

THE INVERSE PROBLEM FOR HAMILTON-JACOBI EQUATIONS AND SEMICONCAVE ENVELOPES

CARLOS ESTEVE AND ENRIQUE ZUAZUA

ABSTRACT. We study the inverse problem, or inverse design problem, for a time-evolution Hamilton-Jacobi equation. More precisely, given a target function u_T and a time horizon $T > 0$, we aim to construct all the initial conditions for which the viscosity solution coincides with u_T at time T . As it is common in this kind of nonlinear equations, the target might not be reachable. We first study the existence of at least one initial condition leading the system to the given target. The natural candidate, which indeed allows determining the reachability of u_T , is the one obtained by reversing the direction of time in the equation, considering u_T as terminal condition. In this case, we use the notion of backward viscosity solution, that provides existence and uniqueness for the terminal-value problem. We also give an equivalent reachability condition based on a differential inequality, that relates the reachability of the target with its semiconcavity properties. Then, for the case when u_T is reachable, we construct the set of all initial conditions for which the solution coincides with u_T at time T . Note that in general, such initial conditions are not unique. Finally, for the case when the target u_T is not necessarily reachable, we study the projection of u_T on the set of reachable targets, obtained by solving the problem backward and then forward in time. This projection is then identified with the solution of a fully nonlinear obstacle problem, and can be interpreted as the semiconcave envelope of u_T , i.e. the smallest reachable target bounded from below by u_T .

1. INTRODUCTION

We consider the initial-value problem for a Hamilton-Jacobi equation of the form

$$(1.1) \quad \begin{cases} \partial_t u + H(D_x u) = 0, & \text{in } [0, T] \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases}$$

where $u_0 \in \text{Lip}(\mathbb{R}^n)$ and the Hamiltonian $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is assumed to satisfy the following hypotheses:

$$(1.2) \quad H \in C^2(\mathbb{R}^n), \quad H_{pp}(p) > 0, \quad \forall p \in \mathbb{R}^n, \quad \text{and} \quad \lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = +\infty.$$

Here, the unknown u is a function $[0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$, the notation $\partial_t u$ stands for the derivative of u with respect to the first variable and $D_x u$, for the vector of partial derivatives with respect to

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the second group of variables. The inequality $H_{pp}(p) > 0$ in (1.2) means that the Hessian matrix of H at p is positive definite. The study of equations such as (1.1) arises in the context of optimal control theory and calculus of variations, where the value function satisfies, in a weak sense, a Hamilton-Jacobi equation like (1.1) (see [4, 7, 8, 13, 19, 20] and the references therein). In this context, Hamilton-Jacobi equations have applications in a wide range of fields such as economics, physics, mathematical finance, traffic flow and geometrical optics.

It is well known that, due to the presence of a nonlinear term in the equation, one cannot in general expect the existence of a classical C^1 solution for problem (1.1), even if the initial datum u_0 is assumed to be very smooth. On the other hand, continuous solutions satisfying the equation almost everywhere might not be unique. In the early 80's, Crandall and Lions solved this problem in [18] by introducing the notion of *viscosity solution* (Definition 4.1), see also [7, 13, 17, 22].

For any initial condition $u_0 \in \text{Lip}(\mathbb{R}^n)$, existence and uniqueness of a viscosity solution $u \in \text{Lip}([0, T] \times \mathbb{R}^n)$ was established in [18]. This solution can be obtained, using the method of the *vanishing viscosity* (see [17, 18, 25]), as the limit when $\varepsilon \rightarrow 0^+$ of the unique solution of the parabolic problem

$$\begin{cases} \partial_t u - \varepsilon \Delta_x u + H(D_x u) = 0, & \text{in } [0, T] \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases}$$

for which, by the standard theory of nonlinear parabolic equations, existence and uniqueness of a classical solution holds as long as $\varepsilon > 0$.

In view of the existence and uniqueness of a viscosity solution for problem (1.1), we can define, for a fixed $T > 0$, the following nonlinear operator, which associates to any initial condition u_0 , the function $u(T, \cdot)$, where u is the viscosity solution of (1.1):

$$\begin{aligned} S_T^+ : \text{Lip}(\mathbb{R}^n) &\longrightarrow \text{Lip}(\mathbb{R}^n) \\ u_0 &\longmapsto S_T^+ u_0 := u(T, \cdot) \end{aligned}$$

Our goal in this work is to study the inverse problem associated to (1.1). More precisely, for a given target function u_T and a time horizon $T > 0$, we want to construct all the initial conditions u_0 such that the viscosity solution of (1.1) coincides with u_T at time T . This type of problems, also known in the literature as data assimilation problems, have relevant importance in any kind of evolution models. For instance in meteorology [21, 27], where the climate prediction must take into account not only the observations at the present time, but also at past times. For related results in the context of Hamilton-Jacobi equations, we refer to [15, 16, 26] and the references therein.

The first thing one notices when addressing this problem is that not all the Lipschitz targets are reachable. Indeed, as it is well known, the viscosity solution to (1.1) is always a semiconcave function (Definition 4.3). Therefore, an obvious necessary (but not sufficient) condition for the reachability of u_T is that it must be a semiconcave function. We then split the problem in the following three steps:

- (i) First, we study the reachability of the target, i.e. the existence of at least one u_0 satisfying $S_T^+ u_0 = u_T$. The natural candidate is the one obtained by reversing the direction of time in the equation (1.1), considering the target u_T as terminal condition (see the definition of *backward viscosity solution* in Definition 4.2). Indeed, it turns out that the initial datum recovered by this method, allows one to determine whether the target is reachable

or not (see Theorem 2.1). We also obtain a reachability criterion based on a differential inequality (see Theorem 2.2), that links the reachability of the target with its semiconcavity properties.

- (ii) Secondly, if the target is reachable, we construct all the initial conditions u_0 such that $S_T^+ u_0 = u_T$ (see Theorem 2.3). As we will see, such u_0 is not in general unique. The impossibility of uniquely determining the initial condition from the solution at time T is a main feature in first-order nonlinear evolution equations like (1.1), and can be interpreted as a loss of the initial information due to the nonlinear effects and the loss of regularity.
- (iii) Finally, if the target u_T is not reachable, we project it on the set of reachable targets by solving the problem (1.1) backward in time and then forward. As we will see, this projection has some interesting properties and seems to be the most natural one. For instance, it is the smallest reachable target which is bounded from below by u_T and can be characterized as the viscosity solution of a fully nonlinear obstacle problem (see Theorem 2.5).

The idea of iterating forward and backward resolutions of the model in order to reconstruct previous states has already been exploited in different types of evolution equations, and seems to be an effective method to deal with data assimilation problems like the one treated in this paper. See for example the works [2, 3] where a Back and Forth Nudging algorithm is proposed.

The paper is structured as follows: in Section 2, we present and discuss our main results concerning each one of the three points described above. Section 3 is devoted to some examples that illustrate these results. In Section 4, we introduce the notion of backward viscosity solution, present some of its well-known properties, and then we prove the reachability criterion of Theorem 2.1. In Section 5, we give the proof of the results about the construction of initial conditions for a reachable target, Theorems 2.3 and 2.4. Section 6 is devoted to the study of the composition operator $S_T^+ \circ S_T^-$, that can be viewed as a projection of u_T on the set of reachable targets. In this section we prove Theorem 2.5, that identifies the image of $S_T^+ \circ S_T^-$ with the viscosity solution of a fully nonlinear obstacle problem. Then, we discuss the connection of this obstacle problem with the concave envelope of a function and give a geometrical interpretation of $S_T^+(S_T^- u_T)$ as the semiconcave envelope of u_T . We end the Section 6 with the proof of Theorem 2.2, that gives a criterion for the reachability of a target in terms of a differential inequality. Finally, in Section 7 we discuss the contributions of this work and present some possible extensions and future perspectives.

2. MAIN RESULTS

2.1. Reachability criteria. For a given target $u_T \in \text{Lip}(\mathbb{R}^n)$, our first goal is to determine whether or not there exists at least one initial condition u_0 such that $S_T^+ u_0 = u_T$. That is, we want to give a necessary and sufficient condition for the set

$$(2.1) \quad I_T(u_T) := \{u_0 \in \text{Lip}(\mathbb{R}^n); S_T^+ u_0 = u_T\}$$

to be nonempty. In Theorem 2.1 below, for any $u_T \in \text{Lip}(\mathbb{R}^n)$, we give a reachability condition based on the so-called *backward viscosity solution* (see Definition 4.2). This notion of solution was already used in [9] in order to study the relation between the regularity of solutions and the time-reversibility of problem (1.1).

It is well known (see Section 4 for more details) that in the class of backward viscosity solutions, existence and uniqueness holds for the terminal-value problem

$$(2.2) \quad \begin{cases} \partial_t w + H(D_x w) = 0, & \text{in } [0, T] \times \mathbb{R}^n, \\ w(T, x) = u_T(x), & \text{in } \mathbb{R}^n. \end{cases}$$

Actually, analogously to the (forward) viscosity solutions, the backward viscosity solution can be obtained as the limit when $\varepsilon \rightarrow 0^+$ of the solution to the problem

$$\begin{cases} \partial_t w + \varepsilon \Delta_x w + H(D_x w) = 0, & \text{in } [0, T] \times \mathbb{R}^n, \\ w(T, x) = u_T(x), & \text{in } \mathbb{R}^n. \end{cases}$$

Hence, we can define the nonlinear operator

$$\begin{aligned} S_T^- : \text{Lip}(\mathbb{R}^n) &\longrightarrow \text{Lip}(\mathbb{R}^n) \\ u_T &\longmapsto S_T^- u_T := w(0, \cdot) \end{aligned}$$

which associates to any terminal condition u_T , the function $w(0, \cdot)$, i.e. the unique backward viscosity solution of (2.2) at time 0.

Here we state the first reachability criterion, which identifies the reachable targets with the fix-points of the composition operator $S_T^+ \circ S_T^-$.

Theorem 2.1. *Let H satisfy (1.2), $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$. Then, the set $I_T(u_T)$ defined in (2.1) is nonempty if and only if $S_T^+(S_T^- u_T) = u_T$.*

Reversing the time in the equation in order to find initial conditions conducting to a given target is a natural approach in all kinds of evolution equations. However, in many cases, the obtained initial condition does not lead the system back to the target. In the case of the problem (1.1), Theorem 2.1 ensures that the target is reachable if and only if this technique of reversing the time gives the desired initial condition.

As a drawback of this result, we need to solve first the problem (2.2) and then the problem (1.1) in order to determine if a target is reachable or not. Next, for the one-dimensional case, and for the case of a quadratic Hamiltonian in any space-dimension, i.e.

$$(2.3) \quad H(p) = \frac{\langle Ap, p \rangle}{2} \quad \text{where } A \text{ is a definite positive } n \times n \text{ matrix,}$$

we give a reachability criterion based on a differential inequality.

Theorem 2.2. *Let $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$.*

- (i) *If H satisfies (1.2) and the space-dimension is 1, the set $I_T(u_T)$ defined in (2.1) is nonempty if and only if u_T satisfies the inequality*

$$\partial_{xx} u_T - (T H_{pp}(\partial_x u_T))^{-1} \leq 0, \quad \text{in } \mathbb{R}$$

in the viscosity sense (see Definition 6.2).

- (ii) *If H is given by (2.3), the set $I_T(u_T)$ defined in (2.1) is nonempty if and only if u_T satisfies the inequality*

$$\lambda_n \left[D^2 u_T - \frac{A^{-1}}{T} \right] \leq 0, \quad \text{in } \mathbb{R}^n$$

in the viscosity sense (see Definition 6.2).

Here, D^2u_T denotes the Hessian matrix of u_T , and the expression $\lambda_n[X]$ denotes the largest eigenvalue of the $n \times n$ symmetric matrix X . Analogously, we will use $\lambda_1[X]$ to denote the smallest eigenvalue of X .

