Analysis and numerics solvability of backward-forward conservation laws

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Abstract

In this paper, we study the problem of initial data identification for the one-dimensional Burgers equation. This problem consists in identifying the set of initial data evolving to a given target at a final time. Due to the time-irreversibility of the Burgers equation, some **target** functions are **unattainable** from solutions of this equation, making the inverse problem under consideration ill-posed. To get around this issue, we introduce a **non-smooth optimization problem** which consists in minimizing the difference between the predictions of the Burgers equation and the observations of the system at a final time in $L^2(\mathbb{R})$ norm. The two main contributions of this work are as follows.

- We fully **characterize** the **set of minimizers** of the aforementioned non-smooth optimization problem.
- A wave-front tracking method is implemented to construct numerically all of them.

One of minimizers is the backward entropy solution, constructed using a backward-forward method.

Keywords: Backward-forward method, Identification problems; Conservation Laws; Weak-entropy solutions; Non-smooth optimization problem; Wave-front tracking algorithm.

AMS classification: 35L65, 35F20, 93B30, 35R30.

1 Introduction

1.1 Presentation of the Problem

Initial data identification consists in finding the origin of a physical phenomenon, governed for instance by partial differential equations (PDEs), from a set of observations at a given time. This arises naturally in meteorology, oceanography or climatology [29, 41, 22, 40, 27, 5, 18] to improve

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the forecasts of a model. Finding optimal positions or shapes of sensors [37, 38, 39] also lead to the study of identification problems.

The time-irreversibility of certain PDEs makes some identification problems ill-posed.

- In the case of parabolic PDEs, the high and instant regularization effect induces the non-existence of initial data for which the corresponding solution evolves to given not-necessary regular target functions, and causes numerical instabilities when solving the PDE backwards in time. In [31], the authors solve an identification problem for the heat equation with applications in pollution source localization. Note however that, when the target is attainable, the initial datum whose the corresponding trajectory evolves to this target, is unique as seen in [33].
- In the case of nonlinear hyperbolic PDEs, the backward uniqueness property fails due to the presence of discontinuities (so-called *shocks*), i.e multiple initial data may evolve to the same attainable target function.

Thus, identification problems need to be carefully addressed, depending on each type of PDEs.

In this paper, we consider the one-dimensional scalar conservation laws

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where u is the state, u_0 is the initial state and the flux function f is a strictly convex function. The study of (1) is motivated by the minimization of the sonic boom effects generated by supersonic aircrafts which are modeled by an augmented Burgers equation [14, 3, 2, 32]. Since f is a strictly convex function, we assume that $f : u \mapsto \frac{u^2}{2}$, without loss of generality.

Let $T > 0$ a final time and u^T a target function. As (1) is time-irreversibility, some conditions on u^T need to be imposed for it to be attainable. This is shown in [15, Theorem 3.1, Corollary 3.2], [25, Corollary 1] or [23] where they prove that u^T is truly attainable in an exact manner by a solution of (1) if and only if u^T satisfies the one-sided Lipschitz condition [8, 24, 36, 20], i.e

$$\partial_x u^T \leq \frac{1}{T} \text{ in } \mathcal{D}'(\mathbb{R}). \tag{2}$$

Due to the property of non-backward uniqueness of (1), there may exist multiple initial data leading to the same attainable target u^T, as seen in Figure 1. In [25], the authors prove that the set of initial data evolving to an attainable target u^T is a convex set. Later on, the aforementioned set was fully characterized in [15, 21] using the classical Lax-Hopf formula. In [32], an alternative proof is given using backward generalized characteristics.

Here, we take into account **unattainable target** functions. This leads to the resolution of the following non-smooth optimization problem

$$\inf_{u_0 \in \mathcal{U}_{\text{ad}}^0} J_0(u_0) := \|u^T(\cdot) - u(T,\cdot)\|_{L^2(\mathbb{R})}, \tag{$\mathcal{O}_T$}$$

where u is a solution of (1) defined in Section 2.1 and $\mathcal{U}_{\text{ad}}^0$ is the class of admissible initial data defined in (4).

To solve the optimization problem $(\mathcal{O}_T)$, some difficulties arise from a theoretical and numerical point of view.

- Since the entropy solution u of (1) may contain shocks even if the initial datum is a smooth function, this generates important added difficulties that have been the object of intensive study in the past, see [35, 34, 9, 10, 6, 7, 4] and the references therein. In [9, 10, 6, 7], the derivative of the cost function J_0 in $(\mathcal{O}_T)$ is regarded in a weak sense by requiring strong conditions on the set of initial data. This leads to require that entropy solutions of (1) have a finite number of non-interacting jumps.

- When J_0 is weakly differentiable, gradient descent methods have been implemented in [11, 12, 1] to solve numerically the optimization problem $(\mathcal{O}_T)$. In the cases where it was applied successfully, only one possible initial datum emerges, namely the backward entropy solution, see Remark 1. This is mainly due to the numerical viscosity that numerical schemes introduce to gain stability. To find some multiple minimizers, the authors in [25] use a filtering step in the backward adjoint solution.

In this article, we give a full characterization of the set of minimizers for the optimization problem $(\mathcal{O}_T)$ in two steps.

Step 1. We prove that the backward entropy solution, denoted by $S_T^-(u^T)$, is an optimal solution of $(\mathcal{O}_T)$ using a backward-forward method described in Section 2.1.

Step 2. We show that u_0 is a minimizer of $(\mathcal{O}_T)$ if and only if the weak-entropy solution of (1) with initial datum u_0 coincides, at time T, with the weak-entropy solution of (1) with initial datum $S_T^-(u^T)$ using variational methods.

Contrary to [11, 12, 1], entropy solutions of (1), generated by the class of initial data $\mathcal{U}_{\text{ad}}^0$ in $(\mathcal{O}_T)$, may have a countable number of interacting jumps. Moreover, a wave-front tracking method is implemented to construct numerically, not only the backward entropy solution $S_T^-(u^T)$, but also the set of minimizers of $(\mathcal{O}_T)$. An illustration of these results is given in Figure 2.

The article is organized as follows. In section 2, we describe the backward-forward method by introducing the forward operator S_T^+ and the backward operator S_T^-. Then, we solve the optimization problem $(\mathcal{O}_T)$ in Section 2.2. In Section 3, we construct numerically the set of minimizers of $(\mathcal{O}_T)$. More precisely, Section 3.1 is devoted to the construction of forward entropy solutions using a wave-front algorithm. Section 3.2, Section 3.3 and Section 3.4 explain how we choose a random numerical element among the class of mimimizers of $(\mathcal{O}_T)$. Finally, we prove Theorem 2.1 in Section 4.

1.2 Some related open problems

Let us address some related open questions and possible extensions of this work.

- It would be interesting to study the optimization problem $(\mathcal{O}_T)$ in L^1-norm, which is the natural distance in the framework of conservation laws. This problem leads to additional difficulties since $x \mapsto \|x\|_{L^1(\mathbb{R})}$ is not a differentiable function.

- It would be also interesting to consider a convex-concave function as a flux function in (1) which is, for instance, a more realistic choice to describe the flow of pedestrian [16, 13]. The main difficulty comes from the existence of discontinuities (called non-classical shocks) violating standard admissibility entropy conditions such that the Oleinik inequality.

- We could also study a Burgers equation with source terms. In this case, some suitable conditions on source terms have to be determined to use the backward-froward method

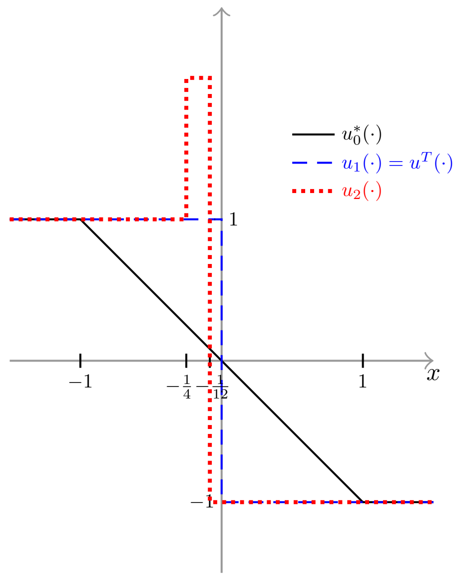


Figure 1: Three initial data $u_0^*(-)$, $u_1(--)$ and $u_2(\cdots)$ leading to a attainable target $u^T(\cdot) := \mathbb{1}_{(-\infty,0)}(\cdot) - \mathbb{1}_{(0,+\infty)}(\cdot)$ at time $T = 1$ along forward entropic evolution.

