Full probabilistic solution of a finite dimensional linear control system with random initial and final conditions

J.-C. Cortés[a,*], A. Navarro-Quiles[a], J.-V. Romero[a], M.-D. Roselló[a], Enrique Zuazua[b,c,d]

[a]*Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022, Valencia, Spain*
[b]*Chair in Applied Analysis, Alexander von Humboldt-Professorship, Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg, 91058 Erlangen, Germany*
[c]*Chair of Computational Mathematics, Fundación Deusto, Av. de las Universidades, 24, 48007 Bilbao, Basque Country, Spain*
[d]*Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain*

Abstract

In this paper we perform a complete probabilistic study of a finite dimensional linear control system with uncertainty. The controllability condition with random initial data and final target is analysed. To conduct this investigation we determine the first probability density function of the control and the solution of the random control problem under different scenarios. To achieve this objective, Random Variable Transformation technique is extensively applied. Several examples illustrate the theoretical results.

Keywords: random linear control system, first probability density function, random variable transformation technique

---

1. Introduction

Control Theory is a branch of Mathematics that studies the behavior of a dynamic system with controllers applied through actuators. Its main objective is to develop models for controlling such systems using a control action in an

*Corresponding author. Phone number: +34–963877000 (ext. 88289)
Email addresses: `jccortes@imm.upv.es` (J.-C. Cortés), `annaqui@posgrado.upv.es` (A. Navarro-Quiles), `jvromero@imm.upv.es` (J.-V. Romero), `drosello@imm.upv.es` (M.-D. Roselló), `enrique.zuazua@fau.de` (Enrique Zuazua)

optimum manner. Applications of Control Theory, in irrigation systems, can be found since the ancient Mesopotamia more than 2000 years B.C. But it was not until the 1868 that the first significant mathematical description of Control Theory was established in the works by J.C. Maxwell [1]. Since then, Control Theory [2] gained importance, becoming nowadays a key tool to develop new technologies [3, 4].

A control problem consists in finding controls, say $u(t)$, such that the solution of a model, $x(t; u)$, coincides or gets close to a target value x^1 at a final time instant T, i.e., $x(T; u) = x^1$ or $\|x(T; u) - x^1\| < \epsilon$. Generally, an optimal control problem is defined via a set of differential equations, ordinary or partial, describing the states which depend on the control variables that minimize a particular cost function of the form

$$J(u) = \frac{1}{2} \left\| x(T; u) - x^1 \right\|^2 + \frac{\beta}{2} \|u\|^2 ,$$

where $\beta \geq 0$ allows us to penalize using too much costly control.

On the other hand, the parameters that appear in this kind of formulations are generally set via experimental data. Therefore, since these values are obtained from certain measurements and samplings, they often contain an intrinsic uncertainty. These facts make more convenient to consider inputs as random variables (RVs) or stochastic processes (SPs) rather than constants or deterministic functions, respectively. In recent years many works dealing with control problems have considered uncertainty in their formulation via Gaussian processes, mainly like the Brownian motion or the Wiener process, see for instance [5, 6]. The key role played by uncertainty in control models has been also shown in numerous areas like mechanics [7, 8], wireless communications [9], etc. With regard to the present contribution where random autonomous linear control problems are studied, it must be pointed out that these class of models have been addressed using different ways to include uncertainties in their formulation, such as switching signals driven by stochastic processes [10], multiplicative noise [11], etc. In the context of partial differential equations (PDEs), interesting advances have been done for different significant PDEs like the heat and wave equations taking control via the averaged and simulataneous behaviour of the problem [12, 13, 14, 15] or applying weak greedy algorithms for parameter dependent PDEs [16, 17].

As it shall see detail in Section 2, the aim of this paper is to provide a full probabilistic analysis, via the computation of the probability densities of the solution and the control, of finite dimensional linear control systems whose initial and final conditions are random variables satisfying certain general assumptions that will be specified later.

The paper is organized as follows. In Section 2, we first formulate the classical (deterministic) version of the linear control problem that will be study from a stochastic standpoint as well as some auxiliary classical results. Afterwards, we randomize the control problem and introduce the stochastic setting where the problem is going to be analysed. In this section, we further include important probabilistic results required to conduct our subsequent study. In Section 3 we give some considerations about the construction of the deterministic solution of the linear control problem that will be necessary to extend it to the random framework. Next, in Section 4 we will compute the complete probabilistic solution given by the 1-PDF of the random linear control problem considering that the initial condition and/or final target are absolutely continuous random variables. Section 5 is devoted to determine the 1-PDF of the control operator, which is a stochastic process. The theoretical results obtained in previous sections are illustrated via several numerical experiments in Section 6. Finally, some conclusions are given.

2. Problem description

In this paper, we are interested in the problem of controllability for finite dimensional linear systems of dimension $n \in \mathbb{N}$. This kind of problems can be formulated as

$$\begin{aligned} x'(t) &= \mathbf{A}x(t) + \mathbf{B}\,u(t), \quad 0 < t \le T, \\ x(0) &= x^0, \end{aligned} \tag{1}$$

where for each $t \in (0, T]$, $x(t) \in \mathbb{R}^n$ is the state of the system, $x^0 \in \mathbb{R}^n$ is the initial data, $\mathbf{A}$ is a square matrix of size $n \times n$ corresponding to the free dynamics part, $\mathbf{B}$ is a $n \times m$ matrix, with $m \in \mathbb{N}$ such that $m \leq n$, and $u(t)$ is an m-dimensional square integrable function termed the control vector, $u(t) \in L^2(0, T; \mathbb{R}^m)$. Furthermore, we will assume that the system (1) is controllable, then every initial state x^0 can be driven to any final state $x^1 \in \mathbb{R}^n$ in a control time $T > 0$, i.e., given an initial condition x^0, $x(T) = x^1$. This controllability property can be established by the so called Kalman's controllability matrix, in terms only of matrices $\mathbf{A}$ and $\mathbf{B}$, as indicated in the next result [2, 16].

Theorem 1 (Kalman). *A necessary and sufficient condition for* (1)*, in terms only on* $\mathbf{A}$ *and* $\mathbf{B}$*, to be controllable is*

$$\operatorname{rank}(\mathbf{C}) = \operatorname{rank}\left(\mathbf{B}|\mathbf{AB}|\cdots|\mathbf{A}^{n-1}\mathbf{B}\right) = n, \tag{2}$$

where $\mathbf{C}$ *is a* $n \times nm$ *matrix, called Kalman's controllability matrix.*

The proof of Theorem 1 [18] can be established by defining the matrices

$$\mathbf{F}(t) = \mathrm{e}^{\mathbf{A}(T-t)}\mathbf{B}, \qquad \mathbf{\Lambda} = \int_0^T \mathbf{F}(t)\mathbf{F}^\top(t)\,\mathrm{d}t, \tag{3}$$

and applying the following Lemmas 1 and 2 that will be used throughout the paper. As usual, $\mathbf{F}^\top$ denotes the transpose of matrix $\mathbf{F}$. The proof of these lemmas can be found in [19, pages 88–89].

Lemma 1. *A necessary and sufficient condition for (*$\mathbf{A}, \mathbf{B}$*) so that* (1) *to be controllable is that* $\mathbf{\Lambda}$ *given by* (3) *is invertible.*

Lemma 2. *Invertibility of* $\mathbf{\Lambda}$ *given by* (3) *is equivalent to Equation* (2)*.*

Remark 1. If problem (1) is controllable, in the proof of Lemma 1 is derived that $\mathbf{F}(t)\mathbf{F}^\top(t)$ is an invertible matrix $\forall t, 0 \leq t \leq T$. Thus, $\int_w^t \mathbf{F}(s)\mathbf{F}^\top(s)\mathrm{d}s$, $0 \leq w < t \leq T$ is an invertible matrix.

The aim of this contribution is to solve, from a probabilistic point of view, the following control problem with uncertainty

$$\begin{aligned} x'(t,\omega) &= \mathbf{A}x(t,\omega) + \mathbf{B}\,u(t,\omega), \quad 0 < t \le T, \\ x(0,\omega) &= x^0(\omega), \end{aligned} \tag{4}$$

where the starting seed $x^0(\omega) = \left[x_1^0(\omega), \ldots, x_n^0(\omega)\right]^\top$, $\omega \in \Omega$, and the final target $x^1(\omega) = \left[x_1^1(\omega), \ldots, x_n^1(\omega)\right]^\top$, $\omega \in \Omega$, are assumed to be absolutely continuous RVs defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In order to provide as much generality as possible throughout our analysis, hereinafter we will assume a joint probability density function (PDF) of the random vector $(x^0(\omega), x^1(\omega))$ given by $f_{x^0,x^1}(x^0, x^1)$. We must point out that this is guaranteed since we are assuming that $x^0(\omega)$ and $x^1(\omega)$ are absolutely continuous RVs. From a practical standpoint, this is not a serious restriction since many important RVs used in real applications are absolutely continuous RVs, some examples include Uniform, Beta, Triangular, Gamma, Weibull, Gaussian, t-Student, etc.

In the stochastic setting, the concepts of space of admissible controls and reachable sets correspond to the generalizations of their corresponding counterpart. The space of admissible controls, $\mathcal{U}$, is made up of SPs, $u = u(t,\omega)$, such as $u \in \mathcal{U} \subset L^2((0,T] \times \Omega; \mathbb{R}^m)$ almost surely (a.s.) or with probability 1 (w.p. 1). Notice that each component of the control operator $u(t,\omega)$, $u_k(t,\omega)$, $1 \le k \le m$, is a SP since for each fixed $\hat{t}$, $u_k(\hat{t},\omega)$ is a RV. From the deterministic theory, as it will be pointed later, $u(t,\omega)$ can be explicitly computed in terms of $\mathbf{A}$, $\mathbf{B}$, $x^0(\omega)$ and $x^1(\omega)$.

In the following, we will assume that the Initial Value Problem (IVP) (4) is controllable in the probabilistic sense, that is

$$x(T,\omega) = x^1(\omega), \quad \forall \omega \in \Omega, \tag{5}$$

i.e., $x(T,\omega) = x^1(\omega)$ a.s. or w.p. 1. The state $x^1(\omega)$, which is a RV, is said to be reachable in time T if there exists an input or control, $u \in \mathcal{U} \subset L^2((0,T] \times \Omega; \mathbb{R}^m)$ in problem (4) such that $x(T,\omega) = x^1(\omega)$ a.s. or w.p. 1. The set of all reachable RVs is called the reachable set. As a main difference wit respect to the deterministic scenario, in the stochastic setting the reachable set changes since depends upon realizations $\omega \in \Omega$.

Notice that as $\mathbf{A}$ and $\mathbf{B}$ are deterministic matrices, the control IVP (4) is controllable iff Kalman's condition given in Theorem 1 is fulfilled.

