

NONUNIQUENESS OF MINIMIZERS FOR SEMILINEAR OPTIMAL CONTROL PROBLEMS

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ABSTRACT. A counterexample to uniqueness of global minimizers of semilinear optimal control problems is given. The lack of uniqueness occurs for a special choice of the state-target in the cost functional. Our arguments show also that, for some state-targets, there exist local minimizers, which are not global. When this occurs, gradient-type algorithms may be trapped by the local minimizers, thus missing the global ones. Furthermore, the issue of convexity of quadratic functional in optimal control is analyzed in an abstract setting.

As a Corollary of the nonuniqueness of the minimizers, a nonuniqueness result for a coupled elliptic system is deduced.

Numerical simulations have been performed illustrating the theoretical results. We also discuss the possible impact of the multiplicity of minimizers on the turnpike property in long time horizons.

1. INTRODUCTION

We produce a counterexample to the uniqueness of the optimal control in semilinear control. Both the case of internal control and boundary control are considered. To fix ideas, we focus on the case of quadratic functional and semilinear governing state equation. However, our techniques are applicable to a wide range of optimal control problems governed by a nonlinear state equation.

1.1. Lack of uniqueness of the minimizer. In the context of boundary control, we consider the control problem

$$(1) \quad \min_{u \in L^\infty(\partial B(0,R))} J(u) = \frac{1}{2} \int_{\partial B(0,R)} |u|^2 d\sigma(x) + \frac{\beta}{2} \int_{B(0,R)} |y - z|^2 dx,$$

where $u = u(x)$ is the control and $y = y(x)$ is the associated state, solution to the semilinear equation

$$(2) \quad \begin{cases} -\Delta y + f(y) = 0 & \text{in } B(0, R) \\ y = u & \text{on } \partial B(0, R). \end{cases}$$

Key words and phrases. Semilinear elliptic equations, nonuniqueness global minimizer, lack of convexity, optimal control.

This project has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 694126-DYCON).

We acknowledge professor Enrique Zuazua for his helpful remarks on the manuscript. We thank professor Martin Gugat for his interesting questions.

The space domain $B(0, R)$ is a ball of $\mathbb{R}^n$ centered at the origin of radius R , with $n = 1, 2, 3$. The nonlinearity $f \in C^2(\mathbb{R})$ is strictly increasing and $f(0) = 0$. The target $z \in L^\infty(B(0, R))$ and $\beta > 0$ is a penalization parameter. As β increases, the distance between the optimal state and the target decreases.

In appendix A we analyze the well-posedness of the state equation (2), while in appendix B we prove the existence of a global minimizer $\bar{u} \in L^\infty(\partial B(0, R))$ for the functional J defined above. As we shall see in the following result, for a special target, the global minimizer is not unique.

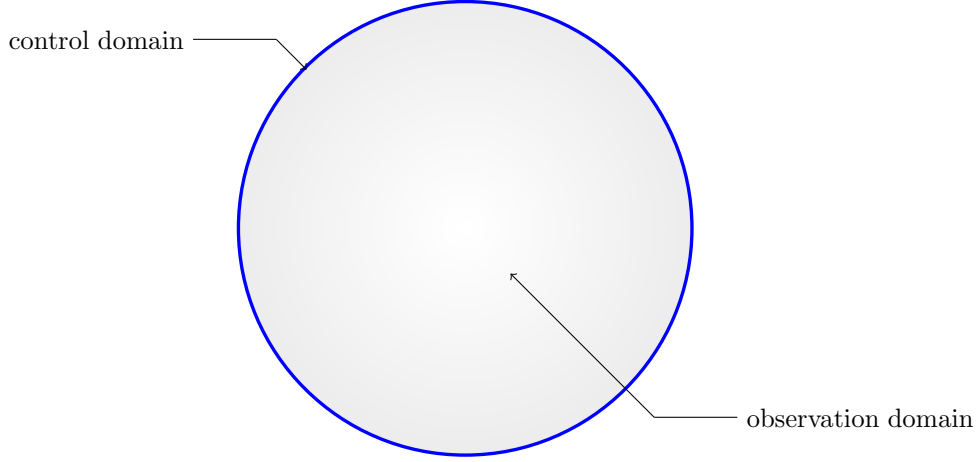


FIGURE 1. control and observation domains. The control domain is the blue boundary of the ball.

Theorem 1.1. *Consider the control problem (2)-(1). Assume, in addition*

$$(3) \quad f''(y) \neq 0 \quad \forall y \neq 0.$$

There exists a target $z \in L^\infty(B(0, R))$ such that the functional J defined in (1) admits (at least) two global minimizers.

To give a first explanation of the above result, we introduce the control-to-state map

$$(4) \quad G : L^\infty(\partial B(0, R)) \longrightarrow L^2(B(0, R))$$

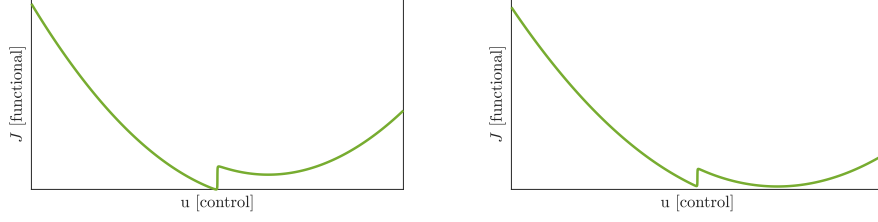
$$u \longmapsto y_u,$$

with y_u solution to (2), with control u . Then, for any control $u \in L^\infty(\partial B(0, R))$, the functional (1) reads as

$$(5) \quad J(u) = \frac{1}{2} \int_{\partial B(0, R)} |u|^2 d\sigma(x) + \frac{\beta}{2} \int_{B(0, R)} |G(u) - z|^2 dx.$$

We have two addenda. The first one is convex, being a squared norm. The second one is a squared norm composed with $u \mapsto G(u) - z$. Now, under the assumption (3), the map $u \mapsto G(u)$ is nonlinear. Then, the term $\int_{B(0, R)} |G(u) - z|^2 dx$, for a special target z , is not convex and generates the lack of uniqueness of the minimizers.

The proof of theorem 1.1 can be found in section 3.1. The main steps for that proof are:



(A) nonuniqueness of the *local* minimizer (B) nonuniqueness of the *global* minimizer

FIGURE 2. functional versus control. This plot is obtained by drawing in MATLAB the graph of J defined in (1), with $R = 1$ and nonlinearity $f(y) = y^3$. Figure 2a and fig. 2b correspond respectively to targets yielding to nonuniqueness of the local and the global minimizers.

- Step 1 **Reduction to constant controls:** by choosing radial targets and using the rotational invariance of $B(0, R)$, we reduce to the case the control set is made of constant controls;
- Step 2 **Existence of two local minimizers:** we look for a target such that there exists two *local* minimizers ($u_1 < 0$ and $u_2 > 0$) for the functional J (see fig. 2);
- Step 3 **Existence of two global minimizers:** by the former step and a bisection argument, we prove the existence of a target such that J admits two *global* minimizers.

The special target yielding nonuniqueness is a step function changing sign in the observation domain, as in fig. 3.

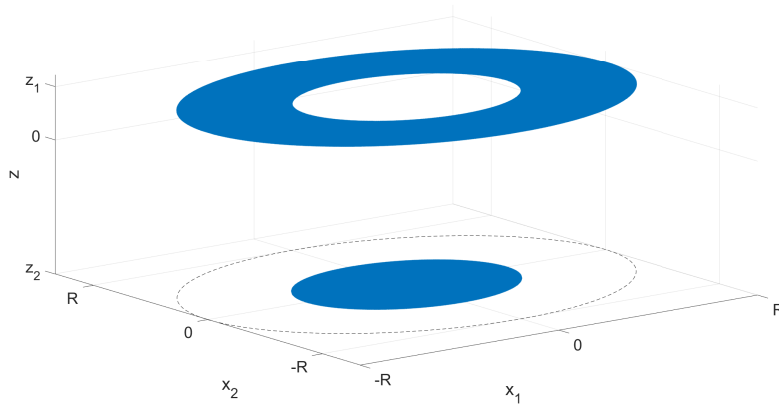


FIGURE 3. target yielding nonuniqueness in boundary control. The constructed target z (in blue) is a step function, taking values z_1 and z_2 .

The above techniques can be applied, with some modifications, to the internal control problem

$$(6) \quad \min_{u \in L^2(B(0,r))} J(u) = \frac{1}{2} \int_{B(0,r)} |u|^2 dx + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |y - z|^2 dx,$$

where

$$(7) \quad \begin{cases} -\Delta y + f(y) = u \chi_{B(0,r)} & \text{in } B(0,R) \\ y = 0 & \text{on } \partial B(0,R). \end{cases}$$

$B(0,R)$ denotes a ball of $\mathbb{R}^n$ centered at the origin of radius R , $n = 1, 2, 3$. The nonlinearity $f \in C^2(\mathbb{R})$ is strictly increasing and $f(0) = 0$. The control acts in $B(0,r)$, with $r \in (0, R)$. We observe in $B(0,R) \setminus B(0,r)$ (see fig. 4). The target $z \in L^2(B(0,R) \setminus B(0,r))$, while $\beta > 0$ is a penalization parameter.

The well-posedness of the state equation follows from [5, Theorem 4.7, page 29], while the existence of a global minimizer in $L^2(B(0,r))$ for (7)-(6) can be shown by the Direct Method of the Calculus of Variations (DMCV).

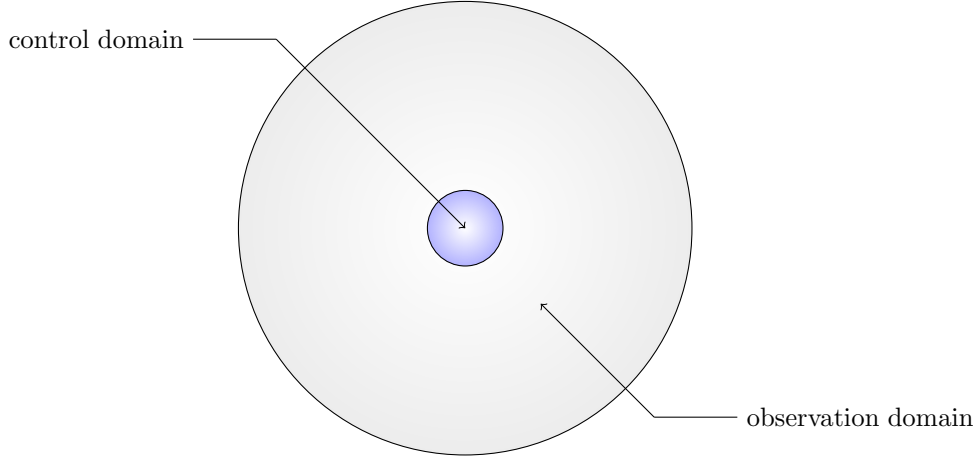


FIGURE 4. control and observation domains

Theorem 1.2. *Consider the control problem (7)-(6). Assume, in addition,*

$$(8) \quad f''(y) \neq 0 \quad \forall y \neq 0.$$

There exists a target $z \in L^\infty(B(0,R) \setminus B(0,r))$ such that the functional J defined in (6) admits (at least) two global minimizers.

The proof can be found in section 3.2.

A by-product of our nonuniqueness results is the lack of uniqueness of solutions $(\bar{y}, \bar{q})$ to the optimality system

$$(9) \quad \begin{cases} -\Delta \bar{y} + f(\bar{y}) = -\bar{q} \chi_{B(0,r)} & \text{in } B(0,R) \\ \bar{y} = 0 & \text{on } \partial B(0,R) \\ -\Delta \bar{q} + f'(\bar{y}) \bar{q} = \beta(\bar{y} - z) \chi_{B(0,R) \setminus B(0,r)} & \text{in } B(0,R) \\ \bar{q} = 0 & \text{on } \partial B(0,R). \end{cases}$$

In the case of internal control, we can deduce the following corollary.

Corollary 1. *Under the assumptions of theorem 1.2, there exists a target $z \in L^\infty(B(0, R) \setminus B(0, r))$, such that (9) admits (at least) two distinguished solutions $(\bar{y}_1, \bar{q}_1)$ and $(\bar{y}_2, \bar{q}_2)$.*

This follows from theorem 1.2, together with the first order optimality conditions for the optimization problem (7)-(6) (see [10]).

