Averaged dynamics and control for heat equations with random diffusion.

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Joint work with Enrique Zuazua.
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Introduction to the problem.

The averaged controllability problem

We study the controllability properties of the system:

$$\begin{cases} y_t - \alpha\Delta y = f 1_{G_0}, & \text{in } (0,T)\times G, \\ y = 0, & \text{on } (0,T)\times \partial G, \\ y(0,\cdot) = y^0, & \text{on } G, \end{cases} \tag{1}$$

for α a positive random variable of density ρ. We cannot expect to control all the possible realizations (consider, for instance, the case in which $\alpha \to 0$), so we seek to control the average. This problem is relevant in applications in which the control has to be chosen independently of the random value, in a robust way.

The averaged observability problem

As usual, there is an equivalence between the averaged controllability of (1) and the averaged observability in G_0 of the time-reversed adjoint system:

$$\begin{cases} u_t - \alpha \Delta u = 0, & \text{in } (0,T) \times G, \\ u = 0, & \text{on } (0,T) \times \partial G, \\ u(0,\cdot) = \phi, & \text{on } G. \end{cases} \tag{2}$$

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We say that (2) have the *averaged unique continuation property* in $G_0 \subset G$ if $\tilde{u} = 0$ in $(0,T) \times G_0$ implies that $\phi = 0$. Similarly, we say that (2) is *averaged null observable* in $G_0 \subset G$ if for all $\phi \in L^2(G)$:

$$\|\tilde{u}(T,\cdot;\phi)\|_{L^2(G)} \leq C \|\tilde{u}(\cdot;\phi)\|_{L^2((0,T)\times G_0)}.$$

Known observability results

Theorem (Coulson, Gharesifard, Lü, Mansouri, Zuazua)

Let α be a random variable with a Riemann integrable density function ρ such that $\text{supp}(\rho) \subset [\alpha_{\min}, +\infty)$ for some $\alpha_{\min} > 0$. Then, system (2) is null observable in average.

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Their result leaves an interesting open question:
What happens if we allow the random variable to vanish; that is, if we allow $0 \in \operatorname{supp}(\rho)$?

The dynamics has a fractional nature when $G = \mathbb{R}^d$

In order to illustrate the effect of averaging in the dynamics, let us study the dynamics of (2) when $G = \mathbb{R}^d$. The Fourier transform of the average of the fundamental solutions is given by:

$$\int_0^{+\infty} \exp(-\alpha|\xi|^2 t)\rho(\alpha)d\alpha;$$

i.e. the Laplace transform of ρ evaluated in $|\xi|^2 t$. In particular, for $r \in (0,1)$ if $\rho(\alpha) \sim_{0^+} e^{-C\alpha^{-\frac{r}{1-r}}}$ we have that:

$$\int_0^{+\infty} \exp(-\alpha|\xi|^2 t)\rho(\alpha)d\alpha \sim \exp(-C|\xi|^{2r} t^r)$$

when $|\xi|^2 t \to +\infty$.

Similarities when $G = \mathbb{R}^d$ and when G is a bounded domain

The dynamics may also be fractional in bounded domains. Indeed, in bounded domains the Laplace transform of the density also appears when considering the Fourier representation:

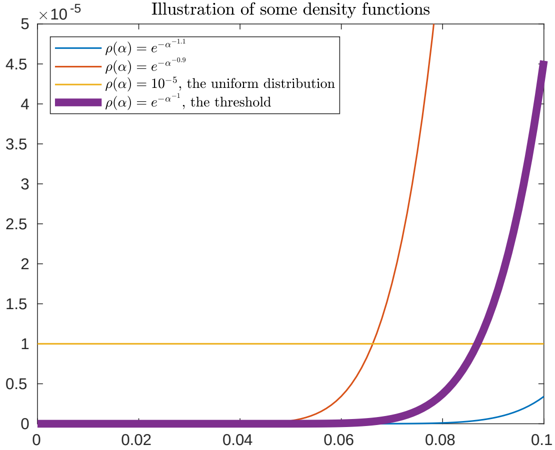
$$\begin{aligned}\tilde{u}(t,x;\phi) &:= \int_0^{+\infty} u(t,x;\alpha,\phi)\rho(\alpha)d\alpha \\ &= \sum_{i\in\mathbb{N}} \int_0^{+\infty} e^{-\alpha\lambda_i t}\rho(\alpha)d\alpha\langle\phi, e_i\rangle_{L^2(G)} e_i(x).\end{aligned}$$