In the deterministic framework, the main objective is to determine the best control and, from it, the solution of system (1). Similarly, in the random scenario, a major objective is to determine the control SP, $u(t,\omega)$, and, from it, the solution SP of the problem, $x(t,\omega)$, can be derived. Unlike the deterministic theory, an important goal in the random setting is the computation of the main statistical functions of the SP, such as the mean and variance functions, and hence confidence intervals, since the average behaviour of that function as well as its variability about the mean are obtained from these two statistical moments. A more ambitious objective consists in the determination of the first probability density function (1-PDF), $f_1(x,t)$, of the solution SP, $x(t,\omega)$, $\omega \in \Omega$, that provides a full probabilistic description of the solution SP in each time instant t. From the 1-PDF the mean and the variance can be easily straightforwardly computed, provided these moments exist, but also other quantities of interest as the asymmetry, the kurtosis and other higher one-dimensional statistical moments

$$\mathbb{E}\left[(x(t,\omega))^l\right] = \int_{\mathbb{R}^n} x^l f_1(x,t)\mathrm{d}x, \quad l = 1,2,\ldots \qquad \left\{ \begin{array}{rcl} \mu_x(t) & = & \mathbb{E}\left[x(t,\omega)\right] \\ & = & \displaystyle\int_{\mathbb{R}^n} x f_1(x,t)\mathrm{d}x, \\ \sigma_x^2(t) & = & \mathbb{E}\left[(x(t,\omega))^2\right] - \mathbb{E}^2\left[x(t,\omega)\right] \\ & = & \displaystyle\int_{\mathbb{R}^n} x^2 f_1(x,t)\mathrm{d}x - \mu_x^2(t). \end{array} \right.$$

As we have previously indicated, there are many works that have introduced uncertainty in control problems. Many of them consider the average controllability [12, 13, 15]. The novelty of this contribution is to work directly with the PDF of the inputs and to obtain closed expressions for the 1-PDFs of the SP solution and the SP operator control. As it has been explained previously, this is advantageous since from these PDFs, we can give a more complete probabilistic description of problem (4) via the computation of the mean, variance, asymmetry, kurtosis, etc., as well as to construct confidence regions at a prescribed confidence level. Additionally, the 1-PDF permits to compute the probability that the solution, $x(t,\omega)$ (analogously, the control) lies in any set, say $\mathcal{D} \in \mathbb{R}^n$, of specific interest

$$\mathbb{P}[\{\omega \in \Omega : x(t,\omega) \in \mathcal{D}\}] = \int_{\mathcal{D}} f_1(x,t)\,\mathrm{d}x.$$

The main goal of this contribution is to compute the 1-PDF of the control $u(t,\omega)$ and the solution SP $x(t,\omega)$ of the random control problem (4). With this aim the Random Variable Transformation method (RVT) will be applied. RVT is a powerful technique to determine the PDF of a RV which comes from mapping another RV whose PDF is known. The multidimensional version of the RVT method is stated in the following result.

Theorem 2 (Random Variable Transformation technique). *[20, pp. 24–25] Let $X(\omega) = (X_1(\omega), \ldots, X_k(\omega))^\top$ and $Y(\omega) = (Y_1(\omega), \ldots, Y_k(\omega))^\top$ be two m-dimensional absolutely continuous random vectors defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $r : \mathbb{R}^k \to \mathbb{R}^k$ be a one-to-one deterministic transformation of $X(\omega)$ into $Y(\omega)$, i.e., $Y(\omega) = r(X(\omega))$, $\omega \in \Omega$. Assume that r is a continuous mapping and has continuous partial derivatives with respect to each component x_i, $1 \le i \le k$. Then, if $f_X(x_1, \ldots, x_k)$ denotes the joint probability density function of the vector $X(\omega)$, and $s = r^{-1} = (s_1(y_1, \ldots, y_k), \ldots, s_k(y_1, \ldots, y_k))$ represents the inverse mapping of $r = (r_1(x_1, \ldots, x_k), \ldots, r_k(x_1, \ldots, x_k))$, the joint probability density function of the random vector $Y(\omega)$ is given by*

$$f_Y(y_1, \ldots, y_k) = f_X\left(s_1(y_1, \ldots, y_k), \ldots, s_k(y_1, \ldots, y_k)\right) |\mathcal{J}_k|,$$

where $|\mathcal{J}_k|$, which is assumed to be different from zero, denotes the absolute *value of the Jacobian defined by the following determinant*

$$\mathcal{J}_k = \det \begin{bmatrix} \dfrac{\partial s_1(y_1,\ldots,y_k)}{\partial y_1} & \cdots & \dfrac{\partial s_k(y_1,\ldots,y_k)}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial s_1(y_1,\ldots,y_k)}{\partial y_k} & \cdots & \dfrac{\partial s_k(y_1,\ldots,y_k)}{\partial y_k} \end{bmatrix}.$$

3. Deterministic theory for linear control problems

As we are interested in the study of the randomized version (4) of the deterministic problem (1), first it is convenient for the sake of completeness to introduce a basic review about the derivation of the solution to problem (1). For further details we refer to [1, 16, 2].

Given a control $u \in L^2(0,T;\mathbb{R}^m)$, according to the variation of parameters formula, the unique solution of control problem (1), $x \in H^1(0,T;\mathbb{R}^n)$, is

$$x(t) = \mathrm{e}^{\mathbf{A}t}x^0 + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\, u(s)\,\mathrm{d}s, \quad t \in (0,T), \tag{6}$$

where $L^2(0,T;\mathbb{R}^m)$ is the space of square integrable $\mathbb{R}^m$-valued functions with domain the interval $[0,T]$ and $H^1(0,T;\mathbb{R}^n)$ is the Sobolev space whose elements, say f, satisfy $f, f' \in L^2(0,T;\mathbb{R}^n)$.

To reach our objective, i.e. the computation of the 1-PDFs of the control and solution SPs, it is more convenient to use the approach based on the so called dual problem of observability of the homogeneous adjoint system

$$\begin{aligned} \rho'(t) &= -\mathbf{A}^\top \rho(t), \quad 0 \le t \le T, \\ \rho(T) &= \rho^0. \end{aligned} \tag{7}$$

In this manner, both the control and the solution SPs can be explicitly represented in terms of coefficients and the initial condition and the final target [16].

The duality principle allows us to reduce the controllability problem of the system (1) into an observability one for the adjoint system (7). Then, the control $u(t)$ is given by

$$u(t) = \mathbf{B}^\top \rho(t), \tag{8}$$

being $\rho(t)$ the solution of the adjoint system (7) taking ρ^0 the minimiser of the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}$

$$J(\rho^0) = \frac{1}{2}\int_0^T |\mathbf{B}^\top \rho(t)|^2 \mathrm{d}t - < x^1, \rho^0 > + < x^0, \rho(0) > .$$

The minimiser ρ^0 can be expressed as

$$\rho^0 = \mathbf{\Lambda}^{-1}(x^1 - \mathrm{e}^{\mathbf{A}T} x^0), \tag{9}$$

where $\mathbf{\Lambda}$ is defined in (3).

Therefore, taking into account (9) the solution of the adjoint system (7) is given by

$$\rho(t) = \mathrm{e}^{\mathbf{A}^\top (T-t)} \rho^0 = \mathrm{e}^{\mathbf{A}^\top (T-t)} \mathbf{\Lambda}^{-1}(x^1 - \mathrm{e}^{T\mathbf{A}} x^0).$$

As it has been pointed out previously, the control is given by (8), so

$$u(t) = \mathbf{B}^\top \mathrm{e}^{\mathbf{A}^*(T-t)} \mathbf{\Lambda}^{-1}(x^1 - \mathrm{e}^{T\mathbf{A}} x^0). \tag{10}$$

Summarizing, the solution of the deterministic control problem (1) with final target, $x(T) = x^1$, is given by

$$x(t) = \mathrm{e}^{\mathbf{A}t} x^0 + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-s)}\, \mathrm{d}s\, \mathbf{\Lambda}^{-1}(x^1 - \mathrm{e}^{\mathbf{A}T} x^0). \tag{11}$$

4. Computing the 1-PDF of the solution SP $x(t, \omega)$

If we randomize the initial condition in IVP (1) and we consider the random final target given in (5), then we obtain the corresponding random problem (4)–(5). Thus, according to (11), the solution of the randomized problem (4)–(5) is

$$x(t,\omega) = \mathrm{e}^{\mathbf{A}t} x^0(\omega) + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-s)}\, \mathrm{d}s \mathbf{\Lambda}^{-1}(x^1(\omega) - \mathrm{e}^{\mathbf{A}T} x^0(\omega)), \tag{12}$$

where $\mathbf{\Lambda}$ is the deterministic matrix given by (3).

To obtain the 1-PDF of the solution SP $x(t, \omega)$, we will apply the RVT method, Theorem 2. First, for sake of clarity, we rewrite the solution as follows

$$x(t,\omega) = \left(\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}\right) x^0(\omega) + \mathbf{H}(t) x^1(\omega),$$

where

$$\mathbf{H}(t) = \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-s)}\,\mathrm{d}s\,\mathbf{\Lambda}^{-1}. \tag{13}$$

In most situations the initial condition and the final target are both random. This case will be studied in Subsection 4.1 in detail. But there are situations in which only the initial or final condition can be considered separately random. These cases will be analysed in Subsections 4.2 and 4.3.

Now, we establish some properties about matrix $\mathbf{H}(t)$ that will be required in the subsequent development.