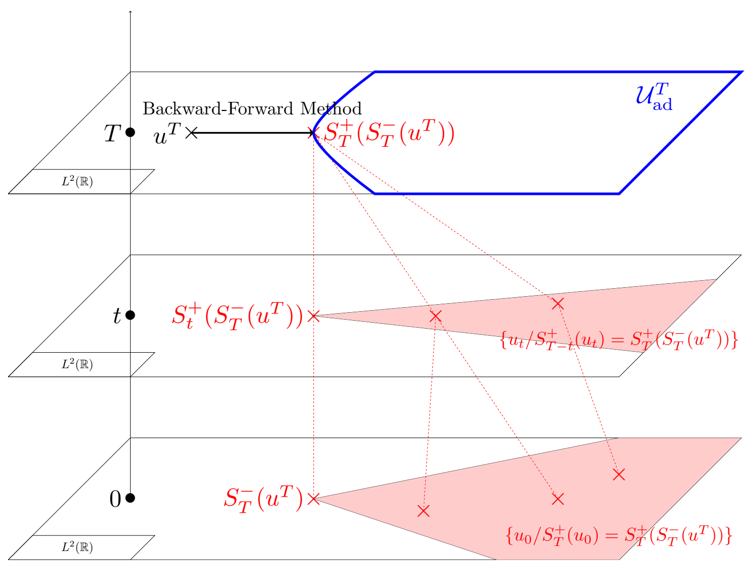


Figure 2: The backward-forward solution $S_T^+(S_T^-(u^T))$ is the projection of u^T onto the set of attainable target functions. The shaded area in red at time $t = 0$ represents the set of minimizers of $(\mathcal{O}_T)$.

described in this paper. For instance, the backward operator $S_t^-(u^T)$ defined in Section 2.1 associated to

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = -u^3(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(T,\cdot) = u^T(x), & x \in \mathbb{R}. \end{cases}$$

may blow up at time $t < T$.

- We can also investigate systems of conservation laws in one dimension (Euler equations, Saint-Venant equations, Aw-Rascle-Zhang traffic flow model). Note that, as soon as the backward-forward operator $S_T^+(S_T^-)$ is well-defined, $S_T^+(S_T^-)(u^T)$ may give a good candidate to solve the inverse design of systems of conservation laws.

- We may consider a multi-dimensional equation of conservation of laws in a numerical point of view. For instance, a fractional steps method [17, 30, 26] (or splitting method) may be implemented to solve an identification problem of a two-dimensional equation of conservation laws.

2 Main results

2.1 The backward-forward method

For a sake of completeness, we recall the definition of a weak-entropy solution of (1).

Definition 2.1
- *We say that $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \cap C^0(\mathbb{R}^+, L^1_{loc}(\mathbb{R}))$ is a weak solution if for all $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R})$,*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (u \partial_t \varphi + f(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} u_0(x) \varphi(0, x) dx = 0.$$

- *We say that $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \cap C^0(\mathbb{R}^+, L^1_{loc}(\mathbb{R}))$ is a weak-entropy solution if u is a weak solution and for every $k \in \mathbb{R}$, for all $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R}^+)$,*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (|u-k| \partial_t \varphi + \operatorname{sgn}(u-k)(f(\rho) - f(k)) \partial_x \varphi) dx dt + \int_{\mathbb{R}} |u_0 - k| \varphi(0, x) dx \geq 0.$$

Kruzkov's theory [28] provides existence and uniqueness of a weak-entropy solution $(t, x) \to S_t^+(u_0)(x)$ of (1) with initial datum $u_0 \in L^\infty(\mathbb{R})$. For a given function u^T, we introduce the function $(t, x) \to S_t^-(u^T)(x)$ as follows: for every $t \in [0, T]$, for a.e $x \in \mathbb{R}$,

$$S_t^-(u^T)(x) = S_t^+(x \to u^T(-x))(-x). \tag{3}$$

Remark 1 *The solutions $S_t^+(u_0)$ and $S_t^-(u^T)$ may be regarded as the zero viscosity limit of the solutions $S_T^{+,\epsilon}(u_0)$ and $S_T^{-,\epsilon}(u^T)$ respectively where $S_t^{+,\epsilon}(u_0)$ and $S_t^{-,\epsilon}(u^T)$ are defined as follows: $S_t^{+,\epsilon}(u_0)$ is the solution of the following viscous Burgers equation*

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = +\epsilon \partial^2_{xx} u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0, \cdot) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

and $S_t^{-,\epsilon}(u^T)$ is the solution of the following backward equation

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = -\epsilon \partial^2_{xx} u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(T, \cdot) = u^T(x), & x \in \mathbb{R}. \end{cases}$$

Using the change of variable $(t, x) \to (T - t, -x)$, we notice that the backward equation above is well-defined. Thus, $S_T^-(u^T)$ is called the backward entropy solution.

The backward-forward method consists in solving backward in time the PDE (1) with final target u^T and then solving it forward in time with initial datum $S_T^-(u^T)$, the solution of the backward PDE.

For any attainable target u^T, we have $S_T^+(S_T^-(u^T)) = u^T$ as seen in [15, Theorem 3.1, Corollary 3.2] and [25, Corollary 1]. However, there exist some target functions u^T verifying $S_T^+(S_T^-(u^T)) \neq u^T$ as seen in Example 1.

Example 1 *Assuming that u^T is defined by $u^T(\cdot) = -\mathbb{1}_{(-\infty,0)}(\cdot) + \mathbb{1}_{(0,\infty)}(\cdot)$ then the weak-entropy solution v of* (1) *with initial datum $v(0,x) = u^T(-x)$ is defined by*

$$v(t,x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0. \end{cases}$$

Thus, $S_{T-t}^-(u^T) : x \to v(T-t, -x)$ is a weak solution of (1) *verifying that $u(T) = u^T$. The weak-entropy solution u_e with initial datum $v(T,-x)$ is defined by*

$$u_e(t,x) = \begin{cases} -1 & \text{if } x < -t, \\ \frac{x}{t} & \text{if } -t \leq x \leq t, \\ 1 & \text{if } t < x. \end{cases}$$

In particular, $S_T^+(S_T^-(u^T)) := u_e(T) \neq u^T$. Note that u^T is an unattainable target.

2.2 An optimization problem

Let $T > 0$ and $u^T \in L^\infty(\mathbb{R})$. We assume that $K^T \subset \mathbb{R}$ is an open bounded set such that $\text{supp}(u^T) \subset K^T$ and the class of admissible initial data $\mathcal{U}_{\text{ad}}^0$ in $(\mathcal{O}_T)$ is defined by

$$\mathcal{U}_{\text{ad}}^0 = \{u_0 \in L^\infty(\mathbb{R}) \ / \ \|u_0\|_{L^\infty(\mathbb{R})} \leq C \text{ and } \text{supp}(u_0) \subset K_0\}, \tag{4}$$

where $C > 0$ is a constant such that $\|u^T\|_{L^\infty(\mathbb{R})} < C$, supp stands for the support of the function u^T and K_0 is an open bounded set large enough, see Lemma 4.1. Theorem 2.1 characterizes the set of optimal solutions for $(\mathcal{O}_T)$.

Theorem 2.1 *Let $T > 0$ and $u^T \in L^\infty(\mathbb{R})$ such that $supp(u^T) \subset K^T$. The optimal control problem $(\mathcal{O}_T)$ admits multiple optimal solutions. Moreover, the initial datum $u_0 \in L^\infty(\mathbb{R})$ is an optimal solution of $(\mathcal{O}_T)$ if and only if $u_0 \in L^\infty(\mathbb{R})$ satisfies $S_T^+(u_0) = S_T^+(S_T^-(u^T))$.*

Remark 2 *The constraints $\|u_0\|_{L^\infty(\mathbb{R})} \leq C$ and $Supp(u_0) \subset K_0$ in* (4) *are used to guarantee the existence of optimal solutions of $(\mathcal{O}_T)$.*

We denote by $BV(\mathbb{R})$ the class of functions of bounded variation, see [20, Definition 1.7.1]. If $g \in BV(\mathbb{R})$, we use the notation $g(x-) := \lim_{\substack{y \to x \\ y<x}} g(y)$ and $g(x+) := \lim_{\substack{y \to x \\ x<y}} g(y)$. Corollary is a direct consequence of Theorem 2.1 and the full characterization of the set $\{u_0 \in L^\infty(\mathbb{R}) / S_T^+(u_0) = S_T^+(S_T^-(u^T))\}$ given in Theorem A.1.

