THE TURNPIKE PROPERTY IN SEMILINEAR CONTROL

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Abstract. An exponential turnpike property for a semilinear control problem is proved. The state-target is assumed to be small, whereas the initial datum can be arbitrary.

Turnpike results have also been obtained for large targets, requiring that the control acts everywhere. In this case, we prove the convergence of the infimum of the averaged time-evolution functional towards the steady one.

Numerical simulations have been performed.

Introduction

In this manuscript, the long time behaviour of semilinear optimal control problems as the time-horizon tends to infinity is analyzed. Our results are global, meaning that we do not require smallness of the initial datum for the governing state equation.

In [33], A. Porretta and E. Zuazua studied turnpike property for control problems governed by a semilinear heat equation, with dissipative nonlinearity. In particular, [33, Theorem 1] yields the existence of a solution to the optimality system fulfilling the turnpike property, under smallness conditions on the initial datum and the target. Our first goal is to

(1) prove that in fact the (exponential) turnpike property is satisfied by the optimal control and state;
(2) remove the smallness assumption on the initial datum.

We keep the smallness assumption on the target. This leads to the smallness and uniqueness of the steady optima (see [33, subsection 3.2]), whence existence and uniqueness of the turnpike follows. We also treat the case of large targets, under the added assumption that control acts everywhere. In this case, we prove a weak turnpike result, which stipulates that the averaged infimum of the time-evolution functional converges towards the steady one. We also provide an L^2 bound of the time derivative of optimal states, uniformly in the time horizon.

Generally speaking, in turnpike theory a time-evolution optimal control problem is considered together with its steady version. The "Turnpike Property" is verified

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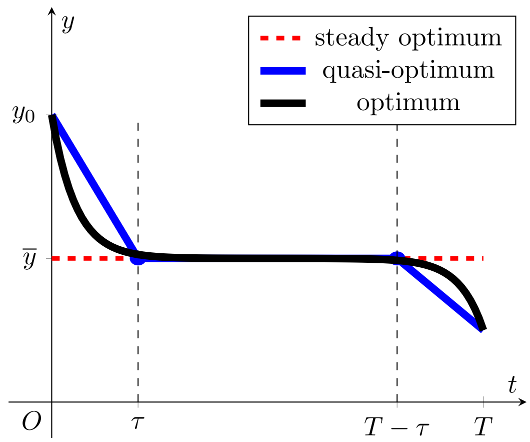

FIGURE 1. quasi-optimal turnpike strategies

if the time-evolution optima remain close to the steady optima up to some thin initial and final boundary layers.

An extensive literature is available on the topic. A pioneer on the topic has been John von Neumann [43]. In econometrics the topic has been widely investigated by several scholars including P. Samuelson and L.W. McKenzie [13, 38, 27, 28, 29, 9, 21]. Long time behaviour of optimal control problems have been studied by P. Kokotovic and collaborators [44, 2], by R.T. Rockafellar [37] and by A. Rapaport and P. Cartigny [35, 36]. A.J. Zaslavski wrote a book [47] on the topic. A turnpike-like asymptotic simplification have been obtained in the context of optimal design of the diffusivity matrix for the heat equation [1]. In the papers [12, 18, 17, 39], the concept of (measure) turnpike is related to the dissipativity of the control problem.

Recent papers on long time behaviour of Mean Field games [7, 8, 31] motivated new research on the topic. A special attention have been paid in providing an exponential estimate, as in the work [32] by A. Porretta and E. Zuazua, where linear quadratic control problems were considered. These results have later been extended in [41, 33, 46, 40, 20, 19] to control problems governed by a nonlinear state equation and applied to optimal control of the Lotka-Volterra system [23]. Recently turnpike property have been studied around nonsteady trajectories [40, 15]. The turnpike property is intimately related to asymptotic behaviour of the Hamilton-Jacobi equation [24].

Note that for a general optimal control problem, even in absence of a turnpike result, we can construct quasi-optimal turnpike strategies (see [22, Remark 7]) as in fig. 1:

(1) in a short time interval $[0, \tau]$ drive the state from the initial configuration y_0 to the turnpike $\overline{y}$;
(2) in a long time arc $[\tau, T-\tau]$, remain on $\overline{y}$;
(3) in a short final arc $[T-\tau, T]$, use to control to match the required terminal condition at time $t = T$.

In general, the corresponding control and state are not optimal, being not smooth. However, they are easy to construct.

The proof of turnpike results is harder than the above construction. In fact, to prove turnpike results, one has to ensure that there is not another time-evolving strategy which is significantly better than the above one. In case the turnpike property is verified, the above strategy is quasi-optimal.

Statement of the main results. We consider the semilinear optimal control problem:

$$\min_{u\in \mathrm{L}^2((0,T)\times\omega)} J_T(u)=\frac{1}{2}\int_0^T\int_\omega |u|^2dxdt+\frac{\beta}{2}\int_0^T\int_{\omega_0}|y-z|^2dxdt, \tag{1}$$

where:

$$\begin{cases} y_t-\Delta y+f\left(y\right)=u\chi_\omega & \text{in } (0,T)\times\Omega\\ y=0 & \text{on } (0,T)\times\partial\Omega\\ y(0,x)=y_0(x) & \text{in } \Omega. \end{cases} \tag{2}$$

As usual, Ω is a regular bounded open subset of $\mathbb{R}^n$, with $n=1,2,3$. The nonlinearity f is C^3 nondecreasing, with $f\left(0\right)=0$. The action of the control is localized by multiplication by χ_ω, characteristic function of the open subregion $\omega\subseteq\Omega$. The target z is assumed to be in $\mathrm{L}^\infty(\omega_0)$. Since that the nonlinearity is nondecreasing, the semilinear problem (2) is well-posed [4, chapter 5] . Namely, given an initial datum $y_0\in\mathrm{L}^2\left(\Omega\right)$ and a control $u\in\mathrm{L}^2((0,T)\times\omega)$, there exists a unique solution

$$y\in\mathrm{C}^0([0,T];\mathrm{L}^2\left(\Omega\right))\cap\mathrm{L}^2(0,T;\mathrm{H}_0^1\left(\Omega\right)).$$

$\omega_0\subseteq\Omega$ is an open subset and $\beta\geq 0$ is a weighting parameter. As β increases, the distance between the optimal state and the target decreases.

By the direct method in the calculus of variations [10, 42], there exists a global minimizer of (1). As we shall see, uniqueness can be guaranteed, provided that the initial datum and the target are small enough in the uniform norm.

Taking the Gâteaux differential of the functional (1) and imposing the Fermat stationary condition, we realize that any optimal control reads as $u^T=-q^T\chi_\omega$, where $\left(y^T,q^T\right)$ solves

$$\begin{cases} y_t^T-\Delta y^T+f(y^T)=-q^T\chi_\omega & \text{in } (0,T)\times\Omega\\ y^T=0 & \text{on } (0,T)\times\partial\Omega\\ y^T(0,x)=y_0(x) & \text{in } \Omega\\ -q_t^T-\Delta q^T+f'(y^T)q^T=\beta(y^T-z)\chi_{\omega_0} & \text{in } (0,T)\times\Omega\\ q^T=0 & \text{on } (0,T)\times\partial\Omega\\ q^T(T,x)=0 & \text{in } \Omega. \end{cases} \tag{3}$$

In order to study the turnpike, we need to study the steady version of (2)-(1):

$$\min_{u_s\in\mathrm{L}^2(\Omega)} J_s(u_s)=\frac{1}{2}\int_\omega |u_s|^2dx+\frac{\beta}{2}\int_{\omega_0}|y_s-z|^2dx, \tag{4}$$

where:

$$\begin{cases} -\Delta y_s+f(y_s)=u_s\chi_\omega & \text{in } \Omega\\ y_s=0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Under the same assumptions required for the problem (2)-(1), for any given control $u_s\in\mathrm{L}^2\left(\Omega\right)$, there exists a unique state $y_s\in\mathrm{H}^2\left(\Omega\right)\cap\mathrm{H}_0^1\left(\Omega\right)$ solution to (5) (see e.g. [5]).

By adapting the techniques of [10], we have the existence of a global minimizer $\overline{u}$ for (4). The corresponding optimal state is denoted by $\overline{y}$. If the target is sufficiently small in the uniform norm, the optimal control is unique (see [33, subsection 3.2]). Furthermore any optimal control $\overline{u} = -\overline{q}\chi_\omega$, where the pair $(\overline{y}, \overline{q})$ satisfies the steady optimality system

$$\begin{cases} -\Delta\overline{y} + f(\overline{y}) = -\overline{q}\chi_\omega & \text{in } \Omega \\ \overline{y} = 0 & \text{on } \partial\Omega \\ -\Delta\overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{\omega_0} & \text{in } \Omega \\ \overline{q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

The analysis in [33, section 3] leads to the following local result.

Theorem 0.1 (Porretta-Zuazua)**.** *Consider the control problem* (5)-(4)*. There exists $\delta > 0$ such that if the **initial datum** and the target fulfill the **smallness condition***

$$\|y_0\|_{L^\infty(\Omega)} \leq \delta \quad \textit{and} \quad \|z\|_{L^\infty(\omega_0)} \leq \delta,$$

there exists a solution $\left(y^T, q^T\right)$ to the Optimality System

$$\begin{cases} y_t^T - \Delta y^T + f(y^T) = -q^T\chi_\omega & \textit{in } (0,T)\times\Omega \\ y^T = 0 & \textit{on } (0,T)\times\partial\Omega \\ y^T(0,x) = y_0(x) & \textit{in } \Omega \\ -q_t^T - \Delta q^T + f'(y^T)q^T = \beta(y^T - z)\chi_{\omega_0} & \textit{in } (0,T)\times\Omega \\ q^T = 0 & \textit{on } (0,T)\times\partial\Omega \\ q^T(T,x) = 0 & \textit{in } \Omega \end{cases}$$

satisfying for any $t \in [0,T]$

$$\|q^T(t) - \overline{q}\|_{L^\infty(\Omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K\left[\exp\left(-\mu t\right) + \exp\left(-\mu(T-t)\right)\right],$$

where K and μ are T-independent.

We observe that the turnpike property is satisfied by one solution to the optimality system. Since our problem may be not convex, we cannot directly assert that such solution of the optimality system is the unique minimizer (optimal control) for (5)-(4).

Large initial data and small targets. We start by keeping the running target small, but allowing the initial datum for (2) to be large.

Theorem 0.2. *Consider the control problem* (2)-(1)*. Let u^T be a minimizer of* (1)*. There exists $\rho > 0$ such that for every initial datum $y_0 \in L^\infty(\Omega)$ and target z verifying*

$$\|z\|_{L^\infty(\omega_0)} \leq \rho, \tag{7}$$

we have

$$\|u^T(t) - \overline{u}\|_{L^\infty(\Omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K\left[\exp\left(-\mu t\right) + \exp\left(-\mu(T-t)\right)\right], \qquad \forall t \in [0,T], \tag{8}$$

the constants K and $\mu > 0$ being independent of the time horizon T.

Note that ρ is smaller than the smallness parameter δ in Theorem 0.1.

The main ingredients our proofs require are:

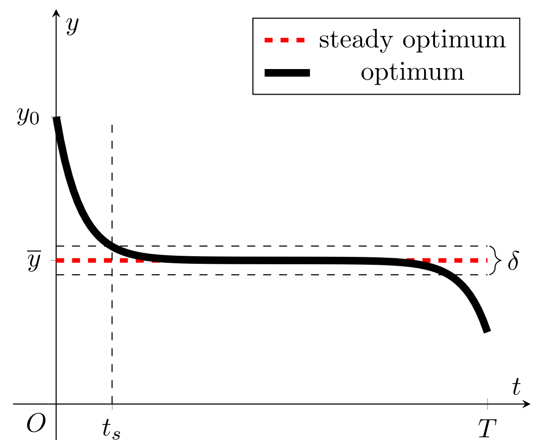

FIGURE 2. global-local argument

(1) prove a L^∞ bound of the norm of the optimal control, uniform in the time horizon $T > 0$ (Lemma 1.1 in section 1.1);
(2) proof of the turnpike property for *small data* and *small targets*. Note that, in Theorem 0.1, the authors prove the existence of a solution to the optimality system enjoying the turnpike property. In this preliminary step, for *small data* and *small targets*, we prove that any optimal control verifies the turnpike property (Lemma 1.2 in section 1.1);
(3) for *small targets* and *any data*, proof of the smallness of $\|y^T(t)\|_{L^\infty(\Omega)}$ in time t large (section 1.2). This is done by estimating the critical time t_s needed to approach the turnpike;
(4) conclude concatenating the two former steps (section 1.2).

Theorem 0.2 ensures that the conclusion of Theorem 0.1 holds for the optimal pair.

Let us outline the proof of 3 (fig. 2), the existence of τ upper bound for the minimal time needed to approach the turnpike t_s.
Suppose, by contradiction, that the critical time t_s to approach the turnpike is very large. Accordingly, the time-evolution optimal strategy obeys the following plan:

(1) stay away from the turnpike for long time;
(2) move close to the turnpike;
(3) enjoy a final time-evolution performance, cheaper than the steady one.

Then, in phase 1, with respect to the steady performance, an extra cost is generated, which should be regained in phase 3. At this point, we realize that this is prevented by validity of the local turnpike property. Indeed, once the time-evolution optima approach the turnpike at some time t_s, the optimal pair satisfies the turnpike property for larger times $t \geq t_s$. Hence, for $t \geq t_s$, the time-evolution performance cannot be significantly cheaper than the steady one. Accordingly, we cannot regain the extra-cost generated in phase 1, so obtaining a contradiction.

Control acting everywhere: convergence of averages for arbitrary targets. In section 2 we deal with large targets, supposing the control acts everywhere (i.e. $\omega = \Omega$). We prove that averages converge. Furthermore, we obtain an L^2 bound for the time derivative of optimal states. The bound is uniform independent of the time horizon T, meaning that, if T is large, the time derivative of the optimal state is small for most of the time.

Theorem 0.3. *Take an arbitrary initial datum $y_0 \in \mathrm{L}^\infty(\Omega)$ and an arbitrary target $z \in \mathrm{L}^\infty(\omega_0)$. Consider the time-evolution control problem* (2)-(1) *and its steady version* (5)-(4)*. Assume $\omega = \Omega$. Then, averages converge*

$$\frac{1}{T} \inf_{\mathrm{L}^2((0,T)\times\omega)} J_T \underset{T\to+\infty}{\longrightarrow} \inf_{\mathrm{L}^2(\Omega)} J_s. \tag{9}$$

Suppose in addition $y_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{H}_0^1(\Omega)$. Let u^T be an optimal control for (2)-(1) *and let y^T be the corresponding state, solution to* (2)*, with control u^T and initial datum y_0. Then, the L^2 norm of the time derivative of the optimal state is bounded uniformly in T*

$$\left\| y_t^T \right\|_{\mathrm{L}^2((0,T)\times\Omega)} \leq K, \tag{10}$$

the constant K being T-independent.

