## The Finite-Time Turnpike Phenomenon for Optimal Control Problems: Stabilization by Non-Smooth Tracking Terms

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#### Abstract

In this paper, problems of optimal control are considered where in the objective function, in addition to the control cost there is a tracking term that measures the distance to a desired stationary state. The tracking term is given by some norm and therefore it is in general not differentiable. In the optimal control problem, the initial state is prescribed. We assume that the system is either exactly controllable in the classical sense or nodal profile controllable. We show that both for systems that are governed by ordinary differential equations and for infinite-dimensional systems, for example for boundary control systems governed by the wave equation, under certain assumptions the optimal system state is steered exactly to the desired state after finite time.

## 1 Introduction

Since the turnpike phenomenon has been studied by P. A. Samuelson in mathematical economics in 1949 (see [2]), it has been analyzed in various contexts, see for example [18], [19] and [1]. For optimal control problems with partial differential equations it has been studied in [13] and [16] where distributed control is considered for linear-quadratic optimal control problems. Problems of optimal boundary control are studied in [8], [6] and [10]. In [15], both integral- and measure-turnpike properties are considered. The turnpike phenomenon for linear quadratic optimal control problems with time-discrete systems is studied in [4]. In [5], linear

quadratic optimal control problems governed by general evolution equations are considered and exponential sensitivity and turnpike analysisis studied. An overview on the turnpike phenomenon is given in the monograph [20].

In this paper, we consider integral turnpike properties for problems where the system is exactly controllable and in the objective function, an  $L^1$ -norm or  $L^2$ -norm tracking term appears. We show that the resulting optimal controls have a finite-time turnpike structure, that is the optimal state reaches the static desired state (that we also refer to as the *turnpike* and that does not depend on time) exactly in finite time.

These turnpike result are also useful for numerical computations since they show that for sufficiently large time horizons T, sufficiently accurate approximations of the optimal state/control pairs should also be identical to the desired state with the corresponding constant control most of the time.

The finite-time (or exact) turnpike property for continuous-time systems has already been discussed in [3] as an assumption in the context of nonlinear model predictive control for a finite-dimensional system that is governed by an ordinary differential equation. Here the aim is to prove convergence in model predictive control. As an application, a problem of optimal fish harvesting control is studied.

This paper has the following structure. In order to illustrate the situation, first we consider optimal control problems that are governed by ordinary differential equations. In these problems the  $L^1$ -norm appears in the tracking term in the objective function. We show that if the weight of the tracking term (i.e. the penalty parameter) is sufficiently large, the optimal states and controls have a finite-time turnpike structure.

In the next section, we present a finite-time turnpike result for optimal control problems with an abstract infinite dimensional system. First we consider the case where the system is exactly controllable. We consider an optimal control problem where the tracking term is given by a certain maximum norm. We show that if the weight of the tracking term is sufficiently large, the solution has a finite-time turnpike structure.

Then we consider the case where the system is nodal profile exactly controllable. We consider an optimal control problem where the tracking term for the nodal profiles is given by an  $L^2$ -norm. We show that if the weight of the tracking term is sufficiently large, the solution has a finite-time turnpike structure for the nodal profiles.

Finally we return to the case where the system is exactly controllable. We consider an optimal control problem where the tracking term is given by a weighted  $L^1$ -norm that has a singularity at  $t_0 > 0$ . We show also in this case, the solution has a finite-time turnpike structure.

In Section 4, examples are presented, where the results from the previous section are applicable. Section 5 contains conclusion.

## 2 Optimal control problems with ordinary differential equation

We start with optimal control problems with systems that are governed by ordinary differential equations. We show that for such systems,  $L^1$ tracking terms in the objective function can lead to finite-time turnpike structures.

**Example 1** We start with a system similar to the motivating example in [9] that is governed by an ordinary differential equation. Let  $\gamma > 0$  be given. For T > 0 sufficiently large (this will be specified later) we consider the problem

$$(\mathbf{OC})_T \begin{cases} \min_{u \in L^2(0,T)} \int\limits_0^T \frac{1}{2} |u(t)|^2 + |u(t)| + \gamma |y(t)| \, dt \text{ subject to} \\ y(0) = -1, \ y'(t) = y(t) + \exp(t) \, u(t). \end{cases}$$

The corresponding optimal control problem where the initial condition does not appear is

$$(\mathbf{OC})^{(\sigma)} \begin{cases} \min_{u \in L^2(0,T)} \int_0^T \frac{1}{2} |u(t)|^2 + |u(t)| + \gamma |y(t)| \, dt \text{ subject to} \\ y'(t) = y(t) + \exp(t) \, u(t). \end{cases}$$

The solution of  $(\mathbf{OC})^{(\sigma)}$  (that we call the turnpike) is zero, that is  $y^{(\sigma)} = 0$ and  $u^{(\sigma)} = 0$ . The results about the solution of  $(\mathbf{OC})_T$  are summarized in the following lemma.

**Lemma 1** For  $\gamma > 0$ , define  $t_0 > 0$  as the minimal value where

$$(t_0 - 1) \exp(t_0) = \frac{1}{\gamma} - 1.$$

Assume that  $T > t_0$  and (even)

$$\gamma e^T \ge 1 + \gamma e^{t_0}. \tag{2.1}$$

Define

 $\hat{u}(t) = \gamma(e^{t_0} - e^t) \ge 0 \text{ for } t \in (0, t_0], \ \hat{u}(t) = 0 \text{ for } t > t_0.$  (2.2)

Then for the state  $\hat{y}$  generated by  $\hat{u}$  for  $t \ge t_0$  we have  $\hat{y}(t) = 0$ . Moreover, for all  $t \in (0,T)$  we have  $\hat{y}(t) \le 0$ .

The control  $\hat{u}$  as defined in (2.2) is the unique solution of  $(\mathbf{OC})_T$ .

**Proof.** Let a control  $u \in L^2(0, T)$  be given. Then for the corresponding state y we have

$$y(t) = e^t \left[ -1 + \int_0^t u(\tau) \, d\tau \right].$$
 (2.3)

Note that for the optimal control we have  $y(t) \leq 0$ . (If  $y(t_0) = 0$ , we can continue with the zero control.) Moreover, we have  $u(t) \geq 0$ . (Otherwise, instead of decreasing the state it is also better to switch off the control).

Hence it suffices to consider the feasible controls  $u(t) \ge 0$  that satisfy the moment inequality

$$\int_0^T u(\tau) \, d\tau \le 1. \tag{2.4}$$

Due to the definition of  $t_0$  and (2.2) we have

$$\int_{0}^{T} \hat{u}(\tau) \, d\tau = 1. \tag{2.5}$$

Then for  $t \in (0, t_0)$  we have

$$\hat{y}(t) = e^t \left[ -1 + \int_0^t u(\tau) \, d\tau \right] = \gamma \, t \, e^{t+t_0} - \gamma \, e^{2t} + (\gamma - 1) e^t \le 0$$

and for  $t \ge t_0$  we have  $\hat{y}(t) = 0$ .