Similarly, in the context of boundary control, the nonuniqueness for (1) leads to nonuniqueness of solution to the optimality system

$$(10) \quad \begin{cases} -\Delta \bar{y} + f(\bar{y}) = 0 & \text{in } B(0, R) \\ \bar{y} = \frac{\partial}{\partial n} \bar{q} & \text{on } \partial B(0, R) \\ -\Delta \bar{q} + f'(\bar{y})\bar{q} = \beta(\bar{y} - z) & \text{in } B(0, R) \\ \bar{q} = 0 & \text{on } \partial B(0, R). \end{cases}$$

To the best of our knowledge, the issue of the uniqueness of the minimizer has not been addressed so far for large targets z . Indeed, the uniqueness of the optimal control has been proved under smallness conditions on the target [24, subsection 3.2] or on the adjoint state [1, Theorem 3.2]. In particular, in [1, Theorem 3.2] the uniqueness holds provided that the adjoint state is strictly smaller than a constant, explicitly determined [1, equation (3.6)].

The issue of uniqueness of the minimizer for elliptic problems is of primary importance when studying the turnpike property for the corresponding time-evolution control problem (see, [28, 24, 27, 26]). Indeed, the existence of multiple global minimizers for the steady problem generates multiple potential attractors for the time-evolution problem.

The control problems we are treating are classical in the literature. General surveys on the topic are [10] by Eduardo Casas and Mariano Mateos and [29, Chapter 4] by Fredi Tröltzsch. The interested reader is referred also to the following articles and books and the references therein: [9, 3, 4, 22, 8, 2, 17, 7, 20, 11, 25].

1.2. Lack of convexity. Before proving our main result on nonuniqueness of global minimizers, we observe that, for some targets, quadratic functionals of the optimal control governed by nonlinear state equations are not convex.

Theorem 1.3. *Consider the optimal control problem introduced in (7)-(6). Then, we have two possibilities:*

- (1) *f is linear. Then, J is convex for any target $z \in L^2(B(0, R) \setminus B(0, r))$.*
- (2) *f is not linear. Then, there exists a target $z \in L^2(B(0, R) \setminus B(0, r))$ such that the corresponding J_s is not convex.*

In the literature, it is well known that convexity cannot be proved by standard techniques, in case the state equation is nonlinear (see, for instance, [1] and [29, section 4]). However, to the best of our knowledge, there are not available counterexamples to convexity. In this work, the lack of convexity can be deduced as a consequence of the lack of uniqueness (theorem 1.1). Anyway, we prefer to prove theorem 1.3 in section 2 as a particular case of the following theorem, which holds in a general functional framework and basically asserts that a quadratic functional of the optimal control is convex for any target if and only if its control-to-state map is affine.

Theorem 1.4. *Let U and H be real Hilbert spaces. Let*

$$G : U \longrightarrow H$$

be a function. Set:

$$(11) \quad J : U \longrightarrow H, \quad J(u) := \frac{1}{2} \|u\|_U^2 + \frac{1}{2} \|G(u) - z\|_H^2,$$

where $z \in H$.

Then, the following are equivalent:

- (1) *for any target $z \in H$, J is convex;*
- (2) *G is affine.*

In the application of theorem 1.4 to optimal control, H is the observation space, U is the control space and G is the control-to-state map. The vector $z \in H$ is the given target for the state. Note that theorem 1.4 applies both to steady and time-evolution control problems. Furthermore, the map G is not required to be smooth.

We sketch the proof of 1. $\implies$ 2.. Namely we show the lack of convexity, in case the control-to-state map G is not affine. For the time being, we assume that G is of class C^2 . In the complete proof in section 2, the smoothness of G is not required.

We start developing the functional (11), for any control $u \in U$

$$\begin{aligned} J(u) &= \frac{1}{2} \|u\|_U^2 + \frac{1}{2} \|G(u) - z\|_H^2 \\ &= \frac{1}{2} \|u\|_U^2 + \frac{1}{2} \|G(u)\|_H^2 + \frac{1}{2} \|z\|_H^2 - \langle G(u), z \rangle \\ &= P(u) + \frac{1}{2} \|z\|_H^2 - \langle G(u), z \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of H and

$$P(u) := \frac{1}{2} \|u\|_U^2 + \frac{1}{2} \|G(u)\|_H^2.$$

Now, since G is not affine, there exists a control $u_1 \in U$ and a direction $v_1 \in U$, such that the second directional derivative of G at u_1 along v_1 does not vanish

$$(12) \quad \frac{d^2}{dv_1^2} G(u_1) \neq 0.$$

Take as target $z^k := k \frac{d^2}{dv_1^2} G(u_1)$, with $k > 0$ to be made precise later and compute the second differential of the functional J at u_1 along direction v_1

$$\begin{aligned} \langle d^2 J(u_1) v_1, v_1 \rangle &= \frac{d^2}{dv_1^2} P(u_1) - \left\langle \frac{d^2}{dv_1^2} G(u_1), z^k \right\rangle \\ &= \frac{d^2}{dv_1^2} P(u_1) - k \left\| \frac{d^2}{dv_1^2} G(u_1) \right\|_H^2 < 0, \end{aligned}$$

choosing k sufficiently large. This shows the lack of convexity in the smooth case. The general nonsmooth case is handled in section 2.

Theorem 1.4 can be applied to internal and boundary control, both in the elliptic and parabolic context.

The lack of convexity and uniqueness of the minimizer is a serious warning for numerics. Indeed, if the problem is not convex the convergence of gradient methods is not guaranteed a priori. Furthermore, by employing our techniques, one can find several counterexamples where there exist local minimizers, which are not global. Then, gradient methods may converge to the local minimizer, thus missing the global ones.

The rest of the manuscript is organized as follows. In section 2, we prove theorem 1.4 and we deduce theorem 1.3. In section 3, we provide the counterexample to uniqueness of the global minimizer, in the context of boundary control (section 3.1) and internal control (section 3.2). In section 4, we perform numerical simulations which explain and confirm our theoretical results. In the appendix, we prove some Lemmas needed for our construction.

2. LACK OF CONVEXITY: PROOF OF THEOREM 1.4 AND THEOREM 1.3

In the proof of theorem 1.4, we need the following lemma.

Lemma 2.1. *Let V_1 and V_2 be two real vector spaces. Take a function*

$$G : V_1 \longrightarrow V_2.$$

Then, G is affine if and only if, for any $\lambda \in [0, 1]$ and $(v, w) \in V_1^2$

$$(13) \quad G((1 - \lambda)v + \lambda w) = (1 - \lambda)G(v) + \lambda G(w).$$

The proof can be deduced by linear algebra theory. We prove now theorem 1.4.

Proof of theorem 1.4. 2. $\implies$ 1. If G is affine, by direct computations and convexity of the square of Hilbert norms, J is convex for any $z \in H$.

1. $\implies$ 2. Assume now G is not affine. We construct a target $z \in H$ such that J is not convex.

In what follows, we denote by $\langle \cdot, \cdot \rangle$ the scalar product of H .

Step 1 Proof of the existence of $\tilde{\lambda} \in [0, 1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that:

$$\left\langle z^0, G\left(\left(1 - \tilde{\lambda}\right)\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2\right) \right\rangle < \left(1 - \tilde{\lambda}\right)\left\langle z^0, G(\tilde{u}_1) \right\rangle + \tilde{\lambda}\left\langle z^0, G(\tilde{u}_2) \right\rangle$$

First of all, we note that, up to change the sign of z^0 , we can reduce to prove the existence of $\tilde{\lambda} \in [0, 1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that:

$$(14) \quad \left\langle z^0, G\left(\left(1 - \tilde{\lambda}\right)\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2\right) \right\rangle \neq \left(1 - \tilde{\lambda}\right)\left\langle z^0, G(\tilde{u}_1) \right\rangle + \tilde{\lambda}\left\langle z^0, G(\tilde{u}_2) \right\rangle.$$

Reasoning by contradiction, if (14) were not true, for any $z \in H$, for every $(u_1, u_2) \in U^2$ and for each $\lambda \in [0, 1]$,

$$\langle z, G((1 - \lambda)u_1 + \lambda u_2) \rangle = (1 - \lambda)\langle z, G(u_1) \rangle + \lambda\langle z, G(u_2) \rangle.$$

By the arbitrariness of z , this leads to:

$$G((1 - \lambda)u_1 + \lambda u_2) = (1 - \lambda)G(u_1) + \lambda G(u_2),$$

for any $\lambda \in [0, 1]$ and $(u_1, u_2) \in U^2$. Then, by lemma 2.1, G is affine, which contradicts our hypothesis. This finishes this step.

Step 2 Conclusion

We remind that in the first step, we have proved the existence of $\tilde{\lambda} \in [0, 1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that:

$$\left\langle z^0, G\left(\left(1 - \tilde{\lambda}\right)\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2\right) \right\rangle < \left(1 - \tilde{\lambda}\right)\left\langle z^0, G\left(\tilde{u}_1\right) \right\rangle + \tilde{\lambda}\left\langle z^0, G\left(\tilde{u}_2\right) \right\rangle.$$

Now, arbitrarily fix $k \in \mathbb{N}^*$. Set as target:

$$z^k := kz^0.$$

We develop J with target z^k , getting for any $u \in U$:

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u) - z^k\|_H^2 \\ &= \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u)\|_H^2 + \frac{1}{2}\|z^k\|_H^2 - \langle z^k, G(u) \rangle \\ &= P(u) + \frac{1}{2}\|z^k\|_H^2 - \langle z^k, G(u) \rangle, \end{aligned}$$

where

$$P : U \longrightarrow \mathbb{R}, \quad u \longmapsto \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u)\|_H^2.$$

At this point, we introduce:

$$c_1 := \left(1 - \tilde{\lambda}\right)P\left(\tilde{u}_1\right) + \tilde{\lambda}P\left(\tilde{u}_2\right) - P\left(\left(1 - \tilde{\lambda}\right)\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2\right)$$

and

$$c_2 := \left(1 - \tilde{\lambda}\right)\left\langle z^0, G\left(\tilde{u}_1\right) \right\rangle + \tilde{\lambda}\left\langle z^0, G\left(\tilde{u}_2\right) \right\rangle - \left\langle z^0, G\left(\left(1 - \tilde{\lambda}\right)\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2\right) \right\rangle.$$

Then, taking as target z^k ,

$$\left(1 - \tilde{\lambda}\right)J\left(\tilde{u}_1\right) + \tilde{\lambda}J\left(\tilde{u}_2\right) - J\left(\left(1 - \tilde{\lambda}\right)\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2\right) = c_1 - kc_2.$$

By the first step, $c_2 > 0$. Then, for k large enough, we have:

$$\left(1 - \tilde{\lambda}\right)J\left(\tilde{u}_1\right) + \tilde{\lambda}J\left(\tilde{u}_2\right) - J\left(\left(1 - \tilde{\lambda}\right)\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2\right) = c_1 - kc_2 < 0,$$

which yields

$$\left(1 - \tilde{\lambda}\right)J\left(\tilde{u}_1\right) + \tilde{\lambda}J\left(\tilde{u}_2\right) < J\left(\left(1 - \tilde{\lambda}\right)\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2\right),$$

i.e. the desired lack of convexity of J . This concludes the proof. $\square$

Theorem 1.4 applies in semilinear control, both in the elliptic case and in the parabolic one. We show how to apply theorem 1.4 for the control problem (7)-(6), thus proving theorem 1.3.

Proof of theorem 1.3. Take

- control space $U = L^2(B(0, r))$;
- $H = L^2(B(0, R) \setminus B(0, r))$ with scalar product $\langle v_1, v_2 \rangle := \beta \int_{B(0, R) \setminus B(0, r)} v_1 v_2 dx$;
- the map

$$G : L^2(B(0, r)) \longrightarrow L^2(B(0, R) \setminus B(0, r))$$

$$u \longrightarrow y_u \upharpoonright_{B(0, R) \setminus B(0, r)},$$

where y_u fulfills (7) with control u .

Then, by theorem 1.4, we have two possibilities:

- (1) f is linear. Then, J is convex for any target $z \in L^2(B(0, R) \setminus B(0, r))$.

- (2) f is not linear. Then, there exists a target $z \in L^2(B(0, R) \setminus B(0, r))$ such that the corresponding J is not convex.

□

3. LACK OF UNIQUENESS

In this section, we prove our nonuniqueness results. We start with boundary control (theorem 1.1), to later deal with internal control (theorem 1.2).

3.1. Boundary control. Hereafter, we will work with radial targets, defined below.

Definition 3.1. A function $z : B(0, R) \rightarrow \mathbb{R}$ is said to be radial if there exists $\phi : [0, R] \rightarrow \mathbb{R}$, such that, for any $x \in B(0, R)$, we have $z(x) = \phi(\|x\|)$.