Corollary 2.1 *Let $T > 0$ and $u^T \in L^\infty(\mathbb{R})$ such that $supp(u^T) \subset K^T$. The map $u_0 \in L^\infty(\mathbb{R})$ is an optimal solution of $(\mathcal{O}_T)$ if and only if the following statements holds. For any $(x,y) \in X(S_T^+(S_T^-(u^T))) \times \mathbb{R}$*

$$\int_{x-Tf'(S_T^+(S_T^-(u^T))(x))}^{y} S_T^-(u^T)(s)\, ds \leq \int_{x-Tf'(S_T^+(S_T^-(u^T))(x))}^{y} u_0(s)\, ds, \tag{5}$$

For any $(x,y) \in X(S_T^+(S_T^-(u^T)))^2$,

$$\int_{x-Tf'(S_T^+(S_T^-(u^T))(x))}^{y-Tf'(S_T^+(S_T^-(u^T))(y))} S_T^-(u^T)(s)\,ds = \int_{x-Tf'(S_T^+(S_T^-(u^T))(x))}^{y-Tf'(S_T^+(S_T^-(u^T))(y))} u_0(s)\,ds, \tag{6}$$

where $X(S_T^+(S_T^-(u^T))) = \{x \in \mathbb{R}, S_T^+(S_T^-(u^T))(x-) = S_T^+(S_T^-(u^T))(x+)\}$ *and* $S_T^-(u^T)$ *is defined in* (3).

Remark 3 *From [20, Theorem 11.2.2],* $S_T^+(S_T^-(u^T)) \in BV_{loc}(\mathbb{R})$*. Thus,* $X(S_T^+(S_T^-(u^T)))$ *is well-defined.*

The proof of Theorem 2.1 is structured as follows. From [15, Theorem 3.1, Corollary 3.2], [25, Corollary 1] or [23], there exists $u_0 \in L^\infty(\mathbb{R})$ such that $S_T^+(u_0) = q$ if and only if $q \in L^\infty(\mathbb{R})$ satisfies the one-sided Lipschitz condition (2). Thus, the optimal problem $(\mathcal{O}_T)$ can be rewritten as follows.

$$\min_{q \in \mathcal{U}_{\text{ad}}^T} J_1(q) := \|u^T - q\|_{L^2(\mathbb{R})}, \tag{7}$$

where the admissible set $\mathcal{U}_{\text{ad}}^T$ is defined by

$$\mathcal{U}_{\text{ad}}^T = \{q \in L^\infty(\mathbb{R}) /\ \partial_x q \leq \frac{1}{T},\ \|q\|_{L^\infty(\mathbb{R})} \leq C \text{ and } \text{Supp}(q) \subset K_1\}. \tag{8}$$

Above, K_1 an open bounded set large enough, see Lemma 4.1. The optimization problem (7) admits a unique minimizer using the Hilbert projection Theorem. We prove that $q = S_T^+(S_T^-(u^T))$ is the minimizer of (7) using the first-order optimality conditions applied to (7) and the full characterization of the set $\{u_0 \in BV(\mathbb{R}) / S_T^-(u_0) = S_T^-(u^T)\}$ given in Theorem A.2.

Remark 4 *The optimal problem* (7) *is not related to the PDE model* (1)*. As a consequence, unlike* J_0 *in* $(\mathcal{O}_T)$*, the cost function* J_1 *in* (7) *is a differentiable function.*

Assuming that the given target u^T is attainable. Since $S_T^+(S_T^-(u^T)) = u^T$, Theorem 2.1 and Corollary 2.1 give a fully characterization of initial data leading to along forward entropic evolution, as in [21, 32]. Note that there exists initial data yielding weak solutions u that coincide with u^T such that the inequalities (5) and (6) do not hold, see Example 2.

Example 2 *Let* $T = 1$ *and assuming that* u^T *is defined by* $u^T(\cdot) = \mathbb{1}_{(-\infty,0)}(\cdot) - \mathbb{1}_{(0,+\infty)}(\cdot)$ *then the weak solution* u *defined by*

$$u(t,x) = \begin{cases} \mathbb{1}_{(-\infty,4t-2)}(x) + 7\mathbb{1}_{(4t-2,3t-\frac{3}{2})}(x) - \mathbb{1}_{(3t-\frac{3}{2},+\infty)}(x) & \text{if } t < \frac{1}{2}, \\ \mathbb{1}_{(-\infty,0)}(x) - \mathbb{1}_{(0,+\infty)}(x) & \text{if } \frac{1}{2} \leq t, \end{cases}$$

satisfies $u(T,\cdot) = u^T$ *and* $S_T^+(u(0,\cdot)) \neq u^T$.

3 Applications and numerical investigations

In this section, a wave-front tracking algorithm is implemented to construct the set of minimizers of $(\mathcal{O}_T)$. As a consequence, we now assume that $u^T \in BV(\mathbb{R})$ and $u_0 \in BV(\mathbb{R})$.

3.1 Wave-front tracking algorithm

Fix $a, b, \underline{u}, \overline{u} \in \mathbb{R}$ such that $a < b$ and $\underline{u} \leq \overline{u}$. We consider the set of initial data $u_0 \in BV(\mathbb{R})$ such that, for every $x \in (-\infty, a)$, $u_0(x) = u_0(a+) \in [\underline{u}, \overline{u}]$, for every $x \in (b, \infty)$, $u_0(x) = u_0(b-) \in [\underline{u}, \overline{u}]$ and for every $x \in \mathbb{R}$, $\underline{u} \leq u_0(x) \leq \overline{u}$.

To solve (1) with initial datum $u_0 \in BV(\mathbb{R})$, we use a wave-front tracking algorithm proposed by Dafermos [19]. This algorithm is based on the tracking in time of the discontinuity of the solution u of (1) with initial datum u_0. Let $\mathcal{M}_n$ the state mesh defined by

$$\mathcal{M}_n := \underline{u} + (\overline{u} - \underline{u})(2^{-n}\mathbb{N} \cap [0, 1]). \tag{9}$$

We construct an approximate piecewise constant function $u_0^n : \mathbb{R} \to \mathcal{M}_n$ of u_0 such that $\underline{u} \leq u_0^n \leq \overline{u}$. The sequence of points $(x_0^i)_{i=1,\cdots,N}$ denotes the N discontinuous points of u_0^n.

- If $u_0^n(x_0^i-) > u_0^n(x_0^i+)$, a shock wave $(u_0^n(x_0^i-), u_0^n(x_0^i+))$ is created with speed given by the Rankine-Hugoniot condition.
- If $u_0^n(x_0^i-) < u_0^n(x_0^i+)$, we split the rarefaction wave $(u_0^n(x_0^i-), u_0^n(x_0^i+))$ into a fan of rarefaction shocks with speeds prescribed by the Rankine-Hugoniot condition.

Thus, solving approximately the Riemann problem at each point of discontinuity of u_0^n as described above and piecing solutions together, we construct a solution u^n until two waves meet at time t_1. The approximate solution $u^n(t_1, \cdot)$ is a piecewise constant function verifying $u^n(t_1, x) \in \mathcal{M}_n$ for a.e $x \in \mathbb{R}$, the corresponding Riemann problems can again be approximately solved within the class of piecewise constant functions and so on.

In the sequel, we denote by $S_t^{+,n}(u_0)$, the approximate solution of (1) with initial datum u_0 at time t constructed using the wave-front tracking algorithm.

Example 3 *We assume that u_0 is a N-wave defined by*

$$u_0(x) = \begin{cases} 0, & \text{if } x < 0, \\ -1 + 2x, & \text{if } 0 < x < 1, \\ 0, & \text{if } 1 < x. \end{cases}$$

In this case, $a = -1$, $b = 2$, $\underline{u} = -1$ and $\overline{u} = 2$. The state mesh $\mathcal{M}_n$ is defined by

$$\mathcal{M}_n := -1 + 3(2^{-n}\mathbb{N} \cap [0, 1]),$$

with $n = 5$. In Figure 3, an approximate function $u_0^n : \mathbb{R} \to \mathcal{M}_n$ of u_0 is constructed. In Figure 5, the values of $S_T^{+,n}(u_0)$ at time $T = 1$ and $T = 2$ are extracted from Figure 4.