Remark 2.1. *In the multidimensional case, Theorem 2.2 only applies to quadratic Hamiltonians. This is due to the fact that for this case, the Hessian matrix of H is constant over \mathbb{R}^n . In the one-dimensional case, the result can be generalized to any strictly convex H , however, the arguments that we use in the proof do not apply to higher dimensions, and a similar necessary and sufficient reachability condition do not seem to be straightforward for the case of a general convex Hamiltonian in any space-dimension (see Remark 6.1).*

Note in addition that, in the one-dimensional case, the transformation

$$(2.4) \quad v(t, x) \longmapsto f'(v(t, x))$$

allows to reduce the study of any scalar conservation law of the form

$$\partial_t v + \partial_x(f(v)) = 0,$$

to the case of Burgers equation (see for example [24])

$$\partial_t w + \partial_x \left(\frac{w^2}{2} \right) = 0.$$

Then, using the relation between Hamilton-Jacobi equations and scalar conservations laws (see for example [16]), in one-space dimension we can reduce the study of (1.1) to the case $H(p) = \frac{p^2}{2}$.

Observe that the reachability criterion of Theorem 2.2 does not involve the operators S_T^+ and S_T^- . Furthermore, this result relates the reachability of a target u_T with its semiconcavity properties. We recall that a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if and only if it is a viscosity solution of $\lambda_n[D^2f] \leq 0$ (see for example the work of Oberman [28]). In view of this, from Theorem 2.2 and the properties of semiconcave functions in Proposition 4.3 below, we can deduce the following result, that relates the reachability of a function u_T with its semiconcavity constant.

Corollary 2.1. *Let H be given by (2.3), $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$.*

- (i) *If $I_T(u_T) \neq \emptyset$, then u_T is semiconcave with linear modulus and constant $\frac{1}{T\lambda_1(A)}$.*
- (ii) *If u_T is semiconcave with linear modulus and constant $\frac{1}{T\lambda_n(A)}$, then $I_T(u_T) \neq \emptyset$.*

See a detailed proof of this corollary in Section 6. For the precise definition of semiconcave function with linear modulus, see Definition 4.3.

Remark 2.2. (i) *Observe that, in the particular case of a Hamiltonian given by (2.3) with $A = cI_n$ and $c > 0$, i.e.*

$$H(p) = c \frac{|p|^2}{2},$$

we have $\lambda_1(A) = \lambda_n(A) = c$. Then, Corollary 2.1 implies that $I_T(u_T) \neq \emptyset$ if and only if u_T is semiconcave with linear modulus and constant $\frac{1}{cT}$.

- (ii) *It can be easily checked that, if a function u_T satisfies the inequality of Theorem 2.2 for some $T > 0$, then the same inequality holds for any $T' \in]0, T]$. This implies that the set of reachable targets becomes smaller as we increase the time horizon T . In the limit case, if we let T go to ∞ , we observe that only the concave functions are reachable for all $T > 0$.*

2.2. Initial data construction. Here, for the case when the target u_T is reachable, our goal is to construct all the initial conditions u_0 in $I_T(u_T)$. Our construction relies on the fact that, in view of Theorem 2.1, $I_T(u_T) \neq \emptyset$ implies that $S_T^- u_T \in I_T(u_T)$.

Theorem 2.3. *Let H satisfy (1.2) and $T > 0$. Let $u_T \in \text{Lip}(\mathbb{R}^n)$ be such that $I_T(u_T) \neq \emptyset$ and set the function $\tilde{u}_0 := S_T^- u_T$. Then, for any $u_0 \in \text{Lip}(\mathbb{R}^n)$, the two following statements are equivalent:*

- (i) $u_0 \in I_T(u_T)$;
(ii) $u_0(x) \geq \tilde{u}_0(x)$, $\forall x \in \mathbb{R}^n$ and $u_0(x) = \tilde{u}_0(x)$, $\forall x \in X_T(u_T)$,

where $X_T(u_T)$ is the subset of \mathbb{R}^n given by

$$X_T(u_T) := \{z - T H_p(\nabla u_T(z)); \forall z \in \mathbb{R}^n \text{ such that } u_T(\cdot) \text{ is differentiable at } z\}.$$

See the Examples 1 and 2 for an illustration of this result. In view of Theorem 2.3, when it is nonempty, the set of initial conditions $I_T(u_T)$ defined in (2.1) can be given in the following way:

$$I_T(u_T) = \{\tilde{u}_0 + \varphi; \varphi \in \text{Lip}(\mathbb{R}^n) \text{ such that } \varphi \geq 0 \text{ and } \text{supp}(\varphi) \subset \mathbb{R}^n \setminus X_T(u_T)\}.$$

All the functions in $I_T(u_T)$ coincide with \tilde{u}_0 in the set $X_T(u_T) \subset \mathbb{R}^n$, while in its complement, they are bigger or equal than \tilde{u}_0 . We can also write

$$I_T(u_T) = \tilde{u}_0 + \mathcal{L}(\mathbb{R}^n \setminus X_T(u_T)),$$

where, for a subset A of \mathbb{R}^n , $\mathcal{L}(A) \subset \text{Lip}(\mathbb{R}^n)$ represents the convex cone defined as

$$\mathcal{L}(A) := \{\varphi \in \text{Lip}(\mathbb{R}^n) \text{ such that } \varphi \geq 0 \text{ and } \text{supp}(\varphi) \subset A\}$$

Remark 2.3. (i) *We observe that in order to construct all the elements in $I_T(u_T)$, we need two ingredients: the function \tilde{u}_0 , that can be obtained as the backward viscosity solution of (2.2) using the formula (4.4); and the set $X_T(u_T)$, which can be deduced from the points of differentiability of u_T . In Theorem 2.4, for the case of a quadratic Hamiltonian of the form (2.3), we give a different characterization of the set $X_T(u_T)$ which does not involve the differentiability points of u_T , so that the set of initial conditions $I_T(u_T)$ can be constructed without knowing the set of points where u_T is differentiable.*

- (ii) *It is important to note that, unlike other models, as for example the heat equation*

$$\partial_t u - \Delta u = 0, \quad \text{in } [0, T] \times \mathbb{R}^n,$$

for which backward uniqueness holds, for the problem (1.1), a target u_T can be reached by considering different initial conditions. Indeed, this is the case whenever $I_T(u_T) \neq \emptyset$ and the set $X_T(u_T)$, introduced in Theorem 2.3, is a proper subset of \mathbb{R}^n . See Example 1 for an illustration of this phenomenon.

- (iii) *Similar results on initial data reconstruction were obtained recently by Colombo and Perrollaz [16] and by Liard and Zuazua [24] for scalar conservation laws and for Hamilton-Jacobi equations in dimension 1. In fact, exploiting the relation between Hamilton-Jacobi equations and hyperbolic systems of conservation laws, our results might be adapted to generalize, to the n -dimensional case, the results given in [16, 24].*

The following theorem gives a different characterization of the set $X_T(u_T)$ introduced in Theorem 2.3, for the case when H is given by (2.3). This result identifies $X_T(u_T)$ with the set of points for which the functions in $I_T(u_T)$ admit a touching paraboloid from below.

Theorem 2.4. *Let H be given by (2.3) and $T > 0$. Let $u_T \in \text{Lip}(\mathbb{R}^n)$ be such that $I_T(u_T) \neq \emptyset$ and take any function $u_0 \in I_T(u_T)$. Then, for any $x_0 \in \mathbb{R}^n$, the following two statements are equivalent:*

- (i) $x_0 \in X_T(u_T)$;
- (ii) There exist $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that

$$u_0(x_0) = -\frac{\langle A^{-1}x_0, x_0 \rangle}{2T} + b \cdot x_0 + c \quad \text{and}$$

$$u_0(x) > -\frac{\langle A^{-1}x, x \rangle}{2T} + b \cdot x + c, \quad \forall x \in \mathbb{R}^n \setminus \{x_0\}.$$

Note that condition (ii) in this theorem is independent of the choice of $u_0 \in I_T(u_T)$. Although the initial condition $\tilde{u}_0 = S_T^- u_T$, obtained by pulling back the target with the operator S_T^- , seems to be a natural choice to identify the points in $X_T(u_T)$, where in view of Theorem 2.3, all the initial conditions in $I_T(u_T)$ coincide, we point out that any element in $I_T(u_T)$ would suffice to carry out this construction. Another important advantage of this theorem is that, unlike in Theorem 2.3, the set $X_T(u_T)$ is characterized independently of the points of differentiability of u_T .

2.3. Projection on the set of reachable targets and semiconcave envelopes. In this subsection, we treat the case when the target u_T is not necessarily reachable. Recall that if, for instance, u_T is not a semiconcave function, then $I_T(u_T) = \emptyset$ (see Proposition 4.2). Let us introduce the following composition operator

$$\begin{aligned} S_T^+ \circ S_T^- : \text{Lip}(\mathbb{R}^n) &\longrightarrow \text{Lip}(\mathbb{R}^n) \\ u_T &\longmapsto S_T^+(S_T^- u_T). \end{aligned}$$

This operator can be viewed as a projection of $\text{Lip}(\mathbb{R}^n)$ onto the set of reachable targets.

Throughout the paper, for any $u_T \in \text{Lip}(\mathbb{R}^n)$, we will denote

$$(2.5) \quad u_T^* := S_T^+(S_T^- u_T),$$

the projection of u_T on the set of reachable targets. Observe that for any $u_T \in \text{Lip}(\mathbb{R}^n)$, the set $I_T(u_T^*)$ is nonempty. Indeed, by definition, the initial condition $u_0 := S_T^- u_T$ belongs to $I_T(u_T^*)$.

As well as Theorem 2.2, our main result in this subsection applies to the one-dimensional case and to the case of a quadratic Hamiltonian H in any space dimension. We prove that for any $u_T \in \text{Lip}(\mathbb{R}^n)$, the function u_T^* defined in (2.5) is the viscosity solution of the following fully nonlinear obstacle problem:

$$(2.6) \quad \min \left\{ v - u_T, -\lambda_n \left[D^2 v - \frac{[H_{pp}(Dv)]^{-1}}{T} \right] \right\} = 0.$$

See Definition 6.1 for the precise definition of viscosity solution to this equation. We recall that for a $n \times n$ symmetric matrix X , $\lambda_1[X]$ and $\lambda_n[X]$ denote respectively the smallest and the largest eigenvalues of X .

Note that in the one-dimensional case, the equation (2.6) can be simply written as

$$(2.7) \quad \min \left\{ v - u_T, -\partial_{xx}v + (T H_{pp}(\partial_x v))^{-1} \right\} = 0,$$

while in the case of a quadratic Hamiltonian given by (2.3) in any space dimension, equation (2.6) can be written as

$$(2.8) \quad \min \left\{ v - u_T, -\lambda_n \left[D^2v - \frac{A^{-1}}{T} \right] \right\} = 0.$$

Analogously to the obstacle problem for the convex envelope, introduced by Oberman in [28], for a given function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the *concave envelope* of f in \mathbb{R}^n is the unique viscosity solution of the obstacle problem

$$(2.9) \quad \min\{v - f, -\lambda_n[D^2v]\} = 0.$$

We recall that the concave envelope of f is the smallest concave function which is bounded from below by f , i.e.

$$(2.10) \quad f^*(x) := \inf\{v(x); v \text{ concave, } v(y) \geq f, \text{ for all } y \in \mathbb{R}^n\}.$$

See Figure 8 for an illustration of a function and its concave envelope.

Remark 2.4. (i) *As we shall prove in Lemma 6.1, for any $T > 0$, $u_T \in \text{Lip}(\mathbb{R}^n)$ and any positive definite matrix A , a function v is a viscosity solution of (2.8) if and only if the function*

$$w(x) := v(x) - \frac{\langle A^{-1}x, x \rangle}{2T}$$

is a viscosity solution of (2.9) with

$$f(x) := u_T(x) - \frac{\langle A^{-1}x, x \rangle}{2T}.$$

In other words, v is a viscosity solution of (2.8) if and only if w is the concave envelope of the function f above defined. It gives an alternative way to obtain the viscosity solution of (2.8) in terms of the concave envelope of the function f .