A qualitative description of the expectable results

- (2) has the unique continuation in average
- (2) is null observable in average if and only if ρ is sufficiently small near 0. In fact, the threshold density functions are those which near 0 satisfy:

$$\rho(\alpha) \sim e^{-\alpha^{-1}}.$$

An illustration of some density functions



Main results: rigorous statements and proofs.

Main result: averaged unique continuation

Theorem

Let $G \subset \mathbb{R}^d$ be a Lipschitz domain, $G_0 \subset G$ be a subdomain, and $\rho = 1_{(0,1)}$ or ρ be a density function which satisfies:

$$-\frac{d}{ds}\ln\left(\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha\right) = \frac{\int_0^{+\infty} e^{-s\alpha}\alpha\rho(\alpha)d\alpha}{\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha} \gtrsim s^{r-1} \quad (3)$$

for some $r \in (0,1]$. Then, system (2) satisfies the averaged unique continuation property in G_0.

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Examples: density functions satisfying $\rho(\alpha) \sim_{0^+} e^{-C\alpha^{-\frac{r}{1-r}}}$ for $r \in (0,1)$

Proof of the unique continuation when $\rho = 1_{(0,1)}$

The proof is based on **explicit computations**:

$$\tilde{u}(t,x;\phi) = \sum_{i\in\mathbb{N}} \int_0^1 e^{-\lambda_i \alpha t} \langle \phi, e_i \rangle e_i(x) d\alpha = \frac{1}{t} \left(\sum_{n\in\mathbb{N}} \frac{1}{\lambda_i} \langle \phi, e_i \rangle e_i(x) - \sum_{n\in\mathbb{N}} \frac{e^{-\lambda_i t}}{\lambda_i} \langle \phi, e_i \rangle e_i(x) \right)$$

$$= \frac{1}{t} \left(-\Delta^{-1}\phi + \sum_{i\in\mathbb{N}} e^{-\lambda_i t} \langle \Delta^{-1}\phi, e_i \rangle e_i(x) \right).$$

Consequently, from $\int_0^T \int_{G_0} |\tilde{u}(t,x;\phi)|^2 = 0$ we find that:

$$-\Delta^{-1}\phi + \sum_{i\in\mathbb{N}} e^{-\lambda_i t} \langle \Delta^{-1}\phi, e_i \rangle e_i(x) = 0 \text{ in } (0,T) \times G_0,$$

which differentiating in time implies that:

$$\sum_{i\in\mathbb{N}} e^{-\lambda_i t} \langle \phi, e_i \rangle e_i(x) = 0 \text{ in } (0,T) \times G_0.$$

Hence, $\phi = 0$ from the unique continuation of the heat equation.

Unique continuation for density functions ρ decaying exponentially when $\alpha \to 0$ (i)

The proof follows from the analiticity of the averaged dynamics:

Proposition

Let G be a Lipschitz domain, α any positive random variable and $\phi \in L^2(G)$. Then, the function:

$$U : t \in (0, \infty) \to \tilde{u}(t, \cdot; \phi) \in L^2(G)$$

is analytic.

This follows from the analiticity of the heat semigroup.

Unique continuation for density functions ρ decaying exponentially when $\alpha \to 0$ (ii)

Let us now prove that from $\tilde{u} = 0$ in $(0, T) \times G_0$ we obtain that $\phi = 0$. From the analiticty we obtain that $\tilde{u} = 0$ in $(0, \infty) \times G_0$. In addition, by considering the limit when $t \to \infty$ we obtain that if $\phi \neq 0$ there is an eigenfunction of the Laplacian null in G_0, which is false.