3.2 A geometrical interpretation of Theorem A.1

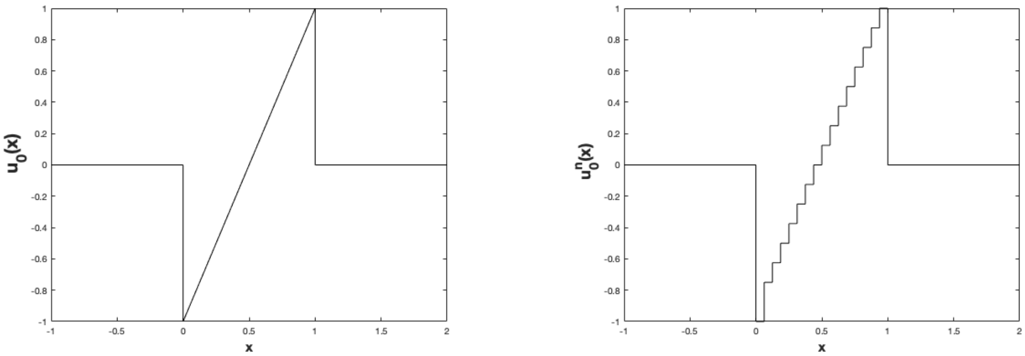
Let $u_L > u_R$, $\bar{x} \in \mathbb{R}$, $T > 0$ and $f : u \mapsto \frac{u^2}{2}$. We introduce the set

$$\Gamma(u_L, u_R, \bar{x}, T) := \left\{\gamma \in W^{1,1}\left([\bar{x} - Tf'(u_L), \bar{x} - Tf'(u_R)], \mathbb{R}\right) / \dot{\gamma} \in BV(\mathbb{R})\right\} \tag{10}$$

defined by $\gamma \in \Gamma(u_L, u_R, \bar{x}, T)$ if

(A1) $\gamma(\bar{x} - Tf'(u_L)) = 0$,

(A2) $\gamma(\bar{x} - Tf'(u_R)) = T(u_L f'(u_L) - f(u_L) - u_R f'(u_R) + f(u_R))$,



Initial datum u_0 Approximate initial datum u_0^n with $n = 5$

Figure 3: Construction of an approximate initial datum $u_0^n : x \to \mathcal{M}_n$ of u_0 with $n = 5$.

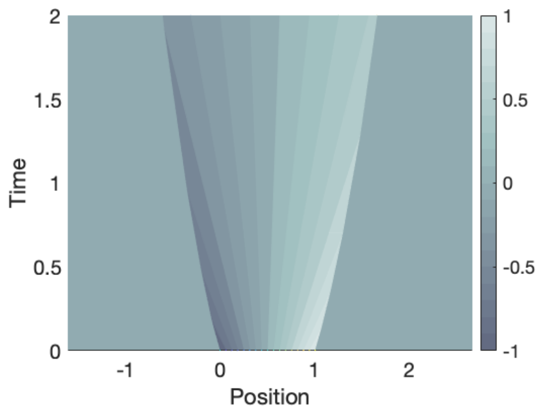


Figure 4: Plotting of $(x,t) \to S_t^{+,n}(u_0)(x)$ with u_0 defined in Example 3

(A3) for every $x \in [\bar{x} - Tf'(u_L), \bar{x} - Tf'(u_R)]$,

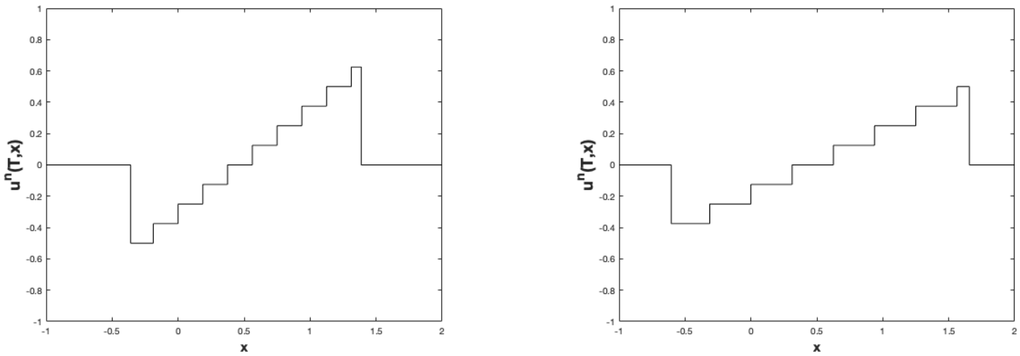
$$\gamma(x) \geq \gamma_*(x) := -T \int_{u_L}^{(f')^{-1}(\frac{\bar{x}-x}{T})} sf''(s)ds.$$

An illustration of the set $\Gamma(u_L, u_R, \bar{x}, T)$ is given in Figure 6. Note that, if $u^T = u_L \mathbb{1}_{(-\infty,\bar{x})} + u_R \mathbb{1}_{(\bar{x},\infty)}$ then for a.e $x \in [\bar{x} - Tf'(u_L), \bar{x} - Tf'(u_R)]$, $\dot{\gamma}_*(x) = S_T^+(S_T^-(u^T))(x)$.

Let $u^T \in BV(\mathbb{R})$, we construct an approximate final target $u^{T,n}$ with N discontinuous points denoted by $(x_i^T)_{i=\{1,\cdots,N\}}$. We denote by $S_t^{-,n}(u^T)$ the approximate backward entropy solution at time t defined by

$$S_t^{-,n}(u^T)(x) = S_t^{+,n}(x \to u^T(-x))(-x). \tag{11}$$

where $S_t^{+,n}$ is the approximate solution of (1) with initial datum $x \mapsto u^T(-x)$ at time t. Theorem A.1 can be written as follows.



Approximate solution $S_T^{+,n}(u_0^n)$ at $T = 1$. Approximate solution $S_T^{+,n}(u_0^n)$ at $T = 2$.

Figure 5: Construction of an approximate solution $u^n(T, \cdot) := S_T^{+,n}(u_0)$ of (1) using a wave-front algorithm with discretization parameter $n = 5$.

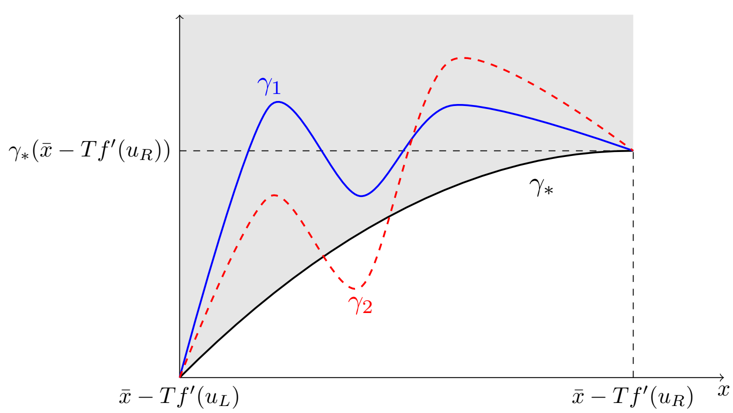


Figure 6: The set $\Gamma(u_L, u_R, \bar{x}, T)$ is illustrated by the white area. The function γ_* is defined by $\gamma_*(x) = -T \int_{u_L}^{(f')^{-1}(\frac{\bar{x}-x}{T})} s f''(s) ds$ with $x \in [\bar{x} - Tf'(u_L), \bar{x} - Tf'(u_R)]$. We have $\gamma_1 \in \Gamma(u_L, u_R, \bar{x}, T)$ and $\gamma_2 \notin \Gamma(u_L, u_R, \bar{x}, T)$.

Corollary 3.1 *Let $u^{T,n} : \mathbb{R} \to \mathcal{M}_n$ a piecewise constant function with N discontinuous points, denoted by $(x_i^T)_{i=\{1,\cdots,N\}}$. The piecewise constant function $u_0^n : \mathbb{R} \to \mathcal{M}_n$ satisfies $S_T^{+,n}(u_0^n) = u^{T,n}$ if and only if the two following statements hold*

- $u_0^n(x) = S_T^{-,n}(u^{T,n})(x)$ *for a.e* $x \in \mathbb{R} \setminus \cup_{i=1}^N [x_i^T - Tf'(u^{T,n}(x_i^T-)), x_i^T - Tf'(u^{T,n}(x_i^T+))]$.
- *For every* $i \in \{1, \cdots, N\}$, *there exists* $\gamma_i \in \Gamma(u^{T,n}(x_i^T-), u^{T,n}(x_i^T+), x_i^T, T)$ *such that* $u_0^n(x) = \dot{\gamma}_i(x)$ *for a.e* $x \in [x_i^T - Tf'(u^{T,n}(x_i^T-)), x_i^T - Tf'(u^{T,n}(x_i^T+))]$.

Let $u^T(\cdot) := \mathbb{1}_{(-\infty,0)}(\cdot) - \mathbb{1}_{(0,+\infty)}(\cdot)$. Example 4 shows that the two initial data u_1 and u_2 defined in Figure 1 verifies $S_T^+(u_1) = S_T^+(u_2) = u^T$ using Corollary 3.1.