With the representation (2.3), for all feasible controls  $u \ge 0$  where  $y \le 0$  integration by parts yields

$$\begin{split} J_{(0,T)}(u,\,y) &= \int_{0}^{T} \left[ \frac{1}{2} |u(t)|^{2} + u(t) - \gamma \, y(t) \right] \, dt \\ &= \int_{0}^{T} \left\{ \frac{1}{2} |u(t)|^{2} + u(t) + \gamma \, \mathrm{e}^{t} \left[ 1 - \int_{0}^{t} u(\tau) \, d\tau \right] \right\} \, dt \\ &= \int_{0}^{T} u(t) \, dt + \int_{0}^{T} \frac{1}{2} |u(t)|^{2} + \gamma \, \mathrm{e}^{t} \left[ 1 - \int_{0}^{t} u(\tau) \, d\tau \right] |_{t=0}^{T} + \int_{0}^{T} \gamma \, \mathrm{e}^{t} \, u(t) \, dt \\ &= \int_{0}^{T} u(t) \, dt - \gamma + \gamma \, \mathrm{e}^{T} \left[ 1 - \int_{0}^{T} u(\tau) \, d\tau \right] + \int_{0}^{T} \left[ \frac{1}{2} |u(t)|^{2} + \gamma \, \mathrm{e}^{t} \, u(t) \right] \, dt \\ &= (\gamma \, \mathrm{e}^{T} - 1) \, \left[ 1 - \int_{0}^{T} u(\tau) \, d\tau \right] + \int_{0}^{T} \left[ \frac{1}{2} |u(t)|^{2} + \gamma \, \mathrm{e}^{t} \, u(t) \right] \, dt + 1 - \gamma. \end{split}$$

If T is sufficiently large in the sense that (2.1) holds, due to the  $L^1$ norm that appears in the objective function, the solution has an exact turnpike structure where the system is steered to zero in the finite time  $t_0$  that is independent of T and remains there for  $t \in (t_0, T)$ . This can be seen as follows. Let  $u(t) = \hat{u}(t) + \delta(t)$  with  $\hat{u}$  as defined in (2.2) and  $\int_0^T \delta(\tau) d\tau \leq 1 - \int_0^T \hat{u}(\tau) d\tau = 0$  where the last equation follows from (2.5) and  $\delta(t) \geq 0$  for  $t \geq t_0$ . Due to (2.1) we have

$$J_{(0,T)}(u, y) = (\gamma e^{T} - 1) \left[ -\int_{0}^{T} \delta(\tau) d\tau \right]$$
$$+ \int_{0}^{T} \left[ \frac{1}{2} |\hat{u}(t) + \delta(t)|^{2} + \gamma e^{t} \left( \hat{u}(t) + \delta(t) \right) \right] dt + 1 - \gamma$$

$$\geq (\gamma e^{T} - 1) \left[ -\int_{0}^{T} \delta(\tau) d\tau \right] + \int_{0}^{T} \frac{1}{2} \hat{u}(t)^{2} + \gamma e^{t} \hat{u}(t) dt + \int_{0}^{T} \left( \hat{u}(t) + \gamma e^{t} \right) \delta(t) dt + 1 - \gamma e^{t}$$

$$= (\gamma e^{T} - 1) \left[ -\int_{0}^{T} \delta(\tau) d\tau \right] + J_{(0,T)}(\hat{u}, \hat{y}) + \int_{0}^{t_{0}} \gamma e^{t_{0}} \delta(t) dt + \int_{t_{0}}^{T} \gamma e^{t} \delta(t) dt$$

$$\geq (\gamma e^{T} - 1) \left[ -\int_{0}^{T} \delta(\tau) d\tau \right] + J_{(0,T)}(\hat{u}, \hat{y}) + \int_{0}^{t_{0}} \gamma e^{t_{0}} \delta(t) dt + \int_{t_{0}}^{T} \gamma e^{t_{0}} \delta(t) dt$$

$$= J_{(0,T)}(\hat{u}, \hat{y}) + (\gamma e^{T} - \gamma e^{t_{0}} - 1) \left[ -\int_{0}^{T} \delta(\tau) d\tau \right].$$

Since  $\gamma e^T - \gamma e^{t_0} - 1 \ge 0$ , this implies that  $\hat{u}$  as defined in (2.2) is the optimal control. Thus we have proved Lemma 1.  $\Box$ 

Consider the value  $t_0$  as a function of  $\gamma$ ,  $t_0 = t_0(\gamma)$ . Then we have  $t_0(1) = 1$  and

$$\lim_{\gamma \to \infty} t_0(\gamma) = 0.$$

In Example 2, we present numerical approximations for the optimal states and controls for three values of  $\gamma$ .

## 2.1 A more general result for scalar ordinary differential equations

Now we consider an optimal control problem with the same objective function and a more general ordinary differential equation. In this problem, we also prescribe a terminal condition. At the end of the section we will present sufficient conditions that imply that if the penalty parameter  $\gamma$  is sufficiently large, the terminal state is reached before the final time.

Let continuous functions f, g from  $[0, \infty)$  to the real numbers be given. Assume that for all  $t \ge 0$  we have f(t) > 0, and g(t) > 0. Let  $\gamma \ge 1$  and  $\alpha < 0$  be given. For a finite time horizon T > 0 we consider the problem

$$(\mathbf{OC})_T \begin{cases} \min_{\substack{u(t) \in L^2(0, T), y(t) \in AC(0, T) \\ y(0) = \alpha, \ y'(t) = f(t) \ y(t) + g(t) \ u(t) \\ y(T) = 0. \end{cases}^T \frac{1}{2} |u(t)|^2 + |u(t)| + \gamma |y(t)| \ dt \text{ subject to}$$

Here again the solution of the corresponding optimal control problem without the initial and the terminal conditions (the turnpike) is zero, that is  $y^{(\sigma)} = 0$  and  $u^{(\sigma)} = 0$ . Note that the turnpike is compatible with the terminal constraint y(T) = 0. In the following theorem we present the optimal control for  $(\mathbf{OC})_T$ , which has a similar structure as in the previous example.

Theorem 1 Define

$$F(t) = \exp(\int_0^t f(s) \, ds), \ H(t) = \int_0^t F(\tau) \, d\tau$$

We have

$$y(t) = F(t) \left[ \alpha + \int_0^t \frac{g(\tau)}{F(\tau)} u(\tau) \, d\tau \right].$$
(2.6)

Define

$$\hat{u}(t) = \max\left\{0, \left[-1 - \gamma \frac{g(t) H(t)}{F(t)} + \lambda \frac{g(t)}{F(t)}\right]\right\}$$
(2.7)

where the number  $\lambda > 0$  is chosen such that

$$\int_0^T \hat{u}(\tau) \, \frac{g(\tau)}{F(\tau)} \, d\tau = -\alpha. \tag{2.8}$$

Then the unique optimal control that solves  $(\mathbf{OC})_T$  is equal to  $\hat{u}(t)$ .