Main results: averaged null observability (i)

Theorem

Let $G \subset \mathbb{R}^d$ be a Lipschitz locally star-shaped domain, $G_0 \subset G$ be a subdomain, $T > 0$ and α be a random variable whose density ρ satisfies that there is some and $r \in (1/2, 1]$ such that:

$$-\frac{d}{ds} \ln \left(\int_0^{+\infty} e^{-s\alpha} \rho(\alpha) d\alpha \right) = \frac{\int_0^{+\infty} e^{-s\alpha} \alpha \rho(\alpha) d\alpha}{\int_0^{+\infty} e^{-s\alpha} \rho(\alpha) d\alpha} \gtrsim s^{r-1}. \tag{4}$$

Then, system (2) is null observable in average. In addition, there are $T_0, C > 0$ such that for all $T \in (0, T_0]$ we have that:

$$K(G, G_0, \rho, T) \leq Ce^{CT^{-(2r-1)^{-1}}}.$$

Here K is the cost of the null averaged observability.

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Examples: $\rho(\alpha) \sim_{0^+} e^{-C\alpha^{-\frac{r}{1-r}}}$ for $r \in (1/2, 1)$.

Proof of the averaged null observability

The result can be proved by following an **iteration method of the type Lebeau-Robbiano**. The averaged dynamics does not satisfy the semigroup property, but this is a minor problem as averaging does not alter the decay rate. Indeed, the inequality (4) implies that:

$$\int_0^{+\infty} e^{-t_2\lambda\alpha}\rho(\alpha)d\alpha \leq e^{-c\lambda^r(t_2-t_1)}\int_0^{+\infty} e^{-t_1\lambda\alpha}\rho(\alpha)d\alpha,$$

which can be proved by an easy casuistic.

Main results: averaged null observability (ii)

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Let $G \subset \mathbb{R}^d$ be a Lipschitz domain, $G_0 \subset G$ be a subdomain such that $G_0 \neq G$ and α be a random variable whose density function ρ satisfies that there are some $C > 0$ and $r \in [0, 1/2)$ such that:

$$\int_0^{+\infty} e^{-s\alpha}\rho(\alpha)d\alpha \gtrsim e^{-Cs^r}. \tag{5}$$

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Examples: $\rho(\alpha) = 1_{(0,1)}$, $\rho(\alpha) = e^{-\alpha}1_{(0,+\infty)}$, $\rho(\alpha) = 2\alpha 1_{(0,1)}$, $\rho(\alpha) \sim_{0^+} e^{-C\alpha^{-\frac{r}{1-r}}}$ for $r \in (0, 1/2)$, ...

Proof of not having null observability (i)

Remark

The problem is not a lack of unique continuation property, as shown before. Instead, the problem is the existence of a sequence ϕ_N satisfying:

$$\lim_{N\to\infty} \frac{\|\tilde{u}(T,\cdot;\phi_N)\|_{L^2(G)}}{\left(\int_0^T \int_{G_0} |\tilde{u}(t,x;\phi_N)|^2 dxdt\right)^{1/2}} = +\infty.$$

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- To ensure that $\tilde{u}$ is small in $(0,T)\times G_0$ we need:

$$\bigcup_{N\geq N_0} \text{supp}(\phi_N) \subset\subset G\setminus\overline{G_0} \quad \text{and } \phi_N \in \langle e_i\rangle^{\perp}_{i\in\Lambda_N}.$$

- To prevent the averaged solution from decaying too fast we need:

$$\|\mathcal{P}_{CN}\phi_N\|_{L^2(G)} \geq \sqrt{3}\|\phi_N\|_{L^2(G)}/2.$$

Proof of not having null observability (ii)