Example 4 *Let $T = 1$ and $u^T(\cdot) := \mathbb{1}_{(-\infty,0)}(\cdot) - \mathbb{1}_{(0,+\infty)}(\cdot)$. The state mesh $\mathcal{M}_n$ is defined in* (9) *with $n = 6$, $\underline{u} = -2$ and $\overline{u} = 2$. In particular, $-1, 1, 2 \in \mathcal{M}_n(\mathbb{R})$. In that case, we have $u^T = u^{T,n}$, $x_i^T = 0$, $u^{T,n}(x_i^T-) = 1$ and $u^{T,n}(x_i^T+) = -1$. We construct two different initial data defined* *by* $u_1^n(x) = \mathbb{1}_{(-\infty,0)}(x) - \mathbb{1}_{(0,+\infty)}(x)$ *and* $u_2^n(x) = \mathbb{1}_{(-\infty,-\frac{1}{4})}(x) + 2\mathbb{1}_{(-\frac{1}{4},-\frac{1}{12})}(x) - \mathbb{1}_{(-\frac{1}{12},+\infty)}(x)$, *see Figure 1. The two* $\gamma_1 : [-1,1] \to \mathbb{R}$ *and* $\gamma_2 : [-1,1] \to \mathbb{R}$ *defined by* $\dot\gamma_1(\cdot) = u_1^n(\cdot)$ *and* $\dot\gamma_2(\cdot) = u_2^n(\cdot)$ *belongs to* $\Gamma(1,-1,0,1)$, *see Figure 7. Thus, from Corollary 3.1,* u_1^n *and* u_2^n *satisfies* $S_T^{+,n}(u_1^n) = S_T^{+,n}(u_2^n) = u^{T,n}$.

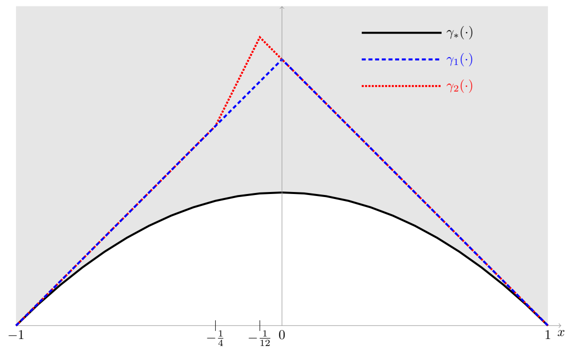


Figure 7: Plotting of γ_1 and γ_2 belonging to $\Gamma(1,-1,0,1)$. For a.e $x \in [-1,1]$, $\gamma_*(x) = -\int_{-1}^{(f')^{-1}(\bar x - x)} sf''(s)ds$, $\dot\gamma_1(x) = u_1^n(x)$ and $\dot\gamma_2(x) = u_2^n(x)$ where u_1^n and u_2^n are defined in Example 4.

3.3 Construction of the set of initial data leading to a shock u^T

We assume that $u^T = u_L\mathbb{1}_{(-\infty,\bar x)} + u_R\mathbb{1}_{(\bar x,\infty)}$ with $u_L > u_R$. Let $n \in \mathbb{N}$, the state mesh $\mathcal{M}_n$ is defined in (9) such that $u_L, u_R \in \mathcal{M}_n$. In that case, $u^T = u^{T,n}$ Our aim is to construct randomly the set of initial data $u_0^n \in \mathcal{M}_n$ such that $S_T^{+,n}(u_0^n) = u^{T,n}$. To that end, we introduce the set $\Gamma^n(u_L,u_R,\bar x,T)$ defined by

$$\Gamma^n(u_L,u_R,\bar x,T) = \{\gamma^n \in \Gamma(u_L,u_R,\bar x,T)/\ \dot\gamma^n(x) \in \mathcal{M}_n, \text{ for a.e } x \in [\bar x - Tf'(u_L), \bar x - Tf'(u_R)]\}.$$

Let $M \in \mathbb{N}$, we construct a set of piecewise constant functions $\gamma^n \in \Gamma^n(u_L,u_R,\bar x,T)$ admitting at most $M \in \mathbb{N}$ discontinuous points $(X_i,Y_i)_{i\in\{1,\cdots,M\}}$ via the following random iterative procedure (see Figure 8). The iterative construction is initiated with $(X_1,Y_1) = (\bar x - Tf'(u_L), 0)$. Then, we construct $(X_i,Y_i)_{i\in\{1,\cdots,M\}}$ such that, for every $i \in \{2,\cdots,M\}$,

$$\gamma_*(X_i) \le Y_i \le \min(\underline{u}(X_i - \bar x + Tf'(u_L)), \overline{u}(X_i - \bar x + Tf'(u_R)) + \gamma_*(\bar x - Tf'(u_R))), \tag{12}$$

and, for every $i \in \{1,\cdots,M\}$,

$$\frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} \in \mathcal{M}_n, \tag{13}$$

with $(X_{M+1},Y_{M+1}) := (\bar x - Tf'(u_R), \gamma^*(\bar x - Tf'(u_R)))$. Above, $\underline{u}$ and $\overline{u}$ are defined in (9) and γ^* is defined in the definition of $\Gamma(u_L,u_R,\bar x,T)$, see (10).

From a point (X_i,Y_i) with $i \in \{1,\cdots,M-1\}$, there exists a discrete set of (X_{i+1},Y_{i+1}) such that (12) and (13) hold as seen in Figure 8 by red crosses. One of them is chosen randomly using the Matlab function rand. The initia data u_0^n defined by

$$u_0^n(x) = \begin{cases} u_L, & \text{for every } x < \bar x - Tf'(u_L), \\ \dot\gamma^n(x), & \text{for a.e } x \in [\bar x - Tf'(u_L), \bar x - Tf'(u_R)], \\ u_R, & \text{for every } \bar x - Tf'(u_R) < x. \end{cases}$$

satisfies $S_T^{+,n}(u_0^n) = u^{T,n}$.

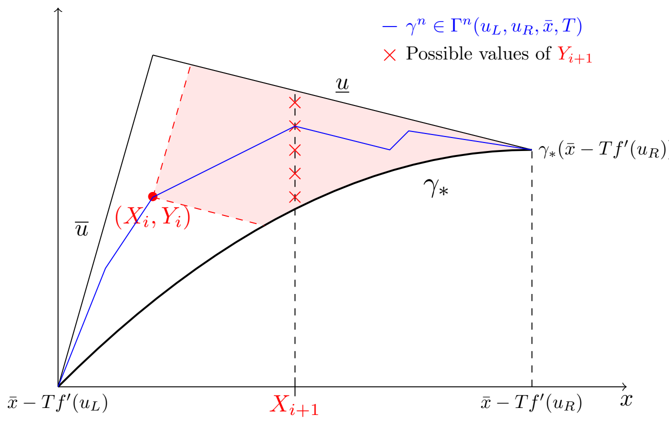


Figure 8: Construction of a random piecewise constant $\gamma^n \in \Gamma^n(u_L, u_R, \bar{x}, T)$

In Figure 9 and Figure 10, six different initial data u_0^n are constructed generating weak-entropy solutions that coincide with the target $u^T(\cdot) = 0.6875\mathbb{1}_{(-\infty,4.6)}(\cdot) - \mathbb{1}_{(4.6,\infty)}(\cdot)$ at time $T = 1$. The discretization parameter is $n = 4$, the state mesh is $\mathcal{M}_n := -1 + 3(2^{-n}\mathbb{N} \cap [0, 1])$ and M stands for the number of discontinuous points of u_0^n. Note that the backward maximum principle is violated in the case $M = 2$, $M = 3$, $M = 5$ and $M = 6$. Since $\mathcal{M}_n := -1 + 3(2^{-n}\mathbb{N} \cap [0, 1])$, the initial data u_0^n constructed verifies $-1 \leq u_0^n \leq 2$.

Remark 5 *If we wish to construct admissible initial data* u_0^n *such that* $M_1 \leq u_0^n \leq M_2$ *with* $M_1 < -1$ *and* $2 < M_2$*, either we modify the state mesh* $\mathcal{M}_n$ *by* $\mathcal{M}_n := M_1 + (M_2 - M_1)(2^{-n}\mathbb{N} \cap [0, 1])$ *or we use that* $\{u_0 \in BV(\mathbb{R})/S_T^+(u_0) = u^T\}$ *is a convex cone having as unique extremal point at its vertex the map* $S_T^-(u^T)$ *as follows. For every* $\eta > 0$*, the initial data* $u_0^{\eta,n} := S_T^{-,n}(u^T) + \eta(u_0^n - S_T^{-,n}(u^T))$ *with* u_0^n *constructed in Figure 10 also leads to* u^T *at time* $T = 1$ *and we notice that* $\lim_{\eta\to\infty} \|u_0^{\eta,n}\|_\infty = +\infty$.