The proof of Theorem 0.3, available in section 2, is based on the following representation formula for the time-evolving functional (Lemma 2.2):

$$\begin{aligned} J_T(u) = & \int_0^T J_s\big(-\Delta y(t,\cdot) + f(y(t,\cdot))\big) dt \\ & + \frac{1}{2}\int_0^T \int_\Omega |y_t(t,x)|^2 \, dx dt \\ & + \frac{1}{2}\int_\Omega \Big[\|\nabla y(T,x)\|^2 + 2F(y(T,x)) - \|\nabla y_0(x)\|^2 - 2F(y_0(x))\Big] dx, \end{aligned} \tag{11}$$

where for a.e. $t \in (0,T)$, $J_s\big(-\Delta y(t,\cdot) + f(y(t,\cdot))\big)$ denotes the evaluation of the steady functional J_s at control $u_s(\cdot) := -\Delta y(t,\cdot) + f(y(t,\cdot))$ and y is the state associated to control u solving

$$\begin{cases} y_t - \Delta y + f(y) = u & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases} \tag{12}$$

Note that the above formula is valid for initial data $y_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{H}_0^1(\Omega)$. However, by the regularizing effect of (12) and the properties of the control problem, one can reduce to the case of smooth initial data.

By means of (11), the functional J_T can be seen as the sum of three terms:

(1) $\int_0^T J_s\big(-\Delta y(t,\cdot) + f(y(t,\cdot))\big) dt$, which stands for the "steady" cost at a.e. time $t \in (0,T)$ integrated over $(0,T)$;
(2) $\frac{1}{2}\int_0^T \int_\Omega |y_t(t,x)|^2 \, dx dt$, which penalizes the time derivative of the functional;
(3) $\frac{1}{2}\int_\Omega \Big[\|\nabla y(T,x)\|^2 + 2F(y(T,x)) - \|\nabla y_0(x)\|^2 - 2F(y_0(x))\Big] dx$, which depends on the terminal values of the state.

Choose now an optimal control u^T for (2)-(1) and plug it in (11). By Lemma 1.1, the term
$\frac{1}{2}\int_\Omega \Big[\|\nabla y(T,x)\|^2 + 2F(y(T,x)) - \|\nabla y_0(x)\|^2 - 2F(y_0(x))\Big] dx$ can be estimated uniformly in the time horizon. At the optimal control, the term $\int_0^T \int_\Omega |y_t(t,x)|^2 \, dx dt$

has to be small and the "steady" cost $\int_0^T J_s\big(-\Delta y(t,\cdot)+f\left(y(t,\cdot)\right)\big)dt$ is the dominant addendum. This is the basic idea of our approach to prove turnpike results for large targets.

The rest of the manuscript is organized as follows. In section 1 we prove Theorem 0.2. In section 2, we prove Theorem 0.3. In section 3 we perform some numerical simulations. The appendix is mainly devoted to the proof of the uniform bound of the optima (Lemma 1.1) and a PDE result needed for Lemma 2.2.

1. Proof of Theorem 0.2

1.1. **Preliminary Lemmas.** As announced, we firstly exhibit an upper bound of the norms of the optima in terms of the data. Note that the Lemma below yields an uniform bound for large targets as well.

Lemma 1.1. *Consider the control problem* (2)-(1). *Let* $R>0$, $y_0\in \mathrm{L}^\infty(\Omega)$ *and* $z\in\mathrm{L}^\infty(\omega_0)$, *satisfying* $\|y_0\|_{\mathrm{L}^\infty(\Omega)}\leq R$ *and* $\|z\|_{\mathrm{L}^\infty(\omega_0)}\leq R$. *Let* u^T *be an optimal control for* (2)-(1). *Then,* u^T *and* y^T *are bounded and*

$$\|u^T\|_{\mathrm{L}^\infty((0,T)\times\omega)}+\left\|y^T\right\|_{\mathrm{L}^\infty((0,T)\times\Omega)}\leq K\left[\|y_0\|_{\mathrm{L}^\infty(\Omega)}+\|z\|_{\mathrm{L}^\infty(\omega_0)}\right], \tag{13}$$

where the constant K *is independent of the time horizon* T, *but it depends on* R.

The proof is postponed to the Appendix.

The second ingredient for the proof of Theorem 0.2 is the following Lemma.

Lemma 1.2. *Consider the control problem* (2)-(1). *Let* $y_0\in\mathrm{L}^\infty(\Omega)$ *and* $z\in\mathrm{L}^\infty(\omega_0)$. *There exists* $\delta>0$ *such that, if*

$$\|z\|_{\mathrm{L}^\infty(\omega_0)}\leq\delta \qquad \textit{and} \qquad \|y_0\|_{\mathrm{L}^\infty(\Omega)}\leq\delta, \tag{14}$$

the functional (1) *admits a unique global minimizer* u^T. *Furthermore, for every* $\varepsilon>0$ *there exists* $\delta_\varepsilon>0$ *such that, if*

$$\|z\|_{\mathrm{L}^\infty(\omega_0)}\leq\delta_\varepsilon \qquad \textit{and} \qquad \|y_0\|_{\mathrm{L}^\infty(\Omega)}\leq\delta_\varepsilon, \tag{15}$$

the functional (1) *admits a unique global minimizer* u^T *and*

$$\|u^T(t)-\overline{u}\|_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)-\overline{y}\|_{\mathrm{L}^\infty(\Omega)}\leq\varepsilon\left[\exp\left(-\mu t\right)+\exp\left(-\mu(T-t)\right)\right], \qquad \forall t\in[0,T], \tag{16}$$

$(\overline{u},\overline{y})$ *being the optimal pair for* (4). *The constants* δ_ε *and* $\mu>0$ *are independent of the time horizon and* μ *is given by*

$$\begin{aligned}\left\|\mathscr{E}(t)-\widehat{E}\right\|_{\mathscr{L}(\mathrm{L}^2(\Omega),\mathrm{L}^2(\Omega))} &\leq C\exp\left(-\mu t\right),\\ \left\|\exp\left(-tM\right)\right\|_{\mathscr{L}(\mathrm{L}^2(\Omega),\mathrm{L}^2(\Omega))} &\leq \exp\left(-\mu t\right), \quad M:=-\Delta+f'\left(\overline{y}\right)+\widehat{E}\chi_\omega.\end{aligned} \tag{17}$$

where $\mathscr{E}$ *and* $\widehat{E}$ *denote respectively the differential and algebraic Riccati operators (see* [33, equation (22)]*) and* $\Delta:\mathrm{H}_0^1(\Omega)\longrightarrow\mathrm{H}^{-1}(\Omega)$ *is the Dirichlet laplacian.*

Proof of Lemma 1.2. We introduce the critical ball

$$B:=\left\{u\in\mathrm{L}^\infty((0,T)\times\omega)\ \Big|\ \|u\|_{\mathrm{L}^\infty((0,T)\times\omega)}\leq K\left[\|y_0\|_{\mathrm{L}^\infty(\Omega)}+\|z\|_{\mathrm{L}^\infty(\omega_0)}\right]\right\}, \tag{18}$$

where K is the constant appearing in (13).

Step 1 **Strict convexity in B for small data**

By [11, section 5] or [10], the second order Gâteaux differential of J reads as

$$\langle d^2 J_T(u)w, w\rangle = \int_0^T \int_\omega w^2 dxdt + \int_0^T \int_{\omega_0} |\psi_w|^2 dxdt - \int_0^T \int_\Omega f''(y)q|\psi_w|^2 dxdt,$$

where y solves (2) with control u and initial datum y_0, ψ_w solves the linearized problem

$$\begin{cases} (\psi_w)_t - \Delta\psi_w + f'(y)\psi_w = w\chi_\omega & \text{in } (0,T)\times\Omega \\ \psi_w = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi_w(0,x) = 0 & \text{in } \Omega \end{cases} \tag{19}$$

and

$$\begin{cases} -q_t - \Delta q + f'(y)q = (y-z)\chi_{\omega_0} & \text{in } (0,T)\times\Omega \\ q = 0 & \text{on } (0,T)\times\partial\Omega \\ q(T,x) = 0 & \text{in } \Omega. \end{cases} \tag{20}$$

Since $f'(y) \geq 0$,

$$\|\psi_w\|_{L^2((0,T)\times\Omega)} \leq K\|w\|_{L^2((0,T)\times\omega)}.$$

Let $u \in B$. By applying a comparison argument to (2) and (20),

$$\|y\|_{L^\infty((0,T)\times\Omega)} + \|q\|_{L^\infty((0,T)\times\Omega)} \leq K\left[\|y_0\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\omega_0)}\right].$$

Hence,

$$\langle d^2 J_T(u)w, w\rangle \geq \int_0^T \int_{\omega_0} |\psi_w|^2 dxdt + \left\{1 - K\left[\|y_0\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\omega_0)}\right]\right\}\int_0^T \int_\omega |w|^2 dxdt,$$

If $\|y_0\|_{L^\infty(\Omega)}$ and $\|z\|_{L^\infty(\omega_0)}$ are small enough, we have

$$\langle d^2 J_T(u)w, w\rangle \geq \frac{1}{2}\int_0^T \int_\omega |w|^2 dxdt,$$

whence the strict convexity of J in the critical ball B. Now, by (13) and (18), if $\|y_0\|_{L^\infty(\Omega)}$ and $\|z\|_{L^\infty(\omega_0)}$ are small enough, any optimal control u^T belongs to B. Then, there exists a unique solution to the optimality system, with control in the critical ball B and such control coincides with u^T the unique global minimizer of (1).

Step 2 **Conclusion**
Let $\varepsilon > 0$. By following the fixed-point argument developed in the proof of [33, Theorem 1 subsection 3.1] and in [33, subsection 3.2], we can find $\delta_\varepsilon > 0$ such that, if

$$\|z\|_{L^\infty(\omega_0)} \leq \delta_\varepsilon \qquad \text{and} \qquad \|y_0\|_{L^\infty(\Omega)} \leq \delta_\varepsilon,$$

there exists a solution $\left(y^T, q^T\right)$ to the optimality system such that

$$\|u^T\|_{L^\infty((0,T)\times\omega)} < \varepsilon$$

and

$$\|u^T(t) - \overline{u}\|_{L^\infty(\Omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K\left[\exp(-\mu t) + \exp(-\mu(T-t))\right], \qquad \forall t \in [0,T].$$

By Step 1, if ε is small enough, $u^T := -q^T\chi_\omega$ is a strict global minimizer for J_T. Then, being strict, it is the unique one. This finishes the proof.

□

In the following Lemma, we compare the value of the time evolution functional (1) at a control u, with the value of the steady functional (4) at control $\overline{u}$, supposing that u and $\overline{u}$ satisfy a turnpike-like estimate.

Lemma 1.3. *Consider the time-evolution control problem* (2)-(1) *and its steady version* (5)-(4). *Fix* $y_0 \in \mathrm{L}^2(\Omega)$ *an initial datum and* $z \in \mathrm{L}^2(\omega_0)$ *a target. Let* $\overline{u} \in \mathrm{L}^\infty(\Omega)$ *be a control and let* $\overline{y}$ *be the corresponding solution to* (5). *Let* $u \in \mathrm{L}^\infty((0,T)\times\omega)$ *be a control and* y *the solution to* (2), *with control* u. *Assume*

$$\|u(t)-\overline{u}\|_{\mathrm{L}^\infty(\Omega)}+\|y(t)-\overline{y}\|_{\mathrm{L}^\infty(\Omega)} \leq K\left[\exp(-\mu t)+\exp(-\mu(T-t))\right], \qquad \forall t\in[0,T], \tag{21}$$

with $K = K(\Omega,\beta,y_0)$ *and* $\mu = \mu(\Omega,\beta)$. *Then,*

$$|J_T(u) - TJ_s(\overline{u})| \leq C\left[1+\|\overline{u}\|_{\mathrm{L}^\infty(\Omega)}+\|z\|_{\mathrm{L}^\infty(\omega_0)}\right], \tag{22}$$

the constant C *depending only on the above constant* K *and* μ.

Proof of Lemma 1.3. We estimate

$$\begin{aligned}
&|J_T(u) - TJ_s(\overline{u})| \\
=\ & \left|\frac{1}{2}\|u\|^2_{\mathrm{L}^2((0,T)\times\omega)}+\frac{\beta}{2}\|y-z\|^2_{\mathrm{L}^2((0,T)\times\omega_0)} - T\left[\frac{1}{2}\|\overline{u}\|^2_{\mathrm{L}^2(\Omega)}+\frac{\beta}{2}\|\overline{y}-z\|^2_{\mathrm{L}^2(\omega_0)}\right]\right| \\
=\ & \left|\frac{1}{2}\|u-\overline{u}\|^2_{\mathrm{L}^2((0,T)\times\omega)}+\frac{\beta}{2}\|y-\overline{y}\|^2_{\mathrm{L}^2((0,T)\times\omega_0)}\right. \\
& \left. +\int_0^T\int_\omega (u-\overline{u})\overline{u}\,dxdt+\beta\int_0^T\int_{\omega_0}(y-\overline{y})(\overline{y}-z)\,dxdt\right| \\
\leq\ & C\left[1+\|\overline{u}\|_{\mathrm{L}^\infty(\omega)}+\|z\|_{\mathrm{L}^\infty(\omega_0)}\right]\left\{\int_0^T\left[\|u-\overline{u}\|^2_{\mathrm{L}^\infty(\omega)}+\|u-\overline{u}\|_{\mathrm{L}^\infty(\omega)}\right]dt\right. \\
& \left.+\int_0^T\left[\|y-\overline{y}\|^2_{\mathrm{L}^\infty(\omega_0)}+\|y-\overline{y}\|_{\mathrm{L}^\infty(\omega_0)}\right]dt\right\} \\
\leq\ & C\left[1+\|\overline{u}\|_{\mathrm{L}^\infty(\omega)}+\|z\|_{\mathrm{L}^\infty(\omega_0)}\right],
\end{aligned}$$

where the last inequality follows from (21). □

The following Lemma (fig. 3) plays a key role in the proof of Theorem 0.2.
Let u^T be an optimal control for (2)-(1). Let y^T be the corresponding optimal state. For any $\varepsilon > 0$, let δ_ε be given by (15). Set

$$t_s := \inf\left\{t\in[0,T] \mid \|y^T(t)\|_{\mathrm{L}^\infty(\Omega)} \leq \delta_\varepsilon\right\},$$

where we use the convention $\inf(\varnothing) = T$.

