Universidad Autónoma de Madrid departamento de matemáticas
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DyCon DYNAMIC CONTROL erc

Some recent results about the controllability of the heat equation.

Jon Asier Bárcena-Petisco

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Overview

1. Null controllability for the heat equation in pseudo-cylinders.
2. Cost of null controllability for parabolic equations with vanishing diffusivity and a transport term.
3. Averaged observability properties of the random heat equation with a random diffusion (in collaboration with Enrique Zuazua).

Null controllability of the heat equation: targeting Lipschitz domains.

Reference:

https://hal.archives-ouvertes.fr/hal-02145122

Accepted in ESAIM:COCV.

Statement of the problem

We study the following classic problem:

$$\begin{cases} y_t - \Delta y + A \cdot \nabla y + ay = v1_\omega & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0 & \text{on } \Omega. \end{cases} \tag{1}$$

Let Ω a (bounded) domain. Does it exists for all $y^0 \in L^2(\Omega)$ a control $v \in L^2((0, T) \times \omega)$ such that the solution of (1) satisfies $y(T, \cdot) = 0$?

An equivalent problem: the observability problem

Let us consider the system:

$$\begin{cases} u_t - \Delta u + \tilde{A} \cdot \nabla u + \tilde{a} u = 0 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0 & \text{on } \Omega. \end{cases} \tag{2}$$

Is there a constant C such that we have for all $u^0 \in L^2(\Omega)$ that:

$$\|u(T,\cdot)\|_{L^2(\Omega)} \leq C \|u\|_{L^2((0,T)\times\omega)}?$$

For linear systems both problems are equivalent (even quantitatively) by the Hilbert Uniqueness Method, and I will mention both indistinctively.

Approaches to prove the null controllability of the heat equation

- Transformations involving the wave equation (Russell, ...) (requires some geometrical hypothesis)
- Spectral inequalities (Lebeau&Robbiano, ...) (known to work for locally star-shaped domains by Apraiz&Euscariaza&Wang&Zhang)
- Carleman estimates (Fursikov&Imanuvilov, ...) (known to work for C^2 domain and many parabolic equations).

A pseudo-cylinder

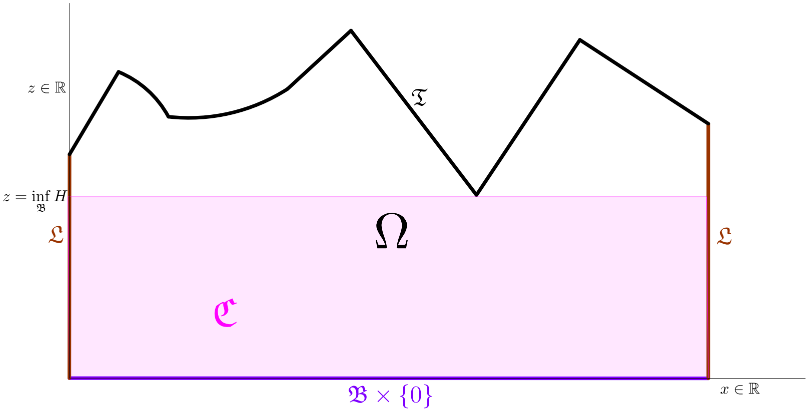

Figure: A canonically oriented pseudo-cylinder

The main result: a usual result but in a different geometry

Theorem

Let Ω *be a* pseudo-cylinder *and* $\omega \subset \Omega$ *be a subdomain. Then, there is* $C > 0$ *such that if* $T > 0$, $A \in (L^\infty((0,T)\times\Omega))^{d+1}$, $a \in L^\infty((0,T)\times\Omega)$, *and* $y^0 \in L^2(\Omega)$, *there is a control* $v \in L^2((0,T)\times\omega)$ *such that the solution of the system:*

$$\begin{cases} y_t - \Delta y + A\cdot\nabla y + ay = v1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y^0 & \text{on } \Omega, \end{cases}$$

satisfies $y(T,\cdot) = 0$, *and such that the control satisfies the estimate:*

$$\|v\|_{L^2((0,T)\times\omega)} \leq Ce^{CK(T,a,A)}\|y^0\|_{L^2(\Omega)},$$

for:

$$K(T,a,A) := 1 + T^{-1} + T\|a\|_{L^\infty} + \|a\|_{L^\infty}^{2/3} + (1+T)\|A\|^2_{(L^\infty)^{d+1}}.$$

Sketch of the proof

- We approximate the domain by regular domains (by taking the contour lines of regularized distances to the boundary).
- It is well-known that the heat equation is observable in those domains.
- We show that the cost of the observability can be bounded uniformly in those domains (Carleman inequalities of the type Fursikov-Imanuvilov).
- We take limits (using compactness results in Sobolev spaces).