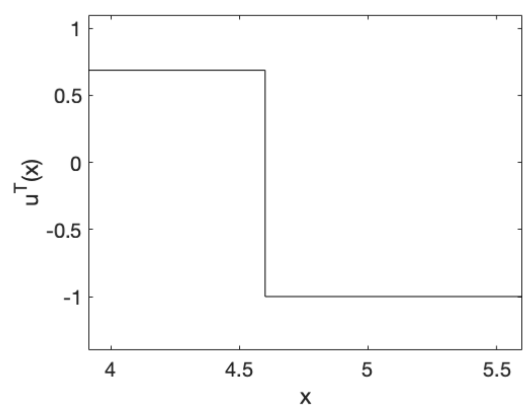
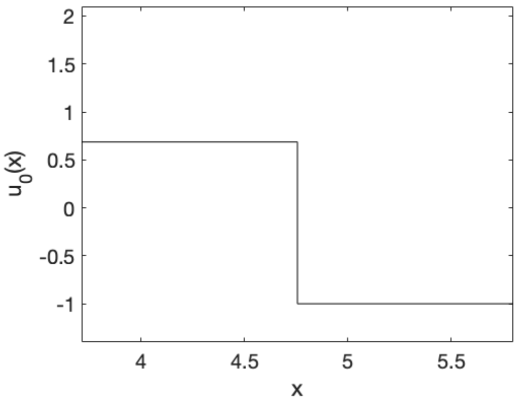
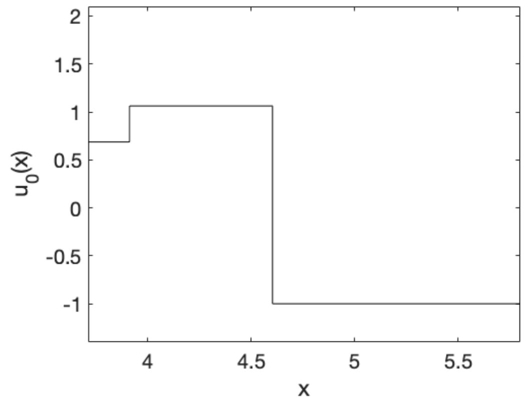


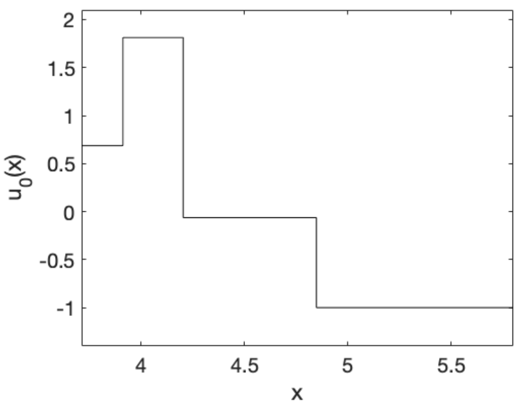
Figure 9: The target u^T defined by $u^T(\cdot) = 0.6875\mathbb{1}_{(-\infty,4.6)}(\cdot) - \mathbb{1}_{(4.6,\infty)}(\cdot)$



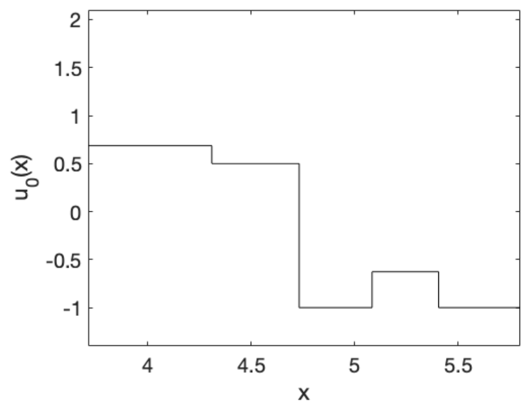
$M = 1$ discontinuous point



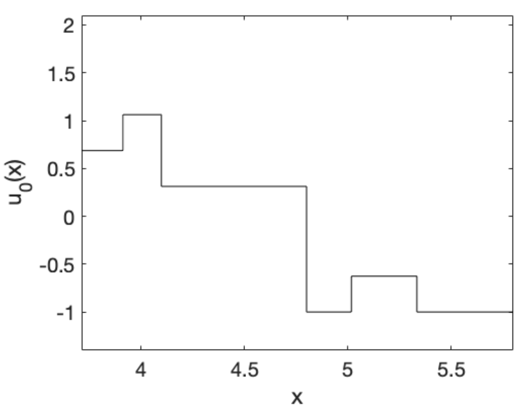
$M = 2$ discontinuous points



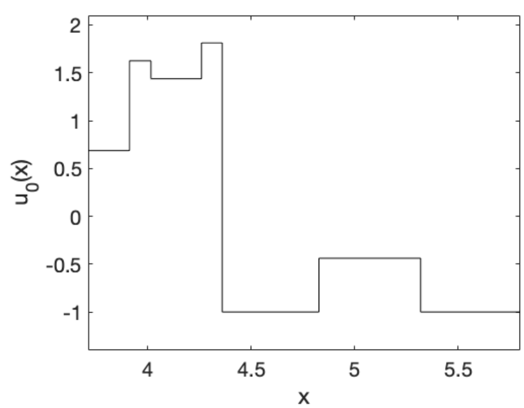
$M = 3$ discontinuous points



$M = 4$ discontinuous points



$M = 5$ discontinuous points



$M = 6$ discontinuous points

Figure 10: Construction of six random initial data $u_0 : \mathbb{R} \to \mathcal{M}_n$ leading to u^T such that u_0 admits $M \in \{1, \cdots, 6\}$ discontinuous points. $T = 1$, $u^T = u_L \mathbb{1}_{(-\infty,\bar{x})} + u_R \mathbb{1}_{(\bar{x},\infty)}$ with $u_L = 0.6875$, $u_R = -1$, $\bar{x} = 4.6$, $\mathcal{M}_n$ defined in (9) with $n = 4$, $\underline{u} = -1$ and $\overline{u} = 2$.

3.4 Construction of the set of minimizers of $(\mathcal{O}_T)$

In this section, we describe the algorithm used to solve the optimal problem $(\mathcal{O}_T)$.

Algorithm. Algorithm for the construction of the set of optimal solutions for $(\mathcal{O}_T)$

Input data:

- The target function u^T.
- The final time T.

Step 0. Construction of a state mesh $\mathcal{M}_n$ and an approximate target $u^{T,n} : \mathbb{R} \to \mathcal{M}_n$ of u^T as described in Section 3.1. The sequence $(x_i^T)_{i=1,\cdots,N}$ denotes de N discontinuous points of $u^{T,n}$.

Step 1.

- Construction of an approximate function $S_T^{-,n}(u^{T,n})$ of $S_T^-(u^T)$ using the wave-front tracking algorithm described in Section 3.1.
- Construction of an approximate function $S_T^{+,n}(S_T^{-,n}(u^{T,n}))$ of $S_T^+(S_T^-(u^T))$ using the wave-front tracking algorithm described in Section 3.1.

Step 2. Construction of all initial data u_0^n such that $S_T^{+,n}(u_0^n) = S_T^{+,n}(S_T^{-,n}(u^{T,n}))$ using Section 3.3.

- We find the set of discontinuous points $(x_j^{*,n})_{i=1,\cdots,N_n^*}$ of $S_T^{+,n}(S_T^{-,n}(u^{T,n}))$ such that for every $i \in \{1, \cdots, N_n^*\}$, $S_T^{+,n}(S_T^{-,n}(u^{T,n}))(x_j^{*,n}-) > S_T^{+,n}(S_T^{-,n}(u^{T,n}))(x_j^{*,n}+)$. To simplify the notations,
$$u_L^j := S_T^{+,n}(S_T^{-,n}(u^{T,n}))(x_j^{*,n}-),$$
and
$$u_R^j := S_T^{+,n}(S_T^{-,n}(u^{T,n}))(x_j^{*,n}+).$$
- At each point $x_j^{*,n}$, we construct a random initial datum $u_j^{*,n}$ such that
$$S_T^{+,n}(u_j^{*,n}) = u_L^j \mathbb{1}_{(-\infty,x_j^{*,n})} + u_R^j \mathbb{1}_{(x_j^{*,n},\infty)},$$
using Section 3.3. More precisely, we construct $\gamma_j^n \in \Gamma(u_L^j, u_R^j, x_j^{*,n}, T)$ such that
$$u_0^n(x) = \begin{cases} u_L^j, & \text{for every } x < x_j^{*,n} - Tf'(u_L^j), \\ \dot{\gamma}_j^n(x), & \text{for a.e } x \in [x_j^{*,n} - Tf'(u_L^j), x_j^{*,n} - Tf'(u_R^j)], \\ u_R^j, & \text{for every } x_j^{*,n} - Tf'(u_R^j) < x. \end{cases}$$
- We construct a random optimal solution $u_0^{\text{rand},n}$ of $(\mathcal{O}_T)$ piecing together every $u_j^{*,n}$ with $S_T^{-,n}(u^{T,n})$ as described in Section 3.3. More precisely, we have, for a.e $x \in \mathbb{R}$,
$$u_0^{\text{rand},n}(x) = \begin{cases} u_j^{*,n}(x) & \text{if } x \in [x_j^{*,n} - Tf'(u_L^j), x_j^{*,n} - Tf'(u_R^j)],\ j \in \{1, \cdots, N_n^*\}, \\ S_T^{-,n}(u^{T,n})(x) & \text{otherwise.} \end{cases}$$

Output data:

- The approximate backward entropy solution $S_T^{-,n}(u^{T,n})$,

- A random approximate optimal solution $u_0^{\text{rand},n}$.