Lemma 1.4 (Global attractor property)**.** *Consider the control problem* (2)-(1). *Let* $y_0 \in \mathrm{L}^\infty(\Omega)$ *and* $z \in \mathrm{L}^\infty(\omega_0)$. *Let* u^T *be an optimal control for* (2)-(1) *and let* y^T *be the corresponding optimal state. For any* $\varepsilon > 0$. *there exist* $\rho_\varepsilon = \rho_\varepsilon(\Omega,\beta,\varepsilon)$ *and* $\tau_\varepsilon = \tau_\varepsilon(\Omega,\beta,y_0,\varepsilon)$, *such that if* $\|z\|_{\mathrm{L}^\infty(\omega_0)} \leq \rho_\varepsilon$ *and* $T \geq \tau_\varepsilon$,

$$\|y^T(t_s)\|_{\mathrm{L}^\infty(\Omega)} \leq \delta_\varepsilon, \qquad \text{with the upper bound } t_s \leq \tau_\varepsilon \tag{23}$$

and

$$\|u^T(t)-\overline{u}\|_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)-\overline{y}\|_{\mathrm{L}^\infty(\Omega)} \leq \varepsilon\left[\exp(-\mu(t-t_s))+\exp(-\mu(T-(t-t_s)))\right], \qquad \forall t\in[t_s,T]. \tag{24}$$

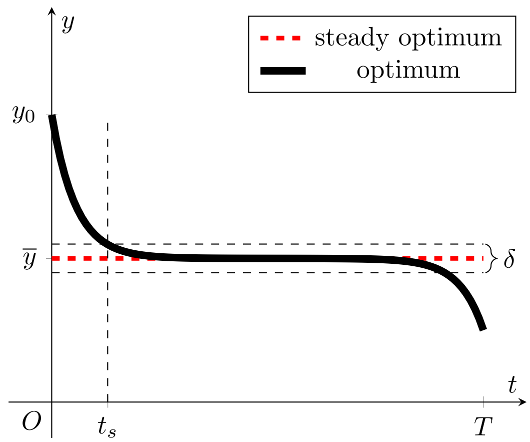

FIGURE 3. global-local argument employed in the proof of Lemma 1.4

The constant μ *is given by* (17) *and* δ_ε *is given by* (15).

Proof of Lemma 1.4. Throughout the proof, constant $K_1 = K_1(\Omega, \beta)$ is chosen as small as needed, whereas constant $K_2 = K_2(\Omega, \beta, y_0)$ is chosen as large as needed.
Step 1 **Estimate of the L^∞ norm of steady optimal controls**
In this step, we follow the arguments of [33, subsection 3.2]. Let $\overline{u} \in \mathrm{L}^2(\Omega)$ be an optimal control for (5)-(4). By definition of minimizer (optimal control),

$$\frac{1}{2}\|\overline{u}\|^2_{\mathrm{L}^2(\omega)} \leq J_s(\overline{u}) \leq J_s(0) = \frac{\beta}{2}\|z\|^2_{\mathrm{L}^2(\omega_0)} \leq \frac{\beta \mu_{leb}(\omega_0)}{2}\|z\|^2_{\mathrm{L}^\infty(\omega_0)}.$$

Now, any optimal control is of the form $\overline{u} = -\overline{q}\chi_\omega$, where the pair $(\overline{y}, \overline{q})$ satisfies the optimality system (6). Since $n = 1, 2, 3$, by elliptic regularity (see, e.g. [14, Theorem 4 subsection 6.3.2]) and Sobolev embeddings (see e.g. [14, Theorem 6 subsection 5.6.3]), $\overline{q} \in \mathrm{C}^0(\overline{\Omega})$ and $\|\overline{q}\|_{\mathrm{L}^\infty(\Omega)} \leq K\|z\|_{\mathrm{L}^\infty(\omega_0)}$, where $K = K(\Omega)$. This yields $\overline{u} \in \mathrm{C}^0(\overline{\omega})$ and

$$\|\overline{u}\|_{\mathrm{L}^\infty(\omega)} \leq K\|z\|_{\mathrm{L}^\infty(\omega_0)}. \tag{25}$$

Step 2 **There exist $\rho_\varepsilon = \rho_\varepsilon(\Omega, \beta, \varepsilon)$ and $\tau_\varepsilon = \tau_\varepsilon(\Omega, \beta, y_0, \varepsilon)$, such that if $\|z\|_{\mathrm{L}^\infty(\omega_0)} \leq \rho_\varepsilon$, then the critical time satisfies $t_s \leq \tau_\varepsilon$**
Let $\overline{u}$ be an optimal control for the steady problem. Then, by definition of minimizer (optimal control),

$$J_T\left(u^T\right) \leq J_T(\overline{u}) \tag{26}$$

and, by Lemma 1.3,

$$J_T(\overline{u}) \leq T \inf_{\mathrm{L}^2(\Omega)} J_s + K_2. \tag{27}$$

Now, we split the integrals in J_T into $[0, t_s]$ and $(t_s, T]$

$$\begin{aligned} J_T\left(u^T\right) &= \frac{1}{2}\int_0^{t_s}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_0^{t_s}\int_{\omega_0} |y^T - z|^2 dxdt \\ &+ \frac{1}{2}\int_{t_s}^{T}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_{t_s}^{T}\int_{\omega_0} |y^T - z|^2 dxdt. \end{aligned} \tag{28}$$

Set:

$$c_y(t,x) := \begin{cases} \dfrac{f(y^T(t,x))}{y^T(t,x)} & y^T(t,x) \neq 0 \\ f'(0) & y^T(t,x) = 0. \end{cases}$$

Since f is nondecreasing and $f(0) = 0$, we have $c_y \geq 0$. Then, Lemma A.1 (with potential c_y and source term $h := u^T \chi_\omega$) yields

$$\frac{1}{2}\int_0^{t_s}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_0^{t_s}\int_{\omega_0} |y^T - z|^2 dxdt \geq K_1 \int_0^{t_s} \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} dt - K_2.$$

Furthermore, by definition of t_s, for any $t \in [0, t_s]$, $\|y^T(t)\|_{\mathrm{L}^\infty(\Omega)} \geq \delta_\varepsilon$. Then,

$$\frac{1}{2}\int_0^{t_s}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_0^{t_s}\int_{\omega_0} |y^T - z|^2 dxdt \geq K_1 t_s \delta_\varepsilon^2 - K_2. \tag{29}$$

Once again, by definition of t_s,

$$\|y^T(t_s)\|_{\mathrm{L}^\infty(\Omega)} = \delta_\varepsilon \quad \text{and} \quad \|z\|_{\mathrm{L}^\infty(\omega_0)} \leq \delta_\varepsilon,$$

where δ_ε is given by (15). Therefore, by Lemma 1.2, the turnpike estimate (16) is satisfied in $[t_s, T]$. Lemma 1.3 applied in $[t_s, T]$ gives

$$\begin{aligned} & \frac{1}{2}\int_{t_s}^{T}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_{t_s}^{T}\int_{\omega_0} |y^T - z|^2 dxdt \\ \geq\ & (T - t_s) \inf_{\mathrm{L}^2(\Omega)} J_s - K_2 \left[1 + \|\overline{u}\|_{\mathrm{L}^\infty(\Omega)} + \|z\|_{\mathrm{L}^\infty(\omega_0)}\right] \\ \geq\ & (T - t_s) \inf_{\mathrm{L}^2(\Omega)} J_s - K_2, \end{aligned} \tag{30}$$

where the last inequality is due to (25) and $\|z\|_{\mathrm{L}^\infty(\omega_0)} \leq \delta_\varepsilon$.

At this point, by section 1.1, (29) and section 1.1

$$J_T\left(u^T\right) \geq K_1 t_s \delta_\varepsilon^2 + (T - t_s) \inf_{\mathrm{L}^2(\Omega)} J_s - K_2. \tag{31}$$

Therefore, by (31), (26) and (27)

$$K_1 t_s \delta_\varepsilon^2 + (T - t_s) \inf_{\mathrm{L}^2(\Omega)} J_s - K_2 \leq T \inf_{\mathrm{L}^2(\Omega)} J_s + K_2,$$

whence

$$t_s \left[K_1 \delta_\varepsilon^2 - \inf_{\mathrm{L}^2(\Omega)} J_s\right] \leq K_2. \tag{32}$$

Now, by (25), there exists $\rho_\varepsilon = \rho_\varepsilon(\Omega, \beta, \varepsilon) \leq \delta_\varepsilon$ such that, if the target $\|z\|_{\mathrm{L}^\infty(\omega_0)} \leq \rho_\varepsilon$, then $\inf_{\mathrm{L}^2(\Omega)} J_s \leq \frac{K_1 \delta_\varepsilon^2}{2}$. This, together with (32), yields

$$t_s \frac{K_1 \delta_\varepsilon^2}{2} \leq K_2,$$

whence

$$t_s \leq \frac{K_2}{\delta_\varepsilon^2}.$$

Set

$$\tau_\varepsilon := \frac{K_2}{\delta_\varepsilon^2} + 1.$$

This finishes this step.
Step 3 **Conclusion**
By Step 2, for any $T \geq \tau_\varepsilon$, there exists $t_s \leq \tau_\varepsilon$ such that

$$\|y^T(t_s)\|_{\mathrm{L}^\infty(\Omega)} \leq \delta_\varepsilon, \tag{33}$$

where δ_ε is given by (16). Now, by Bellman's Principle of Optimality, $u^T\restriction_{(t_s,T)}$ is optimal for (2)-(1), with initial datum $y^T(t_s)$ and target z. We took $\rho_\varepsilon \leq \delta_\varepsilon$, Then, we also have

$$\|z\|_{\mathrm{L}^\infty(\omega_0)} \leq \rho_\varepsilon \leq \delta_\varepsilon. \tag{34}$$

Then, we can apply Lemma 1.2, getting (24). This completes the proof. □

1.2. **Proof of Theorem 0.2.** We now prove Theorem 0.2.

Proof of Theorem 0.2. By Lemma 1.4, there exists $\rho_\varepsilon(\Omega, \beta, \varepsilon) > 0$ such that if

$$\|z\|_{\mathrm{L}^\infty(\omega_0)} \leq \rho_\varepsilon \qquad , \tag{35}$$

any optimal control satisfies the turnpike estimate

$$\|u^T(t)-\overline{u}\|_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)-\overline{y}\|_{\mathrm{L}^\infty(\Omega)} \leq \varepsilon\left[\exp\left(-\mu(t-t_s)\right)+\exp\left(-\mu(T-(t-t_s))\right)\right], \qquad \forall t \in [t_s, T]. \tag{36}$$

Set

$$K_0 := \exp(\mu\tau) K\left[1 + \|y_0\|_{\mathrm{L}^\infty(\Omega)} + \delta\right],$$

with $\mu > 0$ the exponential rate defined in (17) and K is given by (13). Note that $K_0 = K_0(\Omega, \beta, y_0)$ and, in particular, it is independent of the time horizon. By the above definition, for every $T > 0$ and for each $t \in [0, \tau_\varepsilon] \cap [0, T]$

$$\|u^T(t) - \overline{u}\|_{\mathrm{L}^\infty(\Omega)} + \|y^T(t) - \overline{y}\|_{\mathrm{L}^\infty(\Omega)} \leq K_0 \exp(-\mu\tau) \leq K_0 \exp(-\mu t). \tag{37}$$

On the other hand, for $t \geq t_s$, (36) holds. Then, (8) follows. □

2. Control acting everywhere: convergence of averages

In this section, we suppose that the control acts everywhere, namely $\omega = \Omega$ in the state equation (2). Our purpose is to prove Theorem 0.3, valid for any data and targets.

In the following Lemma, we observe that, even in the more general case $\omega \subsetneq \Omega$, we have an estimate from above of the infimum of the time-evolution functional in terms of the steady functional. This is the easier task obtained by plugging the steady optimal control in the time-evolution functional. The complicated task is to estimate from below the infimum of the time-evolution functional, in terms of the steady functional. Indeed, the lower bound indicates that the time-evolution strategies cannot perform significantly better than the steady one and this is in general the hardest task in the proof of turnpike results. The key idea is indicated in Lemma 2.2.

Lemma 2.1. *Consider the time-evolution control problem* (2)-(1) *and its steady version* (5)-(4). *Arbitrarily fix* $y_0 \in \mathrm{L}^\infty(\Omega)$ *an initial datum and* $z \in \mathrm{L}^\infty(\omega_0)$ *a target. We have*

$$\inf_{\mathrm{L}^2((0,T)\times\omega)} J_T \leq T \inf_{\mathrm{L}^2(\Omega)} J_s + K, \tag{38}$$

the constant K *being independent of* $T > 0$.

The proof is available in appendix C.

The main idea for the proof of Theorem 0.3 is in the following Lemma, where an alternative representation formula for the time-evolution functional is obtained.

Lemma 2.2. *Consider the functional introduced in* (1)-(2) *and its steady version* (5)-(4)*. Set* $F(y) := \int_0^y f(\xi)d\xi$*. Assume* $\omega = \Omega$*. Suppose the initial datum* $y_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{H}_0^1(\Omega)$*. Then, for any control* $u \in \mathrm{L}^2((0,T)\times\omega)$*, we can rewrite the functional as*

$$\begin{aligned} J_T(u) = &\int_0^T J_s\big(-\Delta y(t,\cdot) + f(y(t,\cdot))\big)dt \\ &+ \frac{1}{2}\int_0^T\int_\Omega |y_t(t,x)|^2\,dxdt \\ &+ \frac{1}{2}\int_\Omega \Big[\|\nabla y(T,x)\|^2 + 2F(y(T,x)) - \|\nabla y_0(x)\|^2 - 2F(y_0(x))\Big]dx, \end{aligned} \tag{39}$$

where, for a.e. $t \in (0,T)$*,* $J_s\big(-\Delta y(t,\cdot)+f(y(t,\cdot))\big)$ *denotes the evaluation of the steady functional* J_s *at control* $u_s(\cdot) := -\Delta y(t,\cdot) + f(y(t,\cdot))$ *and* y *is the state associated to control* u *solution to*

$$\begin{cases} y_t - \Delta y + f(y) = u & \textit{in } (0,T)\times\Omega \\ y = 0 & \textit{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \textit{in } \Omega. \end{cases} \tag{40}$$

In (39), the term $\int_0^T\int_\Omega |y_t(t,x)|^2\,dxdt$ emerges. This means that the time derivative of optimal states has to be small, whence the time-evolving optimal strategies for (1)-(2) are in fact close to the steady ones.
The proof of Lemma 2.2 is based on the following PDE result, which basically asserts that the squared right hand side of the equation

$$\begin{cases} y_t - \Delta y + f(y) = h & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \end{cases}$$

can be written as

$$\|h\|^2_{\mathrm{L}^2((0,T)\times\Omega)} = \|y_t\|^2_{\mathrm{L}^2((0,T)\times\Omega)} + \|-\Delta y + f(y)\|^2_{\mathrm{L}^2((0,T)\times\Omega)} + \text{remainder}, \tag{41}$$

where the remainder depends on the value of the solution at times $t = 0$ and $t = T$.