Proposition 1. *If* (2) *is fulfilled, then* $\mathbf{H}(t)$ *given by* (13) *is invertible.*

Proof.

$$\begin{aligned}
&\det\left(\mathbf{H}(t)\right) = \det\left(\int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-s)}\mathrm{d}s\mathbf{\Lambda}^{-1}\right) \neq 0\\
\Leftrightarrow\quad &\det\left(\int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-s)}\mathrm{d}s\right) \neq 0\\
\Leftrightarrow\quad &\det\left(\mathrm{e}^{\mathbf{A}(T-t)}\int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-s)}\mathrm{d}s\right)\\
&= \det\left(\int_0^t \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-s)}\mathrm{d}s\right) \neq 0.
\end{aligned}$$

Last expression is true by Remark 1. □

Proposition 2. *If* (2) *is fulfilled, then* $\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}$ *is invertible.*

Proof. Observe that for $t: 0 < t < T$, one gets

$$
\begin{aligned}
& \det\left(\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}\right) = \det\left(\left(\mathrm{e}^{\mathbf{A}(t-T)} - \mathbf{H}(t)\right)\mathrm{e}^{\mathbf{A}T}\right) \neq 0 \\
\Leftrightarrow\ & \det\left(\mathrm{e}^{\mathbf{A}(t-T)} - \mathbf{H}(t)\right) = \det\left(\mathrm{e}^{\mathbf{A}(t-T)} - \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s\boldsymbol{\Lambda}^{-1}\right) \neq 0 \\
\Leftrightarrow\ & \det\left(\mathrm{e}^{\mathbf{A}(t-T)}\boldsymbol{\Lambda} - \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s\right) \\
& = \det\left(\mathrm{e}^{\mathbf{A}(t-T)}\int_0^T \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s - \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s\right) \\
& = \det\left(\int_0^T \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s - \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s\right) \\
& = \det\left(\int_t^T \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s\right) \neq 0 \\
\Leftrightarrow\ & \det\left(\mathrm{e}^{\mathbf{A}(T-t)}\int_t^T \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s\right) \\
& = \det\left(\int_t^T \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top(T-s)}\mathrm{d}s\right) \neq 0.
\end{aligned}
$$

Last expression is true by Remark 1. □

4.1. 1-PDF of $x(t,\omega)$ when $x^0(\omega)$ and $x^1(\omega)$ are random vectors

We will assume a joint PDF of the random vector $(x^0(\omega), x^1(\omega))$ denoted by $f_{x^0,x^1}(x^0,x^1)$. Let $t > 0$ be fixed, and define the mapping $r = (r_1, r_2) : \mathbb{R}^h \to \mathbb{R}^h$ ($h = n+n = 2n$) whose components $r_i : \mathbb{R}^h \to \mathbb{R}^n$, $i = 1, 2$, are defined as follows

$$
\begin{aligned}
y^1 &= r_1(x^0,x^1) = \left(\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}\right)x^0 + \mathbf{H}(t)x^1, \\
y^2 &= r_2(x^0,x^1) = x^0.
\end{aligned}
$$

Then, applying Proposition 1 the components of the inverse mapping of r, $s = r^{-1}$ are given by

$$
\begin{aligned}
x^0 &= s_1(y^1,y^2) = y^2, \\
x^1 &= s_2(y^1,y^2) = H(t)^{-1}y^1 - \left(H(t)^{-1}\mathrm{e}^{\mathbf{A}t} - \mathrm{e}^{\mathbf{A}T}\right)y^2.
\end{aligned}
$$

The Jacobian of the inverse transformation is given by the following determinant

$$
\mathcal{J}_h = \det\begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{H}(t)^{-1} \\ \mathbf{I}_n & \#_{n\times n} \end{bmatrix} = -\det\left(\mathbf{H}(t)^{-1}\right),
$$

where, as usually, $\mathbf{0}_{n_1 \times n_2}$ stands for the null matrix of size $n_1 \times n_2$ and $\mathbf{I}_{n_1}$ for the identity matrix of size n_1. Notice that using the Kalman's condition for controllability, Theorem 1, the transformation is well defined by Proposition 1 and $\mathcal{J}_h \neq 0$.

Then, applying RVT technique, Theorem 2, the PDF of the random vector (y^1, y^2) in terms of the joint PDF of the random vector of input parameters (x^0, x^1), is

$$f_{y^1,y^2}(y^1, y^2) = f_{x^0,x^1}\left(y^2, \mathbf{H}(t)^{-1}y^1 - \left(\mathbf{H}(t)^{-1}\mathrm{e}^{\mathbf{A}t} - \mathrm{e}^{\mathbf{A}T}\right) y^2\right) \left|\det\left(\mathbf{H}(t)^{-1}\right)\right|. \tag{14}$$

Since the solution for every t fixed of the random IVP (4) is given by the first component of mapping r, y^1, we marginalize expression (14) with respect to $y^2 = x^0$

$$f_{y^1}(y^1) = \int_{\mathbb{R}^n} f_{x^0,x^1}\left(x^0, \mathbf{H}(t)^{-1}y^1 - \left(\mathbf{H}(t)^{-1}\mathrm{e}^{\mathbf{A}t} - \mathrm{e}^{\mathbf{A}T}\right) x^0\right) \left|\det\left(\mathbf{H}(t)^{-1}\right)\right| \mathrm{d}x^0,$$

where

$$\mathrm{d}x^0 = \prod_{i=1}^{n} \mathrm{d}x_i^0. \tag{15}$$

Then, taking an arbitrary t, $T > t > 0$ the 1-PDF of the solution SP $x(t,\omega)$ of linear control problem (4)–(5) is given by

$$f_1(x,t) = \int_{\mathbb{R}^n} f_{x^0,x^1}\left(x^0, \mathbf{H}(t)^{-1}x - \left(\mathbf{H}(t)^{-1}\mathrm{e}^{\mathbf{A}t} - \mathrm{e}^{\mathbf{A}T}\right) x^0\right) \left|\det\left(\mathbf{H}(t)^{-1}\right)\right| \mathrm{d}x^0, \tag{16}$$

where $\mathbf{H}(t)$ and $\mathrm{d}x^0$ are defined by (13) and (15), respectively.

4.2. 1-PDF of $x(t,\omega)$ when $x^0(\omega)$ is a random vector

In this section we will obtain the 1-PDF of the solution SP in the case that only the initial condition is a random vector $x^0(\omega)$ whose PDF is denoted by $f_{x^0}(x^0)$. Let us fix $t > 0$, and define the mapping $r : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$y \quad = \quad r(x^0) \quad = \quad \left(\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}\right) x^0 + \mathbf{H}(t)x^1,$$

Then, the inverse mapping, $s = r^{-1}$, is

$$x^0 \quad = \quad s(y) \quad = \quad \left(\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}\right)^{-1} \left(y - \mathbf{H}(t)x^1\right),$$

that is well defined by Proposition 2. Furthermore, the Jacobian of the inverse transformation is given by

$$\mathcal{J}_n = \det\left(\left(\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}\right)^{-1}\right) \neq 0.$$

Since the solution for every t fixed of the random IVP (4) is given by y, applying RVT technique, Theorem 2, taking $T > t > 0$ the 1-PDF of the solution SP $x(t,\omega)$ of the linear control problem when the initial condition is random is given by

$$f_1(x,t) = f_{x^0}\left(\left(\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}\right)^{-1}\left(x - \mathbf{H}(t)x^1\right)\right)\left|\det\left(\left(\mathrm{e}^{\mathbf{A}t} - \mathbf{H}(t)\mathrm{e}^{\mathbf{A}T}\right)^{-1}\right)\right|, \tag{17}$$

where $\mathbf{H}(t)$ is defined by (13).

4.3. 1-PDF of $x(t,\omega)$ when $x^1(\omega)$ is a random vector

Finally, we deal with the scenario where the final condition is a random vector $x^1(\omega)$ whose PDF is denoted by $f_{x^1}(x^1)$. Let us fix $t > 0$. In order to apply RVT technique, we define the following mapping $r : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$y \quad = \quad r(x^1) \quad = \quad \left(\mathrm{e}^{\mathbf{A}t} - H(t)\mathrm{e}^{\mathbf{A}T}\right)x^0 + \mathbf{H}(t)x^1.$$

Reasoning similarly as in Subsection 4.1, we obtain the PDF $f_y(y)$, and then, taking $T > t > 0$, the 1-PDF, $f_1(x,t)$, of the solution SP $x(t,\omega)$ of the linear control problem when the final target is a random vector. This leads to

$$f_1(x,t) = f_{x^1}\left(\mathbf{H}(t)^{-1}x - \left(\mathbf{H}(t)^{-1}\mathrm{e}^{\mathbf{A}t} - \mathrm{e}^{\mathbf{A}T}\right)x^0\right)\left|\det\left(\mathbf{H}(t)^{-1}\right)\right|, \tag{18}$$

where $\mathbf{H}(t)$ is defined by (13). Notice that $\mathbf{H}(t)^{-1}$ exists according to Proposition 1.

5. Computing the 1-PDF of the control SP $u(t,\omega)$

According to the explicit expression of the control $u(t)$ given in (10) in terms of the initial condition x^0 and the target condition x^1, and depending on the random nature of $x^0 = x^0(\omega)$, or $x^1 = x^1(\omega)$, or both, we can take advantage of RVT technique to determine its PDF in each one of these cases. To this end, let us consider the randomization of $u(t)$ given in (10),

$$u(t,\omega) = \mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-t)} \boldsymbol{\Lambda}^{-1} (x^1(\omega) - \mathrm{e}^{\mathbf{A}T} x^0(\omega)). \tag{19}$$

We can rewrite equation (19) as

$$u(t,\omega) = \mathbf{G}(t)(x^1(\omega) - \mathrm{e}^{\mathbf{A}T} x^0(\omega)),$$

where

$$\mathbf{G}(t) = \mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-t)} \boldsymbol{\Lambda}^{-1}, \tag{20}$$

is a matrix of dimensions $m \times n$.

In order to obtain the 1-PDF of the control operator $u(t,\omega)$, we will assume the following hypothesis

H1: The control operator $\mathbf{B}$ has maximun rank, i.e., $\mathrm{rank}(\mathbf{B}) = m$.

Although dependence between several controls is plausible, it is important to remark that hypothesis **H1** is not a restriction. In practice the objective is to find the minimum number of controllers to assure the controllability of the system under study. The ideal case would be that one a unique control is needed, if it is possible.

5.1. 1-PDF of $u(t,\omega)$ when $x^0(\omega)$ and $x^1(\omega)$ are random vectors

Let us consider a joint PDF, $f_{x^0,x^1}(x^0,x^1)$, of the random vector $(x^0(\omega), x^1(\omega))$.