We give the two following examples to illustrate the algorithm described above.

Example 5 *Let $T = 2$. We consider the target u^T defined as*

$$u^T(x) = \begin{cases} 2 & \text{if } x \in (-0.2, 1.1) \bigcup (2, 3.1) \bigcup (4.1, 5.3) \bigcup (6.1, 7.2), \\ -1 & \text{otherwise.} \end{cases}$$

Since for every $x \in \{-0.2, 2, 4.1, 6.1\}$, we have $u^T(x-) + 3 = u^T(x+)$, u^T is an unattainable target. From Theorem 2.1, $S_T^+(S_T^-(u^T))$ is an optimal solution of $(\mathcal{O}_T)$. Firstly, we construct an approximate function $S_T^{+,n}(S_T^{-,n}(u^T))$ of $S_T^+(S_T^-(u^T))$ using the wave-front tracking algorithm described in Section 3.1. The discretization parameter is $n = 8$ and the state mesh $\mathcal{M}_n$ is defined by $\mathcal{M}_n = -1 + 3(2^{-n}\mathbb{N} \cap [0, 1])$. In this example, we have $u^{T,n} = u^T$.

- *In Figure 11a), the function $(x, t) \to S_t^{-,n}(u^T)(-x)$ is plotted.*
- *In Figure 11b), the approximate optimal solution $S_T^{-,n}(u^T)$ of $(\mathcal{O}_T)$ is plotted.*
- *In Figure 11c), the function $(x, t) \to S_t^{+,n}(S_T^{-,n}(u^T))(x)$ is plotted.*
- *In Figure 11d), the function u^T and $x \to S_T^{+,n}(S_T^{-,n}(u^T))(x)$ are plotted.*

*Secondly, we give four different $u_0^{rand,n}$ such that $S_T^{+,n}(u_0^{rand,n}) = S_T^{+,n}(S_T^{-,n}(u^T))$ illustrating the **Step 2.** of the algorithm above, see Figure 12.*

Example 6 *Let $T = 1$. We consider the target u^T defined as*

$$\begin{aligned} u^T = \; & -\mathbb{1}_{(-\infty,0)} + 3\mathbb{1}_{(0,1.1)} + 0.55\mathbb{1}_{(1.1,2)} + 2.11\mathbb{1}_{(2,3.1)} - 0.7\mathbb{1}_{(3.1,5)} \\ & -0.23\mathbb{1}_{(5,5.8)} - \mathbb{1}_{(5.8,6.1)} + 2.89\mathbb{1}_{(6.1,7.2)} - \mathbb{1}_{(7,2,\infty)}. \end{aligned}$$

The function u^T is an unattainable target. The discretization parameter $n = 8$ and the state mesh $\mathcal{M}_n$ is defined by $\mathcal{M}_n = -1 + 4(2^{-n}\mathbb{N} \cap [0, 1])$. In Figure 13, the function u^T and $x \mapsto S_T^{+,n}(S_T^{-,n}(u^T))(x)$ are plotted.

4 Proof of Theorem 2.1

Let $K^T \subset \mathbb{R}$ is an open bounded set such that $\text{supp}(u^T) \subset K^T$. Using the forward maximum principle and the finite velocity of propagation that entropy solutions fulfilled, we immediately deduce the following Lemma

Lemma 4.1

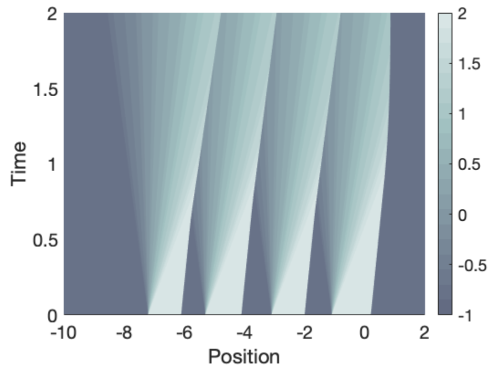
- *There exists an open bounded set $K_0 \subset \mathbb{R}$ such that $supp(S_T^-(u^T)) \subset K_0$.*
- *There exists an open bounded set $K_1 \subset \mathbb{R}$ such that, for every $u_0 \in \mathcal{U}_{ad}^0$, $Supp(S_T^+(u_0)) \subset K_1$.*

The proof of Theorem 2.1 is based on the following Lemma.

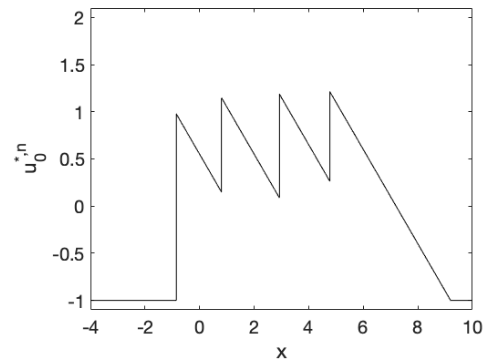
Lemma 4.2 *The optimal problem* (7) *admits a unique minimizer $S_T^+(S_T^-(u^T))$.*

Proof. The proof is divided in two steps.

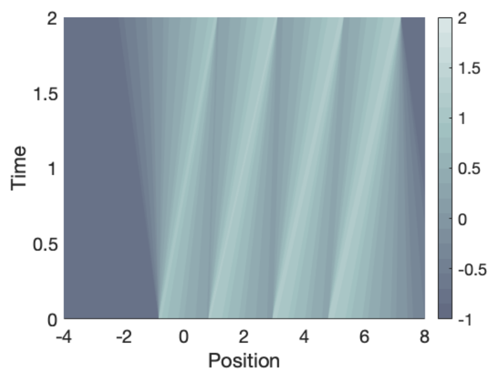
Step 1: Existence and uniqueness of minimizers of (7)**.** By definition of J_1, it is enough to prove that $\mathcal{U}_{\text{ad}}^T$ is a closed convex set of $L^2(\mathbb{R})$ using Hilbert projection Theorem. Assuming that $q_1, q_2 \in \mathcal{U}_{\text{ad}}^T$ we immediately have, for every $\alpha \in [0, 1]$, $\alpha q_1 + (1 - \alpha)q_2 \in \mathcal{U}_{\text{ad}}^T$. Moreover, if



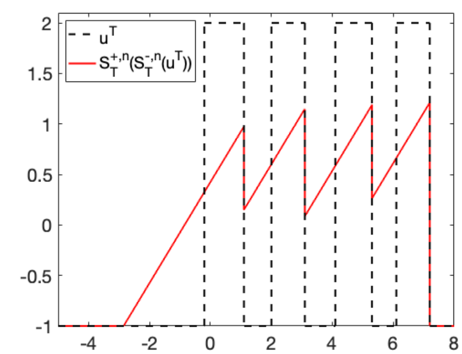
a) $(x,t) \to S_t^{-,n}(u^T)(-x)$



b) $x \to S_T^{-,n}(u^T)(x)$



c) $(x,t) \to S_t^{+,n}(S_T^{-,n}(u^T))(x)$



d) u^T and $x \to S_T^{+,n}(S_T^{-,n}(u^T))(x)$

Figure 11: $T = 2$, $n = 8$. Illustration of the **Algorithm Step 1** described in Section 3.4 where the unattainable target u^T is defined in Example 5.

q_n converges to q in $L^2(\mathbb{R})$ then q_n converges to q in the sense of distributions and by passing to the limit in $\partial_x q_n \leq \frac{1}{T}$, we have $\partial_x q \leq \frac{1}{T}$. Since $\|q_n\|_{L^\infty(\mathbb{R})} \leq C$, q_n converges to $q \in L^\infty(\mathbb{R})$ in the weak* topology of $L^\infty(\mathbb{R})$ and $\|q\|_{L^\infty(\mathbb{R})} \leq \liminf_n \|q_n\|_{L^\infty(\mathbb{R})} \leq C$. Using q_n converges to q in $L^2(\mathbb{R})$, q_n converges a.e to q. Moreover, for a.e $x \in \mathbb{R}\backslash K_1$, $q_n(x) = 0$. Thus,

$$\mathrm{Supp}(q) \subset K_1. \tag{14}$$

We conclude that $q \in \mathcal{U}_{\mathrm{ad}}^T$. Since J_1 is a strictly convex function, there exists a unique minimizer q^* of (7). Note that q^* is the projection of u^T onto $\mathcal{U}_{\mathrm{ad}}^T$.