Lemma 2.3. *Let* Ω *be a bounded open set of* $\mathbb{R}^n$*,* $n \in \{1,2,3\}$*, with* C^∞ *boundary. Let* $f \in \mathrm{C}^3(\mathbb{R};\mathbb{R})$ *be nondecreasing, with* $f(0) = 0$*. Set* $F(y) := \int_0^y f(\xi)d\xi$*. Let* $y_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{H}_0^1(\Omega)$ *be an initial datum and let* $h \in \mathrm{L}^\infty((0,T)\times\Omega)$ *be a source term. Let* y *be the solution to*

$$\begin{cases} y_t - \Delta y + f(y) = h & \textit{in } (0,T)\times\Omega \\ y = 0 & \textit{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \textit{in } \Omega. \end{cases} \tag{42}$$

Then, the following identity holds

$$\begin{aligned} \int_0^T\int_\Omega |h|^2\,dxdt = &\int_0^T\int_\Omega \Big[|y_t|^2 + |-\Delta y + f(y)|^2\Big]dxdt \\ &+ \int_\Omega \Big[\|\nabla y(T,x)\|^2 + 2F(y(T,x)) - \|\nabla y_0(x)\|^2 - 2F(y_0(x))\Big]dx. \end{aligned} \tag{43}$$

Proof of lemma 2.3. We start by proving our assertion for C^∞-smooth data, with compact support. By (42), we have

$$\begin{aligned}\int_0^T\int_\Omega |h|^2\,dxdt &= \int_0^T\int_\Omega |y_t-\Delta y+f(y)|^2\,dxdt\\ &= \int_0^T\int_\Omega \left[|y_t|^2+|-\Delta y+f(y)|^2\right]dxdt\\ &\quad+2\int_0^T\int_\Omega y_t\left[-\Delta y+f(y)\right]dxdt. \end{aligned}\tag{44}$$

We now concentrate on the terms $2\int_0^T\int_\Omega y_t\left[-\Delta y\right]dxdt$ and $2\int_0^T\int_\Omega y_t f(y)\,dxdt$. Integrating by parts in space, we get

$$\begin{aligned}2\int_0^T\int_\Omega y_t\left[-\Delta y\right]dxdt &= \int_0^T\int_\Omega 2\frac{\partial\nabla y}{\partial t}\cdot\nabla y dxdt\\ &= \int_\Omega\left[\|\nabla y(T,x)\|^2-\|\nabla y_0(x)\|^2\right]dx.\end{aligned}\tag{45}$$

By using the chain rule and the definition $F(y) \coloneqq \int_0^y f(\xi)d\xi$, we have

$$\int_0^T\int_\Omega y_t f(y)\,dxdt = \int_0^T\int_\Omega\frac{\partial}{\partial t}\left[F(y)\right]dxdt = \int_\Omega\left[F(y(T,x))-F(y_0(x))\right]dx.\tag{46}$$

By (44), (45) and (46), we get (43).
The conclusion for general data follows from a density argument based on parabolic regularity (see [26, Theorem 7.32 page 182], [25, Theorem 9.1 page 341] or [45, Theorem 9.2.5 page 275]). □

We proceed now with the proof of Lemma 2.2.

Proof of Lemma 2.2. For any control $u\in L^2((0,T)\times\omega)$, by Lemma 2.3 applied to (40), we have

$$\begin{aligned}\frac12\int_0^T\int_\omega |u|^2\,dxdt &= \int_0^T\int_\Omega\left[|y_t|^2+|-\Delta y+f(y)|^2\right]dxdt\\ &\quad+\int_\Omega\left[\|\nabla y(T,x)\|^2+2F(y(T,x))-\|\nabla y_0(x)\|^2-2F(y_0(x))\right]dx.\end{aligned}$$

whence

$$\begin{aligned}J_T(u) &= \frac12\int_0^T\int_\Omega|-\Delta y+f(y)|^2\,dxdt+\frac{\beta}{2}\int_0^T\int_{\omega_0}|y-z|^2\,dxdt\\ &\quad+\frac12\int_0^T\int_\Omega|y_t|^2\,dxdt\\ &\quad+\frac12\int_\Omega\left[\|\nabla y(T,x)\|^2+2F(y(T,x))-\|\nabla y_0(x)\|^2-2F(y_0(x))\right]dx.\end{aligned}$$

By the above equality and the definition of J_s (5)-(4), formula (39) follows. □

The last Lemma needed to prove Theorem 0.3 is the following one.

Lemma 2.4. *Consider the time-evolution control problem* (2)-(1) *and its steady version* (5)-(4). *Assume* $\omega = \Omega$. *Arbitrarily fix* $y_0 \in \mathrm{L}^\infty(\Omega) \cap \mathrm{H}_0^1(\Omega)$ *an initial datum and* $z \in \mathrm{L}^\infty(\omega_0)$ *a target. Let* u^T *be an optimal control for* (2)-(1) *and let* y^T *be the corresponding state, solution to* (2), *with control* u^T *and initial datum* y_0. *Then,*

(1) there exists a T*-independent constant* K *such that*

$$\left| \inf_{\mathrm{L}^2((0,T)\times\omega)} J_T - T \inf_{\mathrm{L}^2(\Omega)} J_s \right| \leq K; \tag{47}$$

(2) the L^2 *norm of the time derivative of the optimal state is bounded uniformly in* T

$$\left\| y_t^T \right\|_{\mathrm{L}^2((0,T)\times\Omega)} \leq K, \tag{48}$$

with K *independent of* $T > 0$.

Proof of Lemma 2.4. Step 1 **Proof of**

$$\inf_{\mathrm{L}^2((0,T)\times\omega)} J_T = J_T\left(u^T\right) \geq T \inf_{\mathrm{L}^2(\Omega)} J_s + \frac{1}{2}\int_0^T \int_\Omega \left|y_t^T(t,x)\right|^2 dx dt - \frac{1}{2}\int_\Omega \left[\|\nabla y_0(x)\|^2 + 2F\left(y_0(x)\right)\right] dx.$$

We start observing that, since the nonlinearity f is nondecreasing and $f(0) = 0$, the primitive F is nonnegative

$$F(y) \geq 0, \qquad \forall\, y \in \mathbb{R}. \tag{49}$$

Let u^T be an optimal control for (2)-(1) and let y^T be the corresponding state, solution to (2), with control u^T and initial datum y_0. By Lemma 2.2 and (49), we have

$$\begin{aligned} J_T\left(u^T\right) &= \int_0^T J_s\big(-\Delta y^T(t,\cdot) + f\left(y^T(t,\cdot)\right)\big) dt \\ &\quad + \frac{1}{2}\int_0^T \int_\Omega \left|y_t^T(t,x)\right|^2 dx dt \\ &\quad + \frac{1}{2}\int_\Omega \left[\left\|\nabla y^T(T,x)\right\|^2 + 2F\left(y^T(T,x)\right) - \|\nabla y_0(x)\|^2 - 2F\left(y_0(x)\right)\right] dx \\ &\geq \int_0^T J_s\big(-\Delta y^T(t,\cdot) + f\left(y^T(t,\cdot)\right)\big) dt \\ &\quad + \frac{1}{2}\int_0^T \int_\Omega \left|y_t^T(t,x)\right|^2 dx dt \\ &\quad - \frac{1}{2}\int_\Omega \left[\|\nabla y_0(x)\|^2 + 2F\left(y_0(x)\right)\right] dx. \end{aligned} \tag{50}$$

Now, for a.e. $t \in (0,T)$, by definition of infimum

$$J_s\big(-\Delta y^T(t,\cdot) + f\left(y^T(t,\cdot)\right)\big) \geq \inf_{\mathrm{L}^2(\Omega)} J_s.$$

The above inequality and (50) yield

$$\begin{aligned}
J_T\left(u^T\right) &\geq \int_0^T J_s\big(-\Delta y^T(t,\cdot)+f\left(y^T(t,\cdot)\right)\big)dt \\
&\quad+\frac{1}{2}\int_0^T\int_\Omega \left|y_t^T(t,x)\right|^2 dxdt-\frac{1}{2}\int_\Omega\left[\|\nabla y_0(x)\|^2+2F\left(y_0(x)\right)\right]dx \\
&\geq \int_0^T\left[\inf_{\mathrm{L}^2(\Omega)} J_s\right]dt+\frac{1}{2}\int_0^T\int_\Omega \left|y_t^T(t,x)\right|^2 dxdt-\frac{1}{2}\int_\Omega\left[\|\nabla y_0(x)\|^2+2F\left(y_0(x)\right)\right]dx \\
&= T\inf_{\mathrm{L}^2(\Omega)} J_s+\frac{1}{2}\int_0^T\int_\Omega \left|y_t^T(t,x)\right|^2 dxdt-\frac{1}{2}\int_\Omega\left[\|\nabla y_0(x)\|^2+2F\left(y_0(x)\right)\right]dx,
\end{aligned}$$

whence

$$\inf_{\mathrm{L}^2((0,T)\times\omega)} J_T=J_T\left(u^T\right)\geq T\inf_{\mathrm{L}^2(\Omega)} J_s+\frac{1}{2}\int_0^T\int_\Omega \left|y_t^T(t,x)\right|^2 dxdt-\frac{1}{2}\int_\Omega\left[\|\nabla y_0(x)\|^2+2F\left(y_0(x)\right)\right]dx. \tag{51}$$

Step 2 **Conclusion**

On the one hand, by Lemma 2.1, we have

$$\inf_{\mathrm{L}^2((0,T)\times\omega)} J_T-T\inf_{\mathrm{L}^2(\Omega)} J_s\leq K, \tag{52}$$

the constant K being independent of $T>0$. On the other hand, by (51), we get

$$\inf_{\mathrm{L}^2((0,T)\times\omega)} J_T-T\inf_{\mathrm{L}^2(\Omega)} J_s\geq -K. \tag{53}$$

By (52) and (53), inequality (47) follows.

It remains to prove (48). By (51) and Lemma 2.1, we have

$$T\inf_{\mathrm{L}^2(\Omega)} J_s+\frac{1}{2}\int_0^T\int_\Omega \left|y_t^T(t,x)\right|^2 dxdt-K\leq \inf_{\mathrm{L}^2((0,T)\times\omega)} J_T\leq T\inf_{\mathrm{L}^2(\Omega)} J_s+K,$$

whence

$$\frac{1}{2}\int_0^T\int_\Omega \left|y_t^T(t,x)\right|^2 dxdt\leq K,$$

as required. □

We are now ready to prove Theorem 0.3.

Proof of Theorem 0.3. Estimate (10) follows directly from Lemma 2.4 (2.).

It remains to prove the convergence of the averages. By the regularizing effect of the state equation (2) and Lemma 1.1, we can reduce to the case of initial datum $y_0\in \mathrm{L}^\infty(\Omega)\cap \mathrm{H}_0^1(\Omega)$. By Lemma 2.4, we have

$$\left|\inf_{\mathrm{L}^2((0,T)\times\omega)} J_T-T\inf_{\mathrm{L}^2(\Omega)} J_s\right|\leq K. \tag{54}$$

Then,

$$\begin{aligned}
\left|\frac{1}{T}\inf_{\mathrm{L}^2((0,T)\times\omega)} J_T-\inf_{\mathrm{L}^2(\Omega)} J_s\right| &= \frac{1}{T}\left|\inf_{\mathrm{L}^2((0,T)\times\omega)} J_T-T\inf_{\mathrm{L}^2(\Omega)} J_s\right| \\
&\leq \frac{K}{T}\underset{T\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

as required. □

3. Numerical simulations

This section is devoted to a numerical illustration of Theorem 0.2. Our goal is to check that turnpike property is fulfilled for small target, regardless of the size of the initial datum.

We deal with the optimal control problem

$$\min_{u\in \mathrm{L}^2((0,T)\times(0,\frac{1}{2}))} J_T(u) = \frac{1}{2}\int_0^T\int_0^{\frac{1}{2}} |u|^2 dxdt + \frac{\beta}{2}\int_0^T\int_0^1 |y-z|^2 dxdt,$$

where:

$$\begin{cases} y_t - y_{xx} + y^3 = u\chi_{(0,\frac{1}{2})} & (t,x)\in(0,T)\times(0,1)\\ y(t,0)=y(t,1)=0 & t\in(0,T)\\ y(0,x)=y_0(x) & x\in(0,1). \end{cases}$$

We choose as initial datum $y_0 \equiv 10$ and as target $z \equiv 1$.

We solve the above semilinear heat equation by using the semi-implicit method:

$$\begin{cases} \dfrac{Y_{i+1}-Y_i}{\Delta t} - \Delta Y_{i+1} + Y_i^3 = U_i\chi_{(0,\frac{1}{2})} & i=0,\dots,N_t-1\\ Y_0 = y_0, \end{cases}$$

where Y_i and U_i denote resp. a time discretization of the state and the control.

The optimal control is determined by a Gradient Descent method, with constant stepsize. The optimal state is depicted in fig. 4.

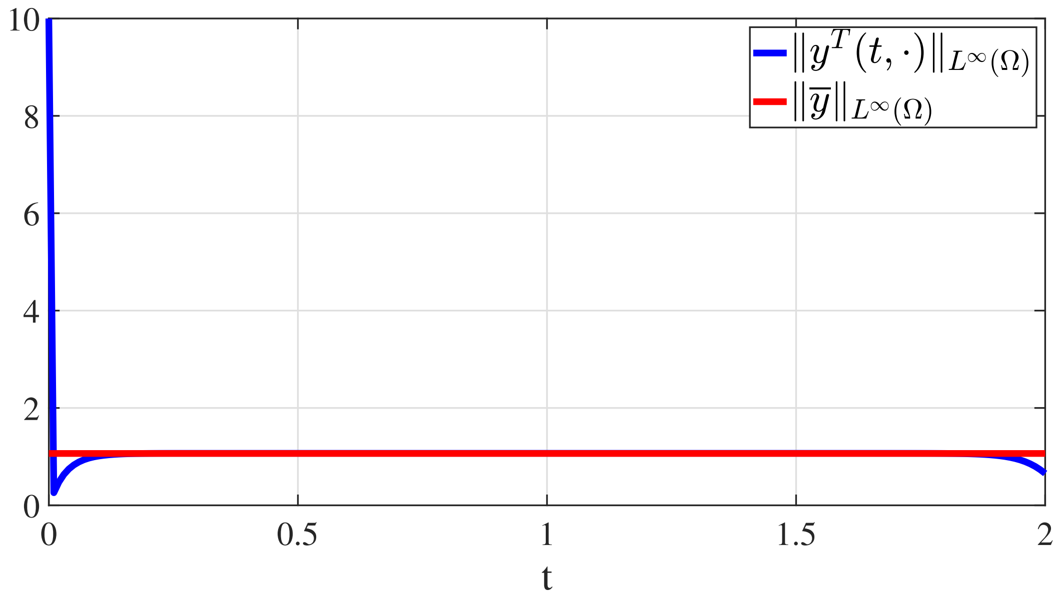

Figure 4. graph of the function $t \longrightarrow \|y^T(t)\|_{\mathrm{L}^\infty(\Omega)}$ (in blue) and $\|\overline{y}\|_{\mathrm{L}^\infty(\Omega)}$ (in red), where y^T denotes an optimal state, whereas $\overline{y}$ stands for an optimal steady state.