**Proof.** Since  $y(0) = \alpha \leq 0$ , for the optimal state we have  $y(t) \leq 0$  for all  $t \geq 0$ . (Since otherwise, instead of increasing the state above zero it is better to switch off the control.) Moreover, for the optimal control we have  $u(t) \geq 0$ . (Since otherwise, instead of decreasing the state it is also better to switch off the control). Hence it suffices to consider the feasible controls  $u(t) \geq 0$  that satisfy the moment inequality

$$\int_{0}^{T} \frac{g(\tau)}{F(\tau)} u(\tau) d\tau \le -\alpha.$$
(2.9)

Due to the choice of  $\lambda$ , for the state  $\hat{y}$  generated by  $\hat{u}$ , we have  $\hat{y}(T) = 0$ . For  $t \in [0, T]$ , consider

$$B(t) = \int_0^t \frac{g(\tau)}{F(\tau)} \hat{u}(\tau) \, d\tau.$$
(2.10)

Then B(0) = 0 and B is increasing. Hence also the function  $[\alpha + B(t)]$  is increasing. We have  $B(0) + \alpha < 0$  and  $B(T) + \alpha = 0$ . Thus there exists a unique point

$$t_0 = \min\{t \in [0, T] : \alpha + B(t) = 0\}.$$

and we have  $t_0 \in (0, T]$ . We have  $B(t_0) = B(T)$  and B is increasing. This implies that for all  $t \in [t_0, T]$ , we have  $B(t) = -\alpha$ . On account of the definition of B as an integral, this is only possible if for all  $t \in [t_0, T]$ , we have  $\hat{u}(t) = 0$ . This implies that for all  $t \in [t_0, T]$  we have

$$-1 - \gamma \, \frac{g(t) \, H(t)}{F(t)} + \lambda \frac{g(t)}{F(t)} \le 0.$$
(2.11)

By (2.6) we have

$$t_0 = \min\{t \in [0, T] : \hat{y}(t) = 0\}.$$

Since  $\hat{y}(t) = F(t) [\alpha + B(t)]$ , for  $t < t_0$  we have  $\hat{y}(t) < 0$ . Since for  $t \ge t_0$ , we have  $\hat{u}(t) = 0$ , this implies that  $\hat{y}(t) = 0$  for all  $t \ge t_0$ . Since  $\hat{u} \ge 0$  and  $\hat{y} \le 0$ , for the objective function we have

$$J(\hat{u}) = \int_0^T \frac{1}{2} (\hat{u}(t))^2 + \hat{u}(t) - \gamma F(t) \left[ \alpha + \int_0^t \hat{u}(\tau) \frac{g(\tau)}{F(\tau)} d\tau \right] dt.$$

Integration by parts yields (since  $B(T) = -\alpha$ )

$$\begin{split} J(\hat{u}) &= \int_0^T \frac{1}{2} \left( \hat{u}(t) \right)^2 + \hat{u}(t) \, dt - \gamma \, H(s) \left[ \alpha + \int_0^s \hat{u}(\tau) \, \frac{g(\tau)}{F(\tau)} \, d\tau \right] |_{s=0}^T \\ &+ \gamma \, \int_0^T H(t) \, \hat{u}(t) \, \frac{g(t)}{F(t)} \, dt \\ &= \int_0^T \frac{1}{2} \left( \hat{u}(t) \right)^2 + \hat{u}(t) \, dt - \gamma \, H(t_0) \left( \alpha + B(T) \right) \\ &+ \gamma \, \int_0^T \hat{u}(\tau) \, \frac{H(\tau) \, g(\tau)}{F(\tau)} \, d\tau \\ &= \int_0^T \frac{1}{2} \left( \hat{u}(t) \right)^2 + \hat{u}(t) \left[ 1 + \gamma \, \frac{H(t) \, g(t)}{F(t)} \right] \, dt. \end{split}$$

Let  $\delta \in L^2(0, T)$  be given. We use  $\delta$  as a perturbation of the control. To make sure that the terminal condition remains valid, we assume that

$$\int_0^T \delta(\tau) \, \frac{g(\tau)}{F(\tau)} \, d\tau = 0. \tag{2.12}$$

Since the optimal control must increase the values of the corresponding trajectory to zero, it can only have positive values. Therefore we assume that for  $t \in [0, T]$  we have  $\hat{u}(t) + \delta(t) \ge 0$ . Thus for  $t \in [0, t_0]$  we have  $\operatorname{sign}(\hat{u}(t) + \delta(t)) = 1$  and for  $t \ge t_0$ , we have  $\delta(t) \ge 0$ . Then we have

$$\begin{split} J(\hat{u} + \delta) &= \int_{0}^{T} \frac{1}{2} \left( \hat{u}(t) + \delta(t) \right)^{2} + \left( \hat{u}(t) + \delta(t) \right) \left[ \text{sign}(\hat{u}(t) + \delta(t)) + \gamma \frac{H(t) g(t)}{F(t)} \right] dt \\ &= J(\hat{u}) + \int_{0}^{T} \frac{1}{2} \delta(t)^{2} dt + \int_{0}^{t_{0}} \delta(t) \left[ \hat{u} + 1 + \gamma \frac{H(t) g(t)}{F(t)} \right] dt \\ &+ \int_{t_{0}}^{T} \delta(t) \left[ \text{sign}(\delta(t)) + \gamma \frac{H(t) g(t)}{F(t)} \right] dt \\ &= J(\hat{u}) + \int_{0}^{T} \frac{1}{2} \delta(t)^{2} dt + \int_{0}^{t_{0}} \delta(t) \lambda \frac{g(t)}{F(t)} dt \\ &+ \int_{t_{0}}^{T} \delta(t) \left[ \text{sign}(\delta(t)) + \gamma \frac{H(t) g(t)}{F(t)} \right] dt \\ &= J(\hat{u}) + \int_{0}^{T} \frac{1}{2} \delta(t)^{2} dt + \int_{0}^{T} \delta(t) \lambda \frac{g(t)}{F(t)} dt \\ &+ \int_{t_{0}}^{T} \delta(t) \left[ -\lambda \frac{g(t)}{F(t)} + \text{sign}(\delta(t)) + \gamma \frac{H(t) g(t)}{F(t)} \right] dt \\ &= J(\hat{u}) + \int_{0}^{T} \frac{1}{2} \delta(t)^{2} dt \\ &+ \int_{t_{0}}^{T} \delta(t) \left[ -\lambda \frac{g(t)}{F(t)} + \text{sign}(\delta(t)) + \gamma \frac{H(t) g(t)}{F(t)} \right] dt \\ &= J(\hat{u}) + \int_{0}^{T} \frac{1}{2} \delta(t)^{2} dt \\ &+ \int_{t_{0}}^{T} \delta(t) \left[ 1 + \gamma \frac{H(t) g(t)}{F(t)} - \lambda \frac{g(t)}{F(t)} \right] dt \\ &\geq J(\hat{u}) \end{split}$$

where the last step follows with (2.11). Thus  $\hat{u}$  is the minimizer of J among all controls that generate states with y(T) = 0. This shows the assertion.  $\Box$ 

The question remains: Do we have  $t_0 < T$  if  $\gamma$  is sufficiently large? Let  $t_1 \in (0, T)$  be given such that

$$-\alpha - \frac{F(t_1)}{g(t_1)} \int_0^{t_1} \frac{g^2}{F^2} dt + \int_0^{t_1} \frac{g}{F} dt > 0.$$
 (2.13)

Note that H is strictly increasing, hence we have the inequality

$$\int_0^{t_1} \left( H(t_1) - H(t) \right) \frac{g^2}{F^2} \, dt > 0.$$

Define the number

$$\gamma(t_1) = \frac{-\alpha - \frac{F(t_1)}{g(t_1)} \int_0^{t_1} \frac{g^2}{F^2} dt + \int_0^{t_1} \frac{g}{F} dt}{\int_0^{t_1} (H(t_1) - H(t)) \frac{g^2}{F^2} dt}.$$
 (2.14)

Then we have  $\gamma(t_1) > 0$ . Define the number

$$\lambda_1 = \frac{-\alpha + \int_0^{t_1} \frac{g}{F} dt + \gamma(t_1) \int_0^{t_1} H \frac{g^2}{F^2} dt}{\int_0^{t_1} \frac{g^2}{F^2} dt}.$$
 (2.15)

