

where α *is a positive constant and* $f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ *is a continuous function decaying to zero as its argument goes to infinity. Then* y *satisfies*

$$y(t) \le \frac{1}{\alpha} \max_{s \in [t/2, t]} |f(s)| + y_0 e^{-\alpha t} + \frac{\|f\|_{L^\infty}}{\alpha} e^{-\frac{\alpha t}{2}}, \quad t \ge 0.$$

(ii) Let $y : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ *be a differentiable function satisfying*

$$y' \le -py + q,$$

where p *and* q *are nonnegative integrable functions. Then* y *satisfies*

$$y(t) \le y_0 e^{-\int_0^t p(\tau) d\tau} + e^{-\int_{\frac{t}{2}}^t p(\tau) d\tau} \int_0^{\frac{t}{2}} q(\tau) d\tau + q\Big(\frac{t}{2}\Big) \int_{\frac{t}{2}}^t e^{-\int_s^t p(\tau) d\tau} ds, \quad t \ge 0.$$

Proof. (i) Note that y satisfies

$$y' + \alpha y \le f.$$

We multiply the above differential inequality by $e^{\alpha t}$ and integrate the resulting relation from $s = 0$ to $s = t$ to find

$$e^{\alpha t} y - y_0 \le \int_0^t f(\tau) e^{\alpha \tau} d\tau$$

$$\begin{aligned}
&= \int_0^{\frac{t}{2}} f(\tau)e^{\alpha\tau}d\tau + \int_{\frac{t}{2}}^t f(\tau)e^{\alpha\tau}d\tau \\
&\leq \|f\|_{L^\infty}\int_0^{\frac{t}{2}} e^{\alpha\tau}d\tau + \max_{\tau\in[\frac{t}{2},t]}|f(\tau)|\int_{\frac{t}{2}}^t e^{\alpha\tau}d\tau \\
&\leq \frac{\|f\|_{L^\infty}}{\alpha}\Big(e^{\frac{\alpha t}{2}}-1\Big) + \frac{1}{\alpha}\max_{\tau\in[\frac{t}{2},t]}|f(\tau)|\Big(e^{\alpha t}-e^{\frac{\alpha t}{2}}\Big). \quad \text{(A.0.1)}
\end{aligned}$$

Hence,

$$y(t) \leq \frac{1}{\alpha}\max_{\tau\in[\frac{t}{2},t]}|f(\tau)| + \Big(y_0 - \frac{\|f\|_{L^\infty}}{\alpha}\Big)e^{-\alpha t} + \Big(\frac{\|f\|_{L^\infty}}{\alpha} - \frac{1}{\alpha}\max_{\tau\in[\frac{t}{2},t]}|f(\tau)|\Big)e^{-\frac{\alpha t}{2}}.$$

Therefore, for $t \geq 0$,

$$y(t) \leq \frac{1}{\alpha}\max_{\tau\in[\frac{t}{2},t]}|f(\tau)| + y_0e^{-\alpha t} + \frac{\|f\|_{L^\infty}}{\alpha}e^{-\frac{\alpha t}{2}}.$$

(ii) Similar to (i), we have

$$y(t) \leq y_0\exp\Big(-\int_0^t p(\tau)d\tau\Big) + \int_0^t \exp\Big(-\int_s^t p(\tau)d\tau\Big)q(s)ds.$$

By splitting the second term on the right-hand side into two integrals over $[0,\frac{t}{2}]$ and $[\frac{t}{2},t]$, we derive the desired result. $\square$

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국문초록

플로킹 현상이란 새들이 떼 지어 나는 현상을 일컫는 말로, 각자의 판단으로 움직임에도 전체적으로 일관적인 집단행동을 할 때를 주로 가리킨다. 이러한 현상을 기술하는 여러 모델이 있는데, 여기에서는 플로킹을 현상적으로 기술하는 쿠커-스메일 모델에 관하여 연구한다. 쿠커-스메일 모델에서는 입자의 동역학이 상대속도 차이에 비례하는 결합 힘으로 기술된다. 이는 모든 객체가 점차적으로 같은 속도를 가지게 되는 현상을 염두에 둔 것인데, 이러한 전역적인 플로킹 현상은 쿠커와 스메일에 의해 모델과 충분조건이 함께 제안되었다. 이 논문에서는 전역적인 플로킹이 일어나지 않을 조건과 그때 일어나는 국소적 플로킹 현상에 집중할 것이다. 특히, 국소적 플로킹 현상을 일으키는 초깃값과 결합의 강도에 대한 조건들을 살펴볼 것이다. 리아프노프 함수적 접근과 시간에 대한 연속성 논리를 통해 국소적 플로킹 현상이 일어날 수 있는 여러 시나리오를 검증하고 그 충분조건을 증명할 것이다. 또한, 기본적인 쿠커-스메일 입자 모델만이 아니라 다른 두 모델, 단위 속력과 각도만으로 속도가 주어질 때, 그리고 입자의 수가 많아 유체와 같은 움직임을 나타낼 때 대해서도 살펴본다. 단위 속력 조건에서는 대칭성이 깨어지는 것이 주된 어려움이며, 유체 모델에서는 라그랑지안 변수를 통해 해의 전역적 존재성을 위협하는 자유경계 문제를 피하는 방법을 살펴볼 것이다.

주요어휘: 쿠커-스메일 모델, 임계 결합력, 유체 모델, 플로킹, 다중 클러스터 플로킹, 동역학계
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