4. Conclusions and open problems

In this manuscript we have obtained some global turnpike results for an optimal control problem governed by a nonlinear state equation. For any data and small targets, we have shown that the exponential turnpike property holds (Theorem 0.2). For arbitrary targets, we have proved the convergence of averages (Theorem 0.3), under the added assumption of controlling everywhere. One of the main tools employed for our analysis is an L^{∞} bound of the norm of the optima, uniform in the time horizon (Lemma 1.1). Numerical simulation have been performed, which confirms the theoretical results.

We present now an interesting open problem in the field.

In Theorem 0.3 we have proved the convergence of averages for large targets, in the context of control everywhere. An interesting challenge is to prove the exponential turnpike property, even if the control is local (namely $\omega \subsetneq \Omega$). The challenge is to prove the following conjecture.

Conjecture 4.1. Consider the control problem (2)-(1). Take any initial datum $y_0 \in L^{\infty}(\Omega)$ and any target $z \in L^{\infty}(\omega_0)$. Let u^T be a minimizer of (1). There exists an optimal pair $(\overline{u}, \overline{y})$ for (5)-(4) such that

$$\|u^T(t)-\overline{u}\|_{L^{\infty}(\Omega)}+\|y^T(t)-\overline{y}\|_{L^{\infty}(\Omega)} \leq K\left[\exp\left(-\mu t\right)+\exp\left(-\mu(T-t)\right)\right], \qquad \forall t \in [0,T], \tag{55}$$

the constants K and $\mu > 0$ being independent of the time horizon T.

In [30] special large targets z are constructed, such that the optimal control for the steady problem (5)-(4) is not unique. For those targets, a question arises: if the turnpike property is satisfied, which minimizer for (5)-(4) attracts the optimal solutions to (2)-(1)?

Note that, in the context of internal control, the counterexample to uniqueness in [30] is valid in case of local control $\omega \subsetneq \Omega$.

Generally speaking a further investigation is required for the linearized optimality system determined in [33, subsection 3.1]. We introduce the problem. As in (3), consider the optimality system for (2)-(1)

$$\begin{cases} y^T_t - \Delta y^T + f(y^T) = -q^T\chi_{\omega} & \text{in } (0,T)\times\Omega \\ y^T = 0 & \text{on } (0,T)\times\partial\Omega \\ y^T(0,x) = y_0(x) & \text{in } \Omega \\ -q^T_t - \Delta q^T + f'(y^T)q^T = \beta(y^T - z)\chi_{\omega_0} & \text{in } (0,T)\times\Omega \\ q^T = 0 & \text{on } (0,T)\times\partial\Omega \\ q^T(T,x) = 0 & \text{in } \Omega. \end{cases} \tag{56}$$

Pick any optimal pair $(\overline{u}, \overline{y})$ for (5)-(4). By the first order optimality conditions, the steady optimal control reads as $\overline{u} = -\overline{q}\chi_{\omega}$, with

$$\begin{cases} -\Delta\overline{y} + f(\overline{y}) = -\overline{q}\chi_{\omega} & \text{in } \Omega \\ \overline{y} = 0 & \text{on } \partial\Omega \\ -\Delta\overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{\omega_0} & \text{in } \Omega \\ \overline{q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{57}$$

As in [33], we introduce the perturbation variables

$$\eta^T := y^T - \overline{y} \quad \text{and} \quad \varphi^T := q^T - \overline{q} \tag{58}$$

and we write down the linearized optimality system around $(\overline{u}, \overline{y})$

$$\begin{cases} \eta_t^T - \Delta \eta^T + f'(\overline{y})\eta^T = -\varphi^T \chi_\omega & \text{in } (0,T)\times\Omega \\ \eta^T = 0 & \text{on } (0,T)\times\partial\Omega \\ \eta^T(0,x) = y_0(x) - \overline{y}(x) & \text{in } \Omega \\ -\varphi_t^T - \Delta\varphi^T + f'(\overline{y})\varphi^T = \left(\beta\chi_{\omega_0} - f''(\overline{y})\,\overline{q}\right)\eta^T & \text{in } (0,T)\times\Omega \\ \varphi^T = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi^T(T,x) = -\overline{q}(x) & \text{in } \Omega. \end{cases} \tag{59}$$

As pointed out in [33, Theorem 1 in subsection 3.1], a key point is to check the validity of the turnpike property for the linearized optimality system (59). This is complicated because of the term $\beta\chi_{\omega_0} - f''(\overline{y})\,\overline{q}$, whose sign is unknown for general large targets. Furthermore, in case of nonuniqueness of steady optimum, it would be interesting to compute the spectrum of the linearized system around any steady optima to check if among them one is a better attractor.

Appendix A. Parabolic regularity results

One of the key tool to carry on the proof of Lemma 1.1 is the following regularity result.

Lemma A.1. *Let Ω be a bounded open set of $\mathbb{R}^n$, $n \in \{1,2,3\}$, with C^2 boundary. Let $c \in L^\infty((0,T)\times\Omega)$ be nonnegative. Let $y_0 \in L^\infty(\Omega)$ be an initial datum and let $h \in L^\infty((0,T)\times\Omega)$ be a source term. Let y be the solution to*

$$\begin{cases} y_t - \Delta y + cy = h & \textit{in } (0,T)\times\Omega \\ y = 0 & \textit{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \textit{in } \Omega. \end{cases}$$

Then, $y \in L^2((0,T); L^\infty(\Omega))$ and we have

$$\|y\|_{L^2((0,T);L^\infty(\Omega))} \leq K\left[\|y_0\|_{L^\infty(\Omega)} + \|h\|_{L^2((0,T)\times\Omega)}\right], \tag{60}$$

where K is independent of the potential $c \geq 0$, the time horizon T and the initial datum y_0.

Proof of Lemma A.1. Step 1 **Comparison**
Let ψ be the solution to:

$$\begin{cases} \psi_t - \Delta\psi = |h| & \text{in } (0,T)\times\Omega \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi(0,x) = |y_0|. & \text{in } \Omega \end{cases} \tag{61}$$

Since $c \geq 0$, a.e. in $(0,T)\times\Omega$, by a comparison argument, for each $t \in [0,T]$:

$$|y(t,x)| \leq \psi(t,x), \quad \text{a.e. } x \in \Omega. \tag{62}$$

Now, since y_0 and h are bounded, again by comparison principle applied to (61), ψ is bounded. Hence, by (62), y is bounded as well and

$$\int_0^T \|y(t)\|^2_{L^\infty(\Omega)} dt \leq \int_0^T \|\psi(t)\|^2_{L^\infty(\Omega)} dt. \tag{63}$$

Then, to conclude it suffices to show

$$\|\psi\|_{L^2(0,T;L^\infty(\Omega))} \leq K\left[\|y_0\|_{L^\infty(\Omega)} + \|h\|_{L^2((0,T)\times\Omega)}\right],$$

the constant K being independent of T.

Step 2 **Splitting**

Split $\psi = \xi + \chi$, where ξ solves:

$$\begin{cases} \xi_t - \Delta\xi = 0 & \text{in } (0,T)\times\Omega \\ \xi = 0 & \text{on } (0,T)\times\partial\Omega \\ \xi(0,x) = |y_0| & \text{in } \Omega \end{cases} \tag{64}$$

while χ satisfies:

$$\begin{cases} \chi_t - \Delta\chi = |h| & \text{in } (0,T)\times\Omega \\ \chi = 0 & \text{on } (0,T)\times\partial\Omega \\ \chi(0,x) = 0 & \text{in } \Omega. \end{cases} \tag{65}$$

First of all, we prove an estimate like (60) for ξ. We start by employing maximum principle (see [34]) to (63), getting

$$\|\xi\|_{L^\infty((0,T)\times\Omega)} \leq \|y_0\|_{L^\infty(\Omega)}. \tag{66}$$

Now, if $T \geq 1$, by the regularizing effect and the exponential stability of the heat equation, for any $t \in [1,T]$, we have

$$\|\xi(t)\|_{L^\infty(\Omega)} \leq K\|\xi(t-1)\|_{L^2(\Omega)} \leq K\exp\left(-\lambda_1(t-1)\right)\|y_0\|_{L^2(\Omega)}, \tag{67}$$

the constant K depending only on the domain Ω. Then, by (66) and (67), for any $T > 0$, for every $t \in [0,T]$,

$$\|\xi(t)\|_{L^\infty(\Omega)} \leq K\min\left\{1, \exp\left(-\lambda_1(t-1)\right)\right\}\|y_0\|_{L^\infty(\Omega)}, \tag{68}$$

with $K = K(\Omega)$.

Now, we focus on (65). By parabolic regularity (see e.g. [14, Theorem 5 subsection 7.1.3]), $\chi \in L^2(0,T;H^2(\Omega))$, with $\chi_t \in L^2((0,T)\times\Omega)$. Then, by multiplying (65) by $-\Delta\chi$ and integrating over $[0,T]\times\Omega$, we obtain

$$\frac{1}{2}\|\nabla\chi(T)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega |\Delta\chi|^2 dxdt \leq \|h\|_{L^2((0,T)\times\Omega)}\|\Delta\chi\|_{L^2((0,T)\times\Omega)}.$$

By Young's Inequality,

$$\int_0^T\int_\Omega |\Delta\chi|^2 dxdt \leq \frac{1}{2}\|h\|^2_{L^2((0,T)\times\Omega)} + \frac{1}{2}\|\Delta\chi\|^2_{L^2((0,T)\times\Omega)},$$

which leads to

$$\int_0^T\int_\Omega |\Delta\chi|^2 dxdt \leq \|h\|^2_{L^2((0,T)\times\Omega)}.$$

Now, by [14, Theorem 6 subsection 5.6.3] and [14, Theorem 4 subsection 6.3.2],

$$\int_0^T \|\chi\|^2_{L^\infty(\Omega)}dt \leq K\int_0^T \|\chi\|^2_{H^2(\Omega)}dt \leq K\int_0^T\int_\Omega |\Delta\chi|^2 dxdt \leq K\|h\|^2_{L^2((0,T)\times\Omega)}. \tag{69}$$

Finally, by (63), (68) and (69),

$$\int_0^T \|y\|^2_{L^\infty(\Omega)}dt \leq 2\int_0^T \|\xi\|^2_{L^\infty(\Omega)}dt + 2K\int_0^T \|\chi\|^2_{L^\infty(\Omega)}dt \leq K\left[\|y_0\|^2_{L^\infty(\Omega)} + \|h\|^2_{L^2((0,T)\times\Omega)}\right],$$

as required. □

The following regularity result is employed in the proof of Lemma 1.1.

Lemma A.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, with $\partial\Omega \in \mathrm{C}^\infty$. Let $c \in \mathrm{L}^\infty((0,T)\times\Omega)$ be nonnegative. Let $y_0 \in \mathrm{L}^\infty(\Omega)$ an initial datum and $h \in \mathrm{L}^\infty((0,T)\times\Omega)$ a source term. Let $\overline{T} \in (0,T)$ and set $N := \lfloor T/\overline{T} \rfloor$. Let y be the solution to*

$$\begin{cases} y_t - \Delta y + cy = h & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases}$$

Then, $y \in \mathrm{L}^\infty((0,T)\times\Omega)$ and we have

$$\|y\|_{\mathrm{L}^\infty((0,T)\times\Omega)} \leq K\left[\|y_0\|_{\mathrm{L}^\infty(\Omega)} + \max_{i=1,\ldots,N}\|h\|_{\mathrm{L}^2(((i-1)\overline{T},i\overline{T});\mathrm{L}^\infty(\Omega))} + \|h\|_{\mathrm{L}^2(N\overline{T},T;\mathrm{L}^\infty(\Omega))}\right], \tag{70}$$

where K is independent of the potential $c \geq 0$ and the time horizon T.