Using the first hypothesis we obtain that:

$$\int_0^T \int_{G_0} \left| \int_0^{+\infty} u(t,x;\alpha,\phi_N)\rho(\alpha)d\alpha \right|^2 dxdt \leq \int_0^T \int_{G_0} \int_0^{+\infty} |u(t,x;\alpha,\phi_N)|^2 \rho(\alpha)d\alpha dxdt$$

$$= \int_0^T \int_{G_0} \int_0^{+\infty} |u(t,x;\alpha,\phi_N)|^2 1_{\alpha t \leq N^{-1/2}}(t,\alpha)\rho(\alpha)d\alpha dxdt$$

$$+ \int_0^T \int_{G_0} \int_0^{+\infty} |u(t,x;\alpha,\phi_N)|^2 1_{\alpha t > N^{-1/2}}(t,\alpha)\rho(\alpha)d\alpha dxdt \leq C\left(e^{-c\sqrt{N}} + e^{-\sqrt{N}}\right)\|\phi_N\|^2_{L^2(G)}.$$

Using the second hypothesis and (5) we obtain that:

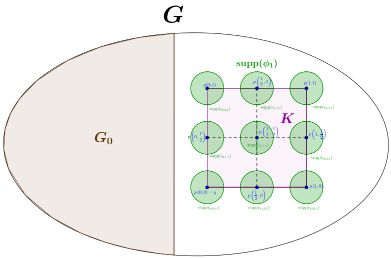
$$\|\tilde{u}(T,\cdot)\|^2_{L^2(G)} = \sum_{i\in\mathbb{N}} \left(\int_0^\infty e^{-\lambda_i \alpha T}\rho(\alpha)d\alpha\right)^2 |\langle \phi_N, e_i\rangle|^2 \geq c\sum_{i\in\mathbb{N}} e^{-C(\lambda_i T)^r}|\langle \phi_N, e_i\rangle|^2$$

$$\geq ce^{-CN^r}\sum_{i\in\Lambda_{CN}} |\langle \phi_N, e_i\rangle|^2 = ce^{-CN^r}\|\mathcal{P}_{CN}\phi_N\|^2_{L^2(G)} \geq ce^{-CN^r}\|\phi_N\|^2_{L^2(G)}.$$

Combining both inequalities, as $r \in [0, 1/2)$ we find that:

$$\lim_{N\to\infty} \frac{\|\tilde{u}(T,\cdot;\phi_N)\|_{L^2(G)}}{\left(\int_0^T \int_{G_0} |\tilde{u}(t,x;\phi_N)|^2 dxdt\right)^{1/2}} = +\infty.$$

Construction of the functions ϕ_N



In fact, we consider as initial values:

$$\phi_N(x) := \sum_{i_1,\ldots,i_d=0}^{\left[\tilde{C}\sqrt{N}\right]} c_{i,N}\varsigma_{i,N}(x), \quad \text{for } \varsigma_{i,N}(x) := \varsigma\left(3\sqrt{N}\frac{x - p\left(\frac{i}{\left[\tilde{C}\sqrt{N}\right]}\right)}{\tilde{C}\ell}\right),$$

for ς a cut-off function and p a parametrization of the cube.

Properties of the functions ϕ_N

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- We can find some coefficients $c_{i,N}$ such that $\langle e_i \rangle^{\perp}_{i \in \Lambda_N}$. This is just solving a linear homogeneous system with more equations than variables by Weyl's Law.
- For the third property it suffices to prove that:

$$\|\Delta \phi_N\|_{L^2(G)} \leq \frac{\overline{C}N}{2} \|\phi_N\|_{L^2(G)}.$$

This is done by linear transformations and because $\text{supp}(\varsigma_{i,N}) \cap \text{supp}(\varsigma_{\tilde{i},N}) = \emptyset$. Indeed:

$$\|\Delta \phi_N\|^2_{L^2(G)} = \sum_{i_1,\dots,i_d=0}^{\left[\tilde{C}\sqrt{N}\right]} c^2_{i,N} \left(\frac{3\sqrt{N}}{\tilde{C}\ell}\right)^4 \int_G |\Delta\varsigma|^2 \left(3\sqrt{N}\frac{x - p\left(\frac{i}{\left[\tilde{C}\sqrt{N}\right]}\right)}{\tilde{C}\ell}\right) dx$$