- (ii) *In the one-dimensional case, the study of problem (1.1) for any strictly convex H can be reduced, after a transformation (see Remark 2.1), to the case $H(p) = p^2/2$. Then, equation (2.7) can also be reduced to equation (2.8) in dimension one with $A = 1$.*
- (iii) *Since the concave envelope of a function is unique, we deduce uniqueness of a viscosity solution for problems (2.7) and (2.8). The existence of a viscosity solution can be obtained from Theorem 2.5 by applying the operator $S_T^+ \circ S_T^-$ to the function u_T . We note that existence and uniqueness can also be deduced directly by means of the Perron's method (see [17]).*

In analogy with the notion of concave envelope, for any $T > 0$ and any positive definite $n \times n$ matrix A , we will refer to the viscosity solution of (2.8) as the $\frac{A^{-1}}{T}$ -semiconcave envelope of u_T in \mathbb{R}^n . Note that being a viscosity solution of (2.8) implies in particular semiconcavity with linear modulus and constant $\frac{\lambda_n(A^{-1})}{T} = \frac{1}{T \lambda_1(A)}$. See Subsection 6.2 for a justification of this fact.

Here we state the main result of this subsection, which ensures that the function u_T^* defined in (2.5) is the $\frac{A^{-1}}{T}$ -semiconcave envelope of u_T in \mathbb{R}^n .

Theorem 2.5. *Let $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$. Then,*

- (i) *If H satisfies (1.2) and the space-dimension is 1, the function $u_T^* := S_T^+(S_T^- u_T)$ is the unique viscosity solution of (2.7).*
- (ii) *If H is given by (2.3), the function $u_T^* := S_T^+(S_T^- u_T)$ is the unique viscosity solution of (2.8).*

See an illustration of this result in Examples 3 and 4. As a consequence of this theorem and Lemma 6.1, we deduce the following corollary which, for the case of a quadratic Hamiltonian, gives an alternative way to obtain u_T^* in terms of the concave envelope of a certain function. This allows one to compute the projection of u_T on the set of reachable targets without applying the operators S_T^- and S_T^+ .

Corollary 2.2. *Let H be given by (2.3), $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$. Let f^* be the concave envelope of*

$$f(x) := u_T(x) - \frac{\langle A^{-1}x, x \rangle}{2T}.$$

Then, the function u_T^ defined in (2.5) satisfies*

$$u_T^*(x) = f^*(x) + \frac{\langle A^{-1}x, x \rangle}{2T}, \quad \text{for all } x \in \mathbb{R}^n.$$

For the particular case

$$H(p) = c \frac{|p|^2}{2}, \quad \text{with } c > 0,$$

the function $u_T^* := S_T^+(S_T^- u_T)$ is the viscosity solution of

$$\min \left\{ v - u_T, -\lambda_n [D^2 v] + \frac{1}{cT} \right\} = 0.$$

This can be deduced from Theorem 2.5 and Property 6.1, together with the fact that $\lambda_1(c^{-1}I_n) = \lambda_n(c^{-1}I_n) = c^{-1}$. In this case, u_T^* can be identified with the $\frac{1}{cT}$ -semiconcave envelope of u_T in \mathbb{R}^n , that is, the smallest semiconcave function with linear modulus and constant $\frac{1}{cT}$ which is bounded from below by u_T .

3. EXAMPLES

Example 1. Here we give a particular example of application of Theorem 2.3. We consider the one-dimensional case and the Hamiltonian $H(p) = |p|^2/2$. As time horizon and reachable target we choose $T = 0.5$ and

$$(3.1) \quad u_T(x) := S_T^+ u_1(x), \quad \text{where } u_1(x) := \begin{cases} 1 - |x + 1| & \text{if } -2 < x \leq 0 \\ 1 - |x - 1| & \text{if } 0 < x < 2 \\ 0 & \text{else.} \end{cases}$$

Note that $I_T(u_T) \neq \emptyset$, indeed, $u_1 \in I_T(u_T)$. In Figure 1a, we can see the function u_T . We observe that u_T is differentiable at all points except for -1 and 1 . Computing the lateral derivatives at these two points, the set $X_T(u_T)$ can be easily determined:

$$X_T(u_T) = \mathbb{R} \setminus ([-1.5, -0.5] \cup [0.5, 1.5]).$$

The function $\tilde{u}_0 := S_T^- u_T$ is represented Figure 1b. The restriction of \tilde{u}_0 to the set $X_T(u_T)$ is marked by a red line. In view of Theorem 2.3, the functions in $I_T(u_T)$ are those which coincide with this function on the red line, while they are bigger or equal than it on the black line. In the same plot, we can also see the function u_1 , represented by a dotted line, as another element in $I_T(u_T)$ different to \tilde{u}_0 .

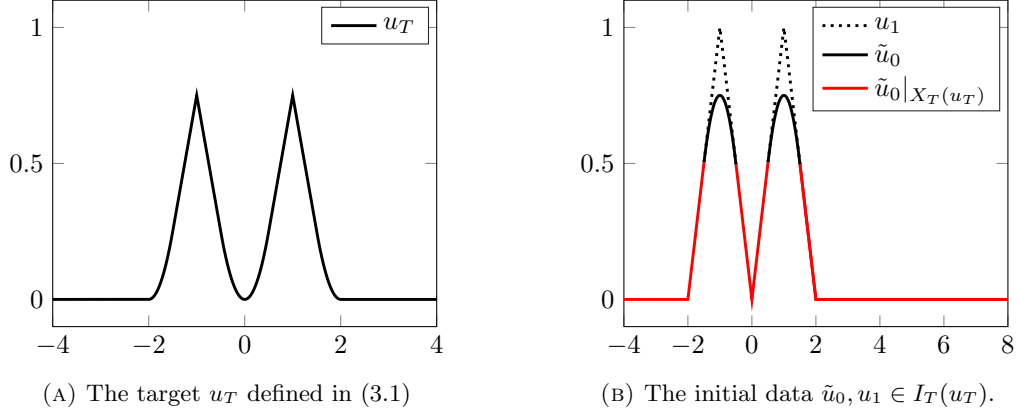


FIGURE 1. The initial data \tilde{u}_0 and u_1 satisfy $S_T^+ \tilde{u}_0 = S_T^+ u_1 = u_T$.

Example 2. In this example, we give an illustration of the result of Theorem 2.3 for the two-dimensional case. We consider the Hamiltonian $H(p) = |p|^2/2$, as time horizon we have chosen $T = 0.5$, and as target, the function

$$(3.2) \quad u_T := S_T^+ u_2, \quad \text{where} \quad u_2(x, y) := \begin{cases} |(x, y) - (-2, 0)| - 1, & \text{if } |(x, y) - (-2, 0)| < 1 \\ 1 - |(x, y) - (2, 0)|, & \text{if } |(x, y) - (2, 0)| < 1 \\ 0, & \text{else.} \end{cases}$$

Note that u_T is reachable, indeed, $u_2 \in I_T(u_T)$. We have computed numerically the target $u_T = S_T^+ u_2$, the function $\tilde{u}_0 = S_T^- u_T$ and the set of points $X_T(u_T)$ defined in Theorem 2.3. In Figure 2b, we can see that the function u_T has a bump and a well. This function is differentiable at all points except for the top of the bump and the circumference around the well. This is due to the semiconcavity of u_T (see Corollary 2.1 and Remark 2.2). We have computed, numerically, the projection on \mathbb{R}^2 of all the points where u_T is differentiable, by the map

$$z \mapsto z - T H_p(\nabla u_T(z)).$$

We have then obtained the set $X_T(u_T)$, represented by the coloured region in Figure 2d. Finally, we have computed the function $\tilde{u}_0 = S_T^- u_T$ by using formula (4.4). It is represented in Figure 2c.

Now, we can apply Theorem 2.3 in order to construct the set of all the initial conditions in $I_T(u_T)$. The functions in $I_T(u_T)$ are those which coincide with \tilde{u}_0 on the set $X_T(u_T)$ (coloured region in Figure 2d), while on its complement (white region) they are bigger or equal than \tilde{u}_0 .

Example 3. Here, we illustrate the result of Theorem 2.5 for the one-dimensional case and the Hamiltonian $H(p) = |p|^2$. We have computed the $\frac{1}{T}$ -semiconcave envelopes of the functions u_3

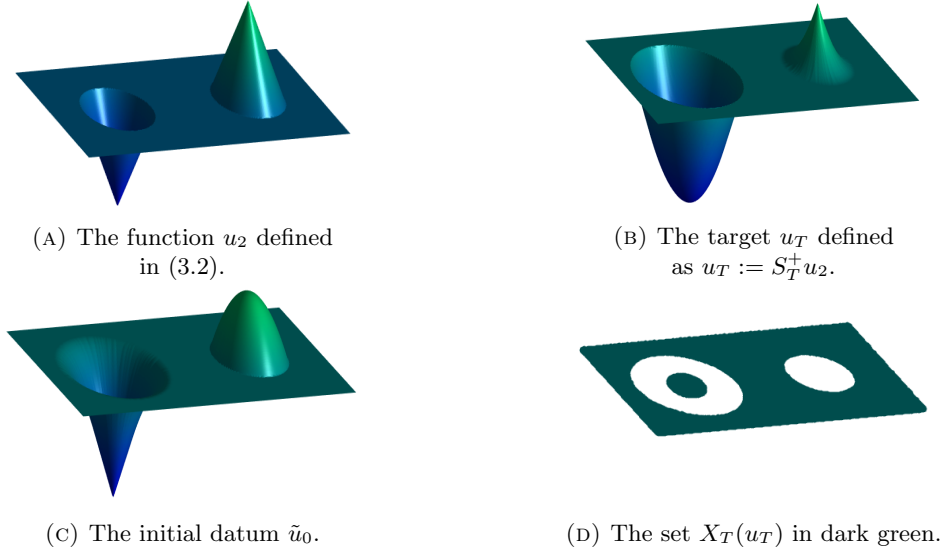


FIGURE 2. The initial conditions in $I_T(u_T)$ are those functions which coincide with \tilde{u}_0 on the blue region while on its complement they are bigger or equal than it.

and u_4 , defined by

$$(3.3) \quad u_3(x) := \begin{cases} |x+1| - 1 & \text{if } -2 < x \leq 0 \\ |x-1| - 1 & \text{if } 0 < x < 2 \\ 0 & \text{else.} \end{cases}$$

$$(3.4) \quad u_4(x) := \begin{cases} 1 - 2|x+1| & \text{if } -1.5 < x \leq 0 \\ 1 - 2|x-1| & \text{if } 0 < x < 1.5 \\ 0 & \text{else.} \end{cases}$$

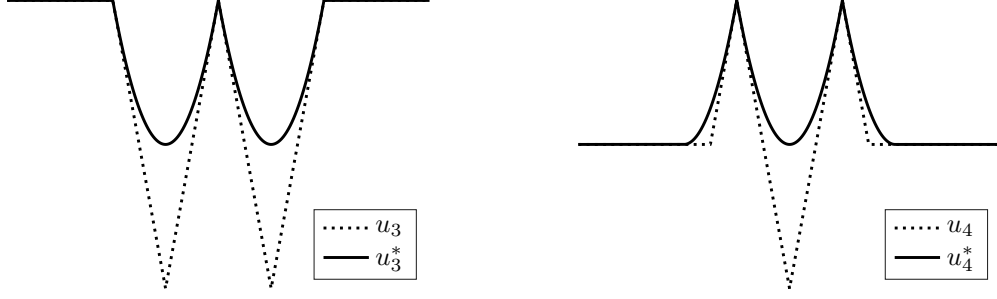
with $T = 1$ and $T = 0.5$ respectively.

In Figure 3a, we can see the function $u_3^* := S_T^+(S_T^- u_3)$, with $T = 1$. In Figure 3b, we can see the function $u_4^* := S_T^+(S_T^- u_4)$, with $T = 0.5$. In both plots, the functions u_3 and u_4 are represented by a dotted line.