Cost of null controllability for parabolic equations with vanishing diffusivity and a transport term.

Reference:

https://hal.archives-ouvertes.fr/hal-02455632

Presentation of the problem

We consider the following problem:

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } (0,T)\times\Omega, \\ \text{boundary conditions} & \\ y(0,\cdot) = y^0, & \text{on } \Omega, \end{cases} \tag{3}$$

The null controllability of (3) is well-known when Ω is C^2 for many boundary conditions. Thus, the next interesting question is to study the cost of the control when $\varepsilon \to 0$. We recall that the cost is given by:

$$K(\Omega,\omega,T,\varepsilon) := \sup_{y^0\in L^2(\Omega)\setminus\{0\}} \inf_{f:y(T,\cdot)=0} \frac{\|f\|_{L^2((0,T)\times\omega)}}{\|y^0\|_{L^2(\Omega)}}.$$

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Known facts with Dirichlet boundary conditions

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- The transport equation is null controllable if and only if $T \geq T^*(\Omega, \omega)$.
- Thus, it looks reasonable that the cost of the control explodes for $T < T^*(\Omega, \omega)$ and decays for $T \geq T^*(\Omega, \omega)$.

The reality is not like that. Coron&Guerrero proved in 2005 that this conjecture is false. However, they prove that there was $\tilde{T}(\omega, \Omega)$ such that for $T \geq \tilde{T}$ we have $K \leq Ce^{-c\varepsilon^{-1}}$ for $\Omega \subset \mathbb{R}$. Afterwards, Guerrero&Lebeau proved in 2007 the same result for $\Omega \subset \mathbb{R}^d$.

Formulation of the problem

We focus on problems of the type:

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } (0,T)\times\Omega, \\ \partial_n y + a^\varepsilon(x) y = 0, & \text{on } (0,T)\times\Gamma, \\ y = 0, & \text{on } (0,T)\times\Gamma^*, \\ y(0,\cdot) = y^0, & \text{on } \Omega. \end{cases}$$

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The control is estimated with the Hilbert Uniqueness Method. Indeed, we focus on the observability properties of:

$$\begin{cases} -\varphi_t - \varepsilon \Delta \varphi - \partial_{x_1} \varphi = 0, & \text{in } (0,T)\times\Omega, \\ \varepsilon \partial_n \varphi + (\varepsilon a^\varepsilon + n_1)\varphi = 0, & \text{on } (0,T)\times\Gamma, \\ \varphi = 0, & \text{on } (0,T)\times\Gamma^*, \\ \varphi(T,\cdot) = \varphi^T, & \text{on } \Omega, \end{cases}$$

as:

$$K(\Omega,\omega,T,\varepsilon) = \sup_{\varphi^T \in L^2(\Omega)\setminus\{0\}} \frac{\|\varphi(0,\cdot)\|_{L^2(\Omega)}}{\|\varphi\|_{L^2((0,T)\times\omega)}}.$$

Positive answer under a geometric condition

We assume the following condition:

$$(a^{\varepsilon} + (2\varepsilon)^{-1} n_1) 1_{\Gamma} \geq 0. \tag{4}$$

The geometrical meaning is that if the flux of the transport enters (if $n_1 < 0$) then we need for a to be large, which implies that the normal variation of y on $\partial\Omega$ opposes to y very strongly if it is not null. In addition, for the points in which the flux comes out (if $n_1 > 0$), we just need for a to be positive or of a small modulus, which implies that the normal variation of y on $\partial\Omega$ either opposes to y or is of the same sign, but not too strongly.