$$= \left(\sum_{i_1,\dots,i_d=0}^{\left[\tilde{C}\sqrt{N}\right]} c^2_{i,N}\right) \left(\frac{3\sqrt{N}}{\tilde{C}\ell}\right)^3 \|\Delta\varsigma\|^2_{L^2(B(0,1))} \leq C \left(\sum_{i_1,\dots,i_d=0}^{\left[\tilde{C}\sqrt{N}\right]} c^2_{i,N}\right) \left(\frac{3\sqrt{N}}{\tilde{C}\ell}\right)^3 \|\varsigma\|^2_{L^2(B(0,1))}$$

$$= CN^2 \sum_{i_1,\dots,i_d=0}^{\left[\tilde{C}\sqrt{N}\right]} c^2_{i,N} \int_G |\varsigma|^2 \left(3\sqrt{N}\frac{x - p\left(\frac{i}{\left[\tilde{C}\sqrt{N}\right]}\right)}{\tilde{C}\ell}\right) dx = CN^2 \|\phi_N\|^2_{L^2(G)}.$$

Some numerical simulations.

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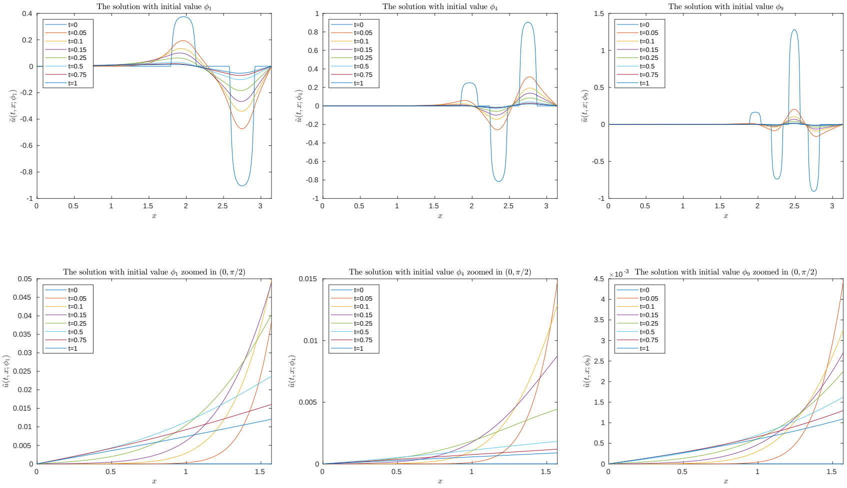

Figure: An example on how the sequence looks like for $G = (0, \pi)$, $G_0 = (0, \pi/2)$ and $\rho(\alpha) = 1_{(0,1)}$.

Numerical illustration of the minimum of the functional associated to the control problem.

On the following slides we minimize the following functional with the help of Matlab:

$$J(\phi) = \frac{1}{2}\int_0^T \int_{G_0} \left| \int_0^{+\infty} \varphi(t,x;\alpha,\phi)\rho(\alpha)d\alpha \right|^2 dxdt \\ + \left\langle y^0, \int_0^{+\infty} \varphi(0;\alpha,\phi)\rho(\alpha)d\alpha \right\rangle,$$

for φ the averaged solution of the adjoint heat equation. In particular, we consider the initial value $y^0 = 1/2$, $G = (0,\pi)$ and $G_0 = (1,2)$ and compare what happens when considering the uniform distributions in $(0,1)$ and $(1,2)$ (i.e. $\rho = 1_{(0,1)}$ and $\rho = 1_{(1,2)}$).