Notice that by hypothesis **H1** we have

$$\mathrm{rank}\,(\mathbf{G}(t)) = \mathrm{rank}\left(\mathbf{B}^\top \mathrm{e}^{\mathbf{A}^\top (T-t)} \boldsymbol{\Lambda}^{-1}\right) = m.$$

Then, we can easily construct a matrix $\mathbf{L}$ of dimensions $(n-m) \times n$ such that

$$\begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}$$

is a non-singular matrix of dimensions $n \times n$. To construct $\mathbf{L}$, we consider the position of the m linear independents columns of matrix $\mathbf{G}(t)$. Then, in the columns of matrix $\mathbf{L}$ associated with this positions we put zero-vectors. The rest of matrix $\mathbf{L}$ is fulfilled with the $n-m$ vectors corresponding to the identity matrix $\mathbf{I}_{n-m}$. Notice that matrix $\mathbf{L}$ is not unique, For example, if the first m columns of matrix $\mathbf{G}(t)$ are linear independent, we can take $\mathbf{L}$ as

$$\mathbf{L} = \begin{bmatrix} \mathbf{0}_{(n-m)\times m} & \mathbf{I}_{n-m} \end{bmatrix}. \tag{21}$$

Now, we will define an adequate mapping in order to apply Theorem 2. Let us fix $t > 0$, we define the mapping $r = (r_1, r_2) : \mathbb{R}^h \to \mathbb{R}^h$ $(h = n + n = 2n)$ whose components $r_i : \mathbb{R}^h \to \mathbb{R}^n$, $i = 1, 2$, are defined as follows

$$\begin{aligned} y^1 &= r_1(x^0, x^1) = \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix} x^1 + \begin{bmatrix} -\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} x^0 \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix}, \\ y^2 &= r_2(x^0, x^1) = x^0. \end{aligned}$$

Then, the components of the inverse mapping $s = r^{-1}$ are

$$\begin{aligned} x^0 &= s_1(y^1, y^2) = y^2, \\ x^1 &= s_2(y^1, y^2) = \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \left(y^1 + \begin{bmatrix} \mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} y^2 \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix} \right). \end{aligned}$$

Notice that the mapping is well defined because of the construction of L and the Jacobian of the inverse transformation is given by the following determinant

$$\mathcal{J}_h = \det \begin{bmatrix} \mathbf{0}_{n\times n} & \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \\ \mathbf{I}_n & \#_{n\times n} \end{bmatrix} = -\det \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \neq 0.$$

Then, applying RVT technique, Theorem 2, the PDF of the random vector (y^1, y^2) in terms of the joint PDF of the random vector of input parameters (x^0, x^1), is

$$f_{y^1,y^2}(y^1, y^2) = f_{x^0,x^1}\left(y^2, \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \left(y^1 + \begin{bmatrix} \mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} y^2 \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix} \right) \right) \left| \det \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \right|. \tag{22}$$

Since the solution for every t fixed of the control operator (19) is given by the m first components of the first component, y^1, of mapping $\mathbf{r}$ we marginalize expression (22) with respect to the $n-m$ latests components of y^1 (this is the $n-m$ latests components of x^1) and $y^2 = x^0$, obtaining the 1-PDF for the control operator as

$$f_1(u,t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-m}} f_{x^0,x^1} \left(x^0, \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \begin{bmatrix} u + \mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} x^0 \\ w \end{bmatrix} \right) \left| \det \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \right| \mathrm{d}w \mathrm{d}x^0, \tag{23}$$

where $\mathbf{G}(t)$, $\mathbf{L}$ and $\mathrm{d}x^0$ are defined by (20), (21) and (15), respectively and

$$w = \left[x^1_{m+1}, \ldots, x^1_n\right]^\top, \qquad \mathrm{d}w = \prod_{i=m+1}^{n} \mathrm{d}x^1_i. \tag{24}$$

5.2. 1-PDF of $u(t,\omega)$ when $x^0(\omega)$ is a random vector

In this subsection, we deal with the computation of the 1-PDF of the control $u(t,\omega)$ in the case that the initial condition $x^0(\omega)$ is a random vector with a PDF, $f_{x^0}(x^0)$. By hypothesis **H1** we have

$$\operatorname{rank}\left(\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T}\right) = \operatorname{rank}\left(\mathbf{G}(t)\right) = m,$$

and matrices $\mathbf{G}(t)$ and $\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T}$ have the m linear independent columns in the same position, so

$$\begin{bmatrix} -\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} \\ \mathbf{L} \end{bmatrix}$$

is a non-singular matrix of dimensions $n \times n$, where $\mathbf{L}$ is constructed as explained in Subsection 5.1, and for simplicity, and without lack of generality, we suppose it is given by expression (21).

Now, let us fix $t > 0$, and define the mapping $r : \mathbb{R}^n \to \mathbb{R}^n$ ($h = n + n = 2n$) as follows

$$y \;=\; r(x^0) \;=\; \begin{bmatrix} \mathbf{G}(t)x^1 \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix} + \begin{bmatrix} -\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} \\ \mathbf{L} \end{bmatrix} x^0.$$

Then, the inverse mapping $s = r^{-1}$ is

$$x^0 \quad = \quad s(y) \quad = \quad \begin{bmatrix} -\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} \\ \mathbf{L} \end{bmatrix}^{-1} \left(y - \begin{bmatrix} \mathbf{G}(t)x^1 \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix} \right).$$

Notice that the mapping is well defined and the Jacobian of the inverse transformation is given by the following determinant

$$\mathcal{J}_n = \det \begin{bmatrix} -\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} \\ \mathbf{L} \end{bmatrix}^{-1} \neq 0.$$

Then, applying RVT technique, Theorem 2, the PDF of the random vector y, $f_y(y)$, in function of the PDF of the random vector of initial condition x^0, is constructed. Since the solution for every t fixed of the control operator (19) is given by the m first components of y, we marginalize $f_y(y)$ with respect to the $n-m$ latests components of y (this is the $n-m$ latests components of x^0), obtaining that the 1-PDF for the control operator is given by

$$f_1(u,t) = \int_{\mathbb{R}^{n-m}} f_{x^0}\left(\begin{bmatrix} -\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} \\ \mathbf{L} \end{bmatrix}^{-1} \begin{bmatrix} u - \mathbf{G}(t)x^1 \\ v \end{bmatrix} \right) \left| \det \begin{bmatrix} -\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} \\ \mathbf{L} \end{bmatrix}^{-1} \right| \mathrm{d}v, \tag{25}$$

where $\mathbf{G}(t)$ and $\mathbf{L}$ are defined by (20), (21) , respectively and

$$v = \left[x^0_{m+1}, \ldots, x^0_n\right]^\top, \qquad \mathrm{d}v = \prod_{i=m+1}^{n} \mathrm{d}x^0_i. \tag{26}$$

5.3. 1-PDF of $u(t,\omega)$ when $x^1(\omega)$ is a random vector

Throughout this subsection we will consider the PDF of random vector $x^1(\omega)$ is given by $f_{x^1}(x^1)$. Let us fix $t > 0$. In order to apply RVT technique, we define the mapping $r : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$y \quad = \quad r(x^1) \quad = \quad \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix} x^1 + \begin{bmatrix} -\mathbf{G}(t)\mathrm{e}^{\mathbf{A}T}x^0 \\ \mathbf{0}_{(n-m)\times 1} \end{bmatrix},$$

where $\mathbf{L}$ is constructed as indicated in Subsection 5.1, and without lack of generality, we consider it is given by (21).

Following a reasoning similar to that performed in Subsection 4.1, we obtain the PDF $f_y(y)$, and then, taking $T > t > 0$, the 1-PDF, $f_1(x,t)$ of the control operator SP $u(t,\omega)$ of the linear control problem when the final target is random is obtained,

$$f_1(u,t) = \int_{\mathbb{R}^{n-m}} f_{x^1}\left(\begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \begin{bmatrix} u + \mathbf{G}(t)\mathrm{e}^{\mathbf{A}T}x^0 \\ w \end{bmatrix}\right) \left|\det\left(\begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1}\right)\right| \mathrm{d}w, \tag{27}$$

where $\mathbf{G}(t)$, $\mathbf{L}$ are defined by (13), (21), respectively and w, $\mathrm{d}w$ are given by (24).

6. Numerical examples

In this section we will illustrate by means of two examples all the theoretical results established in Sections 4 and 5 related to the computation of the 1-PDF of $x(t,\omega)$ and $u(t,\omega)$ depending on the random nature of the initial condition, $x^0(\omega)$ and/or final target $x^1(\omega)$. In both examples, calculations have been carried out using Mathematica© software. In particular, the command NIntegrate, which allows us to perform integrals (16)–(18), (23), (25) and (27) numerically.

Example 1. Let $\mathbf{A}$ and $\mathbf{B}$ be the matrices of the control IVP (1) defined by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{28}$$

By Kalman's rank condition, see Remark 1, $(\mathbf{A},\mathbf{B})$ is controllable

$$\mathrm{rank}(\mathbf{B}|\mathbf{AB}) = \mathrm{rank}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2.$$

Then, the system is controllable and we know the expression for the solution and the control. Therefore, the distribution of both stochastic processes can be computed. In this example we consider randomness in both the initial condition and the final target. To show the capability of the theoretical results previously established we consider that all RVs involved are independent with the following distributions:

- The first component of the initial condition, x_1^0, follows a Gamma distribution with mean 1 and variance 0.005, i.e., $x_1^0(\omega) \sim \mathrm{Ga}(200, 0.005)$.
- The second component of the initial condition, x_2^0, has a Uniform distribution defined in the interval $[0.8, 1.2]$ (with mean 1), i.e., $x_2^0(\omega) \sim \mathrm{U}([0.8, 1.2])$.
- The first component of the final target, x_1^1, is Normal distributed with mean 0 and variance 0.05, i.e., $x_1^1(\omega) \sim \mathrm{N}(0, 0.05)$.
- The second component of the final target, x_2^1, has a Triangular distribution defined in the interval $[-0.05, 0.1]$ with mode -0.05, i.e., $x_2^1(\omega) \sim \mathrm{T}([-0.05, 0.1]; -0.05)$.

With this choice for the distributions, we observe that the system starts at the random vector $x^0(\omega)$ and finishes at the random vector $x^1(\omega)$. The control time considered to carry out the computations shown down below is $T = 1$.

The 1-PDF, $f_1(x, t)$, of the solution SP is shown in Figure 1 for $t = 0.8$. In this plot we have compared Monte Carlo Method with the 1-PDF obtained from expression (16). We can observe a good agreement. We have also displayed confidence regions at levels $1 - \alpha = 0.5$ (blue line) and $1 - \alpha = 0.9$ (red line).

Phase portrait is shown in Figure 2. We have represented confidence regions at 50% (blue curve) and 90% (red curve) confidence levels, respectively, at several time instants. The result of the control is expected since one observes that the solution tends to the random vector $x^1(\omega)$.

As $\mathrm{rank}(\mathbf{B}) = 1$, we can compute the 1-PDF, $f_1(u, t)$, of the control operator through expression (23). In Figure 3 we have represented the 1-PDF of control for several time instants. We can observe that is sharper at intermediate instants.

Since the algebraic expressions of the 1-PDFs, $f_1(x, t)$ and $f_1(u, t)$, are somewhat cumbersome, for the sake of completeness in the Appendix we have detailed how computations could be performed.