We recall that u_3^* and u_4^* are the projection of u_3 and u_4 on the set of reachable targets, and in view of Theorem 2.5, they are the viscosity solution of the obstacle problem (2.8) with $u_T = u_3$ (resp. u_4). The function u_3^* (resp. u_4^*) has been obtained by solving numerically the problem (2.2) with $u_T = u_3$ (resp. $u_T = u_4$), using formula (4.4), and then the problem (1.1) with $u_0 = S_T^- u_3$ (resp. $u_0 = S_T^- u_4$), using formula (4.2).

Example 4. We consider now the two-dimensional case with the Hamiltonian

$$H(p) := \frac{\langle Ap, p \rangle}{2}, \quad \text{with } A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$



(A) The 1-semiconcave envelope of the function u_3 defined in (3.3).

(B) the 2-semiconcave envelope of the function u_4 defined in (3.4).

FIGURE 3. The function u_3^* (resp. u_4^*) is the smallest reachable target, for $T = 1$ (resp. $T = 0.5$), bounded from below by u_3 (resp. u_4).

We have computed numerically the image by $S_T^+ \circ S_T^-$, with $T = 1$, of the function

$$u_5(x, y) := \begin{cases} 1 - |(x, y) - (-1, 0)|, & \text{if } |(x, y) - (-1, 0)| < 1 \\ 0.5(1 - |(x, y) - (1, 0)|), & \text{if } |(x, y) - (1, 0)| < 1 \\ 0, & \text{else.} \end{cases}$$

and with $T = 0.5$ for the function

$$u_6(x, y) = -u_5(x, y).$$

In Figure 4, we can see the function u_5 at the left and its A^{-1} -semiconcave envelope $u_5^* := S_1^+(S_1^- u_5)$ at the right. In Figure 5, we can see the function u_6 at the left and its $2A^{-1}$ -semiconcave envelope $u_6^* := S_{0.5}^+(S_{0.5}^- u_6)$ at the right.

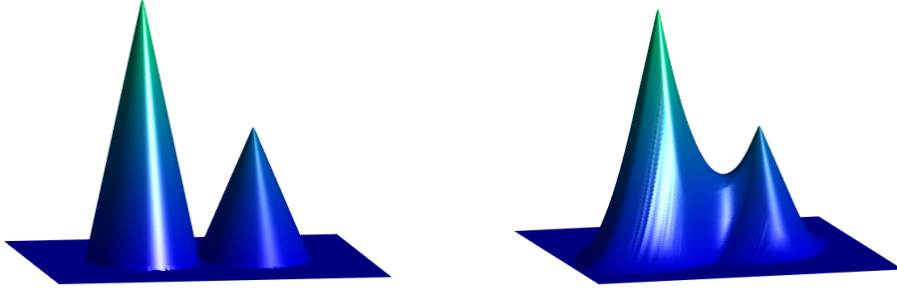


FIGURE 4. At the left, we see the function u_5 from Example 4. At the right, we see the function $u_5^* = S_T^+(S_T^- u_5)$ with $T = 1$.

4. FORWARD AND BACKWARD VISCOSITY SOLUTIONS

We start this section by recalling the definition of viscosity solution for the equation

$$(4.1) \quad \partial_t u + H(D_x u) = 0, \quad \text{in } [0, T] \times \mathbb{R}^n,$$



FIGURE 5. At the left, we see the function u_6 from Example 4. At the right, we see the function $u_6^* = S_T^+(S_T^- u_6)$ with $T = 0.5$.

where H is a continuous function $\mathbb{R}^n \rightarrow \mathbb{R}$.

This notion of solution was introduced by Carndall and Lions in [18] and solved the problem of the lack of uniqueness for generalized solutions to the initial-value problem (1.1), that satisfy the equation (4.1) almost everywhere along with the initial condition $u(0, \cdot) = u_0$.

Definition 4.1. *A uniformly continuous function $u : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called a viscosity solution of (4.1) if the following two statements hold:*

- (i) *u is a viscosity subsolution of (1.1): for each $\varphi \in C^\infty([0, T] \times \mathbb{R}^n)$,*

$$\partial_t \varphi(t_0, x_0) + H(\nabla_x \varphi(t_0, x_0)) \leq 0$$

whenever (t_0, x_0) is a local maximum of $u - \varphi$.

- (ii) *u is a viscosity supersolution of (1.1): for each $\varphi \in C^\infty([0, T] \times \mathbb{R}^n)$,*

$$\partial_t \varphi(t_0, x_0) + H(\nabla_x \varphi(t_0, x_0)) \geq 0$$

whenever (t_0, x_0) is a local minimum of $u - \varphi$.

Throughout the paper we will sometimes refer to viscosity solutions as *forward viscosity solutions*, in contrast with the notion of backward viscosity solution, stated in Definition 4.2. In [18] (see also [7, 13, 25]), it is proved the existence and uniqueness of a viscosity solution for the problem (1.1) for any initial condition $u_0 \in \text{Lip}(\mathbb{R}^n)$. This solution can be obtained by means of the Hopf-Lax formula (see [1, 5, 6, 25]). Therefore, the operator S_T^+ defined in the introduction can be written as

$$(4.2) \quad S_T^+ u_0(x) = \min_{y \in \mathbb{R}^n} \left[u_0(y) + T L \left(\frac{x - y}{T} \right) \right],$$

where, the function $L : \mathbb{R}^n \rightarrow \mathbb{R}$ is the Legendre transform of H , defined as

$$(4.3) \quad L(q) := H^*(q) = \max_{p \in \mathbb{R}^n} [q \cdot p - H(p)].$$

This function corresponds to the Lagrangian in the optimal control problem associated to (1.1). We recall that, under the assumptions (1.2) on H , the function $L = H^*$ is a convex C^2 function satisfying

$$\lim_{|q| \rightarrow \infty} \frac{L(q)}{|q|} = +\infty.$$

See for example Section A.2 in [13]. We then deduce that the minimum in the Hopf-Lax formula (4.2) is always attained.

Observe that in the case of a quadratic Hamiltonian of the form (2.3), an elementary computation gives the Legendre transform of H as

$$L(q) = \frac{\langle A^{-1}q, q \rangle}{2}.$$

As announced in the introduction, a key point in our study is the possibility of reversing the direction of time in problem (1.1). This can be done with the notion of *backward viscosity solution* (see for example [9]).

Definition 4.2. *A function $w : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a backward viscosity solution of (4.1) if the function v obtained from w by “reversing the time”, i.e. $v(t, x) := w(T - t, x)$, is a viscosity solution of*

$$\partial_t v - H(D_x v) = 0, \quad \text{in } [0, T] \times \mathbb{R}^n.$$

It is clear that a function $w \in C^1([0, T] \times \mathbb{R}^n)$ is a backward viscosity solution if and only if it is a (forward) viscosity solution. However, when one deals with non-smooth solutions, both notions of solution are no longer equivalent. Indeed, viscosity solutions are characterized to be semiconcave, while backward viscosity solutions are semiconvex. See Proposition 4.2 and the Example 5 for an illustration of this phenomenon. The following characterization of backward viscosity solutions follows immediately from the definition.

Proposition 4.1. *A uniformly continuous function $w : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a backward viscosity solution to (4.1) if and only if the following two properties hold*

(i) *for each $\phi \in C^\infty([0, T] \times \mathbb{R}^n)$,*

$$\partial_t \phi(t_0, x_0) + H(\nabla_x \phi(t_0, x_0)) \geq 0$$

whenever (t_0, x_0) is a local maximum of $w - \phi$.

(ii) *for each $\phi \in C^\infty([0, T] \times \mathbb{R}^n)$,*

$$\partial_t \phi(t_0, x_0) + H(\nabla_x \phi(t_0, x_0)) \leq 0$$

whenever (t_0, x_0) is a local minimum of $w - \phi$.

Observe that, if we reverse the inequalities in this proposition, we obtain exactly the definition of (forward) viscosity solution (Definition 4.1). We then deduce that any backward solution w satisfies (4.1) at any point where it is differentiable. Using the analogous arguments as for the existence and uniqueness of (forward) viscosity solutions of (1.1), one can prove (see for example [4, 9]) existence and uniqueness of a backward viscosity solution for the terminal-value problem

$$\begin{cases} \partial_t w + H(D_x w) = 0, & \text{in } [0, T] \times \mathbb{R}^n, \\ w(T, x) = u_T(x), & \text{in } \mathbb{R}^n. \end{cases}$$

Hence, for any $T > 0$, the nonlinear operator S_T^- defined in Section 2 is well defined. Recall that this operator associates, to any $u_T \in \text{Lip}(\mathbb{R}^n)$ the backward viscosity solution of (2.2) at time 0.

In addition, $S_T^- u_T$ can be given by the following representation formula, which is the analogous to the Hopf-Lax formula (4.2) for the operator S_T^+ (see [9]):

$$(4.4) \quad S_T^- u_T(x) = \max_{y \in \mathbb{R}^n} \left[u_T(y) - T L \left(\frac{y-x}{T} \right) \right].$$

4.1. Semiconcave and semiconvex functions. Here we recall the following important property of forward and backward viscosity solutions:

Proposition 4.2. *Let H satisfy (1.2), $T > 0$, and $u_0, u_T \in \text{Lip}(\mathbb{R}^n)$. Then, for any bounded set $\mathcal{K} \subset \mathbb{R}^n$,*

- (i) *the function $S_T^+ u_0$ defined in (4.2) is semiconcave in \mathcal{K} with linear modulus;*
- (ii) *the function $S_T^- u_T$ defined in (4.4) is semiconvex in \mathcal{K} with linear modulus.*

See for example Chapter 1 in [13] for a proof of this property. Let us recall the definition of semiconcavity and semiconvexity with linear modulus.

Definition 4.3. (i) *We say that a function $f : \mathcal{K} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is semiconcave with linear modulus if it is continuous and there exists $C \geq 0$ such that*

$$f(x+h) + f(x-h) - 2f(x) \leq C|h|^2, \quad \text{for all } x, h \in \mathbb{R}^n, \text{ such that } [x-h, x+h] \subset \mathcal{K}.$$

The constant C above is called a semiconcavity constant for f in \mathcal{K} .

- (ii) *We say that f is semiconvex if the function $g = -f$ is semiconcave.*

Let us illustrate this property with the following example in dimension 1.

Example 5. We consider the one-dimensional case and the Hamiltonian $H(p) = |p|^2/2$. We apply the operators S_T^+ and S_T^- to the functions u_3 and u_4 from Example 3, with $T = 1$ and $T = 0.5$ respectively.

In Figure 6, we can see the function u_3 represented in both plots by a dotted line. In the plot at the left, we can see the function $S_T^+ u_3$. We observe that it is a semiconcave function (roughly speaking, the second derivative is “bounded from above”). However, in the plot at the right, we see that the function $S_T^- u_3$ is a semiconvex function (the second derivative is “bounded from below”). The same behaviour is observed in Figure 7 for the function u_4 from Example 3.

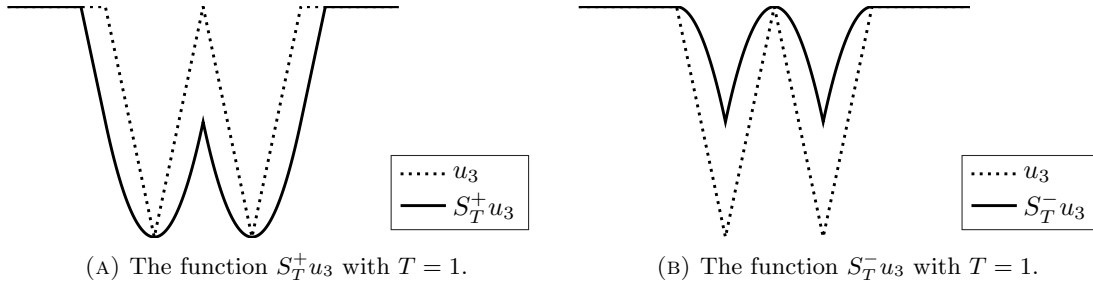


FIGURE 6. The dotted line represents the function u_3 defined in (3.3).