Proof of Lemma A.2. Step 1 **Comparison argument**
Let ψ be the solution to:

$$\begin{cases} \psi_t - \Delta\psi = |h| & \text{in } (0,T)\times\Omega \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi(0,x) = |y_0|. & \text{in } \Omega \end{cases} \tag{71}$$

Since $c \geq 0$, a.e. in $(0,T)\times\Omega$, by a comparison argument, for each $t \in [0,T]$:

$$|y(t,x)| \leq \psi(t,x), \quad \text{a.e. } x \in \Omega. \tag{72}$$

Now, since y_0 and h are bounded, again by comparison principle applied to (71), ψ is bounded. Hence, by (72), y is bounded as well and

$$\|y\|_{\mathrm{L}^\infty((0,T)\times\Omega)} \leq \|\psi\|_{\mathrm{L}^\infty((0,T)\times\Omega)}. \tag{73}$$

Then, to conclude it suffices to show

$$\|\psi\|_{\mathrm{L}^\infty((0,T)\times\Omega)} \leq K\left[\|y_0\|_{\mathrm{L}^\infty(\Omega)} + \max_{i=1,\ldots,N}\|h\|_{\mathrm{L}^2(((i-1)\overline{T},i\overline{T});\mathrm{L}^\infty(\Omega))} + \|h\|_{\mathrm{L}^2(N\overline{T},T;\mathrm{L}^\infty(\Omega))}\right],$$

the constant K being independent of T.
Step 2 **Conclusion**
Let $\{S(t)\}_{t\in\mathbb{R}^+}$ be the heat semigroup on Ω, with zero Dirichlet boundary conditions. Fix $\varepsilon \in (0,\overline{T})$. By the regularizing effect of the heat equation (see, e.g. [6, Theorem 10.1, section 10.1]), for any $t \geq \varepsilon$,

$$\|S(t)y_0\|_{\mathrm{L}^\infty(\Omega)} \leq K\exp(-\mu(t-\varepsilon))\|y_0\|_{\mathrm{L}^2(\Omega)} \leq K\exp(-\mu(t-\varepsilon))\|y_0\|_{\mathrm{L}^\infty(\Omega)}.$$

For $t \in [0,\varepsilon]$, by comparison principle, we have

$$\|S(t)y_0\|_{\mathrm{L}^\infty(\Omega)} \leq K\|y_0\|_{\mathrm{L}^\infty(\Omega)} \leq K\exp(-\mu(t-\varepsilon))\|y_0\|_{\mathrm{L}^\infty(\Omega)},$$

being $\exp(-\mu(t-\varepsilon)) \geq 1$. Hence, for any $t \geq 0$,

$$\|S(t)y_0\|_{\mathrm{L}^\infty(\Omega)} \leq K\exp(-\mu(t-\varepsilon))\|y_0\|_{\mathrm{L}^\infty(\Omega)}. \tag{74}$$

Then, by the Duhamel formula, for any $t \in [0,T]$, we have

$$\psi(t) = S(t)\left(|y_0|\right) + \int_0^t S(t-s)\left(|h(s)|\right)ds. \tag{75}$$

Now, by (74), for any $t \geq 0$,

$$\|S(t)\left(|y_0|\right)\|_{\mathrm{L}^\infty(\Omega)} \leq K \exp\left(-\mu(t-\varepsilon)\right)\|y_0\|_{\mathrm{L}^\infty(\Omega)}. \tag{76}$$

Besides, by applying (74) to the integral term $\eta(t) := \int_0^t S(t-s)\left(|h(s)|\right) ds$ in (75), we obtain

$$\begin{aligned}
\|\eta(t)\|_{\mathrm{L}^\infty} &\leq \int_0^t \|S(t-s)\left(|h(s)|\right)\|_{\mathrm{L}^\infty} ds \\
&\leq K\int_0^t \exp(-\mu(t-s-\varepsilon))\|h(s)\|_{\mathrm{L}^\infty} ds \\
&\leq K\left[\sum_{i=1}^{\lfloor \frac{t}{\overline{T}}\rfloor} \exp(-\mu(t-\varepsilon-i\overline{T}))\int_{(i-1)\overline{T}}^{i\overline{T}} \exp(-\mu(i\overline{T}-s))\|h(s)\|_{\mathrm{L}^\infty} ds\right. \\
&\quad \left. +K\int_{(\lfloor \frac{t}{\overline{T}}\rfloor-1)\overline{T}}^{t} \exp(-\mu(t-s-\varepsilon))\|h(s)\|_{\mathrm{L}^\infty} ds\right] \\
&\leq K\left\{\sum_{i=1}^{\lfloor \frac{t}{\overline{T}}\rfloor} \exp(-\mu(t-\varepsilon-i\overline{T}))\left[\int_{(i-1)\overline{T}}^{i\overline{T}} \exp(-2\mu(i\overline{T}-s))ds\right]^{\frac{1}{2}}\left[\int_{(i-1)\overline{T}}^{i\overline{T}} \|h(s)\|^2_{\mathrm{L}^\infty} ds\right]^{\frac{1}{2}}\right. \\
&\quad \left. +K\left[\int_{(\lfloor \frac{t}{\overline{T}}\rfloor-1)\overline{T}}^{t} \exp(-2\mu(t-s-\varepsilon))ds\right]^{\frac{1}{2}}\left[\int_{(\lfloor \frac{t}{\overline{T}}\rfloor-1)\overline{T}}^{t} \|h(s)\|^2_{\mathrm{L}^\infty} ds\right]^{\frac{1}{2}}\right\} \\
&\leq K\left[\sum_{i=1}^{\lfloor \frac{t}{\overline{T}}\rfloor} \exp(-\mu(t-\varepsilon-i\overline{T}))\|h\|_{\mathrm{L}^2((i-1)\overline{T},i\overline{T};\mathrm{L}^\infty(\Omega))}\right. \\
&\quad \left. +\|h\|_{\mathrm{L}^2(\lfloor \frac{t}{\overline{T}}\rfloor,t;\mathrm{L}^\infty(\Omega))}\right] \\
&\leq K\left[\sum_{i=1}^{+\infty} \exp(-\mu(t-\varepsilon-i\overline{T}))\max_{i=1,\dots,N}\|h\|_{\mathrm{L}^2(((i-1)\overline{T},i\overline{T});\mathrm{L}^\infty(\Omega))}\right. \\
&\quad \left. +\|h\|_{\mathrm{L}^2(\lfloor \frac{t}{\overline{T}}\rfloor,t;\mathrm{L}^\infty(\Omega))}\right] \\
&\leq K\left[\|h\|_{\mathrm{L}^2((i-1)\overline{T},i\overline{T};\mathrm{L}^\infty(\Omega))} + \|h\|_{\mathrm{L}^2(N\overline{T},T;\mathrm{L}^\infty(\Omega))}\right].
\end{aligned} \tag{77}$$

Then, by (76) and appendix A, for each $t \in [0,T]$

$$\|\psi(t)\|_{\mathrm{L}^\infty(\Omega)} \leq K\exp\left(-\mu(t-\varepsilon)\right)\left[\|y_0\|_{\mathrm{L}^\infty(\Omega)} + \|h\|_{\mathrm{L}^2((i-1)\overline{T},i\overline{T};\mathrm{L}^\infty(\Omega))} + \|h\|_{\mathrm{L}^2(N\overline{T},T;\mathrm{L}^\infty(\Omega))}\right]$$

as desired. □

Remark A.3. Lemma A.1 can be applied to a bounded solution y to (2). Indeed, set

$$c_y(t,x) := \begin{cases} \dfrac{f(y(t,x))}{y(t,x)} & y(t,x) \neq 0 \\ f'(0) & y(t,x) = 0. \end{cases}$$

Since f is increasing and $f(0) = 0$, we have $c_y \geq 0$. Hence, we are in position to apply Lemma A.1, with potential c_y.

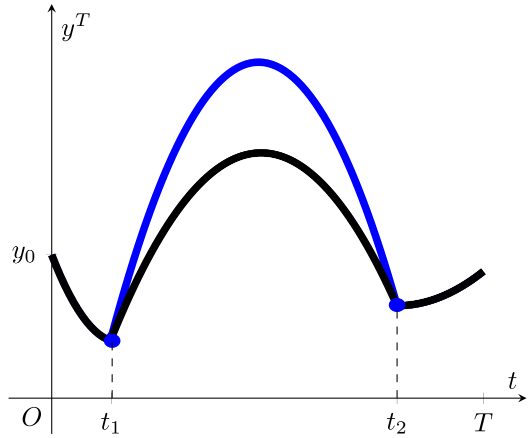

FIGURE 5. The idea of the proof of Lemma 1.1 is to use controllability for (2) to show that optima for (2)-(1) cannot oscillate too much. Indeed, consider a the time interval $[t_1, t_2]$. By controllability, we can link $y^T(t_1, \cdot)$ and $y^T(t_2, \cdot)$ by a controlled trajectory (in blue). By optimality, the optimum (in black) is bounded by the constructed trajectory.

APPENDIX B. UNIFORM BOUNDS OF THE OPTIMA

As pointed out in [33, subsection 3.2], the norms of optimal controls and states can be estimated in terms of the initial datum for (2) and the running target in an averaged sense, using the inequality

$$J_T\left(u^T\right) \leq J_T(0), \tag{78}$$

where u^T is any optimal control for the time-evolution problem. We have to ensure that the bounds actually holds for any time, i.e. we need to show that optimal controls and states do not oscillate too much.

The proof of Lemma 1.1 follows the scheme:

- divide the interval $[0, T]$ into subintervals of T-independent length;
- estimate the magnitude of the optima in each subinterval by using controllability (fig. 5).

In order to carry out the proof of Lemma 1.1, we need some preliminary lemmas. We start by stating some results on the controllability of a dissipative semilinear heat equation.

B.1. **Controllability of dissipative semilinear heat equation.**

Lemma B.1. *Let $y_0 \in L^\infty(\Omega)$ be an initial datum. Let $\hat{y} \in L^\infty((0,+\infty)\times\Omega)$ be a target trajectory, solution to (2), with control $\hat{u} \in L^\infty((0,T)\times\omega)$. Let $R > 0$. Suppose $\|y_0\|_{L^\infty(\Omega)} \leq R$ and $\|\hat{y}\|_{L^\infty((0,+\infty)\times\Omega)} \leq R$. Then, there exists $T_R = T_R(\Omega, f, \omega, R)$, such that for any $T \geq T_R$ there exists $u \in L^\infty((0,T)\times\omega)$ such that the solution y to the controlled equation (2), with initial datum y_0 and control u, verifies the final condition*

$$y(T,x) = \hat{y}(T,x) \quad \text{in } \Omega \tag{79}$$

and

$$\|u-\hat{u}\|_{\mathrm{L}^{\infty}((0,T)\times\omega)} \leq K\|y_0-\hat{y}(0)\|_{\mathrm{L}^{\infty}(\Omega)}, \tag{80}$$

where the constant K depends only on Ω, f, ω and R.

The proof of the above lemma is classical (see, e.g. [16, 3]).

In order to prove Lemma 1.1, we introduce an optimal control problem, with specified terminal states. Let $t_1 < t_2$. Let $\hat{y}$ be a target trajectory, bounded solution to (2) in (t_1,t_2), i.e.

$$\begin{cases} \hat{y}_t-\Delta\hat{y}+f(\hat{y})=\hat{u}\chi_{\omega} & \text{in } (t_1,t_2)\times\Omega \\ \hat{y}=0 & \text{on } (t_1,t_2)\times\partial\Omega \\ \hat{y}(t_1,x)=\hat{y}_0(x) & \text{in } \Omega, \end{cases} \tag{81}$$

with initial datum $\hat{y}_0\in\mathrm{L}^{\infty}(\Omega)$ and control $\hat{u}\in\mathrm{L}^{\infty}((t_1,t_2)\times\omega)$.

For any control $u\in\mathrm{L}^2((t_1,t_2)\times\omega)$, the corresponding state y is the solution to:

$$\begin{cases} y_t-\Delta y+f(y)=u\chi_{\omega} & \text{in } (t_1,t_2)\times\Omega \\ y=0 & \text{on } (t_1,t_2)\times\partial\Omega \\ y(t_1,x)=\hat{y}(t_1,x) & \text{in } \Omega. \end{cases} \tag{82}$$

We introduce the set of admissible controls

$$\mathscr{U}_{\mathrm{ad}} := \left\{u\in\mathrm{L}^2((t_1,t_2)\times\omega) \mid y(t_2,\cdot)=\hat{y}(t_2,\cdot)\right\}.$$

By definition, $\hat{u}\in\mathscr{U}_{\mathrm{ad}}$. Hence, $\mathscr{U}_{\mathrm{ad}}\neq\varnothing$. We consider the optimal control problem

$$\min_{u\in\mathscr{U}_{\mathrm{ad}}} J_{t_1,t_2}(u)=\frac{1}{2}\int_{t_1}^{t_2}\int_{\omega}|u|^2dxdt+\frac{\beta}{2}\int_{t_1}^{t_2}\int_{\omega_0}|y-z|^2dxdt, \tag{83}$$

with running target $z\in\mathrm{L}^{\infty}(\omega_0)$. By the direct methods in the calculus of variations, the functional J_{t_1,t_2} admits a global minimizer in the set of admissible controls $\mathscr{U}_{\mathrm{ad}}$.

We now bound the minimal value of the functional (83), showing that the magnitude of the control $\hat{u}$ in the time interval $[t_1,t_2-T_R]$ can be neglected when estimating the cost of controllability. Namely, what matters is the norm of $\hat{u}$ in the final time interval $[t_2-T_R,t_2]$.

Lemma B.2. *Consider the optimal control problem* (82)-(83)*, with $t_2-t_1\geq 2T_R$. Then,*

$$\begin{aligned}\min_{\mathscr{U}_{ad}} J_{t_1,t_2} \quad\leq\quad & K\left[\|\hat{y}(t_1,\cdot)\|^2_{\mathrm{L}^{\infty}(\Omega)}+(t_2-t_1)\|z\|^2_{\mathrm{L}^{\infty}(\omega_0)}\right.\\ & \left.+\|\hat{u}\|^2_{\mathrm{L}^{\infty}((t_2-T_R,t_2)\times\omega)}+\|\hat{y}(t_2-T_R,\cdot)\|^2_{\mathrm{L}^{\infty}(\Omega)}\right],\end{aligned} \tag{84}$$

the constant K being independent of the time horizon $t_2-t_1\geq 2T_R$.

Proof of Lemma B.2. Step 1 **A quasi-optimal control**
To get the desired bound, we introduce a quasi-optimal control u for (82)-(83), linking $\hat{y}(t_1,\cdot)$ and $y^T(t_2,\cdot)$. The control strategy is the following

(1) employ null control for time $t\in[0,t_2-T_R]$;
(2) match the final condition by control w, for $t\in[t_2-T_R,t_2]$.