The definition of  $\lambda_1$  implies the equation

$$\lambda_1 \int_0^{t_1} \frac{g^2}{F^2} dt = -\alpha + \int_0^{t_1} \frac{g}{F} dt + \gamma(t_1) \int_0^{t_1} H \frac{g^2}{F^2} dt.$$
(2.16)

Moreover, due to the definition of  $\gamma(t_1)$  we have

$$1 + \frac{g(t_1) H(t_1)}{F(t_1)} \gamma(t_1) = \frac{\int_0^{t_1} H \frac{g^2}{F^2} dt + \frac{g(t_1) H(t_1)}{F(t_1)} \left[ \alpha - \int_0^{t_1} \frac{g}{F} dt \right]}{\int_0^{t_1} H \frac{g^2}{F^2} dt - H(t_1) \int_0^{t_1} \frac{g^2}{F^2} dt}.$$

In addition, the definition of  $\lambda_1$  and of  $\gamma(t_1)$  implies

$$\begin{split} \lambda_{1} \frac{g(t_{1})}{F(t_{1})} &= \frac{-\frac{g(t_{1})}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F} \, dt \right] + \frac{g(t_{1})}{F(t_{1})} \gamma(t_{1}) \int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt}{\int_{0}^{t_{1}} \frac{g^{2}}{F^{2}} \, dt} \\ &= \frac{-\frac{g(t_{1})}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F} \, dt \right] + \left[ \frac{\int_{0}^{t_{1}} \frac{g^{2}}{F^{2}} \, dt + \frac{g(t_{1})}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F^{2}} \, dt \right]}{\int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt} \right] \int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt}{\int_{0}^{t_{1}} \frac{g^{2}}{F^{2}} \, dt} \\ &= \frac{1}{\int_{0}^{t_{1}} \frac{g^{2}}{F^{2}} \, dt} \left[ \frac{\int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt}{\int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt} - H(t_{1}) \int_{0}^{t_{1}} \frac{g^{2}}{F^{2}} \, dt}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F} \, dt \right] \right] \\ &+ \frac{\int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt}{\int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt} - H(t_{1}) \int_{0}^{t_{1}} \frac{g^{2}}{F^{2}} \, dt}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F} \, dt \right] \\ &= \frac{g(t_{1}) H(t_{1})}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F} \, dt \right] + \int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt}{f(t_{1})} \left[ \frac{g(t_{1})}{F(t_{1})} \left[ \frac{g(t_{1})}{F(t_{1})} \right] \right] \\ &= \frac{g(t_{1}) H(t_{1})}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F} \, dt \right] + \int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt}{f(t_{1})} \left[ \frac{g(t_{1})}{F(t_{1})} \right] \\ &= \frac{g(t_{1}) H(t_{1})}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F} \, dt \right] + \int_{0}^{t_{1}} H \frac{g^{2}}{F^{2}} \, dt}{f(t_{1})} \left[ \frac{g(t_{1})}{F(t_{1})} \right] \\ &= \frac{g(t_{1}) H(t_{1})}{F(t_{1})} \left[ \alpha - \int_{0}^{t_{1}} \frac{g}{F} \, dt \right] + \int_{0}^{t_{1}} \frac{g^{2}}{F^{2}} \, dt}{f(t_{1})} \left[ \frac{g(t_{1})}{F(t_{1})} \right] \\ &= \frac{g(t_{1}) H(t_{1})}{F(t_{1})} \left[ \frac{g(t_{1})}{F(t_{1})} \left[ \frac{g(t_{1})}{F(t_{1})} \right] \\ &= \frac{g(t_{1}) H(t_{1})}{F(t_{1})} \left[ \frac{g(t_{1})}{F(t_{1})} \right]$$

Hence we have

$$\lambda_1 \, \frac{g(t_1)}{F(t_1)} = 1 + \gamma(t_1) \, \frac{g(t_1) \, H(t_1)}{F(t_1)}.$$

Assume that g is continuously differentiable and we have

$$g'(t) \le f(t) g(t).$$
 (2.17)

Assumption (2.17) implies that the function  $\frac{g}{F}$  is decreasing.

Assumption (2.17) implies that the function  $\frac{g}{F}$  is decreasing. Since the function  $(\lambda_1 - \gamma(t_1) H)$  is decreasing and  $\frac{g}{F} > 0$  this implies that also the product

$$\frac{g}{F}\left(\lambda_1 - \gamma(t_1)\,H\right)$$

is decreasing as a function of time.

Then the optimal control  $\hat{u}$  as defined in (2.7) (with  $\lambda = \lambda_1$  and  $\gamma = \gamma(t_1)$ ) is decreasing,  $\hat{u}(t_1) = 0$  and the support of the optimal control  $\hat{u}$  is contained in  $[0, t_1]$ . With  $\lambda_1$  defined as in (2.15), equation (2.16) holds. This implies that the optimal control  $\hat{u}$  as defined in (2.7) satisfies (2.8). Thus we have shown the following statement:

If (2.17) holds, for all  $t_0 \in (0, T)$  such that (2.13) holds (with  $t_1 = t_0$ ) there is a weight  $\gamma > 0$  such that the support of the corresponding optimal control is contained in  $[0, t_0]$ .

Note that in Example 1, we have f(t) = 1 and  $g(t) = \exp(t) = g'(t)$ , hence (2.17) holds. This explains why in the first example, for sufficiently large values of T no terminal constraint is necessary. As a second example, for the constant function g(t) = 1, (2.17) also holds.

## 3 General results in Hilbert spaces

In this section, we study optimal control problems in a Hilbert space setting. In this way, we obtain results that we can apply to systems that are governed by partial differential equations. Let X and U be Hilbert spaces with the inner products  $\langle \cdot, \cdot \rangle_X$ ,  $\langle \cdot, \cdot \rangle_U$  and the corresponding norms  $\|\cdot\|_X$ ,  $\|\cdot\|_U$  respectively. We use T > 0 to denote the terminal time of our optimal control problems. The space X contains the current state and the space U is used as a framework for the control functions in  $L^2(0,T;U)$ .

Let  $A : \mathcal{D}(A) \subset X \to X$  be the generator of a strongly continuous semigroup, and let *B* denote an admissible control operator. As in [17], Proposition 4.2.5., we consider control systems of the form

$$\begin{cases} x' + Ax = Bu, \\ x(0) = x_0 \end{cases}$$
(3.1)

where  $x_0 \in X$  is a given initial state. For all  $u \in L^2(0,T;U)$ , the Cauchy problem (3.1) has a unique solution  $x \in C([0,T];X)$  (see [12]).

#### 3.1 Exact controllability

Assume that (3.1) is exactly controllable using  $L^2$ -controls in time  $t_0 > 0$ , that is there exists a constant  $C_1 > 0$  such that for all initial states  $x_0 \in X$ and all terminal states  $x_1 \in X$  there is a control  $u \in L^2(0, t_0; U)$  such that the solution  $x \in C([0, t_0]; X)$  of (3.1) satisfies

$$\begin{cases} x(t_0) = x_1. \\ \|u\|_{L^2(0,t_0;U)} \leq C_1(\|x_0\|_X + \|x_1\|_X). \end{cases}$$
(3.2)

Let a desired state  $x_d \in X$  be given. Due to the exact controllability assumption, there exists a control  $u_{\text{exact}} \in L^2(0, t_0; U)$  such that the solution  $x_{\text{exact}} \in C([0, t_0]; X)$  of (3.1) satisfies

$$x_{\text{exact}}(t_0) = x_d. \tag{3.3}$$

We assume that  $x_d$  is a *holdable state* in the sense that we can extend  $u_{\text{exact}}$  to the time interval [0, T] by a constant control  $u_d$  on  $[t_0, T]$  such that for the corresponding state for all  $t \in (t_0, T)$  we have the equation  $x_{\text{exact}}(t) = x_d$  and  $u_{\text{exact}}(t) = u_d$ . Thus on the time-interval  $(t_0, T)$  we have  $A x_{\text{exact}} = A x_d = B u_{\text{exact}}$ .