We present our strategy to prove theorem 1.1:

- Step 1 **Reduction to constant controls:** by choosing radial targets and using the rotational invariance of $B(0, R)$, we reduce to the case the control set is made of constant controls;
- Step 2 **Existence of two local minimizers:** we look for a target $z^0 \in L^\infty(B(0, R))$ such that there exists two *local* minimizers ($u_1 < 0$ and $u_2 > 0$) for the functional J with target z^0 (see fig. 5);
- Step 3 **Existence of two global minimizers:** by a bisection argument, we prove the existence of a target $\tilde{z} \in L^\infty(B(0, R))$ such the functional J with target $\tilde{z}$ admits (at least) two *global* minimizers (see fig. 6).

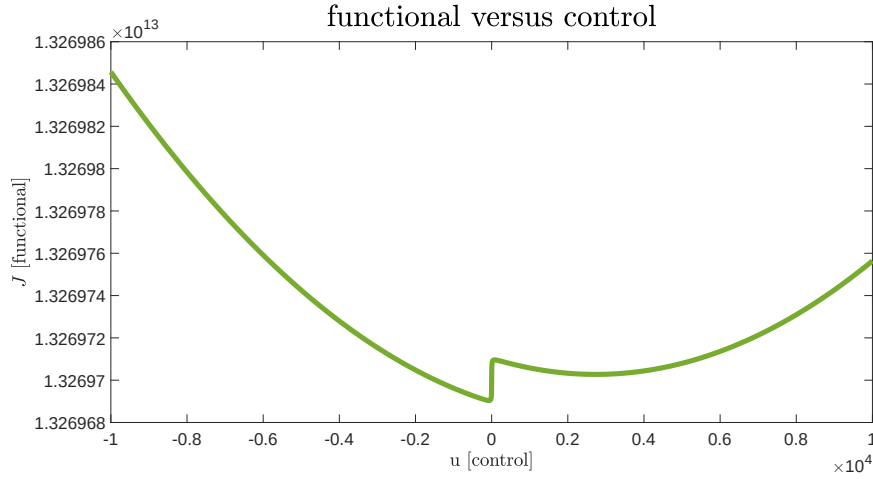


FIGURE 5. functional versus control (nonuniqueness of the *local* minimizer). This plot is obtained by drawing in MATLAB the graph of J defined in (1), with $R = 1$ and nonlinearity $f(y) = y^3$. The target $z = 260000\chi_{(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)} - 10300000\chi_{(\frac{1}{4}, \frac{3}{4})}$.

Notation

First of all, we introduce the control-to-state map

$$(15) \quad G : L^\infty(\partial B(0, R)) \rightarrow L^2(B(0, R))$$

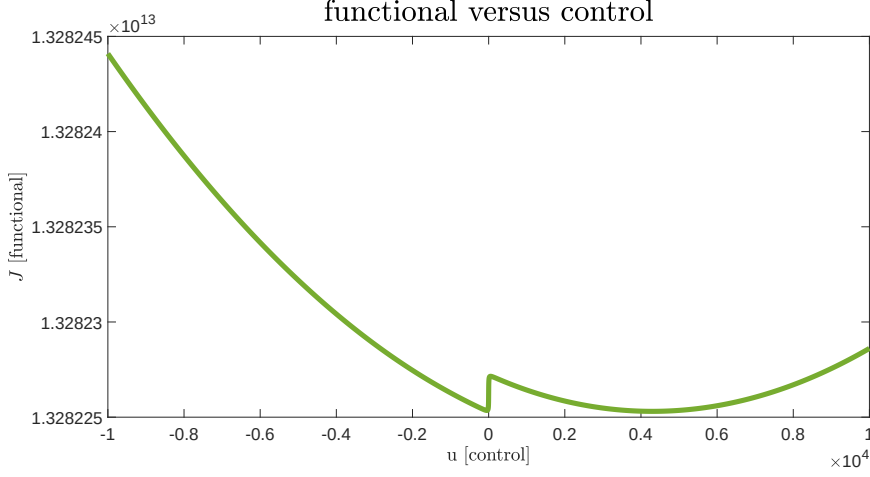


FIGURE 6. functional versus control (nonuniqueness of the *global* minimizer). This plot is obtained by drawing in MATLAB the graph of J defined in (1), with $R = 1$ and nonlinearity $f(y) = y^3$. The target $z = 410000\chi_{(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)} - 10300000\chi_{(\frac{1}{4}, \frac{3}{4})}$.

$$u \mapsto y_u,$$

where y_u is the solution to (2) with control u . Then, set:

$$(16) \quad I : L^\infty(\partial B(0, R)) \times L^2(B(0, R)) \longrightarrow \mathbb{R}$$

$$I(u, z) := \frac{1}{2} \int_{\partial B(0, R)} |u|^2 d\sigma(x) + \frac{\beta}{2} \int_{B(0, R)} |G(u)|^2 dx - \beta \int_{B(0, R)} G(u) z dx,$$

where G is the control-to-state map introduced in (15). One recognizes that, for any target $z \in L^\infty(B(0, R))$, $I(\cdot, z) + \frac{\beta}{2} \|z\|_{L^2(B(0, R))}^2$ coincides with the functional J defined in (1) with target z . Then, for any target $z \in L^\infty(B(0, R))$ minimizing $I(\cdot, z)$ is equivalent to minimizing J with target z . Such translation is convenient, because $I(0, z) = 0$ for any target $z \in L^\infty(B(0, R))$.

We introduce:

$$(17) \quad h_1 : L^\infty(B(0, R)) \longrightarrow \mathbb{R}, \quad h_1(z) := \inf \{I(u, z) \mid u \equiv k, k \in (-\infty, 0]\}$$

and

$$(18) \quad h_2 : L^\infty(B(0, R)) \longrightarrow \mathbb{R}, \quad h_2(z) := \inf \{I(u, z) \mid u \equiv k, k \in [0, +\infty)\}.$$

We formulate the first lemma.

Lemma 3.2. *Let $C \subseteq \{I(u, z) \mid u \equiv k, k \in \mathbb{R}\}$ be a closed subset such that $0 \in C$. Then,*

(1) *for any $z \in L^\infty(B(0, R))$, there exists $u_z \in C$ such that:*

$$I(u_z, z) = \inf_C [I(\cdot, z)].$$

Furthermore, for any minimizer u_z

$$|u_z| \leq \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}} \|z\|_{L^2},$$

where $n\alpha(n)$ is the surface area of $\partial B(0, 1) \subset \mathbb{R}^n$, the unit sphere.
 (2) the map

$$h : L^\infty(B(0, R)) \longrightarrow \mathbb{R}, \quad h(z) := \inf_C [I(\cdot, z)]$$

is continuous.

We prove lemma 3.2 in appendix C.

We now state the second lemma needed to prove theorem 1.1.

Lemma 3.3. *Assume there exists $z^0 \in L^\infty(B(0, R))$ such that*

$$h_1(z^0) < 0 \quad \text{and} \quad h_2(z^0) < 0,$$

where h_1 and h_2 are defined in (17) and (18) resp. Then, there exists a target $\tilde{z} \in L^\infty(B(0, R))$ such that

$$h_1(\tilde{z}) = h_2(\tilde{z}) < 0.$$

The proof of lemma 3.3 can be found in appendix D. The following lemma is the key-point for the proof of the existence of two local minimizers for (1). At this point we employ the nonlinearity of the state equation (2).

Lemma 3.4. *Assume*

$$f''(y) \neq 0 \quad \forall y \neq 0.$$

Fix $r_1 \in (0, R)$. For any $r_2 \in [r_1, R)$, let

$$M := \beta \begin{bmatrix} \int_{B(0, r_1)} G(-1) dx & \int_{B(0, R) \setminus B(0, r_2)} G(-1) dx \\ \int_{B(0, r_1)} G(2) dx & \int_{B(0, R) \setminus B(0, r_2)} G(2) dx \end{bmatrix}.$$

There exists $r_2 \in [r_1, R)$, such that $\text{rank}(M) = 2$.

Note that, due to the nonlinearity of the state equation (2), the states $G(-1)$ and $G(2)$ are not constant.

Proof of lemma 3.4. Let us assume, by contradiction, that for any $r_2 \in \mathbb{R}$, with $0 < r_1 \leq r_2 < R$ there exists $\lambda \in \mathbb{R}$ such that

$$(19) \quad \begin{bmatrix} \int_{B(0, r_1)} G(2) dx \\ \int_{B(0, R) \setminus B(0, r_2)} G(2) dx \end{bmatrix} = \lambda \begin{bmatrix} \int_{B(0, r_1)} G(-1) dx \\ \int_{B(0, R) \setminus B(0, r_2)} G(-1) dx \end{bmatrix}.$$

From (19), we realize that

$$(20) \quad \lambda = \frac{\int_{B(0, r_1)} G(2) dx}{\int_{B(0, r_1)} G(-1) dx}$$

which leads to the independence of λ from r_2 . Note that the denominator does not vanish. Indeed, by applying the strong maximum principle [14, Theorem 8.19 page

198] to the state equation (2), with control $u \equiv -1$, we have $G(-1) < 0$, for any $x \in B(0, R)$, whence

$$(21) \quad \int_{B(0, r_1)} G(-1) dx \neq 0.$$

By (19), for any $r_2 \in [r_1, R)$,

$$\int_{B(0, R) \setminus B(0, r_2)} G(2) dx = \lambda \int_{B(0, R) \setminus B(0, r_2)} G(-1) dx,$$

whence

$$\int_{B(0, R) \setminus B(0, r_2)} [G(2) - \lambda G(-1)] dx, \quad \forall r_2 \in [r_1, R).$$

At this stage, we realize that, since the constant controls -1 and 2 are radial, the corresponding states $G(-1)$ and $G(2)$ are radial as well. Hence, the above equality together with measure theory yields

$$G(2) = \lambda G(-1), \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Then, by definition of the control operator, we have

$$(22) \quad -\Delta(G(-1)) + f(G(-1)) = 0 \quad \text{in } B(0, R) \setminus B(0, r_1)$$

and

$$(23) \quad -\Delta(G(2)) + f(G(2)) = 0 \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Plugging $\lambda G(-1)$ in (23), we obtain

$$(24) \quad -\Delta(\lambda G(-1)) + f(\lambda G(-1)) = 0 \quad \text{in } B(0, R) \setminus B(0, r_1)$$

On the other hand, multiplying (22) by λ , we get

$$(25) \quad -\Delta(\lambda G(-1)) + \lambda f(G(-1)) = 0 \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Subtracting (24) and (25), we obtain

$$(26) \quad f(\lambda G(-1)) = \lambda f(G(-1)) \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Now, by strong maximum principle [14, Theorem 8.19 page 198] applied to (2) with control $u \equiv -1$, we have $G(-1) < 0$ in $B(0, R) \setminus B(0, r_1)$. Hence, using that f is increasing,

$$(27) \quad -\Delta G(-1) = -f(G(-1)) > 0 \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Therefore, $G(-1)$ is not constant in $B(0, R) \setminus B(0, r_1)$. Now, being $G(-1)$ non constant in $B(0, R) \setminus B(0, r_1)$, equation (26) leads to a contradiction with (3). Hence, for some $r_2 \in [r_1, R)$, $\text{rank}(M) = 2$. $\square$

We are now ready to prove theorem 1.1.

Proof of theorem 1.1. Step 1 Reduction to constant controls.

Suppose for some radial target z , the optimal control is not constant. Then, there

exists an orthogonal matrix M , such that $u \circ M \neq u$. Now,

$$\begin{aligned}
 I(u \circ M, z) &= \frac{1}{2} \int_{\partial B(0,R)} |u \circ M|^2 d\sigma(x) + \frac{\beta}{2} \int_{B(0,R)} |G(u \circ M)|^2 dx \\
 &\quad - \beta \int_{B(0,R)} G(u \circ M) z dx \\
 (28) \quad &= \frac{1}{2} \int_{\partial B(0,R)} |u|^2 d\sigma(x) + \frac{\beta}{2} \int_{B(0,R)} |G(u)|^2 dx - \beta \int_{B(0,R)} G(u) z dx \\
 &= I(u, z),
 \end{aligned}$$

where in (28) we have employed the change of variable $\gamma(x) = Mx$. Then, u and $u \circ M$ are two distinguished global minimizers for $I(\cdot, z)$, as desired. It remains to prove the nonuniqueness in case, for any radial targets, all the optimal controls are constants.

Step 2 Existence of a special target $z^0 \in L^\infty(B(0, R))$ such that $I(\cdot, z^0)$ admits (at least) two local minimizers among constant controls.

We start proving the existence of a special target $z^0 \in L^\infty(B(0, R))$ such that $I(-1, z^0) < 0$ and $I(2, z^0) < 0$.