Let us denote by y^0 the solution to the semilinear problem with null control

$$\begin{cases} y^0_t - \Delta y^0 + f\left(y^0\right) = 0 & \text{in } (t_1, t_2) \times \Omega \\ y^0 = 0 & \text{on } (t_1, t_2) \times \partial\Omega \\ y^0(t_1, x) = \hat{y}(t_1, x) & \text{in } \Omega. \end{cases} \tag{85}$$

By Lemma B.1, there exists $w \in \mathrm{L}^\infty((t_2 - T_R, t_2) \times \omega)$, steering (82) from $y^0(t_1, \cdot)$ to $\hat{y}\left(t_2, \cdot\right)$ in the time interval $(t_2 - T_R, t_2)$, with estimate

$$\|w - \hat{u}\|_{\mathrm{L}^\infty((t_2 - T_R, t_2)\times\omega)} \leq K \left\|y^0\left(t_2 - T_R\right) - \hat{y}\left(t_2 - T_R\right)\right\|_{\mathrm{L}^\infty(\Omega)}, \tag{86}$$

Then, set

$$u := \begin{cases} 0 & \text{in } \left(0, t_2 - T_R\right) \\ w & \text{in } \left(t_2 - T_R, t_2\right). \end{cases} \tag{87}$$

By (86), we can bound the norm of the control,

$$\|u\|_{\mathrm{L}^\infty((t_1,t_2)\times\omega)} \leq K\left[\left\|y^0\left(t_2 - T_R\right) - \hat{y}\left(t_2 - T_R\right)\right\|_{\mathrm{L}^\infty(\Omega)} + \|\hat{u}\|_{\mathrm{L}^\infty((t_2 - T_R, t_2)\times\omega)}\right]. \tag{88}$$

Step 2 **Conclusion**
Consider the control u introduced in (87) and let y be the solution to (82), with initial datum y_0 and control u. Then, we have

$$\begin{aligned}
\min_{\mathscr{U}_{\mathrm{ad}}} J_{t_1,t_2} &\leq J_{t_1,t_2}(u) \\
&= \frac{1}{2}\int_{t_1}^{t_2}\int_\omega |u|^2 dxdt + \frac{\beta}{2}\int_{t_1}^{t_2}\int_{\omega_0} |y - z|^2 dxdt \\
&= \frac{1}{2}\int_{t_2 - T_R}^{t_2}\int_\omega |w|^2 dxdt + \frac{\beta}{2}\int_{t_1}^{t_2}\int_{\omega_0} |y - z|^2 dxdt \\
&\leq \frac{1}{2}\int_{t_2 - T_R}^{t_2}\int_\omega |w|^2 dxdt + \beta\int_{t_1}^{t_2}\int_{\omega_0} |y|^2 dxdt + \beta\int_{t_1}^{t_2}\int_{\omega_0} |z|^2 dxdt \\
&\leq \frac{1}{2}\int_{t_2 - T_R}^{t_2}\int_\omega |w|^2 dxdt + \beta\int_{t_1}^{t_2}\int_{\omega_0} |y|^2 dxdt + K(t_2 - t_1)\|z\|^2_{\mathrm{L}^\infty(\omega_0)} \\
&\leq K\Big[\|w\|^2_{\mathrm{L}^\infty((t_2 - T_R, t_2)\times\omega)} + (t_2 - t_1)\|z\|^2_{\mathrm{L}^\infty(\omega_0)} \\
&\quad + \beta\int_{t_1}^{t_2 - T_R}\left\|y^0(t,\cdot)\right\|^2_{\mathrm{L}^2(\Omega)} dt + \|y\|^2_{\mathrm{L}^2((t_2 - T_R, t_2)\times\Omega)}\Big] \\
&\leq K\Big[\left\|y^0\left(t_2 - T_R, \cdot\right) - \hat{y}\left(t_2 - T_R, \cdot\right)\right\|^2_{\mathrm{L}^\infty(\Omega)} + \|\hat{u}\|^2_{\mathrm{L}^\infty((t_2 - T_R, t_2)\times\omega)} \qquad (89) \\
&\quad + (t_2 - t_1)\|z\|^2_{\mathrm{L}^\infty(\omega_0)} + \|\hat{y}(t_1, \cdot)\|^2_{\mathrm{L}^\infty(\Omega)}\Big] \\
&\leq K\Big[\|\hat{y}(t_1, \cdot)\|^2_{\mathrm{L}^\infty(\Omega)} + (t_2 - t_1)\|z\|^2_{\mathrm{L}^\infty(\omega_0)} \qquad (90) \\
&\quad + \|\hat{u}\|^2_{\mathrm{L}^\infty((t_2 - T_R, t_2)\times\omega)} + \|\hat{y}(t_2 - T_R, \cdot)\|^2_{\mathrm{L}^\infty(\Omega)}\Big], \qquad (91)
\end{aligned}$$

where in (89) and in (90) we have employed the dissipativity of (85). This concludes the proof. □

B.2. **A mean value result for integrals.** In the following Lemma we estimate the value of a function at some point, with the value of its integral.

Lemma B.3. *Let $h \in \mathrm{L}^1(c,d) \cap \mathrm{C}^0(c,d)$, with $-\infty < c < d < +\infty$. Assume $h \geq 0$ a.e. in (c,d). Then,*

(1) there exists $t_c \in \left(c, c+\frac{d-c}{3}\right)$, such that

$$h(t_c) \leq \frac{3}{d-c}\int_c^d h dt;$$

(2) there exists $t_d \in \left(d-\frac{d-c}{3}, d\right)$, such that

$$h(t_d) \leq \frac{3}{d-c}\int_c^d h dt.$$

Proof of Lemma B.3. By contradiction, for any $t \in \left(c, c+\frac{d-c}{3}\right)$, $h(t) > \frac{3}{d-c}\int_c^d h ds$. Then, we have

$$\int_c^d h dt \geq \int_c^{c+\frac{d-c}{3}} h dt > \int_c^{c+\frac{d-c}{3}} \left[\frac{3}{d-c}\int_c^d h ds\right] dt = \int_c^d h ds,$$

so obtaining a contradiction. The proof of (2.) is similar. □

B.3. **Proof of Lemma 1.1.** We are now in position to prove Lemma 1.1.

Proof of Lemma 1.1. Step 1 **Estimates on subintervals**
Let T_R be given by Lemma B.1.

The case $T \leq 6T_R$ can be addressed by employing the inequality $J_T\left(u^T\right) \leq J_T(0)$ and bootstrapping in the optimality system (3), as in [33, subsection 3.2].

We address now the case $T > 6T_R$.

Set $N_T := \lfloor \frac{T}{3T_R} \rfloor$. Arbitrarily fix $\theta > 0$, a degree of freedom, to be made precise later. Consider the indexes $i \in \{1, \dots, N_T\}$, such that

$$\int_{(i-1)3T_R}^{i3T_R} \left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right] dt \leq \theta \left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]. \quad (92)$$

Set

$$\mathscr{I}_T := \left\{ i \in \{1, \dots, N_T\} \;\middle|\; \text{the estimate (92) is not verified} \right\}. \quad (93)$$

On the one hand, for any $i \in \{1, \dots, N_T\} \setminus \mathscr{I}_T$, by definition of $\mathscr{I}_T$

$$\int_{(i-1)3T_R}^{i3T_R} \left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right] dt \leq \theta \left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right].$$

On the other hand, for every $i \in \mathscr{I}_T$, we seek to prove the existence of a constant $K_\theta = K_\theta(\Omega, f, R, \theta)$, possibly larger than θ, such that

$$\int_{(i-1)3T_R}^{i3T_R} \left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right] dt \leq K_\theta \left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]. \quad (94)$$

We start by considering the union of time intervals, where (92) is not verified

$$\mathscr{W}_T := \bigcup_{i \in \mathscr{I}_T} [(i-1)3T_R, i3T_R].$$

The above set is made of a finite union of disjoint closed intervals, namely there exists a natural M and $\{(a_j, b_j)\}_{j=1,\dots,M}$, such that

$$b_j < a_{j+1}, \quad j = 1, \dots, M-1$$

and

$$\mathscr{W}_T = \bigcup_{i\in\mathscr{I}_T} [(i-1)3T_R, i3T_R] = \bigcup_{j=1,\dots,M} [a_j, b_j].$$

For any $j = 1, \dots, M$, set

$$C_j := \{i \in \mathscr{I}_T \mid [(i-1)3T_R, i3T_R] \subseteq [a_j, b_j]\}. \tag{95}$$

We are going to prove (94), studying the optima in a neighbourhood of $[a_j, b_j]$, for $j = 1, \dots, M$. Three different cases may occur:

- **Case 1.** $a_1 = 0$ and $b_1 < 3T_R N_T$, namely the left end of the interval $[a_1, b_1]$ coincides with $t = 0$, while the right end is far from $t = T$;
- **Case 2.** $a_j > 0$ and $b_j < 3T_R N_T$, i.e. the left end of the interval $[a_j, b_j]$ is far from $t = 0$ and the right end is far from $t = T$;
- **Case 3.** $a_j > 0$ and $b_j = 3T_R N_T$, i.e. the left end of the interval $[a_j, b_j]$ is far from $t = 0$, while the right end is close to $t = T$.

Case 1. $a_1 = 0$ and $b_1 < 3T_R N_T$.

Since $b_1 < 3T_R N_T$, we have $[b_1, b_1 + 3T_R N_T] \subseteq [0, T] \setminus \mathscr{W}_T$. Hence, by (93),

$$\int_{b_1}^{b_1+3T_R} \left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right] dt \le \theta \left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right].$$

Set $c := b_1$, $d := b_1 + 3T_R$ and $h(t) := \|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}$. By Lemma B.3, there exist t_c and t_d,

$$b_1 < t_c < b_1 + T_R \quad \text{and} \quad b_1 + 2T_R < t_d < b_1 + 3T_R, \tag{96}$$

such that

$$\begin{aligned} \|q^T(t_c)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t_c)\|^2_{\mathrm{L}^\infty(\Omega)} &\le \frac{1}{T_R}\int_{b_1}^{b_1+3T_R} \left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right] dt \\ &\le \frac{\theta}{T_R}\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right] \end{aligned}$$

and

$$\begin{aligned} \|q^T(t_d)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t_d)\|^2_{\mathrm{L}^\infty(\Omega)} &\le \frac{1}{T_R}\int_{b_1}^{b_1+3T_R} \left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right] dt \\ &\le \frac{\theta}{T_R}\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]. \end{aligned}$$

Parabolic regularity in the optimality system (3) in the interval $[t_c, t_d]$ gives

$$\begin{aligned} \left\|y^T\right\|^2_{\mathrm{L}^\infty((t_c,t_d)\times\Omega)} + \left\|q^T\right\|^2_{\mathrm{L}^\infty((t_c,t_d)\times\Omega)} &\le K\Big\{\|q^T(t_d)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t_c)\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)} \\ &\quad + \int_{b_1}^{b_1+3T_R} \left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right] dt\Big\} \\ &\le K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]. \end{aligned} \tag{97}$$

where the constant K_θ is independent of the time horizon T, but it depends on θ. At this point, we want to apply Lemma B.2. To this purpose, we set up a control problem like (82)-(83) with specified final state

$$\begin{aligned} \hat{y} &\coloneqq y^T \\ t_1 &\coloneqq 0 \\ t_2 &\coloneqq t_d. \end{aligned}$$

By (84) and (97),

$$\begin{aligned} \min_{\mathscr{U}_{\text{ad}}} J_{t_1,t_2} &\leq K\Big[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + t_d\|z\|^2_{\mathrm{L}^\infty(\omega_0)} \\ &\quad + \|u^T\|^2_{\mathrm{L}^\infty((t_d-T_R,t_d)\times\omega)} + \|y^T(t_d-T_R)\|^2_{\mathrm{L}^\infty(\Omega)}\Big] \\ &\leq K_\theta\Big[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\Big] + \gamma t_d\|z\|^2_{\mathrm{L}^\infty(\omega_0)}, \end{aligned} \tag{98}$$

where $K_\theta = K_\theta(\Omega, f, R, \theta)$ and $\gamma = \gamma(\Omega, f, R)$. In our case the target trajectory for (82)-(83) is the state y^T associated to an optimal control u^T for (2)-(1). Then, by definition of (82)-(83),

$$J_{t_1,t_2}\left(u^T\right) \leq J_{t_1,t_2}(u), \quad \forall\, u \in \mathscr{U}_{\text{ad}}.$$

Hence, by (98),

$$\begin{aligned} J_{t_1,t_2}\left(u^T\right) &\leq \min_{\mathscr{U}_{\text{ad}}} J_{t_1,t_2} \\ &\leq K_\theta\Big[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\Big] + \gamma t_d\|z\|^2_{\mathrm{L}^\infty(\omega_0)}. \end{aligned} \tag{99}$$

By definition of $\mathscr{I}_T$ (93) and C_1 (95), we have

$$\begin{aligned} \int_0^{b_1}\Big[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\Big]dt &\geq \sum_{i\in C_1}\theta\Big[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\Big] \\ &= \frac{\theta b_1}{3T_R}\Big[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\Big] \\ &> \frac{\theta(t_d-3T_R)}{3T_R}\Big[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\Big], \end{aligned}$$

where in the last inequality we have used (96), which yields $b_1 > t_d - 3T_R$. By the above inequality, Lemma A.1, (97) and (99),

$$\begin{aligned} &\frac{\theta(t_d-3T_R)}{6T_R}\Big[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\Big] \\ &+\frac{1}{2}\int_0^{b_1}\Big[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\Big]dt \leq \int_0^{b_1}\Big[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\Big]dt \\ &\qquad \leq K\Big[J_{t_1,t_2}\left(u^T\right) + \|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|q^T(t_d)\|^2_{\mathrm{L}^\infty(\Omega)}\Big] \\ &\qquad \leq K_\theta\Big[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\Big] + \gamma t_d\|z\|^2_{\mathrm{L}^\infty(\omega_0)}, \end{aligned}$$

whence

$$\begin{aligned}\int_0^{b_1}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt &\leq K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)}+\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\\ &\quad +2\left(\gamma t_d-\frac{\theta(t_d-3T_R)}{6T_R}\right)\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\\ &\leq K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)}+\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\\ &\quad +2t_d\left(\gamma-\frac{\theta}{6T_R}\right)\|z\|^2_{\mathrm{L}^\infty(\omega_0)}.\end{aligned}$$

If θ is large enough, we have $\gamma-\frac{\theta}{6T_R}<0$. Hence, choosing θ large enough, we obtain the estimate

$$\int_0^{b_1}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt\leq K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)}+\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right].$$

Case 2. $a_j>0$ and $b_j<3T_RN_T$.

Since $a_j>0$ and $b_j<3T_RN_T$, we have

$$\int_{a_j-3T_R}^{a_j}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt\leq\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)}+\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right] \tag{100}$$

and

$$\int_{b_j}^{b_j+3T_R}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt\leq\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)}+\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]. \tag{101}$$

In Case 2, we apply Lemma B.3:

- in the interval $[a_j-3T_R,a_j]$;
- in the interval $[b_j,b_j+3T_R]$.

We start by applying Lemma B.3 in $[a_j-3T_R,a_j]$. To this end, set $c:=a_j-3T_R$, $d:=a_j$ and $h(t):=\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}$. By Lemma B.3, there exist $t_{a,c}$ and $t_{a,d}$,

$$a_j-3T_R<t_{a,c}<a_j-2T_R\quad\text{and}\quad a_j-T_R<t_{a,d}<a_j, \tag{102}$$

such that

$$\begin{aligned}\|q^T(t_{a,c})\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t_{a,c})\|^2_{\mathrm{L}^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{a_j-3T_R}^{a_j}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt\\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)}+\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\end{aligned}$$

and

$$\begin{aligned}\|q^T(t_{a,d})\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t_{a,d})\|^2_{\mathrm{L}^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{a_j-3T_R}^{a_j}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}+\|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt\\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)}+\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right].\end{aligned}$$

By parabolic regularity in the optimality system (3) in the interval $[t_{a,c}, t_{a,d}]$, we have

$$\begin{aligned}\left\|y^T\right\|^2_{L^\infty((t_{a,c},t_{a,d})\times\Omega)} + \left\|q^T\right\|^2_{L^\infty((t_{a,c},t_{a,d})\times\Omega)} &\leq K\Big\{\|q^T(t_{a,d})\|^2_{L^\infty(\Omega)} + \|y^T(t_{a,c})\|^2_{L^\infty(\Omega)} \\ &\quad +\|z\|^2_{L^\infty(\omega_0)} \\ &\quad + \int_{a_j-3T_R}^{a_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\Big\} \\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \qquad (103)\end{aligned}$$

where the constant K_θ is independent of the time horizon T, but it depends on θ.