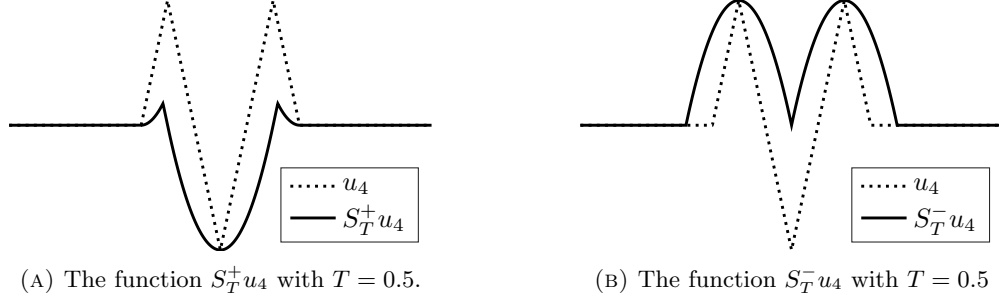


FIGURE 7. The dotted line represents the function u_4 defined in (3.4).

The following proposition, whose proof can be found in Chapter 1 in [13], gives an interesting characterization of semiconcave functions with linear modulus. Combining this characterization with Theorem 6.1 in subsection 6.2, it can be easily deduced that any viscosity solution to the differential inequality

$$\lambda_n \left[D^2 v - \frac{1}{T} A^{-1} \right] \leq 0,$$

where A is a positive definite matrix, are semiconcave with a linear modulus.

Proposition 4.3. *Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a constant $C \geq 0$, the following properties are equivalent:*

- (i) f is semiconcave with linear modulus and constant C ;
- (ii) the function $x \mapsto f(x) - \frac{C}{2}|x|^2$ is concave;
- (iii) there exist two functions $f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f = f_1 + f_2$, f_1 is concave, $f_2 \in C^2(\mathbb{R}^n)$ and satisfies $\|D^2 f_2\|_\infty \leq C$.

4.2. Proof of Theorem 2.1. This result is a consequence of Proposition 4.4 below, which ensures that for any initial condition, the process of taking alternatively forward and backward viscosity solutions stabilizes after the first step.

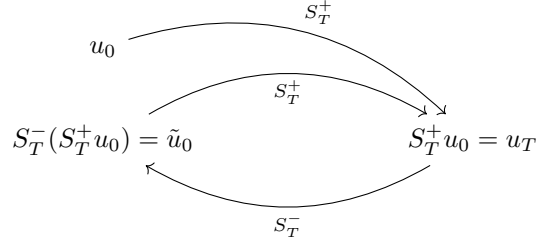
Proposition 4.4. *Let H satisfy (1.2), $u_0 \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$. Set the function*

$$\tilde{u}_0(x) := S_T^-(S_T^+ u_0)(x), \quad \text{for } x \in \mathbb{R}^n.$$

Then it holds

$$(4.5) \quad S_T^+ u_0 = S_T^+ \tilde{u}_0, \quad \text{and} \quad u_0(x) \geq \tilde{u}_0(x), \quad \text{for all } x \in \mathbb{R}^n.$$

The following diagram illustrates this property, that was already proved in [9] for a more general setting. We have included here the proof for completeness.



Proof. By formula (4.2) we have, for any $x \in \mathbb{R}^n$,

$$S_T^+ u_0(x) \leq u_0(y) + T L \left(\frac{x - y}{T} \right), \quad \text{for all } y \in \mathbb{R}^n.$$

By the arbitrariness of x , we deduce

$$u_0(y) \geq S_T^+ u_0(x) - T L \left(\frac{x - y}{T} \right), \quad \text{for all } x, y \in \mathbb{R}^n.$$

Now, for any fixed $y \in \mathbb{R}^n$, taking the maximum over x in the right hand side of the above inequality, and in view of formula (4.4), we obtain $u_0(y) \geq S_T^-(S_T^+ u_0)(y)$ for any $y \in \mathbb{R}^n$. The inequality in (4.5) is then proved.

Now, we use the comparison principle for the viscosity solutions, which can be deduced directly from (4.2) (see also [25] for more details). We obtain

$$S_T^+ u_0(x) \geq S_T^+ \tilde{u}_0(x), \quad \text{for all } x \in \mathbb{R}^n.$$

For the reversed inequality, fix $x_0 \in \mathbb{R}^n$ and let y^* be a minimizer in the right hand side of (4.2). We have

$$(4.6) \quad S_T^+ \tilde{u}_0(x_0) = \tilde{u}_0(y^*) + T L \left(\frac{x_0 - y^*}{T} \right).$$

Then, from the definition of \tilde{u}_0 and formula (4.4), it follows

$$\tilde{u}_0(y^*) = \max_{x \in \mathbb{R}^n} \left[S_T^+ u_0(x) - T L \left(\frac{x - y^*}{T} \right) \right] \geq S_T^+ u_0(x_0) - T L \left(\frac{x_0 - y^*}{T} \right).$$

And combining this inequality with (4.6), we obtain

$$S_T^+ u_0(x_0) \leq \tilde{u}_0(y^*) + T L \left(\frac{x_0 - y^*}{T} \right) = S_T^+ \tilde{u}_0(x_0).$$

□

Using the analogous arguments as in the previous proof, based on the formulas (4.2) and (4.4), we can obtain the following similar result. In this case, we start with a terminal condition $u_T \in \text{Lip}(\mathbb{R}^n)$ and apply first the operator S_T^- , and then the operator S_T^+ . This result will be useful in Section 6 for the proof of Theorem 2.5.

Proposition 4.5. *Let H satisfy (1.2), $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$. Set the function*

$$u_T^*(x) := S_T^+(S_T^-u_T)(x), \quad \text{for } x \in \mathbb{R}^n.$$

Then it holds

$$(4.7) \quad S_T^-u_T = S_T^-u_T^*, \quad \text{and} \quad u_T(x) \leq u_T^*(x), \quad \text{for all } x \in \mathbb{R}^n.$$

We conclude this section with the proof of Theorem 2.1.

Proof of Theorem 2.1. The conclusion follows immediately from Proposition 4.4. Indeed, if $I_T(u_T) \neq \emptyset$, then there exists u_0 such that $S_T^+u_0 = u_T$. Now, setting the function $\tilde{u}_0 = S_T^-(S_T^+u_0)$, after Proposition 4.4 we obtain

$$S_T^+(S_T^-u_T) = S_T^+(S_T^-(S_T^+u_0)) = S_T^+\tilde{u}_0 = S_T^+u_0 = u_T.$$

Reversely, if u_T satisfies $S_T^+(S_T^-u_T) = u_T$, we have $S_T^-u_T \in I_T(u_T)$. And the proof is concluded. \square

5. INITIAL DATA CONSTRUCTION

Our goal in this section is to prove Theorems 2.3 and 2.4 that, for a time horizon $T > 0$ and a reachable target u_T , give a characterization of the set $I_T(u_T)$ defined in (2.1). For the proofs of these two theorems, we will use the following well-known property of viscosity solutions and Hopf-Lax formula. The proof of this property can be found, for example, in [13].

Proposition 5.1. *Let H satisfy (1.2), $u_0 \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$. For any fixed $x \in \mathbb{R}^n$, the minimizer in the right hand side of (4.2) is unique if and only if the function $S_T^+u_0(\cdot)$ is differentiable at x . Moreover, in this case, the minimizer is given by $y^* = x - T H_p(\nabla S_T^+u_0(x))$.*

Let us now proceed with the proof of Theorem 2.3:

Proof of Theorem 2.3. First of all note that, after Theorem 2.1, $I_T(u_T) \neq \emptyset$ implies $S_T^+\tilde{u}_0 = u_T$.

Step 1: (ii) *implies* (i). Let $u_0 \in \text{Lip}(\mathbb{R}^n)$ be any function satisfying the condition (ii) in the enunciate. Since $u_0 \geq \tilde{u}_0$, it follows from the comparison principle (see [25]) that

$$S_T^+u_0(x) \geq S_T^+\tilde{u}_0(x) = u_T(x), \quad \text{for all } x \in \mathbb{R}^n.$$

Let us prove the reversed inequality. Fix any $x_0 \in \mathbb{R}^n$ such that u_T is differentiable at x_0 , and consider $y_0 = x_0 - T H_p(\nabla u_T(x_0))$. Note that $y_0 \in X_T(u_T)$ and then, by the assumption (ii), we have $u_0(y_0) = \tilde{u}_0(y_0)$.

Since $u_T = S_T^+\tilde{u}_0$ is differentiable at x_0 , we know by Proposition 5.1 that y_0 is the unique minimizer in the Hopf-Lax formula (4.2) applied to \tilde{u}_0 . Hence,

$$u_T(x_0) = \tilde{u}_0(y_0) + T L \left(\frac{x_0 - y_0}{T} \right).$$

Applying now the formula (4.2) to the function u_0 , and using $u_0(y_0) = \tilde{u}_0(y_0)$, we obtain

$$S_T^+u_0(x_0) \leq u_0(y_0) + T L \left(\frac{x_0 - y_0}{T} \right) = \tilde{u}_0(y_0) + T L \left(\frac{x_0 - y_0}{T} \right) = u_T(x_0).$$

We have proved that $S_T^+ u_0(x) \leq u_T(x)$ for all x where u_T is differentiable. Since u_T is Lipschitz continuous, by Rademacher's Theorem, u_T is differentiable almost everywhere in \mathbb{R}^n , and then, by the continuity of viscosity solutions, we conclude that $S_T^+ u_0(x) \leq u_T(x)$ for all $x \in \mathbb{R}^n$.

Step 2: (i) implies (ii). Let $u_0 \in I_T(u_T)$. Recall that, by Lemma 4.4, we have

$$(5.1) \quad S_T^+ u_0 = S_T^+ \tilde{u}_0 = u_T \quad \text{and} \quad u_0(x) \geq \tilde{u}_0(x), \text{ for all } x \in \mathbb{R}^n.$$

Let us prove that $u_0 \in I_T(u_T)$ also implies $u_0(x) = \tilde{u}_0(x)$ for any $x \in X_T(u_T)$. Fix any $x_0 \in X_T(u_T)$. By the definition of $X_T(u_T)$, there exists $z_0 \in \mathbb{R}^n$ such that u_T is differentiable at z_0 and $x_0 = z_0 - T H_p(\nabla u_T(z_0))$.

By Proposition 5.1, together with (5.1), it follows that x_0 is the unique minimizer in Hopf-Lax formula (4.2) with $x = z_0$ and initial data u_0 and \tilde{u}_0 . Hence, we deduce that

$$u_T(z_0) = u_0(x_0) + T L \left(\frac{z_0 - x_0}{T} \right) = \tilde{u}_0(x_0) + T L \left(\frac{z_0 - x_0}{T} \right),$$

which implies $u_0(x_0) = \tilde{u}_0(x_0)$. □

We now prove a result that characterizes the set $X_T(u_T)$, introduced in Theorem 2.4, as the set of points x_0 for which there exists a function of the form

$$-T L \left(b + \frac{x_0 - x}{T} \right) + c, \quad \text{with } b \in \mathbb{R}^n, c \in \mathbb{R},$$

touching \tilde{u}_0 from below at x_0 . Note that Theorem 2.4 in Section 2 is a particular case of the following proposition. However, the assumption (2.3) in Theorem 2.4 allows an important simplification of the statement.