Proof of the decay of the cost for a sufficiently large time under the geometric condition (4)

The proof consists on two steps:

- With a "proper" Carleman inequality for the adjoint system assuming (4) we can obtain the estimate:

$$\|\varphi(T-1,\cdot)\|_{L^2(\Omega)} \leq Ce^{C\varepsilon^{-1}}\|\varphi\|_{L^2((T-1,T)\times\omega)}.$$

- Under the geometric condition (4) we can see that the first eigenvalue of $-\varepsilon\Delta - \partial_{x_1}$ is greater than $\frac{1}{4\varepsilon}$ via the its variational formulation. Thus, we obtain for a large T that:

$$\|\varphi(0,\cdot)\|_{L^2(\Omega)} \leq Ce^{-c\varepsilon^{-1}T}\|\varphi(T-1,\cdot)\|_{L^2(\Omega)}.$$

Thus, combining the two previous inequalities we obtain that the cost of the control decays for a sufficiently large time.

Study of the cost when we have Neumann b. c.

In this situation we have that $K \geq Ce^{C/\varepsilon}$. Indeed, $e^{x_1/\varepsilon}$ is a static solution of the adjoint system for any domain Ω.

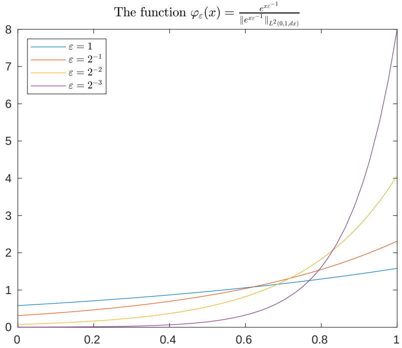

Figure: Normalized eigenfunctions of the adjoint of Neumann b. c.

Averaged dynamics and control for heat equations with random diffusion.

Joint work with Enrique Zuazua.
(https://hal.archives-ouvertes.fr/hal-02958671)

The averaged controllability problem

We study the controllability properties of the system:

$$\begin{cases} y_t - \alpha\Delta y = f1_{G_0}, & \text{in } (0,T)\times G, \\ y = 0, & \text{on } (0,T)\times \partial G, \\ y(0,\cdot) = y^0, & \text{on } G, \end{cases} \tag{5}$$

for α a positive random variable of density ρ. We cannot expect to control all the possible realizations (consider, for instance, the case in which $\alpha \to 0$), so we seek to control the average. This problem is relevant in applications in which the control has to be chosen independently of the random value, in a robust way.

Known controllability results

Theorem (Coulson, Gharesifard, Lü, Mansouri, Zuazua)

Let α be a random variable with a Riemann integrable density function ρ such that $\operatorname{supp}(\rho) \subset [\alpha_{\min}, +\infty)$ *for some* $\alpha_{\min} > 0$*. Then, system* (5) *is null controllable in average.*

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Let α be a random variable with a Riemann integrable density function ρ such that $\operatorname{supp}(\rho) \subset [\alpha_{\min}, +\infty)$ for some $\alpha_{\min} > 0$. Then, system (5) is null controllable in average.

Their result leaves an interesting open question:
What happens if we allow the random variable to vanish; that is, if we allow $0 \in \operatorname{supp}(\rho)$?

Numerical illustration of the minimum of the functional associated to the control problem.

On the following slides we minimize the following functional with the help of Matlab:

$$J(\phi) = \frac{1}{2}\int_0^T \int_{G_0} \left| \int_0^{+\infty} \varphi(t,x;\alpha,\phi)\rho(\alpha)d\alpha \right|^2 dxdt + \left\langle y^0, \int_0^{+\infty} \varphi(0;\alpha,\phi)\rho(\alpha)d\alpha \right\rangle,$$

for φ the averaged solution of the adjoint heat equation. In particular, we consider the initial value $y^0 = 1/2$, $G = (0,\pi)$ and $G_0 = (1,2)$ and compare what happens when considering the uniform distributions in $(0,1)$ and $(1,2)$ (i.e. $\rho = 1_{(0,1)}$ and $\rho = 1_{(1,2)}$).