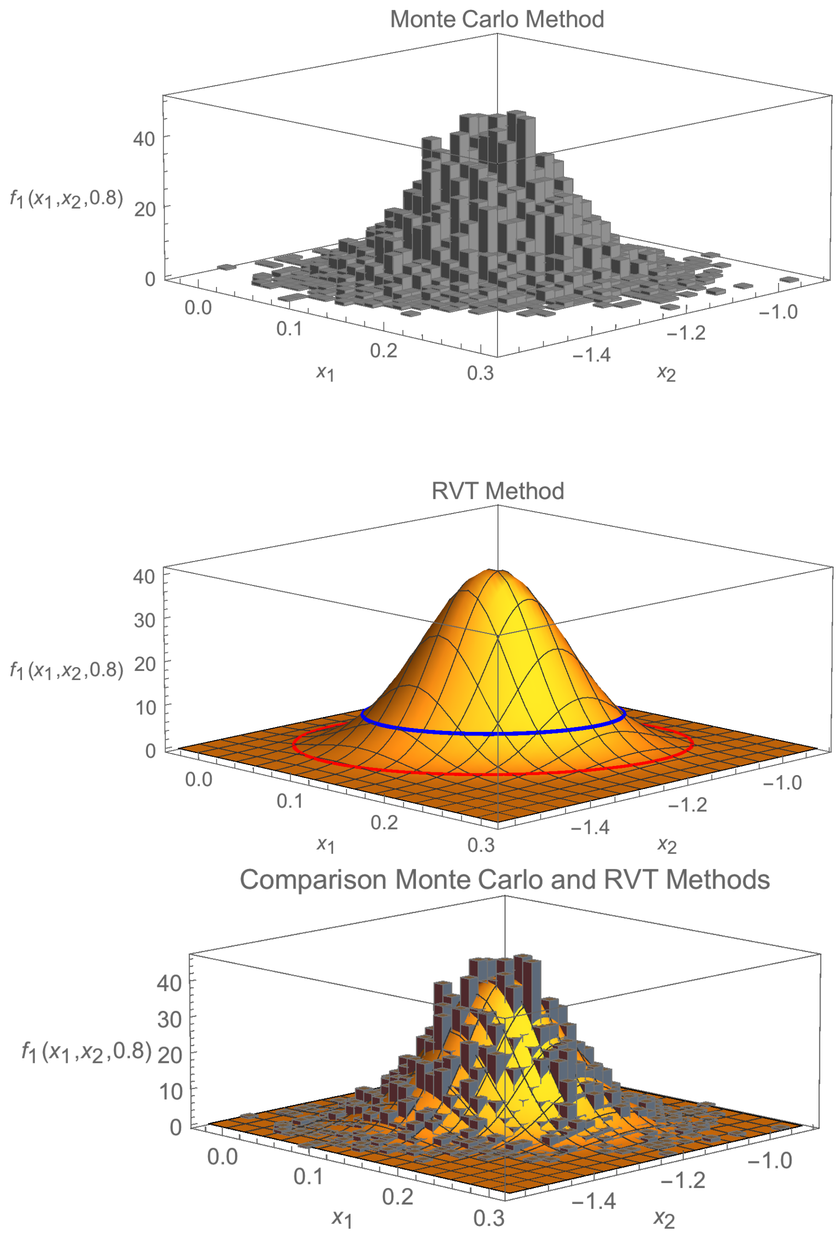

Figure 1: 1-PDF of the solution SP to the random control problem (4) by applying Monte Carlo (top panel), RVT (central panel) and both (down panel) for the time instant $t = 0.8$, Example 1. In the PDF computed via RVT method we highlight confidence regions at different confidence level $1 - \alpha$ (blue, $1 - \alpha = 0.5$ and red, $1 - \alpha = 0.9$).

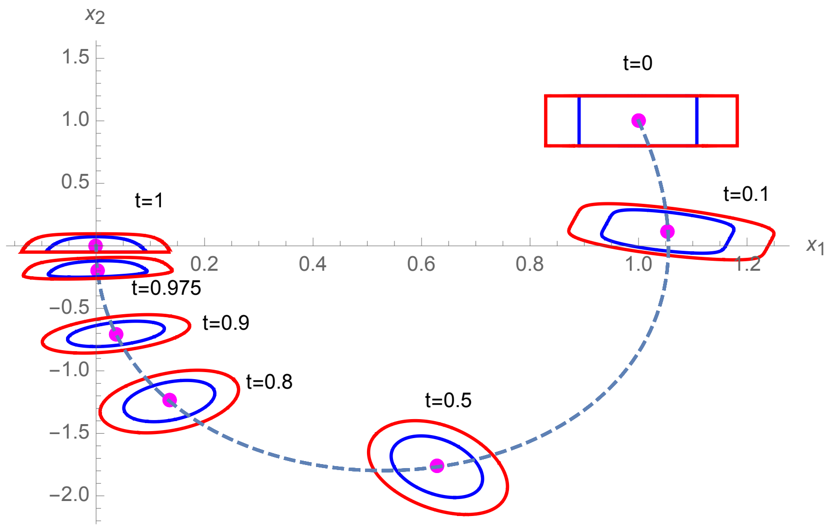

Figure 2: Portrait phase for the random control problem (4). Continuous spiral line represents the expectation of the solution. 50% (blue curve) and 90% (red curve) confidence regions are plotted at the time instants $t \in \{0, 0.1, 0.5, 0.8, 0.9, 0.975, 1\}$. Example 1.

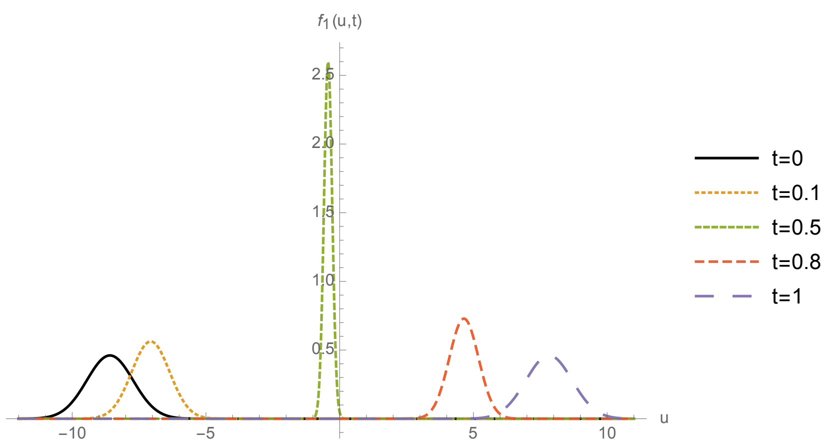

Figure 3: 1-PDF of control plotted at the time instants $t \in \{0, 0.1, 0.5, 0.8, 1\}$. Example 1.

Example 2. Let us consider the matrices $\mathbf{A}$ and $\mathbf{B}$ introduced in Example 1. Thus, we know that the system is controllable and, as $\operatorname{rank}(\mathbf{B}) = 1$ then we can compute the 1-PDF of $x(t,\omega)$ and $u(t,\omega)$. We take the control time $T = 1$. Regarding the randomness of the initial and final conditions we will consider multivariate Normal distributions. In particular, we contemplate the three cases introduced in the theoretical development exhibited in previous Section 4, and in each scenario, we will compute the distribution of the solution and of the control:

- Case 1: $x^0(\omega)$ and $x^1(\omega)$ have multivariate Normal distributions.
- Case 2: The initial condition $x^0(\omega)$ has a multivariate Normal distribution and the target condition x^1 is deterministic.
- Case 3: The initial condition x^0 is deterministic ant the target condition $x^1(\omega)$ has a multivariate Normal distribution.

For the sake of clarity, we will present the same structure to show both computations and results in each one of the above-listed case.

Case 1. We consider that the random vectors $x^0(\omega)$ and $x^1(\omega)$ have multivariate Normal distributions with mean $\mu^0 = (1,1)$ and $\mu^1 = (0,0)$, respectively, and common variance-covariance matrix $\mathbf{\Sigma} = 0.01\mathbf{I}_2$, i.e.

$$x^0 = (x_1^0, x_2^0) \sim \mathrm{N}(\mu^0, \Sigma), \quad x^1 = (x_1^1, x_2^1) \sim \mathrm{N}(\mu^1, \Sigma),$$
$$\mu^0 = (1,1), \quad \mu^1 = (0,0), \quad \mathbf{\Sigma} = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}.$$

The 1-PDF of the solution SP is shown in Figure 4 for $t = 0.5$. In this plot we have compared Monte Carlo Method with the 1-PDF obtained from expression (16). We can observe good agreement between both approaches. We have also displayed confidence regions at confidence levels $1 - \alpha = 0.5$ (blue line) and $1 - \alpha = 0.9$ (red line).

Phase portrait is represented in Figure 5. We have plotted confidence regions at 50% (blue curve) and 90% (red curve) at several time instants. The result of

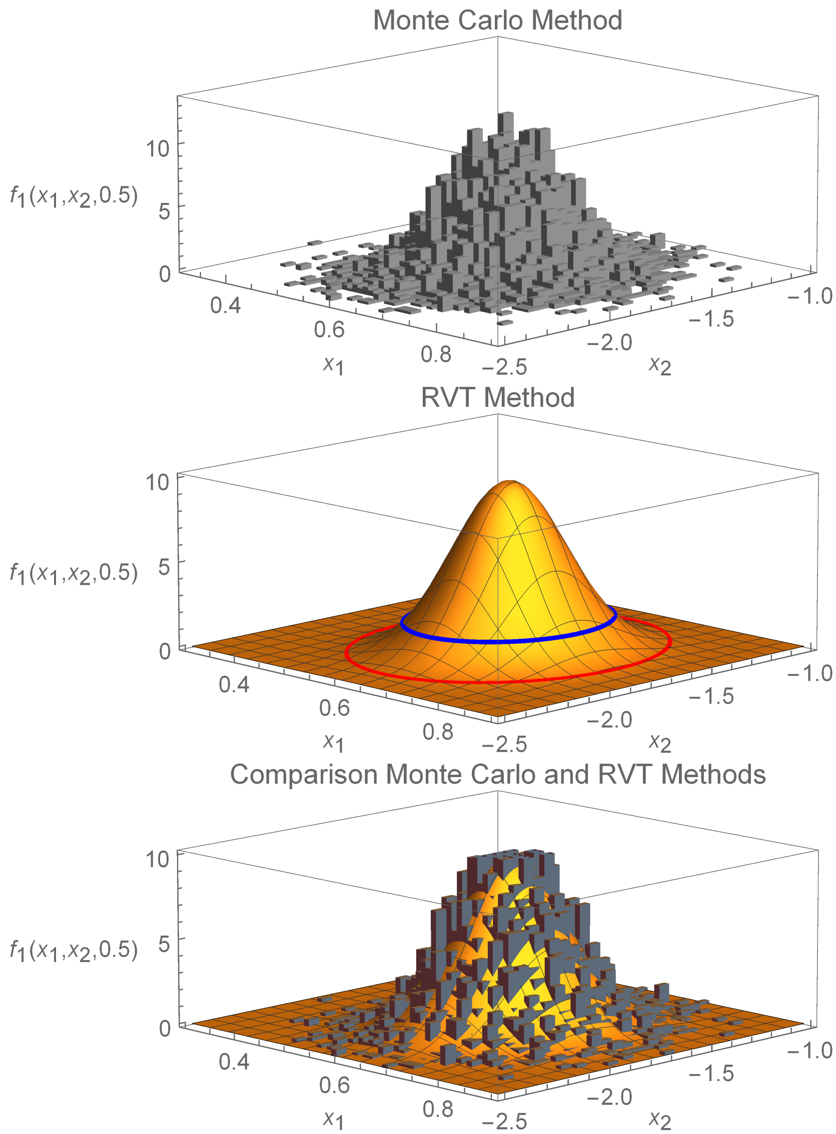

Figure 4: 1-PDF of the solution SP to the random control problem (4) by applying Monte Carlo (top panel), RVT (central panel) and both (down panel) for the time instant $t = 0.5$, Case 1, Example 2. In the PDF computed via RVT method, we have highlighted confidence regions at different levels $1-\alpha$ (blue, $1-\alpha = 0.5$ and red, $1-\alpha = 0.9$).