We apply Lemma B.3 in $[b_j, b_j + 3T_R]$. To this extent, set $c := b_j$, $d := b_j + 3T_R$ and $h(t) := \|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}$. By Lemma B.3, there exist $t_{b,c}$ and $t_{b,d}$,

$$0 < t_{b,c} < b_j + T_R \quad \text{and} \quad b_j + 2T_R < t_{b,d} < b_j + 3T_R, \qquad (104)$$

such that

$$\begin{aligned}\|q^T(t_{b,c})\|^2_{L^\infty(\Omega)} + \|y^T(t_{b,c})\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{b_j}^{b_j+3T_R}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]\end{aligned}$$

and

$$\begin{aligned}\|q^T(t_{b,d})\|^2_{L^\infty(\Omega)} + \|y^T(t_{b,d})\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{b_j}^{b_j+3T_R}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right].\end{aligned}$$

By parabolic regularity in the optimality system (3) in the interval $[t_{b,c}, t_{b,d}]$, we have

$$\begin{aligned}\left\|y^T\right\|^2_{L^\infty((t_{b,c},t_{b,d})\times\Omega)} + \left\|q^T\right\|^2_{L^\infty((t_{b,c},t_{b,d})\times\Omega)} &\leq K\Big\{\|q^T(t_{b,d})\|^2_{L^\infty(\Omega)} + \|y^T(t_{b,c})\|^2_{L^\infty(\Omega)} \\ &\quad +\|z\|^2_{L^\infty(\omega_0)} + \int_{b_j}^{b_j+3T_R}\|q^T(t)\|^2_{L^\infty(\Omega)}dt \\ &\quad + \int_{b_j}^{b_j+3T_R}\|y^T(t)\|^2_{L^\infty(\Omega)}dt\Big\} \\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \qquad (105)\end{aligned}$$

where the constant K_θ is independent of the time horizon T, but it depends on θ.

At this point, we want to apply Lemma B.2. To this purpose, we set up a control problem like (82)-(83) with specified final state

$$\begin{aligned}\hat{y} &:= y^T \\ t_1 &:= t_{a,c} \\ t_2 &:= t_{b,d}.\end{aligned}$$

By (84), (103) and (105),

$$\begin{aligned}\min_{\mathscr{U}_{\text{ad}}} J_{t_1,t_2} &\leq K\left[\|y^T(t_{a,c})\|^2_{\mathrm{L}^\infty(\Omega)} + (t_{b,d}-t_{a,c})\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right.,\\ &\quad \left.+\|u^T\|^2_{\mathrm{L}^\infty((t_{b,d}-T_R,t_{b,d})\times\omega)} + \|y^T(t_{b,d}-T_R)\|^2_{\mathrm{L}^\infty(\Omega)}\right]\\ &\leq K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right] + \gamma(t_{b,d}-t_{a,c})\|z\|^2_{\mathrm{L}^\infty(\omega_0)}, \quad (106)\end{aligned}$$

where $K_\theta = K_\theta(\Omega, f, R, \theta)$ and $\gamma = \gamma(\Omega, f, R)$. In our case the target trajectory for (82)-(83) is the state y^T associated to an optimal control u^T for (2)-(1). Then, by definition of (82)-(83),

$$J_{t_1,t_2}\left(u^T\right) \leq J_{t_1,t_2}(u), \quad \forall\, u \in \mathscr{U}_{\text{ad}}.$$

Hence, by (106),

$$\begin{aligned}J_{t_1,t_2}\left(u^T\right) &\leq \min_{\mathscr{U}_{\text{ad}}} J_{t_1,t_2}\\ &\leq K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right] + \gamma(t_{b,d}-t_{a,c})\|z\|^2_{\mathrm{L}^\infty(\omega_0)}.\end{aligned}$$

By definition of $\mathscr{I}_T$ (93) and C_1 (95), we have

$$\begin{aligned}\int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt &\geq \sum_{i\in C_j}\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\\ &= \frac{\theta(b_j-a_j)}{3T_R}\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\\ &> \frac{\theta(t_{b,d}-t_{a,c}-6T_R)}{3T_R}\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right], \quad (107)\end{aligned}$$

where in the last inequality we have used (102) and (104) to get
$b_j - a_j > t_{b,d} - t_{a,c} - 6T_R$. By the above inequality, Lemma A.1 and (107),

$$\begin{aligned}&\frac{\theta(t_{b,d}-t_{a,c}-6T_R)}{6T_R}\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\\ &\quad +\frac{1}{2}\int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt \leq \int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt\\ &\quad \leq K\left[J_{t_1,t_2}\left(u^T\right) + \|y_0\|^2_{\mathrm{L}^\infty(\Omega)}\right]\\ &\quad \leq K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\\ &\quad\quad +\gamma(t_{b,d}-t_{a,c})\|z\|^2_{\mathrm{L}^\infty(\omega_0)},\end{aligned}$$

whence

$$\begin{aligned}\int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{\mathrm{L}^\infty(\Omega)} + \|y^T(t)\|^2_{\mathrm{L}^\infty(\Omega)}\right]dt &\leq K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\\ &\quad +2\left(\gamma(t_{b,d}-t_{a,c}) - \theta\frac{(t_{b,d}-t_{a,c}-6T_R)}{6T_R}\right)\|z\|^2_{\mathrm{L}^\infty(\omega_0)}\\ &\leq K_\theta\left[\|y_0\|^2_{\mathrm{L}^\infty(\Omega)} + \|z\|^2_{\mathrm{L}^\infty(\omega_0)}\right]\\ &\quad +2(t_{b,d}-t_{a,c})\left(\gamma - \frac{\theta}{6T_R}\right)\|z\|^2_{\mathrm{L}^\infty(\omega_0)}.\end{aligned}$$

If θ is large enough, we have $\gamma - \frac{\theta}{6T_R} < 0$. Hence, choosing θ large enough, we obtain the estimate

$$\int_{a_j}^{b_j} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq K_\theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right].$$

Case 3. $a_j > 0$ and $b_j = 3T_R N_T$.

We now work in case (92) is not satisfied in $[a_j, b_j]$, with $b_j = 3T_R N_T$. We provide an estimate in the final interval $[a_j, T]$. As we shall see, in this case, we will not employ the exact controllability of (2). We shall rather use the stability of the uncontrolled equation.

Since $a_j > 0$, we have

$$\int_{a_j-3T_R}^{a_j} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq \theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \quad (108)$$

We apply Lemma B.3 in $[a_j - 3T_R, a_j]$. To this end, set $c := a_j - 3T_R$, $d := a_j$ and $h(t) := \|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}$. By Lemma B.3, there exist t_c,

$$a_j - 3T_R < t_c < a_j - 2T_R \quad (109)$$

such that

$$\begin{aligned} \|q^T(t_c)\|^2_{L^\infty(\Omega)} + \|y^T(t_c)\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{a_j-3T_R}^{a_j} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \quad (110) \end{aligned}$$

We introduce the control

$$u^* := \begin{cases} u^T & \text{in } (0, t_c) \\ 0 & \text{in } (t_c, T) \end{cases}$$

Let y be the solution to (2), with initial datum y_0 and control u and y^* be the solution to (2), with initial datum y_0 and control u^*. By definition of minimizer, we have

$$\begin{aligned} J_T\left(u^T\right) &\leq J_T(u^*) \\ &\leq \frac{1}{2}\int_0^T\int_\omega |u^*|^2 dxdt + \frac{\beta}{2}\int_0^T\int_{\omega_0} |y^* - z|^2 dxdt \\ &= \frac{1}{2}\int_0^{t_c}\int_\omega \left|u^T\right|^2 dxdt + \frac{\beta}{2}\int_0^{t_c}\int_{\omega_0} \left|y^T - z\right|^2 dxdt \\ &\quad + \frac{\beta}{2}\int_{t_c}^T\int_{\omega_0} |y^* - z|^2 dxdt, \end{aligned}$$

whence,

$$\begin{aligned} \frac{1}{2}\int_{t_c}^T\int_\omega \left|u^T\right|^2 dxdt + \frac{\beta}{2}\int_{t_c}^T\int_{\omega_0} \left|y^T - z\right|^2 dxdt &\leq \frac{\beta}{2}\int_{t_c}^T\int_{\omega_0} |y^* - z|^2 dxdt \\ &\leq K\left[\|y(t_c)\|^2_{L^\infty(\Omega)} + (T - t_c)\|z\|^2_{L^\infty(\omega_0)}\right] \\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\ &\quad + \gamma(T - t_c)\|z\|^2_{L^\infty(\omega_0)}, \end{aligned}$$

where we have used (110) and $K_\theta = K_\theta(\Omega, f, R, \theta)$ and $\gamma = \gamma(\Omega, f, R)$.

Now, on the one hand, by Lemma A.1 applied to the state and the adjoint equation in (3), we have

$$\int_{t_c}^{T} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq K_\theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] + \gamma(T - t_c)\|z\|^2_{L^\infty(\omega_0)}. \tag{111}$$

On the other hand, by (109), $-a_j > -t_c - 3T_R$ and, since $b_j = 3T_R N_T$, $b_j \geq T - 3T_R$. Hence, $b_j - a_j > T - t_c - 6T_R$. Then, by (93),

$$\begin{aligned}
\int_{a_j}^{T} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt &\geq \int_{a_j}^{b_j} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \\
&\geq \sum_{i \in C_j} \theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\
&= \frac{\theta(b_j - a_j)}{3T_R} \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\
&> \frac{\theta(T - t_c - 6T_R)}{3T_R} \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right].
\end{aligned}$$

By the above inequality and Lemma A.1 and (111),

$$\begin{aligned}
&\frac{\theta(T - t_c - 6T_R)}{6T_R} \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\
&+ \frac{1}{2}\int_{a_j}^{T} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq \int_{a_j}^{T} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \\
&\leq K_\theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] + \gamma(T - t_c)\|z\|^2_{L^\infty(\omega_0)},
\end{aligned}$$

whence

$$\begin{aligned}
\int_{a_j}^{T} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt &\leq K_\theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\
&\quad + 2\left(\gamma(T - t_c) - \theta\frac{(T - t_c - 6T_R)}{6T_R}\right)\|z\|^2_{L^\infty(\omega_0)} \\
&\leq K_\theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\
&\quad + 2(T - t_c)\left(\gamma - \frac{\theta}{6T_R}\right)\|z\|^2_{L^\infty(\omega_0)}.
\end{aligned}$$

If θ is large enough, we have $\gamma - \frac{\theta}{6T_R} < 0$. Hence, choosing θ large enough, we obtain the estimate

$$\int_{a_j}^{T} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq K_\theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right].$$

Step 2 **Conclusion**

The proof is concluded, with an application of Lemma A.2 to the state and the adjoint equation in (3). □

Appendix C. Convergence of averages

This section is devoted to the proof of Lemma 2.1.

Proof of lemma 2.1. Let $\overline{u} \in \mathrm{L}^\infty(\Omega)$ be an optimal control for (5)-(4) and let $\overline{y}$ be the corresponding solution to (5) with control $\overline{u}$. Following step 1 of the proof of Lemma 1.4, we obtain $\overline{u} \in \mathrm{C}^0(\overline{\omega})$ and

$$\|\overline{u}\|_{\mathrm{L}^\infty(\omega)} \leq K\|z\|_{\mathrm{L}^\infty(\omega_0)}. \tag{112}$$

Step 1 **Proof of**

$$\left|J_T(\overline{u}) - T \inf_{\mathrm{L}^2(\Omega)} J_s\right| \leq K,$$

with K independent of T
Let $\hat{y}$ be the solution to

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + f(\hat{y}) = \overline{u}\chi_\omega & \text{in } (0,T)\times\Omega \\ \hat{y} = 0 & \text{on } (0,T)\times\partial\Omega \\ \hat{y}(0,x) = y_0(x) & \text{in } \Omega. \end{cases} \tag{113}$$

Set $\eta := \hat{y} - \overline{y}$ solution to

$$\begin{cases} \eta_t - \Delta\eta + f(\hat{y}) - f(\overline{y}) = 0 & \text{in } (0,T)\times\Omega \\ \eta = 0 & \text{on } (0,T)\times\partial\Omega \\ \eta(0,x) = y_0(x) - \overline{y}(x) & \text{in } \Omega. \end{cases} \tag{114}$$

By multiplying (114) by η, since f is increasing, for any $t \in [0,T]$ we have

$$\|\hat{y}(t,\cdot) - \overline{y}\|_{\mathrm{L}^2(\Omega)} \leq \exp(-\lambda_1 t)\|y_0 - \overline{y}\|_{\mathrm{L}^2(\Omega)}, \tag{115}$$

where λ_1 is the first eigenvalue of $-\Delta : \mathrm{H}_0^1(\Omega) \longrightarrow \mathrm{H}^{-1}(\Omega)$.

At this point, let us take the difference

$$\begin{aligned} |J_T(\overline{u}) - T \inf_{\mathrm{L}^2(\Omega)} J_s| &= \frac{1}{2}\left|\int_0^T \int_{\omega_0} \left[|\hat{y} - z|^2 - |\overline{y} - z|^2\right] dxdt\right| \\ &\leq \frac{1}{2}\int_0^T \int_{\omega_0} |\hat{y} - \overline{y}|^2 dxdt + \int_0^T \int_{\omega_0} |\overline{y} - z|\,|\hat{y} - \overline{y}|\,dxdt \\ &\leq K\|y_0 - \overline{y}\|^2_{\mathrm{L}^2(\Omega)} + K\|y_0 - \overline{y}\|_{\mathrm{L}^2(\Omega)} \leq K, \qquad (116) \\ & \qquad (117) \end{aligned}$$

where in appendix C we have used (115) and (112) and the constant K is independent of the time horizon T.
Step 2 **Conclusion**
By the above reasoning, we have

$$\begin{aligned} \inf_{\mathrm{L}^2((0,T)\times\omega)} J_T &\leq J_T(\overline{u}) \\ &= T \inf_{\mathrm{L}^2(\Omega)} J_s + J_T(\overline{u}) - T \inf_{\mathrm{L}^2(\Omega)} J_s \\ &\leq T \inf_{\mathrm{L}^2(\Omega)} J_s + K. \end{aligned}$$

This finishes the proof. □

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