For an arbitrary target $z^0 \in L^\infty(B(0, R))$, we have $I(-1, z^0) < 0$ and $I(2, z^0) < 0$ if and only if the following system of inequalities is fulfilled:

$$(29) \quad \begin{cases} \beta \int_{B(0,R)} G(-1) z^0 dx > \frac{R^{n-1} n \alpha(n)}{2} |-1|^2 + \frac{\beta}{2} \int_{B(0,R)} |G(-1)|^2 dx \\ \beta \int_{B(0,R)} G(2) z^0 dx > \frac{R^{n-1} n \alpha(n)}{2} |2|^2 + \frac{\beta}{2} \int_{B(0,R)} |G(2)|^2 dx, \end{cases}$$

where G is the control-to-state map introduced in (15) and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. In the sequel, we work with *changing-sign* targets

$$z^0 := \begin{cases} z_1^0 & \text{in } B(0, r_1) \\ 0 & \text{in } B(0, r_2) \setminus B(0, r_1) \\ z_2^0 & \text{in } B(0, R) \setminus B(0, r_2). \end{cases}$$

where $(z_1^0, z_2^0) \in \mathbb{R}^2$ and $0 < r_1 \leq r_2 < R$. The radius $r_1 > 0$ is fixed, while r_2 and (z_1^0, z_2^0) are degrees of freedom we need in the remainder of the proof. With the above choice of the target, inequalities (29) are satisfied if the target (z_1^0, z_2^0) satisfies the linear system below

$$(30) \quad \begin{cases} z_1^0 \beta \int_{B(0,r_1)} G(-1) dx + z_2^0 \beta \int_{B(0,R) \setminus B(0,r_2)} G(-1) dx = c_1 \\ z_1^0 \beta \int_{B(0,r_1)} G(2) dx + z_2^0 \beta \int_{B(0,R) \setminus B(0,r_2)} G(2) dx = c_2, \end{cases}$$

with constant terms

$$c_1 := \frac{R^{n-1} n \alpha(n)}{2} |-1|^2 + \frac{\beta}{2} \int_{B(0,R)} |G(-1)|^2 dx + 1$$

and

$$c_2 := \frac{R^{n-1} n \alpha(n)}{2} |2|^2 + \frac{\beta}{2} \int_{B(0,R)} |G(2)|^2 dx + 1.$$

The 2×2 coefficients matrix for the above linear system reads as:

$$M = \beta \begin{bmatrix} \int_{B(0,r_1)} G(-1)dx & \int_{B(0,R) \setminus B(0,r_2)} G(-1)dx \\ \int_{B(0,r_1)} G(2)dx & \int_{B(0,R) \setminus B(0,r_2)} G(2)dx \end{bmatrix}$$

By lemma 3.4, there exists $r_2 \in [r_1, R)$, such that $\text{rank}(M) = 2$. Therefore, by Rouché-Capelli Theorem, there exists a solution to the linear system (30). Such solution (z_1^0, z_2^0) defines a special target

$$z^0 := \begin{cases} z_1^0 & \text{in } B(0, r_1) \\ 0 & \text{in } B(0, r_2) \setminus B(0, r_1) \\ z_2^0 & \text{in } B(0, R) \setminus B(0, r_2). \end{cases}$$

such that $I(-1, z^0) < 0$ and $I(2, z^0) < 0$.

We show now that $I(\cdot, z^0)$ admits (at least) two local minimizers. Indeed, by lemma 3.2 (1.), there exist:

$$u_1 \leq 0 \quad \text{such that} \quad I(u_1, z^0) = \inf \{I(u, z) \mid u \equiv k, \ k \leq 0\}$$

and

$$u_2 \geq 0 \quad \text{such that} \quad I(u_2, z^0) = \inf \{I(u, z) \mid u \equiv k, \ k \geq 0\}.$$

Now,

$$I(u_1, z^0) = \{I(u, z) \mid u \equiv k, \ k \leq 0\} \leq I(-1, z^0) < 0 = I(0, z^0)$$

and

$$I(u_2, z^0) = \{I(u, z) \mid u \equiv k, \ k \geq 0\} \leq I(2, z^0) < 0 = I(0, z^0).$$

Then, the control u_1 minimizes $I(\cdot, z^0)$ in the half line $(-\infty, 0)$, while u_2 minimizes $I(\cdot, z^0)$ in the half line $(0, +\infty)$. We have found u_1 and u_2 two distinct local minimizers for $I(\cdot, z^0)$ in $\mathbb{R}$.

Step 3 Conclusion

We remind the definition of h_1 and h_2 given by (17) and (18) resp. In Step 2, we have determined $z^0 \in L^\infty(B(0, R))$ such that $h_1(z^0) < 0$ and $h_2(z^0) < 0$. To finish our proof it suffices to find $\tilde{z} \in \mathbb{R}^n$ such that $h_1(\tilde{z}) = h_2(\tilde{z}) < 0$. This follows from lemma 3.3. $\square$

Remark 1. The thesis of theorem 1.1 holds with nonsmooth nonlinearities, like

$$(31) \quad f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(y) := y|y|.$$

The proofs of theorem 1.1 and related lemmas remains unchanged, except the proof of lemma 3.4. Indeed in this case, since f is not C^2 , we cannot use condition (3) to conclude from (26). However, for any $y \in \mathbb{R}$ and for any $\lambda \in \mathbb{R}$,

$$f(\lambda y) = \lambda y |\lambda y| = \lambda |\lambda| y |y| = |\lambda| [\lambda f(y)].$$

Then, $f(\lambda y) = \lambda f(y)$ if and only if $|\lambda| [\lambda f(y)] = \lambda f(y)$. Now,

$$\{(y, \lambda) \in \mathbb{R}^2 \mid |\lambda| [\lambda f(y)] = \lambda f(y)\} = \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\} \cup \{\pm 1\} \times \mathbb{R}.$$

Therefore,

$$(32) \quad \{(y, \lambda) \in \mathbb{R}^2 \mid f(\lambda y) = \lambda f(y)\} = \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\} \cup \{\pm 1\} \times \mathbb{R}.$$

In (26), because of (20), $\lambda \notin \{0, \pm 1\}$ and $G(-1) \neq 0$. Hence, (32) together with (26) leads to a contradiction. The rest of the proof of theorem 1.1 remains unchanged.

3.2. Internal control. We introduce the concept of radial control.

Definition 3.5. A control $u : B(0, r) \rightarrow \mathbb{R}$ is said to be radial if there exists $\psi : [0, r] \rightarrow \mathbb{R}$, such that, for any $x \in B(0, r)$, we have $u(x) = \psi(\|x\|)$.

We present our strategy to prove theorem 1.2:

- Step 1 **Reduction to radial controls:** by choosing radial targets and using the rotational invariance of $B(0, R)$, we reduce to the case the control set is made of radial controls;
- Step 2 **Existence of two local minimizers:** we look for a target $z^0 \in L^\infty(B(0, R) \setminus B(0, r))$ such that there exists two *local* minimizers for the functional J with target z^0 ;
- Step 3 **Existence of two global minimizers:** by a bisection argument, we prove the existence of a target $\tilde{z} \in L^\infty(B(0, R) \setminus B(0, r))$ such the functional J with target $\tilde{z}$ admits (at least) two *global* minimizers.

Notation

First of all, we define the control-to-state map

$$(33) \quad G : L^2(B(0, r)) \rightarrow L^2(B(0, R))$$

$$u \mapsto y_u,$$

where y_u is the solution to (7) with control u . Then, set:

$$(34) \quad I : L^2(B(0, r)) \times L^\infty(B(0, R) \setminus B(0, r)) \rightarrow \mathbb{R}$$

$$I(u, z) := \frac{1}{2} \int_{B(0, r)} |u|^2 dx + \frac{\beta}{2} \int_{B(0, R) \setminus B(0, r)} |G(u)|^2 dx - \beta \int_{B(0, R) \setminus B(0, r)} G(u) z dx,$$

where G is the control-to-state map introduced in (33). One recognizes that, for any target $z \in L^\infty(B(0, R) \setminus B(0, r))$, $I(\cdot, z) + \frac{\beta}{2} \|z\|_{L^2(B(0, R) \setminus B(0, r))}^2$ coincides with the functional J defined in (6) with target z . Then, for any target $z \in L^\infty(B(0, R) \setminus B(0, r))$ minimizing $I(\cdot, z)$ is equivalent to minimizing J with target z . Such translation is convenient, because $I(0, z) = 0$ for any target $z \in L^\infty(B(0, R) \setminus B(0, r))$.

We define

$$(35) \quad \mathcal{U}_r := \{u \in L^2(B(0, r)) \mid u \text{ is radial}\}.$$

We have

$$(36) \quad \mathcal{U}_r = \mathcal{U}_r^- \cup \mathcal{U}_r^+,$$

with

$$(37) \quad \begin{aligned} \mathcal{U}_r^- &:= \{u \in \mathcal{U}_r \mid G(u)|_{\partial B(0, r)} \leq 0\} \\ \mathcal{U}_r^+ &:= \{u \in \mathcal{U}_r \mid G(u)|_{\partial B(0, r)} \geq 0\}. \end{aligned}$$

We introduce:

$$(38) \quad h_1 : L^\infty(B(0, R) \setminus B(0, r)) \rightarrow \mathbb{R}, \quad h_1(z) := \inf \{I(u, z) \mid u \in \mathcal{U}_r^-\}$$

and

$$(39) \quad h_2 : L^\infty(B(0, R) \setminus B(0, r)) \rightarrow \mathbb{R}, \quad h_2(z) := \inf \{I(u, z) \mid u \in \mathcal{U}_r^+\}.$$

We formulate the first Lemma.

Lemma 3.6. *Let $C = \mathcal{U}_r^-$ or $C = \mathcal{U}_r^+$. Then,*

(1) for any $z \in L^\infty(B(0, R) \setminus B(0, r))$, there exists $u_z \in C$ such that:

$$I(u_z, z) = \inf_C [I(\cdot, z)].$$

Furthermore, for any minimizer u_z

$$\|u_z\|_{L^2(B(0, r))} \leq \sqrt{\beta} \|z\|_{L^2}.$$

(2) the map

$$\begin{aligned} h : L^\infty(B(0, R) \setminus B(0, r)) &\longrightarrow \mathbb{R} \\ z &\longmapsto \inf_C [I(\cdot, z)] \end{aligned}$$

is continuous.

We prove lemma 3.6 in appendix E.

We now state the second lemma needed to prove theorem 1.2.

Lemma 3.7. Assume there exists $z^0 \in L^\infty(B(0, R) \setminus B(0, r))$ such that

$$h_1(z^0) < 0 \quad \text{and} \quad h_2(z^0) < 0,$$

where h_1 and h_2 are defined in (38) and (39) resp. Then, there exists $\tilde{z} \in L^\infty(B(0, R) \setminus B(0, r))$ such that

$$h_1(\tilde{z}) = h_2(\tilde{z}) < 0.$$

The proof of this lemma can be found in appendix F. The next lemma is the foundation of the proof of the existence of two local minimizers for (6). The nonlinearity of the state equation (7) will play a key role in the proof.

Lemma 3.8. Suppose

$$f''(y) \neq 0 \quad \forall y \neq 0.$$

Arbitrarily fix r_1 , such that $r < r_1 < R$. For any $r_2 \in [r_1, R)$, let

$$M = \beta \begin{bmatrix} \int_{B(0, r_1) \setminus B(0, r)} G(-1) dx & \int_{B(0, R) \setminus B(0, r_2)} G(-1) dx \\ \int_{B(0, r_1) \setminus B(0, r)} G(2) dx & \int_{B(0, R) \setminus B(0, r_2)} G(2) dx \end{bmatrix}.$$

There exists $r_2 \in [r_1, R)$, such that $\text{rank}(M) = 2$.

The state equation (7) is nonlinear. Then, the states $G(-1)$ and $G(2)$ are not constant.