Proposition 5.2. *Let H satisfy (1.2) and $T > 0$. Let $u_T \in \text{Lip}(\mathbb{R}^n)$ be such that $I_T(u_T) \neq \emptyset$ and take any function $u_0 \in I_T(u_T)$. Then, for any $x_0 \in \mathbb{R}^n$, the following two statements are equivalent:*

- (i) $x_0 \in X_T(u_T)$;
- (ii) There exist $p_0 \in D^- u_0(x_0)$ and $c_0 \in \mathbb{R}$ such that

$$\begin{aligned} u_0(x_0) &= -T L(H_p(p_0)) + c_0 \quad \text{and} \\ u_0(x) &> -T L \left(H_p(p_0) + \frac{x_0 - x}{T} \right) + c_0, \quad \forall x \in \mathbb{R}^n \setminus \{x_0\}. \end{aligned}$$

Here we recall the definition of subdifferential of a function $u_0 \in \text{Lip}(\mathbb{R}^n)$:

$$(5.2) \quad D^- u_0(x_0) = \{p \in \mathbb{R}^n; \exists \varphi \in C^1(\mathbb{R}^n), \nabla \varphi(x_0) = p, u_0 - \varphi \geq 0, (u_0 - \varphi)(x_0) = 0\}.$$

Proof. Let u_0 be any initial condition in $I_T(u_T)$, i.e. u_0 satisfies $S_T^+ u_0 = u_T$. Let us recall the definition of $X_T(u_T)$ from Theorem 2.3:

$$X_T(u_T) = \{z - T H_p(\nabla_x u_T(z)); \forall z \in \mathbb{R}^n \text{ such that } u_T \text{ is differentiable at } z\}.$$

Consider any $x_0 \in X_T(u_T)$. There exists $z_0 \in \mathbb{R}^n$ such that u_T is differentiable at z_0 and

$$(5.3) \quad x_0 = z_0 - T H_p(\nabla_x u_T(z_0)).$$

From Proposition 5.1, we deduce that x_0 is the unique minimizer in the Hopf-Lax formula (4.2) with $x = z_0$. Therefore, we have

$$u_0(x_0) = u_T(z_0) - T L \left(\frac{z_0 - x_0}{T} \right)$$

and

$$u_0(x) > u_T(z_0) - T L \left(\frac{z_0 - x}{T} \right), \quad \text{for all } x \in \mathbb{R}^n \setminus \{x_0\}.$$

Taking into account these two relations and the identity (5.3), the statement (ii) follows with $c_0 = u_T(z_0)$ and $p_0 = \nabla_x u_T(z_0)$. We observe that, in view of the definition of subdifferential in (5.2), we have $L_q(H_p(p_0)) \in D^-u_0(x_0)$, and using the properties of the Legendre transform (see Property 5.1 below), it follows that $p_0 \in D^-u_0(x_0)$.

Conversely, if statement (ii) holds, we see that x_0 is the unique minimizer in formula (4.2) with $x = z_0$, where $z_0 = x_0 + T H_p(p_0)$. From Proposition 5.1, it follows that u_T is differentiable at z_0 and $\nabla_x u_T(z_0) = p_0$. Then $x_0 \in X_T(u_T)$. \square

Here we recall the following elementary property of the Legendre transform that has been used in the previous proof. See for example Section A.2 in [13] for the proof of this property.

Property 5.1. *Under the assumptions (1.2) on H , the function L defined by*

$$L(q) = \max_{p \in \mathbb{R}^n} [q \cdot p - H(p)]$$

is a strictly convex C^2 function, and its gradient L_q is a C^1 diffeomorphism from \mathbb{R}^n to \mathbb{R}^n satisfying

$$H_p(p) = (L_q)^{-1}(p) \quad \text{and} \quad [H_{pp}(p)]^{-1} = L_{qq}(H_p(p)), \quad \text{for all } p \in \mathbb{R}^n.$$

We end the section with the proof of Theorem 2.4.

Proof of Theorem 2.4. The result is a particular case of Proposition 5.2. Note that in this case H is given by (2.3). Hence, we have

$$(5.4) \quad H_p(p) = A p \quad \text{and} \quad L(q) = H^*(q) = \frac{\langle A^{-1}q, q \rangle}{2}.$$

Consider any $u_0 \in I_T(u_T)$. By Proposition 5.2, if $x_0 \in X_T(u_T)$, then we have

$$u_0(x_0) = -T L(H_p(p_0)) + c_0,$$

and

$$u_0(x) > -T L \left(H_p(p_0) + \frac{x_0 - x}{T} \right) + c_0, \quad \forall x \in \mathbb{R}^n \setminus \{x_0\},$$

for some $p_0 \in D^-u_0(x_0)$ and $c_0 \in \mathbb{R}$. The identities in (5.4) and a simple computation give

$$u_0(x_0) = -T \frac{\langle A p_0, p_0 \rangle}{2} + c_0,$$

and

$$u_0(x) > -\frac{\langle A^{-1}(x_0 - x), x_0 - x \rangle}{2T} - \langle p_0, x_0 - x \rangle - T \frac{\langle A p_0, p_0 \rangle}{2} + c_0, \quad \forall x \in \mathbb{R}^n \setminus \{x_0\}.$$

The statement (ii) in Theorem 2.4 then holds with

$$b := \frac{1}{T}A^{-1}x_0 + p_0 \quad \text{and} \quad c := -\frac{\langle A^{-1}x_0, x_0 \rangle}{2T} - \langle p_0, x_0 \rangle - T\frac{\langle Ap_0, p_0 \rangle}{2} + c_0.$$

Now, let $x_0 \in \mathbb{R}^n$ be such that condition (ii) in Theorem 2.4 holds for some $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. In view of the definition of subdifferential (5.2), we see that

$$p_0 := -\frac{1}{T}A^{-1}x_0 + b \in D^-u_0(x_0).$$

With this choice of p_0 and taking $c_0 := u_0(x_0) + T\frac{\langle Ap_0, p_0 \rangle}{2}$, it is not difficult to verify that condition (ii) in Proposition 5.2 holds, and then $x_0 \in X_T(u_T)$.

□

6. SEMICONCAVE ENVELOPES

In this section, we study the following fully nonlinear obstacle problem

$$(6.1) \quad \min \left\{ v - u_T, -\lambda_n \left[D^2v - \frac{[H_{pp}(Dv)]^{-1}}{T} \right] \right\} = 0,$$

where H is a function satisfying (1.2), and $u_T \in \text{Lip}(\mathbb{R}^n)$ and $T > 0$ are given.

We first prove Theorem 2.5, which ensures that, for the one-dimensional case and when H is quadratic (i.e. given by (2.3)), the function $u_T^* := S_T^+(S_T^-u_T)$ is a viscosity solution of (6.1). This function is obtained by solving the terminal-value problem (2.2) and then the initial-value problem (1.1) with initial condition $S_T^-u_T$. Afterwards, we will discuss the connection of equation (6.1) with the notion of semiconcave envelope and we will give the proofs of Theorem 2.2 and Corollary 2.1.

6.1. Proof of Theorem 2.5. We start by recalling the definition of viscosity solution to problem (6.1), in terms of sub- and supersolutions.

Definition 6.1. *Let $u_T \in \text{Lip}(\mathbb{R}^n)$ be a given function.*

- (i) *The upper semicontinuous function v is a viscosity subsolution of (6.1) if for every twice-differentiable function $\phi(\cdot)$,*

$$\phi(x_0) - u_T(x_0) \leq 0 \quad \text{or} \quad -\lambda_n \left[D^2\phi(x_0) - \frac{[H_{pp}(\nabla\phi(x_0))]^{-1}}{T} \right] \leq 0,$$

whenever x_0 is a local maximum of $v - \phi$ and $v(x_0) = \phi(x_0)$.

- (ii) *The lower semicontinuous function v is a viscosity supersolution of (6.1) if for every twice-differentiable function $\phi(\cdot)$,*

$$\phi(x_0) - u_T(x_0) \geq 0 \quad \text{and} \quad -\lambda_n \left[D^2\phi(x_0) - \frac{[H_{pp}(\nabla\phi(x_0))]^{-1}}{T} \right] \geq 0,$$

whenever x_0 is a local minimum of $v - \phi$ and $v(x_0) = \phi(x_0)$.

- (iii) *The continuous function v is a viscosity solution of (6.1) if it is at the same time a viscosity super- and subsolution.*

See [17, 22] for more details on the theory of viscosity solutions, and [28] for the particular case of viscosity solutions to degenerate elliptic obstacle problems similar to (6.1).

Proof of Theorem 2.5. Since most of the steps in the proof are in common for the one-dimensional case and the quadratic case in any space dimension, we shall treat both cases at the same time, considering a general convex H in any space-dimension. We will treat both cases separately only when it is needed. In both cases, the uniqueness of a viscosity solution can be deduced from Remark 2.4.

Consider $n \geq 1$, H satisfying (1.2) and let $L = H^*$. Let us start by proving that $u_T^* := S_T^+(S_T^- u_T)$ is a viscosity supersolution. Fix $x_0 \in \mathbb{R}^n$ and let ϕ be a twice-differentiable function such that $u_T^*(x_0) = \phi(x_0)$ and $\phi(x) \leq u_T^*(x)$ for any $x \in B_\delta(x_0)$, with $\delta > 0$ small.

Since $u_T^*(x_0) = \phi(x_0)$, using the inequality in Proposition 4.5 it follows that $\phi(x_0) \geq u_T(x_0)$. By the definition of u_T^* and formula (4.2), there exists $y^* \in \mathbb{R}^n$ such that

$$(6.2) \quad \phi(x_0) = u_T^*(x_0) = S_T^- u_T(y^*) + T L \left(\frac{x_0 - y^*}{T} \right).$$

In the other hand, using the equality in Proposition 4.5, together with formula (4.4), we have

$$S_T^- u_T(y^*) = S_T^- u_T^*(y^*) \geq u_T^*(x) - T L \left(\frac{x - y^*}{T} \right), \quad \text{for all } x \in \mathbb{R}^n,$$

and this implies

$$\phi(x) \leq u_T^*(x) \leq S_T^- u_T(y^*) + T L \left(\frac{x - y^*}{T} \right) \quad \text{for all } x \in B_\delta(x_0).$$

This inequality, combined with (6.2), implies that the function

$$\Psi(x) := S_T^- u_T(y^*) + T L \left(\frac{x - y^*}{T} \right)$$

satisfies $\phi(x_0) = \Psi(x_0)$ and $\phi(x) \leq \Psi(x)$ for all $x \in B_\delta(x_0)$.

Hence, we deduce that $\nabla \phi(x_0) = \nabla \Psi(x_0)$ and the Hessian matrix $D^2(\phi(x_0) - \Psi(x_0))$ is semidefinite negative.

We then have

$$\nabla \phi(x_0) = L_q \left(\frac{x_0 - y^*}{T} \right), \quad \text{and} \quad -\lambda_n \left[D^2 \phi(x_0) - \frac{1}{T} L_{qq} \left(\frac{x_0 - y^*}{T} \right) \right] \geq 0.$$

Now, using Property 5.1, we obtain $\frac{x_0 - y^*}{T} = H_p(\nabla \phi(x_0))$, and then

$$-\lambda_n \left[D^2 \phi(x_0) - \frac{L_{qq}(H_p(\nabla \phi(x_0)))}{T} \right] \geq 0.$$

Using again Property 5.1, we conclude

$$-\lambda_n \left[D^2 \phi(x_0) - \frac{[H_{pp}(\nabla \phi(x_0))]^{-1}}{T} \right] \geq 0.$$

Next, we prove that u_T^* is also a viscosity subsolution. Fix $x_0 \in \mathbb{R}^n$ and let ϕ be a twice-differentiable function such that $u_T^*(x_0) = \phi(x_0)$ and $\phi(x) \geq u_T^*(x)$ for any $x \in B_\delta(x_0)$, with $\delta > 0$ small. We shall prove that

$$\text{if } -\lambda_n \left[D^2 \phi(x_0) - \frac{[H_{pp}(\nabla \phi(x_0))]^{-1}}{T} \right] > 0, \text{ then } \phi(x_0) = u_T(x_0).$$

Let us suppose that $\lambda_n \left[D^2 \phi(x_0) - \frac{[H_{pp}(\nabla \phi(x_0))]^{-1}}{T} \right] < 0$. If we take $y_0 = x_0 - T H_p(\nabla \phi(x_0))$, using Property 5.1, we obtain

$$\lambda_n \left[D^2 \phi(x_0) - \frac{1}{T} L_{qq} \left(\frac{x_0 - y_0}{T} \right) \right] < 0.$$

This implies that the function

$$\psi(x) := \phi(x) - T L \left(\frac{x - y_0}{T} \right),$$

is strictly concave in a neighbourhood of x_0 . In addition, by the choice of y_0 , we can use Property 5.1 to prove that $\nabla \psi(x_0) = 0$.