Some numerical simulations

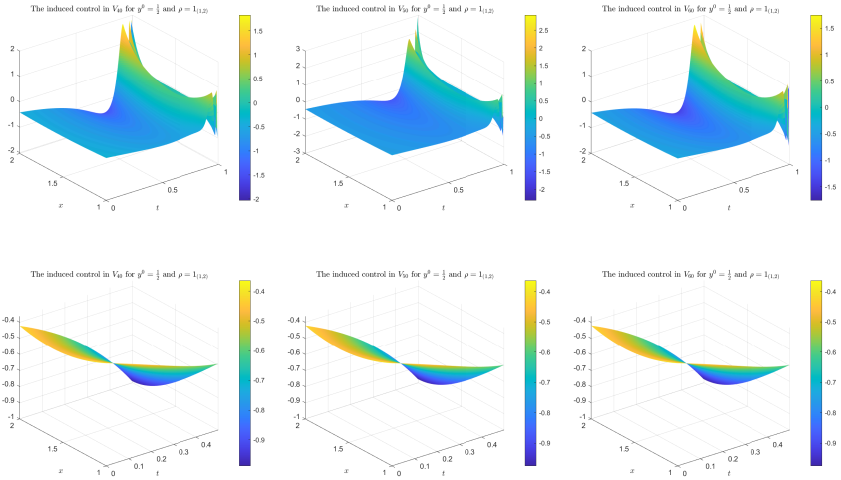

Figure: The optimal control for $\rho = 1_{(1,2)}$ and $y^0 = \frac{1}{2}$ induced by the minimum of the functional J in V_{40}, V_{50} and V_{60}, for $V_M := \langle e_i \rangle_{i=1}^M$.

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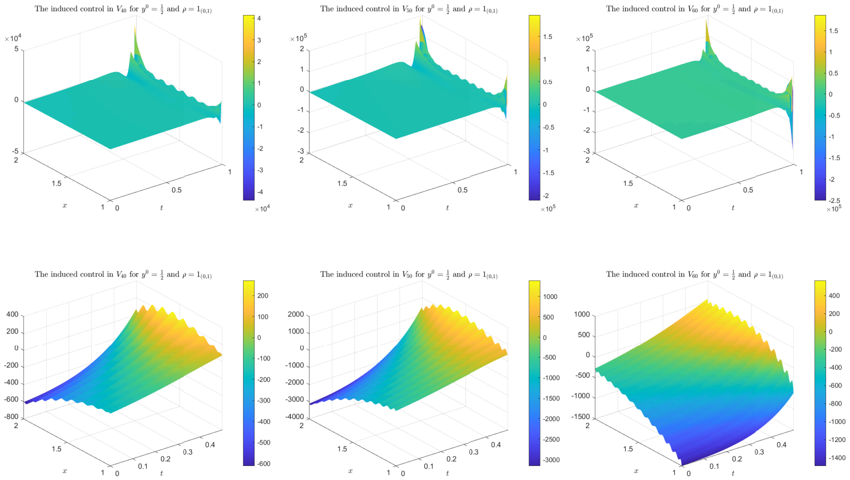

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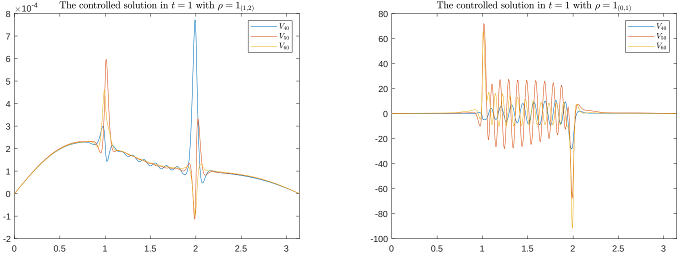

Figure: The state in time $t = 1$ of the averaged solutions of the heat equation after applying the control induced by the minimum of J in V_{40}, V_{50} and V_{60} with $y^0 = \frac{1}{2}$. In the left figure we have considered $\rho = 1_{(1,2)}$ and in the right one $\rho = 1_{(0,1)}$.

Open problems

- Study the averaged controllability of more general heat equations:

$$y_t - \mathrm{div}(\sigma(x,\alpha)\nabla y) + A(x,\alpha)\cdot\nabla y + a(x,\alpha)y = 0.$$

  The difficulty is that in the general case the eigenfunctions of the elliptic operator depend on α.
- Study the averaged wave and Schrödinger equations with arbitrary random diffusion.
- Determine how to minimize the variance when we control the average of the heat equation.
- Determine if we have the averaged unique continuation property for all random diffusions α. We have to deal with the difficulty that the frequencies do not decay hierarchically.

Thank you for your attention!

Is there any question?