Proof of lemma 3.8. Let us assume, by contradiction, that for any $r_2 \in \mathbb{R}$, with $0 < r_1 \leq r_2 < R$ there exists $\lambda \in \mathbb{R}$ such that

$$(40) \quad \begin{bmatrix} \int_{B(0, r_1) \setminus B(0, r)} G(2) dx \\ \int_{B(0, R) \setminus B(0, r_2)} G(2) dx \end{bmatrix} = \lambda \begin{bmatrix} \int_{B(0, r_1) \setminus B(0, r)} G(-1) dx \\ \int_{B(0, R) \setminus B(0, r_2)} G(-1) dx \end{bmatrix}.$$

From (40), we realize that

$$(41) \quad \lambda = \frac{\int_{B(0,r_1) \setminus B(0,r)} G(2) dx}{\int_{B(0,r_1) \setminus B(0,r)} G(-1) dx},$$

which yields the independence of λ from r_2 . In the above expression, the denominator does not vanish. Indeed, by strong maximum principle [14, Theorem 8.19 page 198] applied to the state equation (7), with control $u \equiv -1$, we have $G(-1) < 0$ for any $x \in B(0, R)$, whence

$$(42) \quad \int_{B(0,r_1) \setminus B(0,r)} G(-1) dx \neq 0.$$

By (40), for any $r_2 \in [r_1, R)$,

$$\int_{B(0,R) \setminus B(0,r_2)} G(2) dx = \lambda \int_{B(0,R) \setminus B(0,r_2)} G(-1) dx,$$

whence

$$\int_{B(0,R) \setminus B(0,r_2)} [G(2) - \lambda G(-1)] dx = 0, \quad \forall r_2 \in [r_1, R).$$

Now, the constant controls -1 and 2 are radial, whence the states $G(-1)$ and $G(2)$ are radial as well. Then, the above inequality together with measure theory yields

$$G(2) = \lambda G(-1), \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Then, by definition of the control operator, we have

$$(43) \quad -\Delta(G(-1)) + f(G(-1)) = 0 \quad \text{in } B(0, R) \setminus B(0, r_1)$$

and

$$(44) \quad -\Delta(G(2)) + f(G(2)) = 0 \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Plugging $\lambda G(-1)$ in (44), we obtain

$$(45) \quad -\Delta(\lambda G(-1)) + f(\lambda G(-1)) = 0 \quad \text{in } B(0, R) \setminus B(0, r_1)$$

On the other hand, multiplying (43) by λ , we get

$$(46) \quad -\Delta(\lambda G(-1)) + \lambda f(G(-1)) = 0 \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Subtracting (45) and (46), we obtain

$$(47) \quad f(\lambda G(-1)) = \lambda f(G(-1)) \quad \text{in } B(0, R) \setminus B(0, r_1).$$

By strong maximum principle [14, Theorem 8.19 page 198] applied to the state equation (7) with control $u \equiv -1$, we have $G(-1) < 0$ in $B(0, R)$. Hence, using that f is increasing,

$$(48) \quad -\Delta G(-1) = -f(G(-1)) > 0 \quad \text{in } B(0, R) \setminus B(0, r_1).$$

Therefore, $G(-1)$ is not constant in $B(0, R) \setminus B(0, r_1)$. Now, being $G(-1)$ non constant in $B(0, R) \setminus B(0, r_1)$, (47) leads to a contradiction with (8). $\square$

We are now ready to prove theorem 1.2.

Proof of theorem 1.2. Step 1 Reduction to radial controls.

Suppose for some radial target z , the optimal control u is not radial, that is there exists an orthogonal matrix M , such that $u \circ M \neq u$. Now,

$$\begin{aligned}
 I(u \circ M, z) &= \frac{1}{2} \int_{B(0,r)} |u \circ M|^2 dx + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |G(u \circ M)|^2 dx \\
 &\quad - \beta \int_{B(0,R) \setminus B(0,r)} G(u \circ M) z dx \\
 (49) \quad &= \frac{1}{2} \int_{B(0,r)} |u|^2 dx + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |G(u)|^2 dx \\
 &\quad - \beta \int_{B(0,R) \setminus B(0,r)} G(u) z dx \\
 &= I(u, z),
 \end{aligned}$$

where in the last equality (49) we have employed the change of variable $\gamma(x) = Mx$. Then, u and $u \circ M$ are two distinguished global minimizers for $I(\cdot, z)$, as desired. It remains to prove the nonuniqueness in case, for any radial target, all the optimal controls are radial. Hereafter, for a radial target z , we will consider the restriction of the functional $I(\cdot, z)$ to $\mathcal{U}_r$.

Step 2 Existence of a special target $z^0 \in L^\infty(B(0, R) \setminus B(0, r))$ such that $I(\cdot, z^0)$ admits (at least) two local minimizers, among radial controls.

We start proving the existence of a special target $z^0 \in L^\infty(B(0, R) \setminus B(0, r))$ such that $I(-1, z^0) < 0$ and $I(2, z^0) < 0$.

For an arbitrary target $z^0 \in L^\infty(B(0, R) \setminus B(0, r))$, we have $I(-1, z^0) < 0$ and $I(2, z^0) < 0$ if and only if the following system of inequalities is fulfilled:

$$(50) \quad \begin{cases} \beta \int_{B(0,R) \setminus B(0,r)} G(-1) z^0 dx > \frac{r^n \alpha(n)}{2} |-1|^2 + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |G(-1)|^2 dx \\ \beta \int_{B(0,R) \setminus B(0,r)} G(2) z^0 dx > \frac{r^n \alpha(n)}{2} |2|^2 + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |G(2)|^2 dx, \end{cases}$$

where G is the control-to-state map introduced in (33) and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. In the sequel, we work with *changing-sign* targets

$$z^0 := \begin{cases} z_1^0 & \text{in } B(0, r_1) \setminus B(0, r) \\ 0 & \text{in } B(0, r_2) \setminus B(0, r_1) \\ z_2^0 & \text{in } B(0, R) \setminus B(0, r_2). \end{cases}$$

where $(z_1^0, z_2^0) \in \mathbb{R}^2$ and $0 < r < r_1 \leq r_2 < R$. The pair of radii (r_1, r) is fixed, while r_2 and (z_1^0, z_2^0) are degrees of freedom we need in the remainder of the proof. With the above choice of the target, inequalities (50) are satisfied if the target (z_1^0, z_2^0) satisfies the linear system below

$$(51) \quad \begin{cases} z_1^0 \beta \int_{B(0,r_1) \setminus B(0,r)} G(-1) dx + z_2^0 \beta \int_{B(0,R) \setminus B(0,r_2)} G(-1) dx = c_1 \\ z_1^0 \beta \int_{B(0,r_1) \setminus B(0,r)} G(2) dx + z_2^0 \beta \int_{B(0,R) \setminus B(0,r_2)} G(2) dx = c_2, \end{cases}$$

with constant terms

$$c_1 := \frac{r^n \alpha(n)}{2} |-1|^2 + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |G(-1)|^2 dx + 1$$

and

$$c_2 := \frac{r^n \alpha(n)}{2} |2|^2 + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |G(2)|^2 dx + 1.$$

The 2×2 coefficients matrix for the above linear system reads as:

$$M = \beta \begin{bmatrix} \int_{B(0,r_1) \setminus B(0,r)} G(-1) dx & \int_{B(0,R) \setminus B(0,r_2)} G(-1) dx \\ \int_{B(0,r_1) \setminus B(0,r)} G(2) dx & \int_{B(0,R) \setminus B(0,r_2)} G(2) dx \end{bmatrix}$$

By lemma 3.8 there exists $r_2 \in [r_1, R)$, such that $\text{rank}(M) = 2$. Therefore, by Rouché-Capelli Theorem, there exists a solution to the linear system (51). Such solution (z_1^0, z_2^0) defines a special target

$$z^0 := \begin{cases} z_1^0 & \text{in } B(0, r_1) \setminus B(0, r) \\ 0 & \text{in } B(0, r_2) \setminus B(0, r_1) \\ z_2^0 & \text{in } B(0, R) \setminus B(0, r_2). \end{cases}$$

such that $I(-1, z^0) < 0$ and $I(2, z^0) < 0$.

We show now that $I(\cdot, z^0)$ admits (at least) two local minimizers in $\mathcal{U}_r$. Indeed, the set $\mathcal{U}_r$ (introduced in (35)) splits

$$\mathcal{U}_r = \mathcal{U}_r^- \cup \mathcal{U}_r^+,$$

with

$$(52) \quad \begin{aligned} \mathcal{U}_r^- &= \{u \in \mathcal{U}_r \mid G(u)|_{\partial B(0,r)} \leq 0\} \\ \mathcal{U}_r^+ &= \{u \in \mathcal{U}_r \mid G(u)|_{\partial B(0,r)} \geq 0\}. \end{aligned}$$

By lemma 3.6 (1.), there exist:

$$u_1 \in \mathcal{U}_r^- \quad \text{such that} \quad I(u_1, z^0) = \inf_{\mathcal{U}_r^-} [I(\cdot, z^0)]$$

and

$$u_2 \in \mathcal{U}_r^+ \quad \text{such that} \quad I(u_2, z^0) = \inf_{\mathcal{U}_r^+} [I(\cdot, z^0)].$$

Now, for any control $u \in \{u \in \mathcal{U}_r \mid G(u)|_{\partial B(0,r)} = 0\}$, we have

$$I(u_1, z^0) = \inf_{\mathcal{U}_r^-} [I(\cdot, z^0)] \leq I(-1, z^0) < 0 \leq I(u, z^0)$$

and

$$I(u_2, z^0) = \inf_{\mathcal{U}_r^+} [I(\cdot, z^0)] \leq I(2, z^0) < 0 \leq I(u, z^0).$$

Then, necessarily u_1 is a local minimizer for $I(\cdot, z^0)$ in the open set $\{u \in \mathcal{U}_r \mid G(u)|_{\partial B(0,r)} < 0\}$ and u_2 is a local minimizer for $I(\cdot, z^0)$ in the open set $\{u \in \mathcal{U}_r \mid G(u)|_{\partial B(0,r)} > 0\}$. Hence, we have found u_1 and u_2 two distinct local minimizers for $I(\cdot, z^0)$ in $\mathcal{U}_r$.

Step 3 Conclusion

We remind the definition of h_1 and h_2 given by (38) and (39) resp. In Step 2, we

have determined $z^0 \in L^\infty(B(0, R) \setminus B(0, r))$ such that $h_1(z^0) < 0$ and $h_2(z^0) < 0$. To finish our proof it suffices to find $\tilde{z} \in \mathbb{R}^n$ such that $h_1(\tilde{z}) = h_2(\tilde{z}) < 0$. This follows from lemma 3.7. $\square$

The conclusions of Remark 1 holds as well in internal control.

4. NUMERICAL SIMULATIONS

We have performed a numerical simulation in the context of boundary control

- space dimension $n = 1$ and radius $R = 1$;
- nonlinearity $f(y) = y^3$;
- weighting parameter $\beta = 1$;
- step target

$$z(x) := \begin{cases} 410000 & \text{for } 0 < x < \frac{1}{4} \text{ and } \frac{3}{4} < x < 1 \\ -10300000 & \text{for } \frac{1}{4} < x < \frac{3}{4}. \end{cases}$$

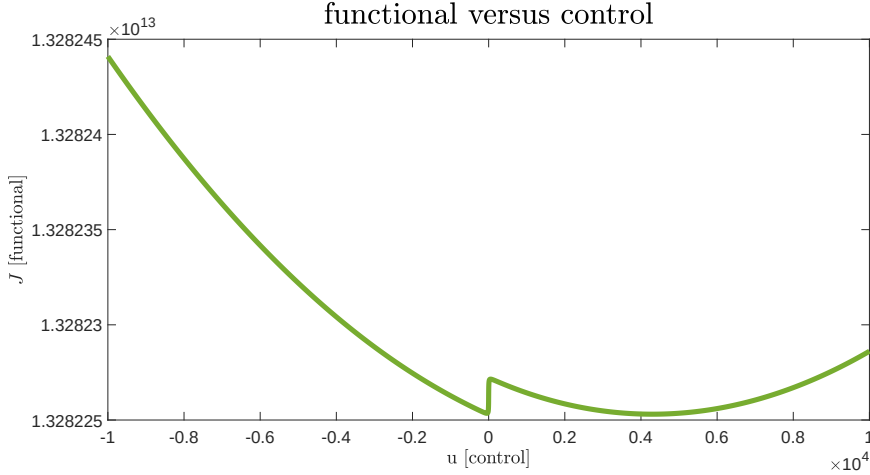


FIGURE 7. functional versus control (nonuniqueness of the *global* minimizer). This plot is obtained by drawing in MATLAB the graph of J defined in (1), with $R = 1$ and nonlinearity $f(y) = y^3$. The target $z = 410000\chi_{(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)} - 10300000\chi_{(\frac{1}{4}, \frac{3}{4})}$.