This implies that ψ has a strict local maximum at x_0 , i.e. there exist $\delta' > 0$ small and a constant $C \in \mathbb{R}$, such that

$$\psi(x_0) = C \quad \text{and} \quad \psi(x) < C \quad \text{for all } x \in B_{\delta'}(x_0) \setminus \{x_0\}.$$

Taking $\delta'' = \min\{\delta, \delta'\}$, since $\phi \geq u_T^*$ in $B_\delta(x_0)$ and $\phi(x_0) = u_T^*(x_0)$, we deduce that the function

$$\Psi(x) := u_T^*(x) - T L \left(\frac{x - y_0}{T} \right)$$

also satisfies

$$(6.3) \quad \Psi(x_0) = C \quad \text{and} \quad \Psi(x) < C, \quad \text{for all } x \in B_{\delta''}(x_0) \setminus \{x_0\}.$$

Now, we shall prove that for the one-dimensional case and for the case when H is quadratic in any space-dimension, x_0 is in fact the unique maximizer of Ψ over \mathbb{R}^n . Here, we treat both cases separately:

Case 1: One-dimensional case. Let us suppose, for a contradiction, that there exists $x_1 \in \mathbb{R} \setminus \{x_0\}$ such that $\Psi(x_1) \geq \Psi(x_0)$. Assume without loss of generality that $x_0 < x_1$, and let

$$x_2 := \operatorname{argmin}_{x \in [x_0, x_1]} \Psi(x).$$

From (6.3), we deduce that x_2 is an interior point of $[x_0, x_1]$ and satisfies $\Psi(x_2) < \Psi(x_0)$.

Then, using the definition of subdifferential in (5.2), it is not difficult to prove that $0 \in D^- \Psi(x_2)$, and by the definition of Ψ , we have

$$(6.4) \quad L_q \left(\frac{x_2 - y_0}{T} \right) \in D^- u_T^*(x_2).$$

Now, observe that, by Proposition 4.2, we have that $u_T^* = S_T^+(S_T^- u_T)$ is a semiconcave function. From a well-known property of semiconcave functions (Proposition 3.3.4 in [13]), the superdifferential of a semiconcave function is nonempty for any x .

Hence, we have

$$D^- u_T^*(x_2) \neq \emptyset \quad \text{and} \quad D^+ u_T^*(x_2) \neq \emptyset,$$

and by Proposition 3.1.5 in [13], we deduce that u_T^* is differentiable at x_2 . Indeed, in view of (6.4), we have

$$\nabla u_T^*(x_2) = L_q \left(\frac{x_2 - y_0}{T} \right).$$

Now, by Proposition 5.1, the unique minimizer in the Hopf-Lax formula (4.2), with $u_0 = S_T^- u_T$ and $x = x_2$, is given by

$$y^* = x_2 - T H_p(\nabla u_T^*(x_2)) = x_2 - T H_p \left(L_q \left(\frac{x_2 - y_0}{T} \right) \right) = y_0.$$

Here we used Property 5.1. Therefore, we have

$$u_T^*(x_2) = S_T^- u_T(y_0) + T L \left(\frac{x_2 - y_0}{T} \right).$$

This implies that $\Psi(x_2) = S_T^- u_T(y_0)$, and using Proposition 4.5, we deduce

$$\begin{aligned} \Psi(x_2) &= S_T^- u_T(y_0) = S_T^- u_T^*(y_0) \\ &= \max_{x \in \mathbb{R}} \left\{ u_T^*(x) - T L \left(\frac{x - y_0}{T} \right) \right\} \\ &\geq u_T^*(x_0) - T L \left(\frac{x_0 - y_0}{T} \right) = \Psi(x_0), \end{aligned}$$

and this contradicts $\Psi(x_2) < \Psi(x_0)$. We then deduce that x_0 is the unique maximizer of Ψ .

Case 2: H quadratic. Suppose that H is given by (2.3) for some positive definite matrix A . As we have proved in the first part of this proof, u_T^* is a viscosity supersolution of (6.1). Since in this case $H_{pp}(p) = A$ for all $p \in \mathbb{R}^n$. We have that u_T^* satisfies the inequality $-\lambda_n \left[D^2 u_T^* - \frac{A^{-1}}{T} \right] \geq 0$ in a viscosity sense.

Then, since $L_{qq}(q) = A^{-1}$ for all $q \in \mathbb{R}^n$, we deduce that Ψ satisfies $-\lambda_n [D^2 \Psi] \geq 0$ in a viscosity sense, and this implies that Ψ is a concave function (see Theorem 6.1 below). From the concavity of Ψ , together with (6.3), we obtain that x_0 is the unique maximizer of Ψ .

In both cases, we have proved that

$$(6.5) \quad \Psi(x_0) = C \quad \text{and} \quad \Psi(x) < C, \quad \text{for all } x \in \mathbb{R}^n \setminus \{x_0\}.$$

Therefore, we have

$$(6.6) \quad C = u_T^*(x_0) - T L \left(\frac{x_0 - y_0}{T} \right)$$

and

$$(6.7) \quad C > u_T^*(x) - T L \left(\frac{x - y_0}{T} \right), \text{ for all } x \in \mathbb{R}^n \setminus \{x_0\},$$

and after the equality in Proposition 4.5 and formula (4.4), we obtain

$$(6.8) \quad S_T^- u_T(y_0) = S_T^- u_T^*(y_0) = C.$$

Finally, applying (6.7) and the inequality in Proposition 4.5, we obtain

$$S_T^- u_T(y_0) = C > u_T(x) - T L \left(\frac{x - y_0}{T} \right), \quad \text{for all } x \in \mathbb{R}^n \setminus \{x_0\},$$

which, combined with (6.8), implies $u_T(x_0) - T L \left(\frac{x_0 - y_0}{T} \right) = C$, since the maximum in formula (4.4) is always attained. The equality $\phi(x_0) = u_T^*(x_0) = u_T(x_0)$ then follows from (6.6) and the choice of ϕ . \square

Remark 6.1. *Observe that in Case 1 of this proof, we used that in the one-dimensional case, between two local maxima of a continuous function there is always a local minimum. In higher dimension, this property is no longer true since there can be saddle points. Therefore, the same argument cannot be used to treat the multidimensional case. However, by assuming that the Hessian matrix of H is constant over \mathbb{R}^n , we can prove that Ψ is concave and then deduce that x_0 is the unique maximizer of Ψ .*

6.2. Semiconcave envelopes and proof of Theorem 2.2. It is well known that twice-differentiable convex functions are characterized by the property that the Hessian is everywhere positive semi-definite, and analogously, twice-differentiable concave functions are those such that the Hessian is everywhere negative semidefinite. This result can be generalized to continuous functions by the following result proved by Oberman in [28].

Theorem 6.1 (Oberman [28]).

- (i) *The continuous function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if and only if it is a viscosity solution of $\lambda_1[D^2v] \geq 0$.*
- (ii) *The continuous function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if and only if it is a viscosity solution of $\lambda_n[D^2v] \leq 0$.*

The definition of viscosity solution for these two inequalities is the same as Definition 6.2 below, adapted in an obvious way. In [28], it was proved that the convex envelope of a given function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the viscosity solution of the obstacle problem

$$(6.9) \quad \max\{v - f, -\lambda_1[D^2v]\} = 0.$$

In [28], the solution of (6.9) is also identified with the value function of a stochastic optimal control problem.

Analogously to the equation for the convex envelope, the concave envelope of f is the unique viscosity solution of

$$\min\{v - f, -\lambda_n[D^2v]\} = 0.$$

For related literature concerning equations involving operators like $\lambda_1[\cdot]$ and $\lambda_n[\cdot]$ and its relation with convex/concave envelopes and with stochastic control theory we refer to [10, 11, 12, 20, 29, 30] and the references therein.

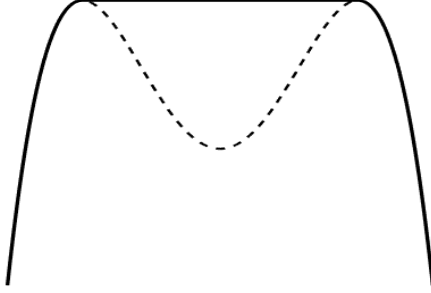


FIGURE 8. A function f , represented by a dotted line, and its concave envelope f^* .

In our case, the viscosity solution v of the obstacle problem (6.1) is not necessarily concave. However, if H is given by (2.3), the solution of (6.1) satisfies the inequality

$$\lambda_n \left[D^2v - \frac{1}{T}A^{-1} \right] \leq 0,$$

in the viscosity sense of Definition 6.2. This implies, using Theorem 6.1, that the function

$$x \mapsto v(x) - \frac{\langle A^{-1}x, x \rangle}{2T}$$

is concave. Hence, applying the characterization of semiconcave functions in Proposition 4.3, we deduce that v is semiconcave with linear modulus and constant

$$C = \lambda_n \left[\frac{A^{-1}}{T} \right] = \frac{1}{T\lambda_1[A]}.$$

Next we prove that, when H is quadratic, the solution to the problem (6.1) can be given in terms of the concave envelope of a certain auxiliary function f . This result proves the conclusion of Corollary 2.2.

Lemma 6.1. *Let H be given by (2.3) for some definite positive $n \times n$ matrix A . Let $T > 0$ and $u_T \in \text{Lip}(\mathbb{R}^n)$. The function v is the viscosity solution of (6.1) if and only if the function*

$$w(x) := v(x) - \frac{\langle A^{-1}x, x \rangle}{2T}$$

is the viscosity solution of

$$(6.10) \quad \min \{w - f, -\lambda_n[D^2w]\} = 0,$$

where

$$f(x) := u_T(x) - \frac{\langle A^{-1}x, x \rangle}{2T}.$$

Proof. We recall that the definition of viscosity solution of (6.10) is the same as Definition 6.1 replacing H by 0 and u_T by f .

We need to prove that v is a viscosity supersolution (resp. subsolution) of (6.1) if and only if w is a viscosity supersolution (resp. subsolution) of (6.10). We only give the proof for the equivalence of viscosity supersolution since the same argument works for the viscosity subsolutions.

Let v be a viscosity supersolution of (6.1). For any $x_0 \in \mathbb{R}^n$, take ϕ any twice-differentiable function such that $w(x_0) = \phi(x_0)$ and x_0 is a local minimum for $w - \phi$. Using the definition of w in the enunciate of the lemma, this implies that

$$v(x_0) - \left(\frac{\langle A^{-1}x_0, x_0 \rangle}{2T} + \phi(x_0) \right) = 0$$

and

$$0 \leq v(x) - \underbrace{\left(\frac{\langle A^{-1}x, x \rangle}{2T} + \phi(x) \right)}_{\varphi(x)}, \quad \text{for all } x \in B_\delta(x_0),$$

for some $\delta > 0$ small. Observe that φ is a twice differentiable function such that $v(x_0) = \varphi(x_0)$ and x_0 is a local minimum for $v - \varphi$. Then, since v is a viscosity supersolution of (6.1), we have

$$\varphi(x_0) \geq u_T(x_0) \quad \text{and} \quad -\lambda_n \left[D^2\varphi(x_0) - \frac{A^{-1}}{T} \right] \geq 0.$$

Using the definition of f and the choice of φ , this implies that

$$\phi(x_0) \geq u_T(x_0) - \frac{\langle A^{-1}x_0, x_0 \rangle}{2T} = f(x_0) \quad \text{and} \quad -\lambda_n [D^2\phi(x_0)] \geq 0.$$

We have then proved that w is a viscosity supersolution of (6.10). Similarly, we can prove that if w is a viscosity supersolution of (6.10), then v is a viscosity supersolution of (6.1). \square

We now turn our attention to the reachability condition given in Theorem 2.2: if H is quadratic or if the space dimension is 1, then the target u_T is reachable if and only if u_T satisfies the inequality

$$(6.11) \quad \lambda_n \left[D^2u_T - \frac{[H_{pp}(Du_T)]^{-1}}{T} \right] \leq 0$$

in the viscosity sense. Let us make precise the definition of viscosity solution of the above inequality.