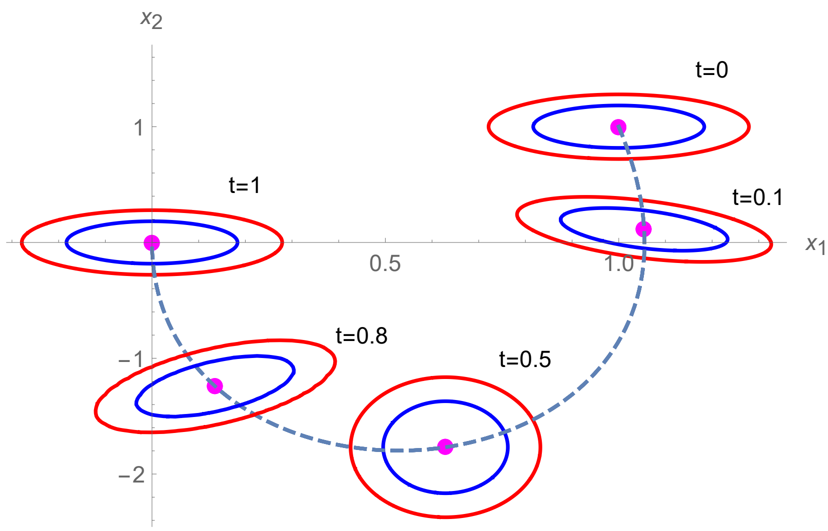

Figure 5: Portrait phase for the random control problem (4). Continuous spiral line represents the expectation of the solution. 50% (blue curve) and 90% (red curve) confidence regions are plotted at the time instants $t \in \{0, 0.1, 0.5, 0.8, 1\}$. Case 1, Example 2.

the control is expected since we observe that the solution tends to the random vector $x^1(\omega)$.

In Figure 6, we have represented the 1-PDF of control for several time instants. We can observe that it is sharper at intermediate instants.

Case 2. Now we will suppose that the initial condition is random and the target condition is deterministic, specifically, we consider that $x^0(\omega)$ has a multivariate Normal distribution with mean μ^0 and variance-covariance matrix $\boldsymbol{\Sigma}$, i.e., $x^0 = (x_1^0, x_2^0) \sim \mathrm{N}(\mu^0, \Sigma)$, where

$$\mu^0 = (1, 1) \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix},$$

and the final target has the deterministic value $x^1 = (1, 1)$.

The 1-PDF of the solution SP is shown in Figure 7 for $t = 0.1$. Observe that analytical computations from expression (17) and Monte Carlo simulations are in full agreement.

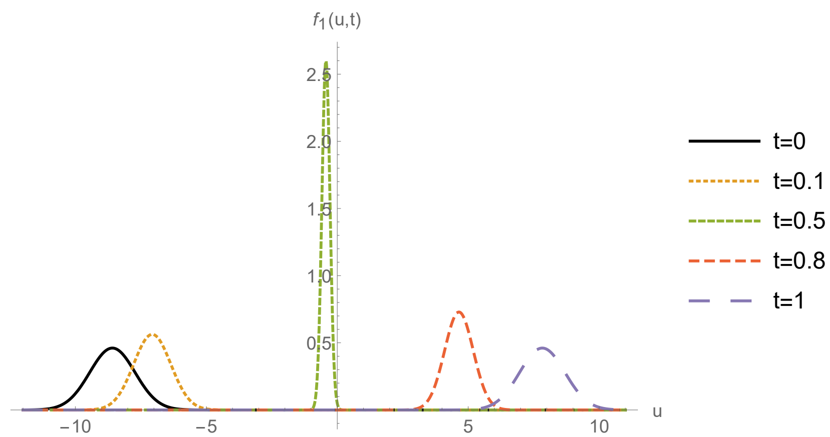

Figure 6: 1-PDF of control plotted at the time instants $t \in \{0, 0.1, 0.5, 0.8, 1\}$. Case 1, Example 2.

Phase portrait is represented in Figure 8. We can observe that the solution tends to the target point $x^1 = (0, 0)$. As this point is deterministic, the variability vanishes as time increases.

In Figure 9 we have represented the 1-PDF of control given by (25) for several time instants. Again, we can observe that it is sharper at intermediate times.

Case 3. Finally, we suppose that the initial condition is deterministic, $x^0 = (0, 0)$, and the target condition is a random vector following a multivariate Normal distribution with mean μ^1 and variance-covariance matrix $\mathbf{\Sigma}$, i.e., $x^1 = (x_1^1, x_2^1) \sim \mathrm{N}(\mu^1, \Sigma)$, where

$$\mu^1 = (1, 1), \qquad \mathbf{\Sigma} = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}.$$

The 1-PDF of the solution SP is shown in Figure 10 for $t = 0.8$. As in previous examples, analytical computations and Monte Carlo simulations agree.

Phase portrait is represented in Figure 11. We can observe how the random target is reached from an initial deterministic point.

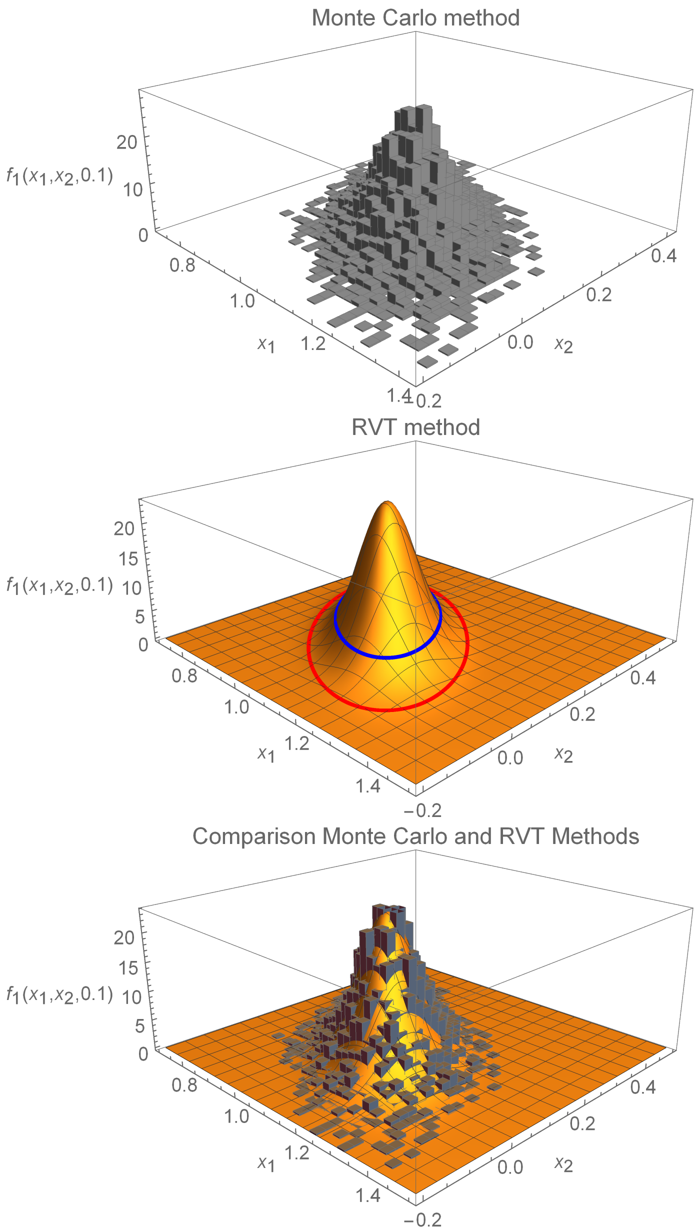

Figure 7: 1-PDF of the solution SP to the random control problem (4) by applying Monte Carlo (top panel), RVT (central panel) and both (down panel) for the time instant $t = 0.1$, Case 2, Example 2. In the PDF computed via RVT method, we have highlighted confidence regions at different confidence levels $1-\alpha$ (blue, $1-\alpha = 0.5$ and red, $1-\alpha = 0.9$).

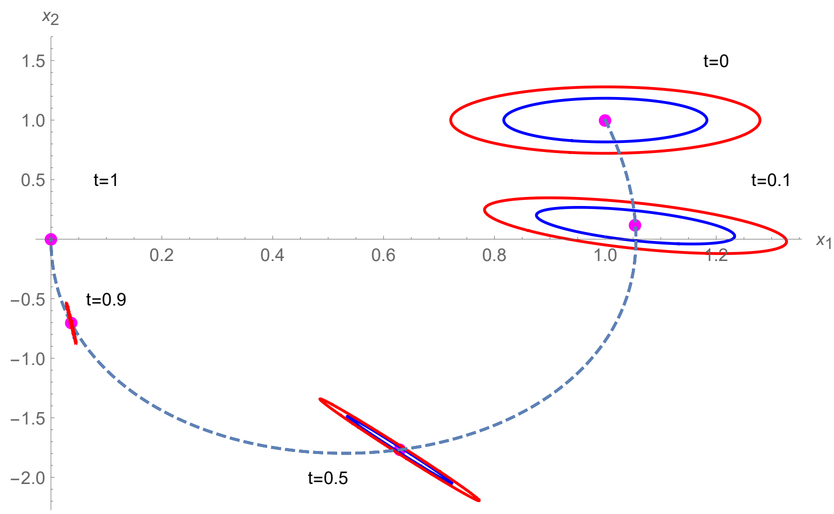

Figure 8: Portrait phase for the random control problem (4). Continuous spiral line represents the expectation of the solution. 50% (blue curve) and 90%(red curve) confidence regions are plotted at the time instants $t \in \{0, 0.1, 0.5, 0.9, 1\}$. Case 2, Example 2.