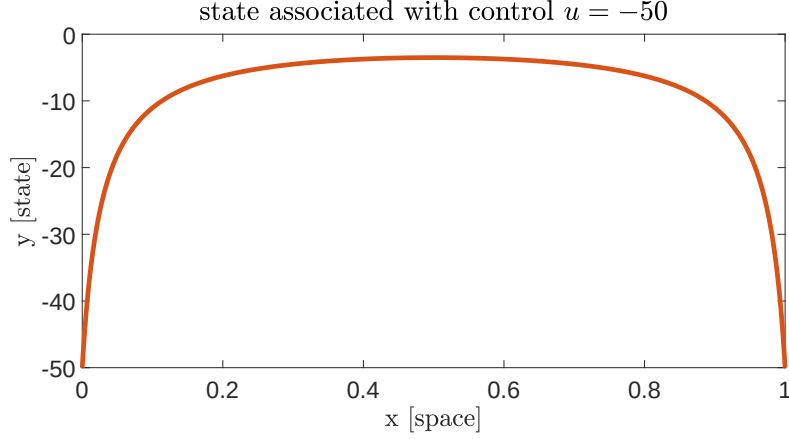
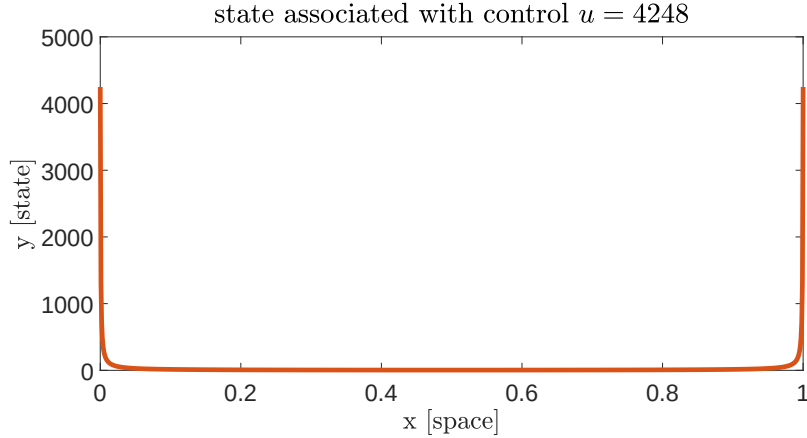
As we have seen in the proof of theorem 1.1, we can reduce to the case of constant controls on the boundary. In our case, the space dimension is $n = 1$. Then, we have reduced to the case the same control acts on both endpoints $x = 0$ and $x = 1$. Hence, we plot in fig. 7 the restriction $J|_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$, the functional J being defined in (1).

There exist two distinguished global minimizers:

- a negative one $u_1 \cong -50$;
- a positive one $u_2 \cong 4298$.

The corresponding optimal states are depicted in figures fig. 8 and fig. 9.

The idea behind this example is that two optimal strategies are available:

FIGURE 8. state associated with control $u = -50$.FIGURE 9. state associated with control $u = 4298$.

- take a large positive control u_2 to better approximate the target in $(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$;
- take a negative control u_1 to keep the state closer to the target in $(\frac{1}{4}, \frac{3}{4})$.

Note that $|u_1| < |u_2|$. Indeed, the control acts at the endpoints $x = 0$ and $x = 1$ of the space domain. Then, the effect of the control is stronger in $(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$ than in $(\frac{1}{4}, \frac{3}{4})$. For this reason, it is worth to take a large positive control to better approximate the target in $(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$. On the other hand, it is less convenient to take a very negative control to approximate the target in $(\frac{1}{4}, \frac{3}{4})$ (see the local estimates for semilinear equations [18] and [13, proof of Theorem 1.3]).

In fig. 7 we observe that the functional has a different behaviour close to zero and away from zero. This can be explained by studying the behaviour of the control-to-state map (15):

- close to zero (15) is closed to its linearization around zero;

- far from zero (15) is strongly influenced by the nonlinearity $f(y) = y^3$, thus producing a drastic change in the shape of the functional.

Numerical simulations have been performed in MATLAB. We explain now the numerical methods employed.

Firstly choose an interval of controls $[-M, M]$, where to study the functional J . Then, our goal is to plot $J|_{[-M, M]}: [-M, M] \rightarrow \mathbb{R}$.

For the interval $[-M, M]$, we choose an equi-spaced grid $v_i = -M + (i-1)\frac{2M}{N_c-1}$, with $i = 1, \dots, N_c$ and $N_c \in \mathbb{N} \setminus \{0\}$.

Now, for each control v_i , we need to find numerically the corresponding state y_i , solution to the following PDE with cubic nonlinearity

$$(53) \quad \begin{cases} -(y_i)_{xx} + (y_i)^3 = 0 & x \in (0, 1) \\ y_i(0) = y_i(1) = v_i. \end{cases}$$

Following [6, subsubsection 4.3.2], we solve (53) by a fixed-point type algorithm with relaxation. Namely, in any iteration k , we determine the solution $y_{i,k}$ to the linear PDE

$$(54) \quad \begin{cases} -(y_{i,k})_{xx} + (\theta_{i,k-1})^2 y_{i,k} = 0 & x \in (0, 1) \\ y_{i,k}(0) = y_{i,k}(1) = v_i \end{cases}$$

and we set $\theta_k := \frac{1}{2}\theta_{i,k-1} + \frac{1}{2}y_k$. The initial guess $\theta_{i,0}$ is taken to be y_{i-1} , i.e. the solution to (53), with control v_{i-1} .

To compute the solution to the linear PDE (54), we choose a finite difference scheme with uniform space grid $x_j = \frac{j-1}{\Delta x}$, where $j = 1, \dots, N_x$, $N_x \in \mathbb{N} \setminus \{0\}$ and $\Delta x := \frac{1}{N_x-1}$. Then, $y_{i,k} = (y_{i,k,j})_j$ is a N_x -dimensional discrete vector solution to

$$\begin{cases} \frac{-y_{i,k,j-1} + 2y_{i,k,j} - y_{i,k,j+1}}{(\Delta x)^2} + (\theta_{i,k-1,j})^2 y_{i,k,j} = 0 & j = 2, \dots, N_x - 1 \\ y_{i,k,1} = y_{i,k,N_x} = v_i. \end{cases}$$

Once we have determined the state y_i , we evaluate the functional J at the control v_i . The integral appearing in (1) can be computed by quadrature methods. We are now in position to plot the functional $J|_{[-M, M]}: [-M, M] \rightarrow \mathbb{R}$.

Note that, as long as we know, the actual convergence of the fixed-point method described has not been proved. However, for any control v_i , we are able to check that the state computed solves the finite difference version of the nonlinear problem (53) up to a small error.

An extensive literature is available on the numerical approximation of solutions to (53) (see, for instance, [15] for a survey). Let us mention two alternative numerical methods.

The first one is a finite difference-Newton method presented in [19, subsection 2.16.1]. The idea is to discretize directly (53). This leads to a nonlinear equation in finite dimension, solved by a Newton method.

Another option is to find the solution to (53), as minimizer of the convex functional

$$K(y) = \frac{1}{2} \int_0^1 |y_x|^2 dx + \frac{1}{4} \int_0^1 y^4 dx$$

over the affine space

$$\mathcal{A} := \{y \in H^1(0, 1) \mid y(0) = y(1) = v\}.$$

5. CONCLUSIONS AND OPEN PROBLEMS

We have illustrated a general methodology to show lack of convexity for quadratic functionals with nonlinear state equations (theorem 1.4). Furthermore, we have developed a counterexample to uniqueness of the global minimizer in optimal control of semilinear elliptic equations (theorem 1.1 and theorem 1.2).

We list some interesting problems, which, to the best of our knowledge, have not been addressed in the literature so far.

5.1. General space domain. Our counterexample to uniqueness of the minimizer in semilinear control relies on the rotational invariance of the space domain $B(0, R)$ to reduce to constant/radial controls. It would be interesting to enhance the developed techniques to more general space domains.

5.2. Relations with the turnpike property. Consider the time-evolution control problem associated to (7)-(6)

$$(55) \quad \min_{u \in \mathcal{U}_T} J_T(u) = \frac{1}{2} \int_0^T \int_{B(0,r)} |u|^2 dx dt + \frac{\beta}{2} \int_0^T \int_{B(0,R) \setminus B(0,r)} |y - z|^2 dx dt,$$

where $\mathcal{U}_T := L^2((0, T) \times B(0, r))$ and the state y associated to control u is solution to the semilinear heat equation

$$(56) \quad \begin{cases} y_t - \Delta y + f(y) = u \chi_{B(0,r)} & \text{in } (0, T) \times B(0, R) \\ y = 0 & \text{on } (0, T) \times \partial B(0, R) \\ y(0, x) = y_0(x) & \text{in } B(0, R). \end{cases}$$

The nonlinearity f is C^3 and nondecreasing, with $f(0) = 0$. The assumptions on the state equation are the same of [24, section 3]. An optimal control for the above problem is denoted by u^T , while the corresponding optimal state by y^T .

We rewrite (7)-(6) with an “s” subscript to stress the steady-state character of the problem

$$(57) \quad \min_{u_s \in L^2(B(0,r))} J_s(u_s) = \frac{1}{2} \int_{B(0,r)} |u_s|^2 dx + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |y_s - z|^2 dx,$$

where:

$$(58) \quad \begin{cases} -\Delta y_s + f(y_s) = u_s \chi_{B(0,r)} & \text{in } B(0, R) \\ y_s = 0 & \text{on } \partial B(0, R). \end{cases}$$

We denote by $(\bar{u}, \bar{y})$ an optimal pair, where $\bar{u}$ is an optimal control and $\bar{y}$ the corresponding optimal state.

Consider a target z , such that J_s has two distinguished global minimizers, as in theorem 1.2. Choose any initial datum $y_0 \in L^\infty(B(0, R))$ for the evolution equation (56). Let u^T be a minimizer for (55). Then, a question arises: if the turnpike property is satisfied, which minimizer for (58)-(57) attracts the optimal solutions to (56)-(55)? Namely, for which optimal pair $(\bar{u}, \bar{y})$ for (58)-(57) we have the estimate

$$\|u^T(t) - \bar{u}\|_{L^\infty(B(0,r))} + \|y^T(t) - \bar{y}\|_{L^\infty(B(0,R))} \leq K \left[e^{-\mu t} + e^{-\mu(T-t)} \right], \quad \forall t \in [0, T],$$

where the constants K and $\mu > 0$ are independent of the time horizon T .

According to [24, Theorem 1, section 3], this depends on the sign of the second differential of the functional J_s computed at the minima, which in turns is linked to the sign of the term $\beta \chi_{B(0,R) \setminus B(0,r)} - f''(\bar{y}) \bar{q}$.

APPENDIX A. WELL-POSEDNESS OF (2)

In this section, we justify the well-posedness of the state equation (2). We accomplish this task in a general space domain Ω .

Let Ω be a bounded open subset of $\mathbb{R}^n$, with boundary $\partial\Omega \in C^2$. The nonlinearity $f \in C^1(\mathbb{R})$ is increasing and $f(0) = 0$. We introduce the class of test functions

$$\mathcal{C} := \{\varphi \in C^2(\overline{\Omega}) \mid \varphi(x) = 0, \forall x \in \partial\Omega\}$$

and the notion of solution.

Definition A.1. Let $u \in L^\infty(\partial\Omega)$. Then, $y \in L^\infty(\Omega)$ is said to be a solution to the boundary value problem

$$(59) \quad \begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega \\ y = u & \text{on } \partial\Omega. \end{cases}$$

if for any test function $\varphi \in \mathcal{C}$, we have

$$\int_{\Omega} [-y\Delta\varphi + f(y)\varphi] dx + \int_{\partial\Omega} u \frac{\partial\varphi}{\partial n} d\sigma(x) = 0,$$

where n is the outward normal to $\partial\Omega$.

We have the following existence and uniqueness result, inspired by the proof of [12, Proposition 5.1].

Proposition 1. *Let $u \in L^\infty(\partial\Omega)$. There exists a unique solution $y \in L^\infty(\Omega) \cap H^{\frac{1}{2}}(\Omega)$ to (59), with estimate*

$$(60) \quad \|y\|_{L^{2^*}(\Omega)} \leq K \|u\|_{L^2(\partial\Omega)},$$

the constant $K = K(\Omega)$ being independent of the nonlinearity f and $2^* = \frac{2n}{n-1}$.

One of the key points of the proof will be the increasing character of the nonlinearity.

Proof of proposition 1. Step 1 **Solve a non-homogeneous linear problem**

By [21, Theorem 7.4, page 202], there exists a unique solution $y_1 \in H^{\frac{1}{2}}(\Omega)$ to the non-homogeneous boundary value problem

$$(61) \quad \begin{cases} -\Delta y_1 = 0 & \text{in } \Omega \\ y_1 = u & \text{on } \partial\Omega. \end{cases}$$

The boundary value $u \in L^\infty(\partial\Omega)$. Hence, by a comparison argument, we have $y_1 \in L^\infty(\Omega)$.