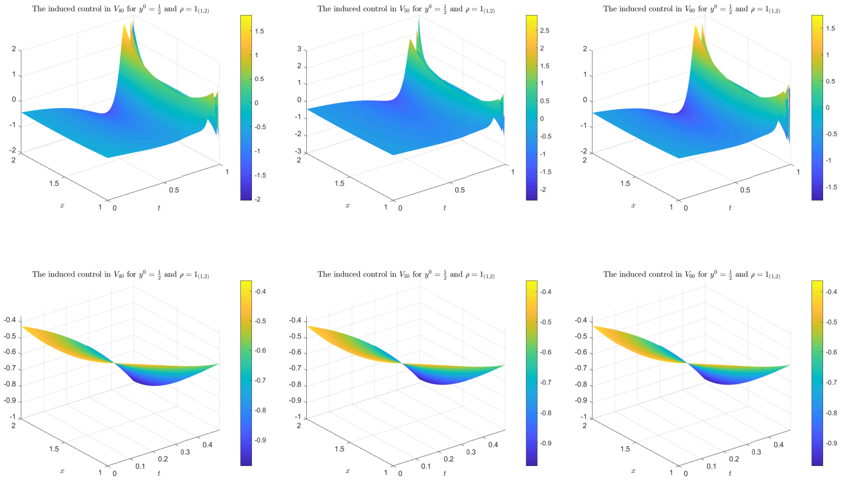

Figure: The optimal control for $\rho = 1_{(1,2)}$ and $y^0 = \frac{1}{2}$ induced by the minimum of the functional J in V_{40}, V_{50} and V_{60}, for $V_M := \langle e_i \rangle_{i=1}^{M}$.

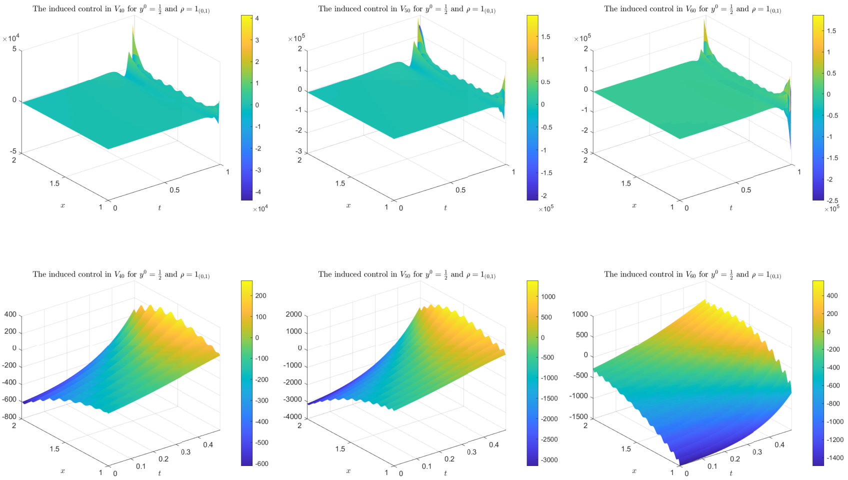

Figure: The optimal control for $\rho = 1_{(0,1)}$ and $y^0 = \frac{1}{2}$ induced by the minimum of the functional J in V_{40}, V_{50} and V_{60}, for $V_M := \langle e_i \rangle_{i=1}^{M}$.

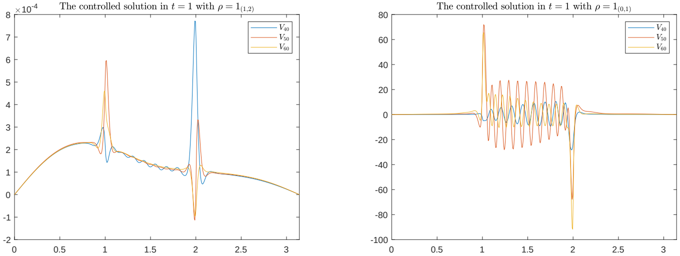

Figure: The state in time $t = 1$ of the averaged solutions of the heat equation after applying the control induced by the minimum of J in V_{40}, V_{50} and V_{60} with $y^0 = \frac{1}{2}$. In the left figure we have considered $\rho = 1_{(1,2)}$ and in the right one $\rho = 1_{(0,1)}$.

A qualitative description of the main results

(5) is null controllable in average if and only if ρ is sufficiently small near 0. In fact, the threshold density functions are those which near 0 satisfy:

$$\rho(\alpha) \sim e^{-\alpha^{-1}}.$$

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The proof is based in studying the averaged observability of its dual problem and in analogies with the fractional heat equation.

Thank you for your attention!

Is there any question?