## 3.2 An optimal control problem with max-norm penalization

First we consider a tracking term with the maximum-norm. For systems that are exactly controllable, the optimal control steers the system to the desired state after the prescribed time  $t_0$ .

For  $\gamma > 0$  we consider the following optimization problem:

$$\mathbf{P}(T,\gamma) \begin{cases} \min_{\substack{u \in L^2(0,t;U) \\ u \in L^2(0,t;U)}} \frac{1}{2} \|u - u_d\|_{L^2(0,T;U)}^2 + \gamma \max_{t \in [t_0,T]} \|x(s) - x_d\|_X \\ \text{subject to} \\ x' + Ax = Bu, \ x(0) = x_0. \end{cases}$$

In problem  $\mathbf{P}(T, \gamma)$  the end condition  $x(t) = x_d$  does not appear. Note that problem  $\mathbf{P}(T, \gamma)$  has a unique solution.

Our goal is to show that, due to the property of exact controllability using  $L^2$ -controls of the system, for  $\gamma$  sufficiently large the optimal state  $x_T$  satisfies the condition

$$x_T(t) = x_d$$

for all  $t \in [t_0, T]$ . A precise statement is given in the following theorem:

**Theorem 2** Assume that  $T > t_0$  and that the system (3.1) is exactly controllable. If  $\gamma > 0$  is sufficiently large, for all  $t \in [t_0, T]$  the solution  $(u_T, x_T)$  of problem  $\mathbf{P}(T, \gamma)$  satisfies the equation

$$x_T(t) = x_d$$

**Proof:** An application of the Direct Method of the Calculus of Variations shows that a solution of  $\mathbf{P}(T, \gamma)$  exists. The strict convexity of the control cost  $\frac{1}{2} \| \cdot \|_{L^2(0,T;U)}^2$  implies that the solution of  $\mathbf{P}(T, \gamma)$  is uniquely determined. Choose

$$\gamma > C_1 \| u_{\text{exact}} - u_d \|_{L^2(0, t_0; U)}.$$
(3.4)

Similarly as in [9], consider the optimal control problem

$$\mathbf{Q}(T,\gamma) \begin{cases} \min_{\substack{u \in L^2(0,T;U) \\ \text{subject to} \\ x' + Ax = Bu, \ x(0) = x_0. \end{cases}} \frac{1}{2} \|u - u_d\|_{L^2(0,T;U)}^2 + \gamma \|x(t_0) - x_d\|_X$$

Let  $(u^*, x^*)$  denote the solution of  $\mathbf{Q}(T, \gamma)$ . Now similarly as in Theorem 1 in ([9]), we show that  $x^*(t_0) = x_d$  by an indirect proof.

Suppose that  $x^*(t_0) \neq x_d$ . Then the objective functional of  $\mathbf{Q}(T, \gamma)$  is differentiable at  $(u^*, x^*)$  and the necessary optimality conditions imply

$$\int_{0}^{T} \langle u^{*} - u_{d}, v \rangle_{U} \, dt + \gamma \, \frac{\langle x^{*}(t_{0}) - x_{d}, y \rangle_{X}}{\|x^{*}(t_{0}) - x_{d}\|_{X}} = 0 \tag{3.5}$$

for all  $v \in L^2(0, t_1; U)$  where y solves

$$y' + Ay = Bv, \ y(0) = 0.$$

Due to the exact controllability of the system, we can choose a control  $\tilde{v} \in L^2(0, t_0; U)$  such that for the corresponding state  $\tilde{y}$  we have

$$\tilde{y}(t_0) = \frac{x^*(t_0) - x_d}{\|x^*(t_0) - x_d\|_X}$$

and

$$\|\tilde{v}\|_{L^2(0,t_0;U)} \le C_1. \tag{3.6}$$

We extend  $\tilde{v}$  to an element of  $L^2(0,T;U)$  by the definition  $\tilde{v}(s) = 0$  for all  $s \in (t_0, T)$ . Then the necessary optimality condition yields the equation

$$\int_{0}^{T} \langle u^{*} - u_{d}, \tilde{v} \rangle_{U} dt + \gamma \, \frac{\langle x^{*}(t_{0}) - x_{d}, \frac{x^{*}(t_{0}) - x_{d}}{\|x^{*}(t_{0}) - x_{d}\|_{X}} \rangle_{X}}{\|x^{*}(t_{0}) - x_{d}\|_{X}} = 0.$$
(3.7)

This implies the equation

$$\left| \int_0^T \langle u^* - u_d, \tilde{v} \rangle_U \, dt \right| = \gamma. \tag{3.8}$$

On the other hand, we have the inequality

$$\left| \int_0^T \langle u^* - u_d, \tilde{v} \rangle_U \, dt \right| \le \| u^* - u_d \|_{L^2(0,T;U)} \, \| \tilde{v} \|_{L^2(0,T;U)}.$$

Since the control  $u_{\text{exact}}$  is feasible for  $\mathbf{Q}(T, \gamma)$ , we have the inequality

$$\frac{1}{2} \|u^* - u_d\|_{L^2(0,T;U)}^2 \le \frac{1}{2} \|u_{\text{exact}} - u_d\|_{L^2(0,T;U)}^2 + \gamma \|y_{\text{exact}}(t_0) - y_d\|_X$$
$$= \frac{1}{2} \|u_{\text{exact}} - u_d\|_{L^2(0,T;U)}^2.$$

Hence

 $||u^* - u_d||_{L^2(0,T;U)} \le ||u_{\text{exact}} - u_d||_{L^2(0,T;U)}.$ 

Moreover, (3.6) implies

 $\|\tilde{v}\|_{L^2(0,T;U)} \le C_1.$ 

Hence (3.8) implies

$$\gamma \le C_1 \| u_{\text{exact}} - u_d \|_{L^2(0,T;U)},$$

which is a contradiction to (3.4). Thus we have shown that  $x^*(t_0) = x_d$ . This implies that for  $s \in (t_0, T]$  we have  $u^*(s) = u_d$  and  $x^*(s) = x_d$ . Let  $v_Q$  denote the optimal value of  $\mathbf{Q}(T, \gamma)$  and  $v_P$  denote the optimal value of  $\mathbf{P}(T, \gamma)$ . Then the definition of the corresponding objective functionals implies the inequality

$$v_Q \leq v_P$$
.