Step 2 **Solve an homogeneous semilinear problem**

Since the nonlinearity f is increasing, by adapting the techniques of [5, Theorem 4.7, page 29], there exists a unique $y_2 \in H_0^1(\Omega)$ solution to

$$(62) \quad \begin{cases} -\Delta y_2 + f(y_1 + y_2) = 0 & \text{in } \Omega \\ y_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

By a comparison argument, since $y_1 \in L^\infty(\Omega)$, we have $y_2 \in L^\infty(\Omega)$. Then, $y = y_1 + y_2 \in L^\infty(\Omega) \cap H^{\frac{1}{2}}(\Omega)$ is the unique solution to (59).

Step 3 **Proof of the estimate (60)**

By a comparison argument, we have

$$(63) \quad |y| \leq \hat{y}, \quad \text{a.e. } \Omega,$$

with

$$(64) \quad \begin{cases} -\Delta \hat{y} = 0 & \text{in } \Omega \\ \hat{y} = |u| & \text{on } \partial\Omega. \end{cases}$$

Now, by [21, Theorem 7.4, page 202], the solution $\hat{y} \in H^{\frac{1}{2}}(\Omega)$, with estimate

$$(65) \quad \|\hat{y}\|_{H^{\frac{1}{2}}(\Omega)} \leq K \|u\|_{L^2(\partial\Omega)}.$$

The above inequality, together with the fractional Sobolev embedding $H^{\frac{1}{2}}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ (see e.g. [23, Theorem 6.7]), yields

$$\|\hat{y}\|_{L^{2^*}(\Omega)} \leq \|\hat{y}\|_{H^{\frac{1}{2}}(\Omega)} \leq K \|u\|_{L^2(\partial\Omega)},$$

whence by (63), we have

$$\|y\|_{L^{2^*}(\Omega)} \leq \|\hat{y}\|_{L^{2^*}(\Omega)} \leq K \|u\|_{L^2(\partial\Omega)},$$

with $K = K(\Omega)$, as required. $\square$

APPENDIX B. EXISTENCE OF THE OPTIMAL CONTROL FOR (2)-(1) IN L^∞

The purpose of this section is to prove the existence of a global minimizer for the functional J , defined in (2)-(1). This will be given by the coercivity in L^2 of J , enhanced by employing the regularity of the solutions to the optimality system. As we did in the former section, we are going to accomplish this task in a general space domain Ω . Consider the optimal control problem

$$(66) \quad \min_{u \in L^\infty(\partial\Omega)} J(u) = \frac{1}{2} \int_{\partial\Omega} |u|^2 d\sigma(x) + \frac{\beta}{2} \int_{\Omega} |y - z|^2 dx,$$

where:

$$(67) \quad \begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega \\ y = u & \text{on } \partial\Omega. \end{cases}$$

Ω is a bounded open subset of $\mathbb{R}^n$, with $n = 1, 2, 3$ and $\partial\Omega \in C^\infty$. The nonlinearity $f \in C^2(\mathbb{R})$ is strictly increasing and $f(0) = 0$. The target $z \in L^\infty(\Omega)$ and $\beta > 0$ is a penalization parameter.

Proposition 2. *Let $z \in L^\infty(\Omega)$ be target for the state and let J be the corresponding functional, defined in (67)-(66). There exists $\bar{u} \in L^\infty(\partial\Omega)$ a global minimizer for J .*

Proof of proposition 2. Step 1 **Existence of the minimizer for a constrained problem**

Let $a, b \in \mathbb{R}$, with $a < 0 < b$ and let the convex set

$$\mathbb{K} := \{u \in L^\infty(\partial\Omega) \mid a \leq u \leq b, \text{ a.e. } \partial\Omega\}.$$

Under the same assumptions of (67)-(66), we consider the constrained optimal control problem:

$$(68) \quad \min_{u \in \mathbb{K}} J(u) = \frac{1}{2} \int_{\partial\Omega} |u|^2 d\sigma(x) + \frac{\beta}{2} \int_{\Omega} |y - z|^2 dx,$$

where:

$$(69) \quad \begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega \\ y = u & \text{on } \partial\Omega. \end{cases}$$

By using the techniques in [10], we have the existence of an optimal control $\bar{u}_{(a,b)} \in \mathbb{K}$ and any optimal control is given by $\bar{u}_{(a,b)} = \mathbb{P}_{[a,b]} \left(\frac{\partial \bar{q}_{(a,b)}}{\partial n} \right)$, with

$$(70) \quad \begin{cases} -\Delta \bar{y}_{(a,b)} + f(\bar{y}_{(a,b)}) = 0 & \text{in } \Omega \\ \bar{y}_{(a,b)} = \mathbb{P}_{[a,b]} \left(\frac{\partial \bar{q}_{(a,b)}}{\partial n} \right) & \text{on } \partial\Omega \\ -\Delta \bar{q}_{(a,b)} + f'(\bar{y}_{(a,b)}) \bar{q}_{(a,b)} = \beta (\bar{y}_{(a,b)} - z) & \text{in } \Omega \\ \bar{q}_{(a,b)} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathbb{P}_{[a,b]}$ is the projector

$$(71) \quad \mathbb{P}_{[a,b]}(\xi) := \begin{cases} a & \text{if } \xi \leq a \\ \xi & \text{if } a < \xi < b \\ b & \text{if } \xi \geq b. \end{cases}$$

Step 2 L^∞ **bounds for optimal controls uniform on** $(a, b) \in \mathbb{R}^2$, **with** $a < 0 < b$

Since $a < 0 < b$, the null control $0 \in \mathbb{K}$. Then, for any optimal control $\bar{u}_{(a,b)}$ for (69)-(68), we have

$$\frac{1}{2} \int_{\partial\Omega} |\bar{u}_{(a,b)}|^2 d\sigma(x) \leq J(\bar{u}_{(a,b)}) \leq J(0) \leq K,$$

whence

$$(72) \quad \|\bar{u}_{(a,b)}\|_{L^2(\partial\Omega)} \leq K,$$

where $K = K(\Omega, f, \beta, z)$ is independent of (a, b) .

We now bootstrap in the optimality system (70), to get the desired L^∞ bound, given the above L^2 bound.

First of all, by a comparison argument, we have

$$(73) \quad |\bar{y}_{(a,b)}| \leq \hat{y}_{(a,b)}, \quad \text{a.e. } \Omega,$$

with

$$(74) \quad \begin{cases} -\Delta \hat{y}_{(a,b)} = 0 & \text{in } \Omega \\ \hat{y}_{(a,b)} = |\bar{u}_{(a,b)}| & \text{on } \partial\Omega. \end{cases}$$

Comparison gives also

$$(75) \quad |\bar{q}_{(a,b)}| \leq \hat{q}_{(a,b)} \quad \text{and} \quad \left| \frac{\partial \bar{q}_{(a,b)}}{\partial n} \right| \leq \left| \frac{\partial \hat{q}_{(a,b)}}{\partial n} \right|, \quad \text{a.e. } \Omega$$

with

$$(76) \quad \begin{cases} -\Delta \hat{q}_{(a,b)} = \beta |\bar{y}_{(a,b)} - z| & \text{in } \Omega \\ \hat{q}_{(a,b)} = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, by [21, Theorem 7.4, page 202], the solution $\hat{y}_{(a,b)} \in H^{\frac{1}{2}}(\Omega) \hookrightarrow L^3(\Omega)$ and

$$\|\hat{y}_{(a,b)}\|_{L^3(\Omega)} \leq K \|\hat{y}_{(a,b)}\|_{H^{\frac{1}{2}}(\Omega)} \leq K \|\bar{u}_{(a,b)}\|_{L^2(\partial\Omega)} \leq K.$$

where the first inequality is given by the Sobolev embedding $H^{\frac{1}{2}}(\Omega) \hookrightarrow L^3(\Omega)$ valid for space dimension $n = 1, 2, 3$ (see e.g. [23, Theorem 6.7]) and the last inequality is justified by (72). By (73),

$$\|\bar{y}_{(a,b)}\|_{L^3(\Omega)} \leq \|\hat{y}_{(a,b)}\|_{L^3(\Omega)} \leq K.$$

We now concentrate on the adjoint equation. By [16, Theorem 2.4.2.5 page 124] applied to (76), we have $\hat{q}_{(a,b)} \in W^{2,3}(\Omega)$, with estimate

$$\|\hat{q}_{(a,b)}\|_{W^{2,3}(\Omega)} \leq K \|\bar{y}_{(a,b)} - z\|_{L^3(\Omega)} \leq K \left[\|\bar{y}_{(a,b)}\|_{L^3(\Omega)} + \|z\|_{L^\infty(\Omega)} \right] \leq K.$$

By the trace Theorem ([16, Theorem 1.5.1.3 page 38]) applied to $\nabla \hat{q}_{(a,b)}$,

$$\left\| \frac{\partial \hat{q}_{(a,b)}}{\partial n} \right\|_{L^4(\partial\Omega)} \leq K \|\hat{q}_{(a,b)}\|_{W^{2,3}(\Omega)} \leq K.$$

By (75), we have then

$$(77) \quad \left\| \frac{\partial \bar{q}_{(a,b)}}{\partial n} \right\|_{L^4(\partial\Omega)} \leq \left\| \frac{\partial \hat{q}_{(a,b)}}{\partial n} \right\|_{L^4(\partial\Omega)} \leq K,$$

whence

$$\|\bar{u}_{(a,b)}\|_{L^4(\partial\Omega)} = \left\| \mathbb{P}_{[a,b]} \left(\frac{\partial \bar{q}_{(a,b)}}{\partial n} \right) \right\|_{L^4(\partial\Omega)} \leq \left\| \frac{\partial \bar{q}_{(a,b)}}{\partial n} \right\|_{L^4(\partial\Omega)} \leq K.$$

By using the definition of solution by transposition for (74) and the above estimate, we get

$$\|\hat{y}_{(a,b)}\|_{L^4(\Omega)} \leq K \|\bar{u}_{(a,b)}\|_{L^4(\partial\Omega)} \leq K,$$

whence, by (73)

$$\|\bar{y}_{(a,b)}\|_{L^4(\Omega)} \leq \|\hat{y}_{(a,b)}\|_{L^4(\Omega)} \leq K.$$

In conclusion, we employ the elliptic regularity ([16, Theorem 2.4.2.5 page 124]) in (76), to get

$$\|\hat{q}_{(a,b)}\|_{W^{2,4}(\Omega)} \leq K \|\bar{y}_{(a,b)} - z\|_{L^4(\Omega)} \leq K,$$

whence, by Sobolev embeddings in space dimension $n = 1, 2, 3$,

$$\|\hat{q}_{(a,b)}\|_{C^1(\bar{\Omega})} \leq \|\hat{q}_{(a,b)}\|_{W^{2,4}(\Omega)} \leq K \|y - z\|_{L^4(\Omega)} \leq K.$$

Now, (75) yields

$$(78) \quad \left\| \frac{\partial \bar{q}_{(a,b)}}{\partial n} \right\|_{C^0(\partial\Omega)} \leq \left\| \frac{\partial \hat{q}_{(a,b)}}{\partial n} \right\|_{C^0(\partial\Omega)} \leq \|\hat{q}_{(a,b)}\|_{C^1(\bar{\Omega})} \leq K,$$

which in turn implies

$$\|\bar{u}_{(a,b)}\|_{L^\infty(\partial\Omega)} = \left\| \mathbb{P}_{[a,b]} \left(\frac{\partial \bar{q}_{(a,b)}}{\partial n} \right) \right\|_{L^\infty(\partial\Omega)} \leq \left\| \frac{\partial \bar{q}_{(a,b)}}{\partial n} \right\|_{L^\infty(\partial\Omega)} \leq K,$$

where the last inequality follows from (78). We have then, the estimate

$$(79) \quad \|\bar{u}_{(a,b)}\|_{L^\infty(\partial\Omega)} \leq K, \quad \forall a, b \in \mathbb{R}, \text{ with } a < 0 < b,$$

the constant $K = K(\Omega, f, \beta, z)$ being independent of (a, b) . This finishes this step.

Step 3 Conclusion

Let K be the upper bound appearing in (79). We want to show that, for any control $u \in L^\infty(\partial\Omega)$, with $\|u\|_{L^\infty(\partial\Omega)} > K$, the value of the functional

$$J(u) > \inf_{B^{L^\infty}(0, K)} J,$$

Indeed, for any control $u \in L^\infty(\partial\Omega)$, with $\|u\|_{L^\infty(\partial\Omega)} > K$, set $b := \|u\|_{L^\infty(\partial\Omega)} + 1$, $a := -b$ and set accordingly the control set

$$\mathbb{K} := \{u \in L^\infty(\partial\Omega) \mid a \leq u \leq b, \text{ a.e. } \partial\Omega\}.$$

By definition of a and b , the control $u \in \mathbb{K}$ and, by (79)

$$(80) \quad J(u) > \inf_{B^{L^\infty}(0, K)} J,$$

as desired. Now, by step 1, there exists $\bar{u} \in \overline{B^{L^\infty}(0, K)}$ minimizing J in $\overline{B^{L^\infty}(0, K)}$. By (80), such control $\bar{u}$ is in fact a global minimizer for J in $L^\infty(\partial\Omega)$, thus concluding the proof. $\square$

APPENDIX C. PROOF OF LEMMA 3.2

We prove lemma 3.2.