Definition 6.2. *The upper semicontinuous function u_T is a viscosity solution of (6.11) if for every twice-differentiable function $\phi(\cdot)$,*

$$\lambda_n \left[D^2\phi(x) - \frac{[H_{pp}(\nabla\phi(x))]^{-1}}{T} \right] \leq 0, \quad \text{whenever } x \text{ is a local minimum of } u_T - \phi.$$

Observe that it is equivalent to say that u_T is a viscosity supersolution of (6.1) (see (ii) in Definition 6.1).

Proof of Theorem 2.2. The result can be deduced from Theorems 2.1 and 2.5. Let $T > 0$ be fixed. From Theorem 2.1, a function u_T satisfies $I_T(u_T) \neq \emptyset$ if and only if

$$u_T^* = S_T^+(S_T^-u_T) = u_T.$$

In view of Theorem 2.5, if $n = 1$ or if H is given by (2.3), this is equivalent to say that u_T is the viscosity solution of the obstacle problem (6.1)

Observe that, in view of Definition 6.1, u_T is always a viscosity subsolution of (6.1). In order to be also a viscosity supersolution, it only needs to verify the inequality

$$\lambda_n \left[D^2 u_T(x) - \frac{[H_{pp}(Du_T)]^{-1}}{T} \right] \leq 0$$

in the viscosity sense of Definition 6.2. We then deduce that u_T is a viscosity solution of the obstacle problem (6.1) if and only if u_T verifies this inequality in the viscosity sense. \square

We end this section with the proof of Corollary 2.1, that links the reachability of a target u_T with its semiconcavity constant. At some point in the proof, it will be necessary to use the following elementary property from linear algebra:

Property 6.1. *For any two $n \times n$ symmetric matrices X and Y , the following inequality holds:*

$$\lambda_1(X) + \lambda_n(Y) \leq \lambda_n(X + Y) \leq \lambda_n(X) + \lambda_n(Y).$$

This property can be deduced from the formula

$$\lambda_1(X) = \inf_{p \in \mathbb{R}^n} \frac{\langle X p, p \rangle}{|p|^2} \quad \text{and} \quad \lambda_n(X) = \sup_{p \in \mathbb{R}^n} \frac{\langle X p, p \rangle}{|p|^2}.$$

Proof of Corollary 2.1. Let u_T be a Lipschitz function such that $I_T(u_T) \neq \emptyset$. From Theorem 2.2 we have that u_T is a viscosity solution of

$$\lambda_n \left[D^2 u_T - \frac{1}{T} A^{-1} \right] \leq 0.$$

Hence, in view of Theorem 6.1, the function $x \mapsto u_T(x) - \frac{\langle A^{-1}x, x \rangle}{2T}$ is concave. Using the characterization of semiconcave functions in Proposition 4.3, we deduce that u_T is semiconcave with linear modulus and constant $\frac{1}{T\lambda_1[A]}$.

Let us prove now the second statement. Suppose that u_T is semiconcave with linear modulus and constant

$$C = \frac{1}{T\lambda_n[A]} = \lambda_1 \left[\frac{A^{-1}}{T} \right].$$

Using the equivalence (i)-(ii) in Proposition 4.3, we have that the function

$$f(x) := u_T(x) - \frac{\lambda_1[A^{-1}]}{2T} |x|^2$$

is concave. Hence, applying Theorem 6.1, we have that f is a viscosity solution of $\lambda_n[D^2 f] \leq 0$.

We then have that u_T satisfies

$$(6.12) \quad \lambda_n \left[D^2 u_T - \frac{\lambda_1[A^{-1}]}{T} I_n \right] \leq 0$$

in the viscosity sense, where I_n is the $n \times n$ identity matrix. Now, we combine (6.12) with Property 6.1, along with the fact that $\lambda_1[cI_n] = c$, to obtain

$$\begin{aligned} 0 &\geq \lambda_n \left[D^2 u_T - \frac{\lambda_1[A^{-1}]}{T} I_n \right] \\ &\geq \lambda_n [D^2 u_T] + \lambda_1 \left[-\frac{\lambda_1[A^{-1}]}{T} I_n \right] \\ &= \lambda_n [D^2 u_T] - \frac{\lambda_1[A^{-1}]}{T}. \end{aligned}$$

Finally, combining this inequality again with Property 6.1, we obtain

$$\begin{aligned} \lambda_n \left[D^2 u_T - \frac{1}{T} A^{-1} \right] &\leq \lambda_n [D^2 u_T] + \frac{\lambda_n[-A^{-1}]}{T} \\ &= \lambda_n [D^2 u_T] - \frac{\lambda_1[A^{-1}]}{T} \leq 0. \end{aligned}$$

And from Theorem 2.2, we conclude that $I_T(u_T) \neq \emptyset$. \square

7. CONCLUSIONS AND PERSPECTIVES

The first goal in this work was to characterize the set of reachable targets u_T for a time-evolution Hamilton-Jacobi equation of the form (1.1). Using the inversion of the direction of time in the equation and the notion of backward viscosity solution we proved, in Theorem 2.1, that the reachable targets are the fix-points of the composition operator $S_T^+ \circ S_T^-$ (see the precise definition of S_T^+ and S_T^- in Section 4). We note that this result can be extended, using similar arguments, to Hamilton-Jacobi equations with a Hamiltonian depending on x , i.e.

$$\partial_t u + H(x, D_x u) = 0.$$

In addition, for the one-dimensional case and for quadratic Hamiltonians in any space-dimension we characterize, in Theorem 2.2, the set of reachable targets as the viscosity solutions to the differential inequality

$$(7.1) \quad \lambda_n \left[D^2 u_T - \frac{(H_{pp}(Du_T))^{-1}}{T} \right] \leq 0.$$

Since H is assumed to be strictly convex (see hypothesis (1.2)), inequality (7.1) implies in particular that the reachable targets are semiconcave functions with linear modulus and constant

$$C = \frac{1}{T K_{u_T}}, \quad \text{where } K_{u_T} = \min_{|p| \leq \text{Lip}(u_T)} H_{pp}(p) > 0.$$

Here, $\text{Lip}(u_T)$ is the Lipschitz constant of u_T .

Although the semiconcavity of the viscosity solutions is a well-known property of Hamilton-Jacobi equations, we point out that inequality (7.1) gives a necessary and sufficient condition for the reachability of the target. Inequality (7.1) describes exactly the semiconcavity that is needed for a target to be reachable. In one space dimension, inequality (7.1) can be written as

$$\partial_{xx}u_T - (T H_{pp}(\partial_x u_T))^{-1} \leq 0,$$

and is the analogous to the one-sided Lipschitz condition for scalar conservation laws (see for example [16, 24]). We recall that the transformation

$$u(t, x) \longmapsto v(t, x) := \partial_x u(t, x),$$

transforms any viscosity solution of the Hamilton-Jacobi equation (1.1) in one-space dimension, into an entropy solution of the scalar conservation law

$$\partial_t v + \partial_x(H(v)) = 0.$$

In higher dimension, the hypothesis of $H_{pp}(p)$ being constant seems to be crucial in our proof, and we do not know if the inequality (7.1) also provides a necessary and sufficient reachability condition for general strictly convex Hamiltonians in dimension higher than 1. For the case of a Hamiltonian $H(x, p)$ depending on x , the existence of an inequality similar to (7.1) characterizing the set of reachable targets is a probably difficult problem.

Our second goal in this work was to construct, for any reachable target u_T , the set of all the initial conditions u_0 satisfying $S_T^+ u_0 = u_T$. In Theorem 2.3, we proved that this construction can be carried out by obtaining two elements:

- (i) the function $\tilde{u}_0 := S_T^- u_T$, which in fact happens to be the smallest initial condition satisfying $S_T^+ u_0 = u_T$;
- (ii) and the set $X_T(u_T) \subset \mathbb{R}^n$, obtained from the differentiability points of u_T , where all the initial conditions u_0 satisfying $S_T^+ u_0 = u_T$ coincide.

A similar construction was done recently in [16] for the one-dimensional case, using the equivalence with scalar conservation laws. However, our approach is based only on the Hamilton-Jacobi setting and applies to any space-dimension. We stress that we also give, in Theorem 2.4 and Proposition 5.2, a rather geometrical identification of the points in $X_T(u_T)$, which allows to construct the set $X_T(u_T)$ without the necessity of knowing the differentiability points of u_T .

The problem of identifying initial sources has been addressed in the context of many different time-evolution models, and has a relevant interest from both a theoretical and a practical viewpoint. We refer to [14, 23] for works on initial data identification for parabolic equations, where the fast decay of the solution and the regularizing effect make difficult the reconstruction of the solution at the initial time, specially if the time horizon T is large. In this setting, the usual approach is to approximate the final target by considering initial sources constituted by a finite number of Dirac deltas, where the location of the Dirac deltas as well as the weight for each of them must be carefully determined.

Finally, our last main result concerns the composition operator $S_T^+ \circ S_T^-$, which applied to any target u_T (reachable or not), represents a projection of u_T on the set of reachable targets. For the one-dimensional case and for the case of a quadratic Hamiltonian in any space-dimension we prove,

in Theorem 2.5, that the function $u_T^* := S_T^+(S_T^-u_T)$ is the unique viscosity solution to the fully nonlinear obstacle problem

$$(7.2) \quad \min \left\{ v - u_T, -\lambda_n \left[D^2v - \frac{[H_{pp}(Dv)]^{-1}}{T} \right] \right\} = 0.$$

In analogy with the notion of concave envelope of a function, we call the solution to (7.2) the semiconcave envelope of u_T . Note that, if we make T go to infinity, equation (7.2) can be viewed as an approximation of the equation for the concave envelope (2.9).

For a fixed $T > 0$, the projection u_T^* is in fact the smallest reachable target bounded from below by u_T . For the case of a general strictly convex Hamiltonian in any space-dimension, we have not been able to prove the validity of Theorem 2.5, and the existence of an equation similar to (7.2) for the case of a more general Hamiltonian $H(x, p)$ depending on x is unknown up to the best of our knowledge. We would like to end this discussion by noting that projections like $S_T^+ \circ S_T^-$, obtained by a backward-forward resolution of the equation, might be of interest in time-evolution problems different from Hamilton-Jacobi equations, and its identification with the solution of an equation like (7.2) could be useful to better understand the nature and properties of this projection.

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CARLOS ESTEVE
 DEPARTAMENTO DE MATEMÁTICAS,
 UNIVERSIDAD AUTÓNOMA DE MADRID,
 28049 MADRID, SPAIN

AND

CHAIR OF COMPUTATIONAL MATHEMATICS, FUNDACIÓN DEUSTO
 AV. DE LAS UNIVERSIDADES, 24
 48007 BILBAO, BASQUE COUNTRY, SPAIN

Email address: `carlos.esteve@uam.es`

ENRIQUE ZUAZUA
 CHAIR IN APPLIED ANALYSIS, ALEXANDER VON HUMBOLDT-PROFESSORSHIP
 DEPARTMENT OF MATHEMATICS,
 FRIEDRICH-ALEXANDER-UNIVERSITÄT ERLANGEN-NÜRNBERG
 91058 ERLANGEN, GERMANY

AND

CHAIR OF COMPUTATIONAL MATHEMATICS, FUNDACIÓN DEUSTO
 AV. DE LAS UNIVERSIDADES, 24
 48007 BILBAO, BASQUE COUNTRY, SPAIN

AND

DEPARTAMENTO DE MATEMÁTICAS,
 UNIVERSIDAD AUTÓNOMA DE MADRID,
 28049 MADRID, SPAIN

Email address: `enrique.zuazua@fau.de`