Since the control  $u^*$  is feasible for  $\mathbf{P}(T, \gamma)$ , we also have the inequality

$$v_{P} = \frac{1}{2} \|u_{T} - u_{d}\|_{L^{2}(0,T;U)}^{2} + \gamma \max_{t \in [t_{0},T]} \|x_{T}(s) - x_{d}\|_{X}$$
  
$$\leq \frac{1}{2} \|u^{*} - u_{d}\|_{L^{2}(0,T;U)}^{2} + \gamma \max_{t \in [t_{0},T]} \|x^{*}(s) - x_{d}\|_{X}$$
  
$$= \frac{1}{2} \|u^{*} - u_{d}\|_{L^{2}(0,t_{0};U)}^{2} = v_{Q}.$$

Thus we have  $v_P = v_Q$ , and  $(u^*, x^*)$  is an optimal control/state pair for  $\mathbf{Q}(T, \gamma)$ . Since the solution is unique, this implies the assertion.  $\Box$ .

### 3.3 An optimal control problem for nodal profile exactly controllable systems

Motivated by application problems in the operation of gas piplines, the exact controllability of nodal profiles has been introduced in [7], see also [11]. The assumption of exact controllability of nodal profiles also allows to derive a result about the exactness of an  $L^2$ -norm penalty term.

Let a Hilbert space Z,  $t_0 \in (0, T)$  and a linear map  $\Pi : L^2(0, T; X) \to L^2(t_0, T; Z)$  be given. In the applications, typically  $\Pi$  will be some trace operator, for example the boundary trace of the system state restricted to the time-interval  $[t_0, T]$ , see [7].

Assume that (3.1) is nodal profile exactly controllable using  $L^2$ -controls in time  $t_0 > 0$ , that is there exists a constant  $C_1 > 0$  such that for all initial states  $x_0 \in X$  and all nodal profiles  $z \in L^2(t_0, T; Z)$  there is a control  $u \in L^2(0, T; U)$  such that the solution  $x \in C([0, T]; X)$  of (3.1) satisfies for all  $t \in [t_0, T]$ 

$$\begin{cases} \Pi x(t) = z(t), \\ \|u\|_{L^2(0,T;U)} \leq C_1 (\|x_0\|_X + \|z\|_{L^2(t_0,T;Z)}). \end{cases} (3.9)$$

**Remark 1** The exact boundary controllability of nodal profile for hyperbolic systems is discussed in [11].

For  $\gamma > 0$  we consider the following optimization problem:

$$\mathbf{S}(T, \gamma) \begin{cases} \min_{u \in L^{2}(0,t;U)} \frac{1}{2} \|u - u_{d}\|_{L^{2}(0,T;U)}^{2} + \gamma \sqrt{\int_{t_{0}}^{T} \|\Pi x(s) - \Pi x_{d}\|_{Z}^{2}} ds \\ \text{subject to} \\ x' + Ax = Bu, \ x(0) = x_{0} \end{cases}$$

where as before,  $x_d \in X$  is the desired holdable state. In problem  $\mathbf{S}(T, \gamma)$  the end condition  $x(t) = x_d$  does not appear. Note that problem  $\mathbf{S}(T, \gamma)$  has a unique solution.

**Remark 2** Optimization problems of a similar structure with a differentiable tracking term have been considered in [6] and [10].

Due to the nodal profile exact controllability assumption, there exists a control  $v_{\text{exact}} \in L^2(0, t_0; U)$  such that the solution  $p_{\text{exact}} \in C([0, t_0]; X)$ of (3.1) satisfies

$$\Pi p_{\text{exact}}(t) = \Pi x_d \tag{3.10}$$

for all  $t \in [t_0, T]$ .

Our goal is to show that, due to the property of nodal profile exact controllability using  $L^2$ -controls of the system, for  $\gamma$  sufficiently large the optimal state  $x_T$  satisfies the condition

$$\Pi x_T(t) = \Pi x_d$$

for all  $t \in [t_0, T]$ . In the application in supply systems, this means that on the time interval  $[t_0, T]$ , the nodal profile that is desired by the customer is attained exactly. A precise statement is given in the following theorem:

**Theorem 3** Assume that  $T > t_0$  and that the system (3.1) is nodal profile exactly controllable. If  $\gamma > C_1 ||u_{\text{exact}} - u_d||_{L^2(0,t_0;U)}$ , for all  $s \in [t_0, T]$ the solution  $(u_T, x_T)$  of problem  $\mathbf{S}(T, \gamma)$  satisfies the equation

$$\Pi x_T(s) = \Pi x_d$$

**Proof:** An application of the Direct Method of the Calculus of Variations shows that a solution of  $\mathbf{S}(T, \gamma)$  exists. The strict convexity of the control cost  $\frac{1}{2} \| \cdot \|_{L^2(0,T;U)}^2$  implies that the solution of  $\mathbf{S}(T, \gamma)$  is uniquely determined. Choose

$$\gamma > C_1 \| v_{\text{exact}} - u_d \|_{L^2(0, t_0; U)}.$$
(3.11)

Suppose that there exists  $\tau \in [t_0, T]$  such that  $\Pi x^*(\tau) \neq \Pi x_d$ . Then  $\|\Pi x^* - \Pi x_d\|_{L^2(t_0,T;Z)} \neq 0$ . Hence the objective functional of  $\mathbf{S}(T,\gamma)$  is differentiable in  $(u^*, x^*)$  and the necessary optimality conditions imply

$$\int_{0}^{T} \langle u^{*} - u_{d}, v \rangle_{U} dt + \gamma \int_{t_{0}}^{T} \frac{\langle \Pi x^{*}(\tau) - \Pi x_{d}, \Pi y(\tau) \rangle_{Z}}{\|\Pi x^{*} - \Pi x_{d}\|_{L^{2}(t_{0}, T; Z)}} d\tau = 0$$
(3.12)

for all  $v \in L^2(0,T;U)$  where y solves

$$y' + Ay = Bv, \ y(0) = 0.$$

Due to the nodal profile exact controllability of the system, we can choose a control  $\tilde{v} \in L^2(0, t_0; U)$  such that for the corresponding state  $\tilde{y}$ we have for all  $\tau \in [t_0, T]$ 

$$\Pi \tilde{y}(\tau) = \frac{\Pi x^*(\tau) - \Pi x_d}{\|\Pi x^* - \Pi x_d\|_{L^2(t_0, T; Z)}}$$

and

$$\|\tilde{v}\|_{L^2(0,T;U)} \le C_1.$$

Then the necessary optimality condition (3.12) yields the equation

$$\int_{0}^{T} \langle u^{*} - u_{d}, \tilde{v} \rangle_{U} dt + \gamma \int_{t_{0}}^{T} \frac{\langle \Pi x^{*}(\tau) - \Pi x_{d}, \frac{\Pi x^{*}(\tau) - \Pi x_{d}}{\|\Pi x^{*} - \Pi x_{d}\|_{L^{2}(t_{0}, T; Z)}} \rangle_{Z}}{\|\Pi x^{*} - \Pi x_{d}\|_{L^{2}(t_{0}, T; Z)}} d\tau = 0.$$
(3.13)

This implies the equation

$$\left|\int_{0}^{T} \langle u^{*} - u_{d}, \tilde{v} \rangle_{X} dt\right| = \gamma.$$
(3.14)

On the other hand, we have the inequality

$$\left| \int_0^T \langle u^* - u_d, \tilde{v} \rangle_U \, dt \right| \le \| u^* - u_d \|_{L^2(0,T;U)} \, \| \tilde{v} \|_{L^2(0,T;U)}.$$

Since the control  $v_{\text{exact}}$  is feasible for  $\mathbf{S}(T, \gamma)$ , we have the inequality

$$\begin{split} \frac{1}{2} \|u^* - u_d\|_{L^2(0,T;U)}^2 &\leq \frac{1}{2} \|v_{\text{exact}} - u_d\|_{L^2(0,T;U)}^2 + \gamma \int_{t_0}^T \|\Pi p_{\text{exact}}(\tau) - \Pi x_d\|_Z \, d\tau \\ &= \frac{1}{2} \|v_{\text{exact}} - u_d\|_{L^2(0,T;U)}^2. \end{split}$$

Hence

$$||u^* - u_d||_{L^2(0,T;U)} \le ||v_{\text{exact}} - u_d||_{L^2(0,T;U)}$$

Moreover, we have

$$\|\tilde{v}\|_{L^2(0,T;U)} \le C_1.$$

Hence (3.14) implies

$$\gamma \leq C_1 \| v_{\text{exact}} - u_d \|_{L^2(0,T;U)},$$

which is a contradiction to (3.11). Thus we have shown that  $\Pi x^* = \Pi x_d$  on  $[t_0, T]$ . This implies the assertion.  $\Box$ .