Proof of lemma 3.2. Step 1 **Proof of 1.**

Arbitrarily fix $z \in L^\infty(B(0, R))$. The existence of a minimizer u_z is a consequence of the direct methods in the Calculus of Variations. Moreover, by (16), definition of minimizer and $G(0) = 0$:

$$\begin{aligned} \frac{1}{2} R^{n-1} n \alpha(n) |u_z|^2 &\leq I(u_z, z) + \frac{\beta}{2} \int_{B(0, R)} |z|^2 dx \\ &\leq I(0, z) + \frac{\beta}{2} \int_{B(0, R)} |z|^2 dx = \frac{\beta}{2} \int_{B(0, R)} |z|^2 dx, \end{aligned}$$

which yields $\frac{1}{2} |u_z|^2 \leq \frac{\beta}{2 R^{n-1} n \alpha(n)} \int_{B(0, R)} |z|^2 dx$, as required.

Step 2 **Proof of 2.**

Arbitrarily fix $M \in \mathbb{R}^+$. For any pair of targets $(z_1, z_2) \in L^\infty(B(0, R))^2$ such that:

$$\|z_1\|_{L^2} \leq M \quad \text{and} \quad \|z_2\|_{L^2} \leq M.$$

For each control $u \in C$ such that $|u| \leq \sqrt{\frac{\beta}{R^{n-1} n \alpha(n)}} M$, we have:

$$\begin{aligned} I(u, z_2) - I(u_{z_1}, z_1) &= I(u, z_2) - I(u, z_1) + I(u, z_1) - I(u_{z_1}, z_1) \\ &\geq -|I(u, z_2) - I(u, z_1)| + 0 = -\beta \left| \int_{B(0, R)} G(u)(z_1 - z_2) dx \right| \\ &\geq -K \|z_2 - z_1\|_{L^\infty}, \end{aligned}$$

where the last inequality is justified by $|u| \leq \sqrt{\frac{\beta}{R^{n-1} n \alpha(n)}} M$ and the continuity of the control-to-state map G .

Then, one has that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that:

$$I(u, z_2) - I(u_{z_1}, z_1) > -\varepsilon,$$

whenever $\|z_2 - z_1\|_{L^\infty} < \delta_\varepsilon$.

Now, by the first step, any minimizer u_{z_2} for $I(\cdot, z_2)$ verifies $|u_{z_2}| \leq \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}} \|z_2\|_{L^2} \leq \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}} M$. Then, we have proved that:

$$\inf_C [I(\cdot, z_2)] - \inf_C [I(\cdot, z_1)] = I(u_{z_2}, z_2) - I(u_{z_1}, z_1) > -\varepsilon.$$

Exchanging the role of z_1 and z_2 , one can get:

$$\inf_C [I(\cdot, z_1)] - \inf_C [I(\cdot, z_2)] > -\varepsilon.$$

This yields the continuity of h . $\square$

APPENDIX D. PROOF OF LEMMA 3.3

We prove lemma 3.3.

Proof of lemma 3.3. If $h_1(z^0) = h_2(z^0)$, we take $\tilde{z} := z^0$, thus concluding. Let us now suppose $h_1(z^0) \neq h_2(z^0)$.

We start by considering the case $h_1(z^0) < h_2(z^0)$.

Step 1 Proof of the existence of $\mu_0 \geq 0$ such that:

- $\forall \mu \in [0, \mu_0], h_2(z^0 + \mu) < 0$;
- $h_1(z^0 + \mu_0) = 0$.

First of all, we observe that for any $\mu \geq 0$, $h_2(z^0 + \mu) < 0$. Indeed, since $h_2(z^0) < 0$, there exists $u_2 > 0$ such that $I(u_2, z^0) < 0$. Then,

$$\begin{aligned} h_2(z^0 + \mu) &\leq I(u_2, z^0 + \mu) \\ &= \frac{R^{n-1}n\alpha(n)}{2} |u_2|^2 + \frac{\beta}{2} \int_{B(0,R)} |G(u_2)|^2 dx - \beta \int_{B(0,R)} (z^0 + \mu) G(u_2) dx \\ &= I(u_2, z^0) - \mu \beta \int_{B(0,R)} G(u_2) dx \leq I(u_2, z^0) < 0, \end{aligned}$$

where we have used that $G(u_2) \geq 0$ a.e. in $B(0, r)$.

We prove now that $h_1(z^0 + \mu_0) = 0$, for $\mu_0 = \|z^0\|_{L^\infty}$. Indeed, for any $v \leq 0$:

$$I(v, z^0 + \mu_0) = \frac{R^{n-1}n\alpha(n)}{2} |v|^2 + \frac{\beta}{2} \int_{B(0,R)} |G(v)|^2 dx - \beta \int_{B(0,R)} (z^0 + \mu_0) G(v) dx \geq 0,$$

since $z^0 + \mu_0 \geq 0$ and $G(v) \leq 0$ a.e. in $B(0, r)$. This finishes the first step.

Step 2 Conclusion

Set:

$$g : [0, \mu_0] \longrightarrow \mathbb{R}$$

$$\mu \longmapsto h_2(z^0 + \mu) - h_1(z^0 + \mu).$$

Since $h_1(z^0) < h_2(z^0)$, $g(0) > 0$ and by Step 1 $g(\mu_0) < 0$. Then, by continuity, there exists $\mu_1 \in (0, \mu_0)$ such that $g(\mu_1) = 0$. Hence,

$$\tilde{z} := z^0 + \mu_1$$

is the desired target. Indeed, by definition of g and μ_1 , $h_1(\tilde{z}) = h_2(\tilde{z})$. Furthermore, since $\mu_1 \in (0, \mu_0)$, by Step 1, $h_2(\tilde{z}) < 0$. This concludes the proof for the case $h_1(z^0) < h_2(z^0)$. The proof for the remaining case $h_1(z^0) > h_2(z^0)$ is similar. $\square$

APPENDIX E. PROOF OF LEMMA 3.6

We prove lemma 3.6.

Proof of lemma 3.6. Step 1 **Proof of 1.**

Arbitrarily fix $z \in L^\infty(B(0, R) \setminus B(0, r))$. The existence of a minimizer u_z is a consequence of the direct methods in the Calculus of Variations. Moreover, by (34), definition of minimizer and $G(0) = 0$:

$$\begin{aligned} \frac{1}{2} \int_{B(0, r)} |u_z|^2 dx &\leq I(u_z, z) + \frac{\beta}{2} \int_{B(0, R) \setminus B(0, r)} |z|^2 dx \\ &\leq I(0, z) + \frac{\beta}{2} \int_{B(0, R) \setminus B(0, r)} |z|^2 dx = \frac{\beta}{2} \int_{B(0, R) \setminus B(0, r)} |z|^2 dx, \end{aligned}$$

which yields $\frac{1}{2} \int_{B(0, r)} |u_z|^2 dx \leq \frac{\beta}{2} \int_{B(0, R) \setminus B(0, r)} |z|^2 dx$, as required.

Step 2 **Proof of 2.**

Arbitrarily fix $M \in \mathbb{R}^+$. For any pair of targets $(z_1, z_2) \in L^\infty(B(0, R) \setminus B(0, r))^2$ such that:

$$\|z_1\|_{L^2} \leq M \quad \text{and} \quad \|z_2\|_{L^2} \leq M.$$

For each control $u \in C$ such that $\|u_z\|_{L^2(B(0, r))} \leq \sqrt{\beta}M$, we have:

$$\begin{aligned} I(u, z_2) - I(u_{z_1}, z_1) &= I(u, z_2) - I(u, z_1) + I(u, z_1) - I(u_{z_1}, z_1) \\ &\geq -|I(u, z_2) - I(u, z_1)| + 0 = -\beta \left| \int_{B(0, R) \setminus B(0, r)} G(u)(z_1 - z_2) dx \right| \\ &\geq -K \|z_2 - z_1\|_{L^\infty}, \end{aligned}$$

where the last inequality is justified by $\|u_z\|_{L^2(B(0, r))} \leq \sqrt{\beta}M$, the continuous embedding $H^2(B(0, r)) \hookrightarrow C^0(\overline{B(0, r)})$ and the continuity of the control-to-state map $G : L^2(B(0, r)) \rightarrow H^2(B(0, r))$.

Then, one has that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that:

$$I(u, z_2) - I(u_{z_1}, z_1) > -\varepsilon,$$

whenever $\|z_2 - z_1\|_{L^\infty} < \delta_\varepsilon$.

Now, by the first step, any minimizer u_{z_2} for $I(\cdot, z_2)$ verifies $\|u_{z_2}\|_{L^2} \leq \sqrt{\beta} \|z_2\|_{L^2} \leq \sqrt{\beta}M$. Then, we have proved that:

$$\inf_C [I(\cdot, z_2)] - \inf_C [I(\cdot, z_1)] = I(u_{z_2}, z_2) - I(u_{z_1}, z_1) > -\varepsilon.$$

Exchanging the role of z_1 and z_2 , one can get:

$$\inf_C [I(\cdot, z_1)] - \inf_C [I(\cdot, z_2)] > -\varepsilon.$$

This yields the continuity of h . □

APPENDIX F. PROOF OF LEMMA 3.7

We prove lemma 3.7.

Proof of lemma 3.7. If $h_1(z^0) = h_2(z^0)$, we take $\tilde{z} := z^0$, thus concluding. Let us now suppose $h_1(z^0) \neq h_2(z^0)$.

We start by considering the case $h_1(z^0) < h_2(z^0)$.

Step 1 **Proof of the existence of $\mu_0 \geq 0$ such that:**

- $\forall \mu \in [0, \mu_0], h_2(z^0 + \mu) < 0$;
- $h_1(z^0 + \mu_0) = 0$.

First of all, we observe that for any $\mu \geq 0$, $h_2(z^0 + \mu) < 0$. Indeed, since $h_2(z^0) < 0$, there exists $u_2 \in \mathcal{U}_r^+ \setminus \{0\}$ such that $I(u_2, z^0) < 0$. Then,

$$\begin{aligned} h_2(z^0 + \mu) &\leq I(u_2, z^0 + \mu) \\ &= \frac{1}{2} \int_{B(0,r)} |u_2|^2 dx + \frac{\beta}{2} \int_{B(0,R)} |G(u_2)|^2 dx - \beta \int_{B(0,R) \setminus B(0,r)} (z^0 + \mu) G(u_2) dx \\ &= I(u_2, z^0) - \mu \beta \int_{B(0,R) \setminus B(0,r)} G(u_2) dx \leq I(u_2, z^0) < 0, \end{aligned}$$

where we have used that $G(u_2) \geq 0$ a.e. in $B(0, R) \setminus B(0, r)$.

We prove now that $h_1(z^0 + \mu_0) = 0$, for $\mu_0 = \|z^0\|_{L^\infty}$. Indeed, for any $u \in \mathcal{U}_r^-$:

$$\begin{aligned} I(u, z^0 + \mu_0) &= \frac{1}{2} \int_{B(0,r)} |u|^2 dx + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |G(u)|^2 dx \\ &\quad - \beta \int_{B(0,R) \setminus B(0,r)} (z^0 + \mu_0) G(u) dx \geq 0, \end{aligned}$$

since $z^0 + \mu_0 \geq 0$ and $G(u) \leq 0$ a.e. in $B(0, R) \setminus B(0, r)$. This finishes the first step.

Step 2 Conclusion

Set:

$$\begin{aligned} g : [0, \mu_0] &\longrightarrow \mathbb{R} \\ \mu &\longmapsto h_2(z^0 + \mu) - h_1(z^0 + \mu). \end{aligned}$$

Since $h_1(z^0) < h_2(z^0)$, $g(0) > 0$ and by Step 1 $g(\mu_0) < 0$. Then, by continuity, there exists $\mu_1 \in (0, \mu_0)$ such that $g(\mu_1) = 0$. Hence,

$$\tilde{z} := z^0 + \mu_1$$

is the desired target. Indeed, by definition of g and μ_1 , $h_1(\tilde{z}) = h_2(\tilde{z})$. Furthermore, since $\mu_1 \in (0, \mu_0)$, by Step 1, $h_2(\tilde{z}) < 0$. This concludes the proof for the case $h_1(z^0) < h_2(z^0)$. The proof for the remaining case $h_1(z^0) > h_2(z^0)$ is similar. $\square$

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