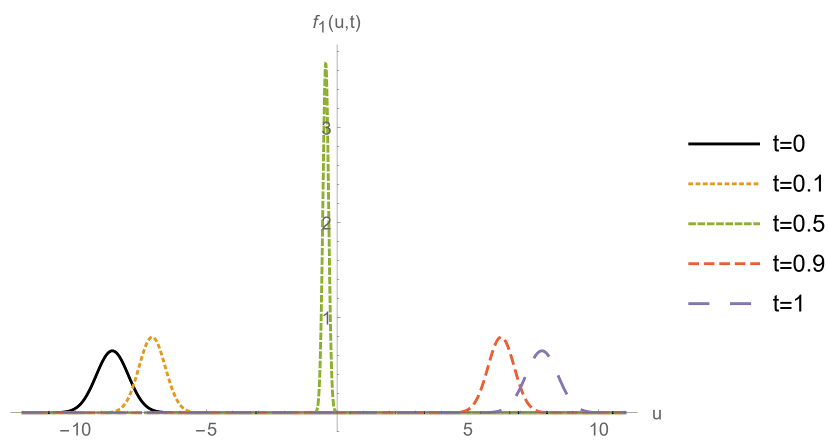

Figure 9: 1-PDF of control plotted at the time instants $t \in \{0, 0.1, 0.5, 0.9, 1\}$. Case 2, Example 2.

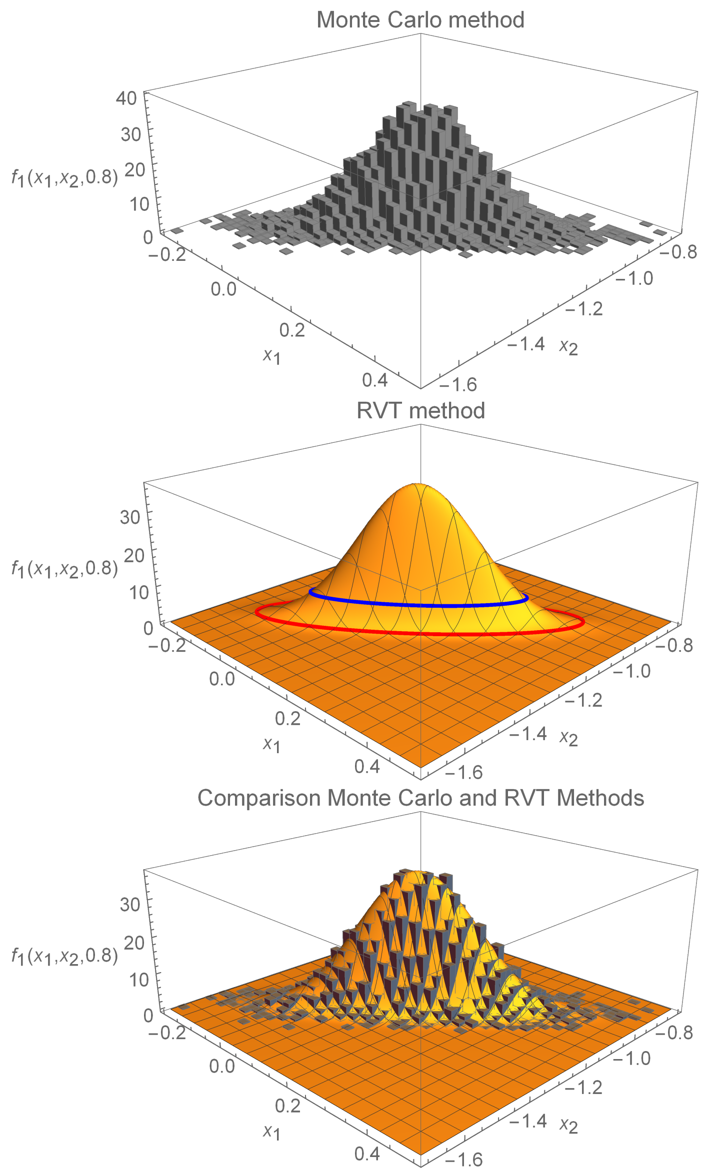

Figure 10: 1-PDF of the solution SP to the random control problem (4) by applying Monte Carlo (top panel), RVT (central panel) and both (down panel) for the time instant $t = 0.8$, Case 3, Example 2. In the PDF computed via RVT method, we have highlighted confidence regions at different confidence levels $1 - \alpha$ (blue, $1 - \alpha = 0.5$ and red, $1 - \alpha = 0.9$).

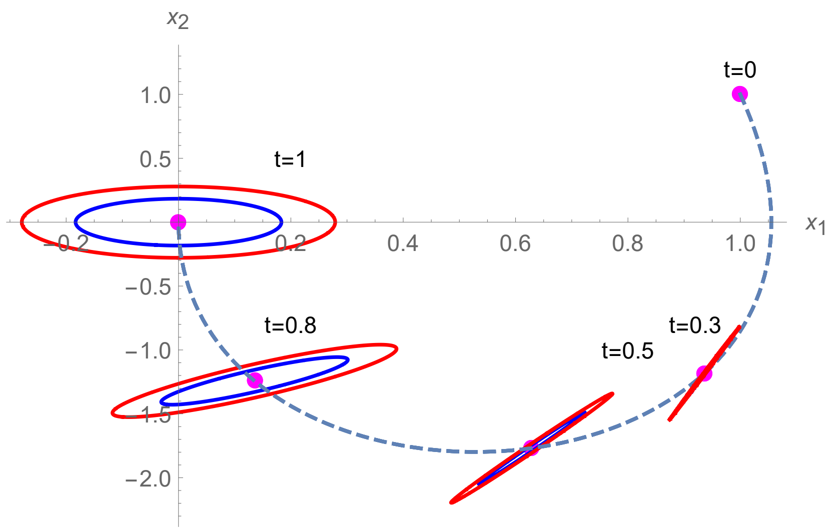

Figure 11: Portrait phase for the random control problem (4). Continuous spiral line represents the expectation of the solution. 50% (blue curve) and 90% (red curve) confidence regions are plotted at the time instants $t \in \{0, 0.3, 0.5, 0.8, 1\}$. Case 3, Example 2.

In Figure 12 we have plotted the 1-PDF of control for several time instants. As in previous examples, we can observe that it is sharper at intermediate times.

7. Conclusions

Controllability of systems in the presence of uncertainty is a topic of interest in recent years. The novelty of this paper is that we work directly with random variables for the initial condition and final target, and provide a complete probabilistic solution, unlike other contributions that work with the average. The complete probabilistic solution is given via the first probability density function, both for the stochastic solution process and for the control operator. This type of solutions allow us, for example, to calculate mean, variance and confidence regions for both the solution and the control operator in any specific time instant of interest. In our future research we plan to extend the analysis to the case that inputs involved in the random differential equation are random matrices.

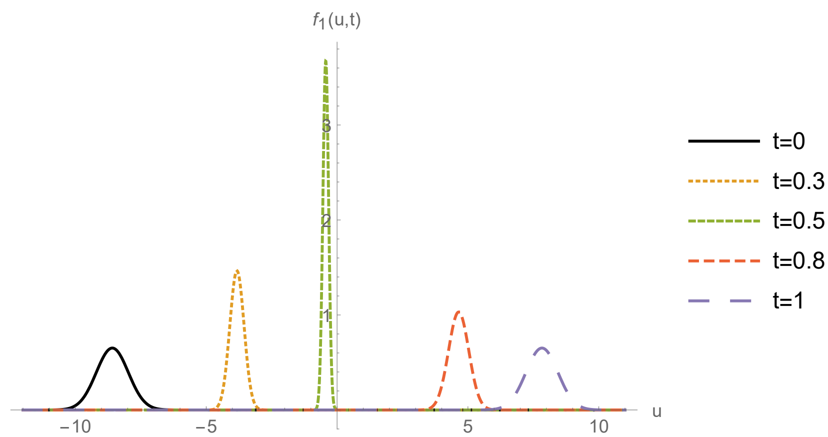

Figure 12: 1-PDF of control plotted at the time instants $t \in \{0, 0.1, 0.5, 0.7, 0.8, 1\}$. Case 3, Example 2.

Appendix

In this appendix we obtain explicit expressions of the 1-PDFs, $f_1(x,t)$ and $f_1(u,t)$, of the solution SP $x(t,\omega)$ and the control $u(t,\omega)$, respectively, in the context of Example 1. In this example we have considered independence among the random input parameters, $x^0(\omega) = (x_1^0(\omega), x_2^0(\omega))$ (initial condition) and $x^1(\omega) = (x_1^1(\omega), x_2^1(\omega))$ (terminal condition), therefore the joint PDF can be written as the product of the marginals PDFs, i.e.,

$$f_{x^0,x^1}(x_1^0, x_2^0, x_1^1, x_2^1) = f_{x_1^0}(x_1^0) f_{x_2^0}(x_2^0) f_{x_1^1}(x_1^1) f_{x_2^1}(x_2^1).$$

Below we indicate the distributions chosen for each random variable and their density:

- $x_1^0(\omega) \sim \mathrm{Ga}(200, 0.005)$. Then, the PDF is

$$f_{x_1^0}(x_1^0) = \begin{cases} 4.07512 \times 10^{87}\, \mathrm{e}^{-200\, x_1^0} \left(x_1^0\right)^{199}, & x_1^0 > 0, \\ 0, & \text{in other case.} \end{cases}$$

- $x_2^0(\omega) \sim \mathrm{U}([0.8, 1.2])$. Then, the PDF is

$$f_{x_2^0}(x_2^0) = \begin{cases} 2.5, & 0.8 \leq x_2^0 \leq 1.2, \\ 0, & \text{in other case.} \end{cases}$$

- $x_1^1(\omega) \sim \mathrm{N}(0, 0.05)$. Then, the PDF is

$$f_{x_1^1}(x_1^1) = 7.97885\, \mathrm{e}^{-200\,(x_1^1)^2}\,.$$

- $x_2^1(\omega) \sim \mathrm{T}\left([-0.05, 0.1]; -0.05\right)$. Then, the PDF is

$$f_{x_2^1}(x_2^1) = \begin{cases} 88.8889(0.1 - x_2^1), & -0.05 \leq x_2^1 \leq 0.1, \\ 0, & \text{in other case.} \end{cases}$$