# 3.4 An optimal control problem with $L^1$ -norm tracking term

In this section we present a result about the finite-time turnpike structure of the optimal state and the optimal control that we have shown under the assumption of exact controllability (3.2) for an optimal control problem with an  $L^1$ -norm tracking term with a singular weight in the objective function.

For  $\gamma>0$  we consider the following optimal control problem  ${\bf R}(T,\gamma)$  with  $L^1\text{-norm}$  tracking term:

$$\mathbf{R}(T,\gamma) \begin{cases} \min_{\substack{u \in L^2(0,T;U) \\ \text{subject to} \\ x' + Ax = Bu, \ x(0) = x_0.}} \frac{1}{2} \|u - u_d\|_{L^2(0,T;U)}^2 + \gamma \int_{t_0}^T \frac{1}{s - t_0} \|x(s) - x_d\|_X \, ds \end{cases}$$

In problem  $\mathbf{R}(T, \gamma)$  the end condition  $x(T) = x_d$  does not appear. Problem  $\mathbf{R}(T, \gamma)$  has a unique solution.

Our goal is to show that, due to the property of exact controllability using  $L^2$ -controls of the system, for  $\gamma$  and T sufficiently large the optimal state  $x_T$  for  $\mathbf{R}(T, \gamma)$  satisfies the condition

$$x_T(t) = x_d$$

for all  $t \in (t_0, T]$ . A precise statement is given in the following theorem:

**Theorem 4** Assume that  $T > t_0$  and that the system (3.1) is exactly controllable. If  $\gamma > 0$ , the solution  $(u_T, x_T)$  of problem  $\mathbf{R}(T, \gamma)$  satisfies the equation

$$x_T(t) = x_d$$

for all  $t \in [t_0, T]$ .

**Proof:** Since  $u_{\text{exact}}$  is a feasible control for  $\mathbf{R}(T, \gamma)$ , evaluating the objective function of  $\mathbf{R}(T, \gamma)$  at  $u_{\text{exact}}$  yields the inequality

$$\|u_T - u_d\|_{L^2(0,T;U)}^2 \le \|u_T - u_d\|_{L^2(0,t_0;U)}^2 + 2\gamma \int_{t_0}^T \frac{1}{s - t_0} \|x_{\text{exact}}(s) - x_d\|_X ds$$

$$= \|u_{\text{exact}} - u_d\|_{L^2(0,T;U)}^2 \le \|u_T - u_d\|_{L^2(0,t_0;U)}^2 + 2\gamma \int_{t_0}^T \frac{1}{s - t_0} \|x_{\text{exact}}(s) - x_d\|_X ds$$

$$(3.15)$$

$$= \|u_{\text{exact}} - u_d\|_{L^2(0,t_0;U)}^2.$$

An application of the Direct Method of the Calculus of Variations shows that a solution of  $\mathbf{R}(T, \gamma)$  exists. For the optimal control/state pair we use the notation  $(u_T, x_T)$ .

If there exists  $\hat{t} \in (0, T)$  with  $x_T(\hat{t}) = x_d$ , the optimal way to continue the control for  $s \in (\hat{t}, T]$  is with  $(u_d, x_d)$ , hence for all  $s \in (\hat{t}, T]$  we have  $x_T(s) = x_d$ .

Suppose that there exists a number  $t_1 \in (t_0, T]$  such that  $x_T(t_1) \neq x_d$ . Then for all  $t \in [t_0, t_1)$ , we also have  $x_T(t) \neq x_d$ . In particular, for all  $t \in [t_0, t_1]$ , we have  $||x_T(t) - x_d||_X > 0$ . Since  $x_T$  is continuous, this implies that

$$\inf_{t \in [t_0, t_1]} \|x_T(t) - x_d\|_X = \varepsilon > 0.$$

This implies

$$\int_{t_0}^{t_1} \frac{1}{s - t_0} \|x(s) - x_d\| \, ds \ge \varepsilon \int_{t_0}^{t_1} \frac{1}{s - t_0} = \infty.$$

Hence  $x_T$  cannot be optimal, and this is a contradiction.  $\Box$ 

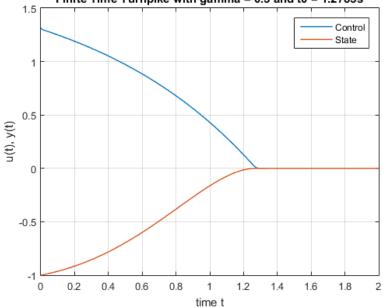
### 4 Examples

In this section we present some examples to illustrate our results about the finite-time turnpike phenomenon. We start with one example with a system that is governed by an ordinary differential equation and then we present examples with partial differential equations.

**Example 2** Let us first return to Example 1. Here we present numerical results that illustrate that the numerical solution for the discretized optimal control problem where for T = 2 the interval [0,2] has been replaced

with a grid of 201 equidistant points and the ordinary differential equation has been replaced by a discrete time-system with the Euler backwards discretization.

The resulting optimization problem has been solved numerically with a standard method from matlab. To improve the performance, in the numerical experiments the constraints  $u \ge 0$  and  $y \le 0$  have been included in the problem. (As shown in Example 1, they do not change the solution). The numerical results are presented in Figure 1 for  $\gamma = \frac{1}{2}$ , Figure 2 for  $\gamma = 1$  and Figure 3 for  $\gamma = 2$ .



Finite Time Turnpike with gamma = 0.5 and t0 = 1.2785s

Figure 1: The figure shows the optimal control and the optimal state as approximate solutions of problem  $(\mathbf{OC})_T$  for T = 2 and  $\gamma = \frac{1}{2}$  defined in Example 2.

Now we present examples of optimal control problems where Theorem 2 or Theorem 4 is applicable. These theorems assume that the system is exactly controllable.

**Example 3** Now we consider a problem of optimal torque control for an Euler-Bernoulli beam. Let  $y_0 \in H^2(0,1)$  and  $y_1 \in H^1(0,1)$  be given. We study the following optimal control problem:

$$\begin{cases} \min_{u \in L^{2}(0,T)} \frac{1}{2} \|u^{2}(t)\|^{2} dt + \gamma \max_{t \in [t_{0},T]} \|y(t,\cdot)\|_{L^{2}(0,1)} \text{ subject to} \\ y(0,x) = y_{0}(x), \ y_{t}(0,x) = y_{1}(x), \ x \in (0,1) \\ y(t,0) = 0, \ y_{xx}(t,0) = u(t), \ t \in (0,T) \\ y(t,1) = y_{xx}(t,1) = 0, \\ y_{tt}(t,x) = -y_{xxxx}(t,x), \ (t,x) \in (0,T) \times (0,1). \end{cases}$$

~

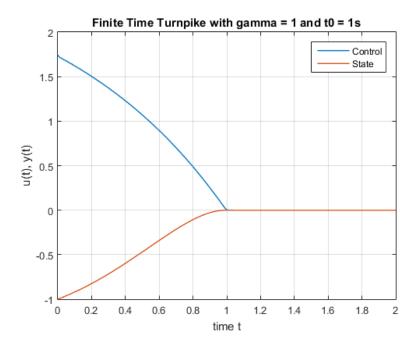


Figure 2: The figure shows the optimal control and the optimal state as approximate solutions of problem  $(\mathbf{OC})_T$  for T = 2 and  $\gamma = 1$  defined in Example 2.

We have  $U = L^2(0,1)$  and  $X = L^2(0,1)$ . Note that the Euler-Bernoulli beam is exactly controllable in arbitrarily short times (see [17], Example 11.2.8), so in this case  $t_0 > 0$  can be chosen arbitrarily small. Theorem 2 implies that if  $\gamma$  is chosen sufficiently large the beam is steered to a position of rest in the time  $t_0 > 0$ ,

**Example 4** Consider the problem of optimal Neumann boundary control of the wave equation. Define  $Q = (0, T) \times (0, 1)$ . Here we have  $U = L^2(0, 1), X = L^1(0, 1),$ 

Let  $y_d \in X$  and  $u_d \in U$  be given. Consider the optimal control problem

$$\begin{cases} \min_{u \in U} \frac{1}{2} \int_{0}^{T} (u(t) - u_d)^2 dt + \int_{2}^{T} \frac{1}{t-2} \int_{0}^{1} |y(t, x) - y_d| dx dt \text{ subject to} \\ y(0, x) = 0, \ y_t(0, x) = 0, \ x \in (0, 1) \\ y(t, 0) = 0, \ y_x(t, 1) = u(t), \ t \in (0, T) \\ y_{tt}(t, x) - y_{xx}(t, x) = 0, \ (t, x) \in Q. \end{cases}$$

Our results show that the solution has a turnpike structure as described in Theorem 4. The optimal control problem is similar to the Neumann optimal boundary control problem with a differentiable objective function considered in [8].

Now we present an example where Theorem 3 is applicable, that assumes that the system is nodal profile exactly controllable.

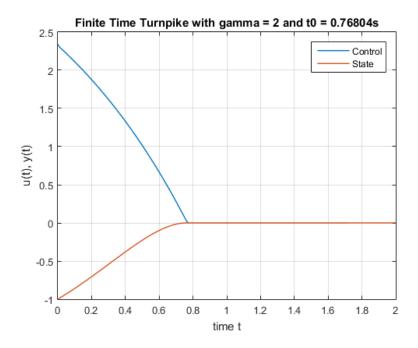


Figure 3: The figure shows the optimal control and the optimal state as approximate solutions of problem  $(\mathbf{OC})_T$  for T = 2 and  $\gamma = 2$  defined in Example 2.

**Example 5** Now we consider a problem or optimal control where Theorem 3 is applicable. The problem is similar as in [6], but in the tracking term instead of the squared  $L^2$ -norm we take the  $L^2$ -norm. The motivation for this type of problem where the boundary trace of the state is driven to a desired profile comes from the operation of networks of gas pipelines, where the aim is to satisfy customer demands in an optimal way.

We consider a  $2 \times 2$  system in diagonal form. Let a length L > 0 and a time interval [0,T] be given. Let  $d_{-}$  and  $d_{+}$  be real numbers such that

$$d_{-} < 0 < d_{+}.$$

Define the diagonal matrices

$$D = \left(\begin{array}{cc} d_+ & 0\\ 0 & d_- \end{array}\right).$$

For all  $x \in [0, L]$ , let M(x) denote a 2×2 matrix that depends continuously on x. Assume that for all  $x \in [0, L]$  the matrix M(x) is positive semidefinite. Let  $\eta_0 \leq 0$  be a real number.

Consider the linear hyperbolic partial differential equation

$$r_t + D r_x = \eta_0 M r \tag{4.1}$$

where for  $x \in (0, L)$  and  $t \in (0, T)$ , the state is given by  $r(t, x) = \begin{pmatrix} r_+(t, x) \\ r_-(t, x) \end{pmatrix}$ . Let real numbers  $R^d_+$  and  $R^d_-$  be given. To obtain an initial bound-

Let real numbers  $R_{+}^{d}$  and  $R_{-}^{d}$  be given. To obtain an initial boundary value problem, in addition to (4.1) we consider the initial condition r(0, x) = 0 for  $x \in (0, L)$  at the time t = 0 and for  $t \in (0, T)$  the Dirichlet boundary conditions  $r_{+}(t, 0) = u_{+}(t)$ ,  $r_{-}(t, L) = R_{-}^{d}$ , with a boundary control  $u_{+}$  in  $L^{2}(0, T)$ . The resulting initial boundary value problem

$$\begin{cases} r(0, x) = 0, \\ r_t + D r_x = \eta_0 M r, \\ r_+(t, 0) = u_+(t), \\ r_-(t, L) = R_-^d \end{cases}$$
(4.2)

has a solution  $r \in C([0,T], L^2((0,L); \mathbb{R}^2))$ . Moreover, for the boundary traces of the solution we have  $r_+(\cdot,L)$ ,  $r_-(\cdot,0) \in L^2(0,T)$ .

For  $x = (x_+, x_-) \in \mathbb{R}^2$ , we use the notation  $||x||_{\mathbb{R}^2} = \sqrt{x_+^2 + x_-^2}$ . For  $u = (u_+, u_-) \in (L^2(0, T))^2$  and  $R = (R_+, R_-) \in (L^2(0, T))^2$ , define the objective function

$$= \int_0^T \frac{1}{2} \left( u_+(t) \right)^2 dt + \gamma \int_{t_0}^{T-t_0} \| (R_+(t) - R_+^d, R_-(t) - R_-^d \|_{\mathbb{R}^2} dt.$$
(4.3)

Then if L is sufficiently small and T and  $t_0 < T$  are sufficiently large, the system is nodal profile exactly controllable and Theorem 3 is applicable for the optimal control problem

$$\begin{cases} \min_{u_+ \in L^2(0,T)} J(u_+, (r_+(\cdot, L), r_-(\cdot, L))) \\ \text{subject to (4.2).} \end{cases}$$
(4.4)

In fact the result of Theorem 3 can be interpreted as a finite-time turnpike result (or exact turnpike), where the system is driven to a desired stationary state in finite time

## 5 Conclusion

We have shown that a finite-time turnpike phenomenon occurs for problems of optimal control with nondifferentiable norm tracking terms.

We have first considered systems that are governed by ordinary differential equations. In the objective functions,  $L^1$ -norm tracking terms are used. The finite-time turnpike means that after finite time the optimal state reaches the desired state. For infinite-dimensional systems, we have shown that a finite-time turnpike phenomenon occurs for problems of optimal control for systems that are exactly controllable with a max-norm type tracking term and a weighted  $L^1$ -norm tracking term. For systems that are nodal profile exactly controllable, we have shown that a finitetime turnpike phenomenon occurs with an  $L^2$ -norm tracking term.

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