From (16) the 1-PDF of $x(t, \omega)$ is:

$$f_1(x_1, x_2, t) = \int_0^{\infty} \int_{0.8}^{1.2} f_{x_1^0}(x_1^0) f_{x_2^0}(x_2^0) f_{x_1^1}(\mathbf{M}[1]) f_{x_2^1}(\mathbf{M}[2]) \left|\det\left(\mathbf{H}(t)^{-1}\right)\right| \mathrm{d}x_2^0\, \mathrm{d}x_1^0,$$

where $\mathbf{M}[i]$, $i = \{1, 2\}$, denotes the i-th component of vector

$$\mathbf{M} \equiv \mathbf{M}(x_1, x_2, x_1^0, x_2^0, t) = \mathbf{H}(t)^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \left(\mathbf{H}(t)^{-1}\mathrm{e}^{\mathbf{A}t} - \mathrm{e}^{\mathbf{A}T}\right) \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}.$$

From matrices $\mathbf{A}$ and $\mathbf{B}$ (see (28)), and the final instant time $T = 1$, we can calculate, using expressions in (3), the auxiliary matrices $\mathbf{F}(t)$ and $\mathbf{F}(t)$

$$\mathbf{F}(t) = \begin{pmatrix} \sin(1-t) \\ \cos(1-t) \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} 0.272676 & 0.354037 \\ 0.354037 & 0.727324 \end{bmatrix},$$

required to determine matrix $\mathbf{H}(t)$, using expression (13),

$$\mathbf{H}(t) = \begin{bmatrix} 4.73329t\cos(t) + (2.88247t - 4.73329)\sin(t) & (2.88247 - 1.03166t)\sin(t) - 2.88247t\cos(t) \\ 2.88247t\cos(t) + (2.88247 - 4.73329t)\sin(t) & (2.88247t - 1.03166)\sin(t) - 1.03166t\cos(t) \end{bmatrix}.$$

Then, $\mathbf{H}(t)^{-1}$ is given by

$$\begin{bmatrix} \frac{(0.841471t - 0.301169)\sin(t) - 0.301169t\cos(t)}{t^2 + 0.5\cos(2t) - 0.5} & \frac{(0.301169t - 0.841471)\sin(t) + 0.841471t\cos(t)}{t^2 + 0.5\cos(2t) - 0.5} \\ \frac{(1.38177t - 0.841471)\sin(t) - 0.841471t\cos(t)}{t^2 + 0.5\cos(2t) - 0.5} & \frac{(0.841471t - 1.38177)\sin(t) + 1.38177t\cos(t)}{t^2 + 0.5\cos(2t) - 0.5} \end{bmatrix}$$

and its determinant by

$$\det\left(\mathbf{H}(t)^{-1}\right) = \frac{0.291927}{t^2 + 0.5\cos(2t) - 0.5}.$$

Substituting the densities previously indicated for each RV, we obtain

$$f_1(x_1, x_2, t) = \int_0^{\infty} \int_{0.8}^{1.2} 7.2255{\cdot}10^{90}\, \mathrm{e}^{-200\left(x_1^0 + (\mathbf{M}[1])^2\right)} \left(x_1^0\right)^{199} (0.1 - \mathbf{M}[2]) \left|\det\left(\mathbf{H}(t)^{-1}\right)\right| \mathrm{d}x_2^0\, \mathrm{d}x_1^0.$$

Analogously, we calculate the 1-PDF of the control $u(t, \omega)$ by means of formula (23)

$$f_1(u_1, u_2, t) = \int_0^{\infty} \int_{0.8}^{1.2} \int_{-0.05}^{0.1} f_{x_1^0}(x_1^0) f_{x_2^0}(x_2^0) f_{x_1^1}(\mathbf{N}[1]) f_{x_2^1}(\mathbf{N}[2]) \left|\det \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1}\right| \mathrm{d}x_2^1 \mathrm{d}x_2^0\, \mathrm{d}x_1^0,$$

where $\mathbf{N}[i]$, $i = \{1, 2\}$, denotes the i-th component of vector

$$\mathbf{N} \equiv \mathbf{N}(u_1, u_2, x_1^0, x_2^0, x_2^1, t) = \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \mathbf{G}(t)\mathrm{e}^{\mathbf{A}T} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} \\ x_2^1 \end{bmatrix},$$

and matrices $\mathbf{G}(t)$ and $\mathbf{L}$ have been computed by (20), resulting

$$\mathbf{G}(t) = \begin{bmatrix} 9.96585\sin(1-t) - 4.85104\cos(1-t) \\ 3.73622\cos(1-t) - 4.85104\sin(1-t) \end{bmatrix}^{\top}, \quad \mathbf{L} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\top}.$$

Furthermore, the determinant is

$$\det \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1} = \frac{0.540302}{1.11482\cos(t) - 3.11482\sin(t)}.$$

Finally, substituting the densities of input parameters as well as the auxiliary calculations previously shown, we obtain the following expression for 1-PDF of the control SP

$$f_1(u_1, u_2, t) = \int_0^{\infty} \int_{0.8}^{1.2} \int_{-0.05}^{0.1} 7.2255{\times}10^{90}\, \mathrm{e}^{-200\left(x_1^0 + (\mathbf{N}[1])^2\right)} \left(x_1^0\right)^{199} (0.1 - \mathbf{N}[2]) \left|\det \begin{bmatrix} \mathbf{G}(t) \\ \mathbf{L} \end{bmatrix}^{-1}\right| \mathrm{d}x_2^1 \mathrm{d}x_2^0\, \mathrm{d}x_1^0.$$

The integrals defining the 1-PDFs $f_1(x_1, x_2, t)$ and $f_1(u_1, u_2, t)$ have been calculated by Mathematica© software. We skip explicitly showing their respective algebraic expressions because are very cumbersome.

Declarations of interest

None.

Acknowledgements

Authors acknowledge professor Martin Lazar for fruitful discussions.

The authors express their deepest thanks and respect to the four reviewers for their valuable comments that have improve the final version of the paper.

Funding

This work has been partially supported by the European Research Council (ERC) under the European Unions's Horizon 2020 research and innovation programme (grant agreement No. 694126-DyCon) and the Ministerio de Economía y Competitividad grant MTM2017–89664–P. Enrique Zuazua has also been funded by the Alexander von Humboldt-Professorship program, the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 765579-ConFlex, grant MTM2017-92996 of MINECO (Spain), ELKARTEK project KK-2018/00083 ROAD2DC of the Basque Government, ICON-ANR-16-ACHN-0014 of the French ANR and Nonlocal PDEs: Analysis, Control and Beyond, AFOSR Grant FA9550-18-1-0242. Ana Navarro Quiles acknowledges the postdoctoral contract financed by DyCon project funding from the European Research Council(ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 694126-DYCON) and the funding received from Generalitat Valenciana through a postdoctoral contract (APOSTD/2019/128).

References

References

[1] E. Fernández-Cara, E. Zuazua, Control theory: History, mathematical achievements and perspectives, Boletín de la Sociedad Española de Matemáticas (2003) 79–140doi:10.1016/S1874-5717(07)80010-7.

[2] E. Zuazua, Chapter 7 - Controllability and observability of partial differential equations: Some results and open problems, Vol. 3 of Handbook of Differential Equations: Evolutionary Equations, North-Holland, 2007, pp. 527 – 621. doi:10.1016/S1874-5717(07)80010-7.

[3] K. Ogata, Modern Control Engineering, Instrumentation and Controls Series, Prentice Hall, 2010.

[4] S. Sethi, Optimal Control Theory: Applications to Management Science and Economics, Springer International Publishing, 2018.

[5] L. Wu, Y. Gao, J. Liu, H. Li, Event-triggered sliding mode control of stochastic systems via output feedback, Automatica 82 (2017) 79 – 92. doi:10.1016/j.automatica.2017.04.032.

[6] X. Li, O. D. Zhu, Quanxin, Optimal h^2 filtering for periodic linear stochastic systems with multiplicative white noise perturbations and sampled measurements, Journal of the Franklin Institute 351 (9) (2014) 4435–4456. doi:10.1016/j.jfranklin.2014.04.008.

[7] L. Shaikhet, Improved condition for stabilization of controlled inverted pendulum under stochastic perturbations, Discrete and Continuous Dynamical Systems 24 (4) (2009) 1335–1343. doi:10.3934/dcds.2009.24.1335.

[8] L. Deori, S. Garatti, M. Prandin, Trading performance for state constraint feasibility in stochastic constrained control: A randomized approach, Journal of the Franklin Institute 354 (1) (2017) 501 – 529. doi:10.1016/j.jfranklin.2016.10.019.

[9] J. Song, Y. Niu, S. Wang, Robust finite-time dissipative control subject to randomly occurring uncertainties and stochastic fading measurements, Journal of the Franklin Institute 354 (9) (2017) 3706 – 3723. doi:10.1016/j.jfranklin.2016.07.020.

[10] A. N. Vargas, C. F. Caruntu, J. a. Y. Ishihara, Stability of switching linear systems with switching signals driven by stochastic processes, Journal of the Franklin Institute 356 (1) (2017) 31 – 41. doi:10.1016/j.jfranklin.2018.09.029.

[11] V. Dragan, S. Aberkane, I.-L. Popa, Optimal h^2 filtering for periodic linear stochastic systems with multiplicative white noise perturbations and sampled measurements, Journal of the Franklin Institute 352 (12) (2015) 985 – 6010. doi:10.1016/j.jfranklin.015.10.010.

[12] E. Zuazua, Averaged control, Automatica 50 (12) (2014) 3077–3087. doi:10.1016/j.automatica.2014.10.054.

[13] M. Lazar, E. Zuazua, Averaged control and observation of parameter-depending wave equations, Comptes Rendus Mathematique 352 (6) (2014) 497 – 502. doi:10.1016/j.crma.2014.04.007.

[14] J. Lohéac, E. Zuazua, From averaged to simultaneous controllability, Annales de la Faculté des Sciences de Toulouse: Mathématiques Ser. 6, 25 (4) (2016) 785–828. doi:10.5802/afst.1511.

[15] J. Coulson, B. Gharesifard, A.-R. Mansouri, On average controllability of random heat equations with arbitrarily distributed diffusivity, Automatica 103 (2019) 46 – 52. doi:https://doi.org/10.1016/j.automatica.2019.01.014.

[16] M. Lazar, E. Zuazua, Greedy controllability of finite dimensional linear systems, Automatica 74 (2016) 327 – 340. doi:10.1016/j.automatica.2016.08.010.

[17] G. Fabrini, L. Lapichino, S. Volkwein, Reduced-order greedy controllability of finite dimensional linear systems, IFAC-PapersOnLine 51 (2) (2018) 296–301.

[18] S. Micu, E. Zuazua, An introduction to the controllability of partial differential equations, in: T. Sari (Ed.), Quelques questions de théorie du contròle, Collection Travaux en Cours, Hermann, 2004, pp. 69 – 157.

[19] J. Levine, Analysis and Control of Nonlinear Systems, Springer, Berlin Heidelberg, 2009.

[20] T. T. Soong, Random Differential Equations in Science and Engineering, Academic Press, New York, 1973.