Step 2: First-order optimality conditions. Our aim is to prove that, for any admissible perturbation $h \in \mathcal{T}_{S_T^+(S_T^-(u^T))}$[1], we have

$$-\int_{\mathbb{R}} \left(u^T(x) - S_T^+(S_T^-(u^T))(x)\right) h(x)\, dx \geq 0. \tag{15}$$

[1]That is a set of functions $h \in L^2(\mathbb{R})$ such that, for any sequence of positive real numbers ϵ_n decreasing to 0, there exists a sequence of functions $h_n \in L^2(\mathbb{R})$ converging to h as $n \to \infty$ and $S_T^+(S_T^-(u^T)) + \epsilon_n h_n \in \mathcal{U}_{\mathrm{ad}}^T$ for every $n \in \mathbb{N}$.

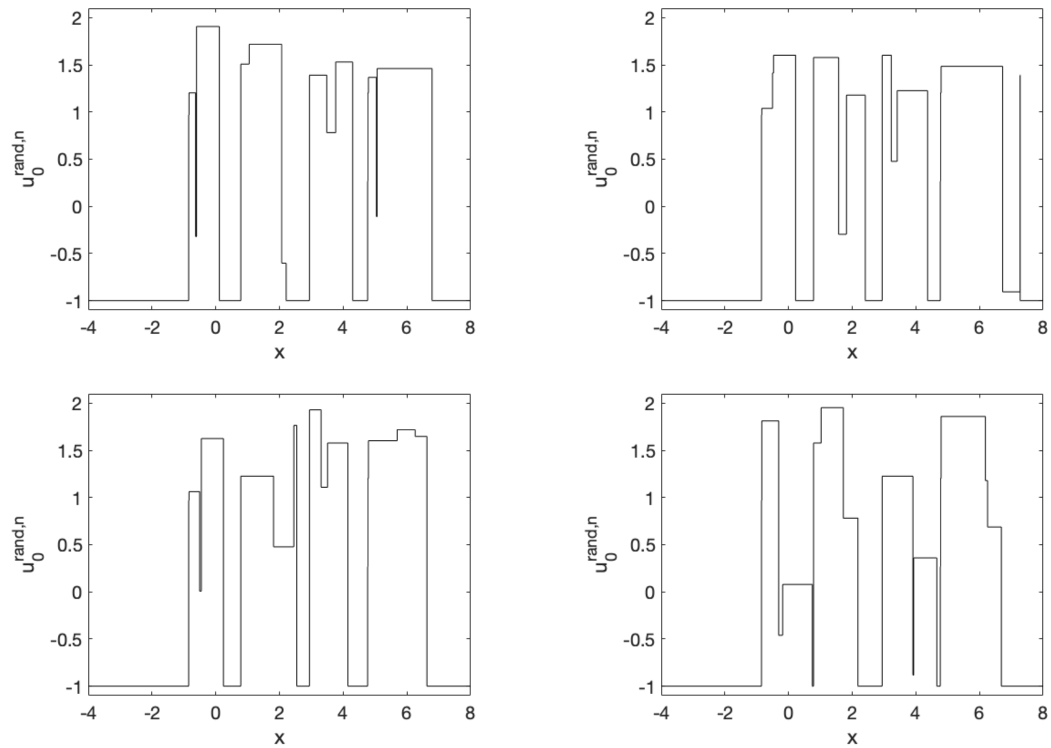
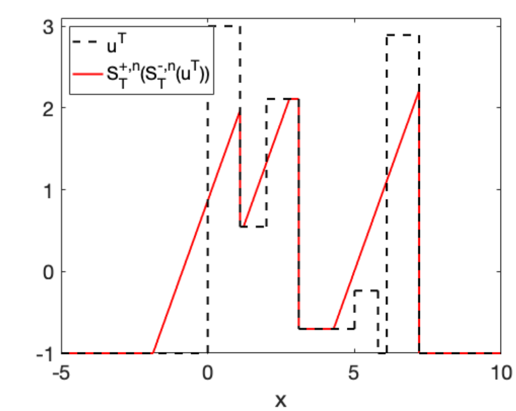


Figure 12: $T = 2$, $n = 8$. Four different $u_0^{\text{rand},n}$ such that $S_T^{+,n}(u_0^{\text{rand},n}) = S_T^{+,n}(S_T^{-,n}(u^T))$ with u^T defined in Example 5.



x

Figure 13: u^T and $x \to S_T^{+,n}(S_T^{-,n}(u^T))(x)$ with u^T defined in Example 6

We recall that the set $X(f)$ is defined by $X(f) = \{x \in \mathbb{R} / f(x-) = f(x+)\}$ when $f \in BV(\mathbb{R})$. Let $x \in X(S_T^-(u^T))$, we introduce the function F defined by, for every $y \in X(S_T^+ S_T^-(u^T))$,

$$F(y) = \int_{x+Tf'(S_T^-(u^T)(x))}^{y} \left(u^T(s) - S_T^+(S_T^-(u^T))(s)\right) ds. \tag{16}$$

Since $u^T \in L^\infty(\mathbb{R})$ and $\text{supp}(u^T) \subset K^T$ then $u^T \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By definition of S_T^- and S_T^+ (see Section 2.1) and [20, Theorem 11.3.5], for any $T > 0$, we have $S_T^-(u^T) \in BV_{\text{loc}}(\mathbb{R})$. Using the finite speed of propagation and $\text{supp}(u^T) \subset K^T$ then $S_T^-(u^T) \in BV(\mathbb{R})$. Similarly, we have $S_T^+ S_T^-(u^T) \in BV(\mathbb{R})$. We conclude that

$$F \in W^{1,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}). \tag{17}$$

Since $S_T^+ S_T^-(u^T) \in BV(\mathbb{R})$, the set $\mathbb{R} \backslash X(S_T^+ S_T^-(u^T))$ has zero Lebesgue measure. Thus, (16) is defined for a.e $y \in \mathbb{R}$. We introduce function $p : X(S_T^+ S_T^-(u^T)) \to \mathbb{R}$ defined by

$$p : y - Tf'(S_T^+(S_T^-(u^T))(y)). \tag{18}$$

From [20, Theorem 11.1.3] associated with the solution $S_t^+(S_T^-(u^T))$, the function p is well-defined. More precisely, for any $y \in X(S_T^+ S_T^-(u^T))$, there exists a unique $z \in \mathbb{R}$ such that

$$z = y - Tf'(S_T^+(S_T^-(u^T))(y)). \tag{19}$$

Note that p cannot be extend to $\mathbb{R}$ since $S_T^+ S_T^-(u^T)$ may have discontinuous points. Nevertheless, the set $X(S_T^+ S_T^-(u^T))$ has full measure in $\mathbb{R}$.

- Let $y \in X(S_T^+ S_T^-(u^T))$ and $z \in X(S_T^-(u^T))$ with z defined in (19). From [20, Theorem 11.1.3, Theorem 11.3.2] associated with the solution $S_t^+(S_T^-(u^T))$, then $y = z + Tf'(S_T^-(u^T)(z))$. Applying Theorem A.2 with $u_0 = S_T^-(u^T)$, we conclude that for any $y \in p^{-1}(X(S_T^-(u^T)))$,

$$F(y) = 0. \tag{20}$$

  Above, the set $p^{-1}(X(S_T^-(u^T))) := \{y \in X(S_T^+ S_T^-(u^T)) \ / \ p(y) \in X(S_T^-(u^T))\}$. From Lebesgue differentiation theorem, for a.e $y \in p^{-1}(X(S_T^-(u^T)))$,

$$F'(y) = 0. \tag{21}$$

  Note that the set $X(S_T^-(u^T))$ has full measure in $\mathbb{R}$ since $S_T^-(u^T) \in BV(\mathbb{R})$.

- Let $y \in X(S_T^+ S_T^-(u^T))$ and $z \in \mathbb{R} \backslash X(S_T^-(u^T))$ with z defined in (19). From [20, Theorem 11.1.3] associated with the solution $S_t^+(S_T^-(u^T))$, $S_T^-(u^T)(z-) < S_T^-(u^T)(z+)$ which implies that a rarefaction wave is created at time $t = 0$ and at the position z. By definition of $S_T^-(u^T)$ and [20, Theorem 11.1.3] associated with the solution $S_t^-(u^T)$,

$$X(S_T^+ S_T^-(u^T)) \setminus p^{-1}(X(S_T^-(u^T))) = \cup_k [z_k + Tf'(S_T^-(u^T)(z_k-)), z_k + Tf'(S_T^-(u^T)(z_k+))], \tag{22}$$

  where $(z_k)_k$ are the countable number of discontinuous points of $S_T^-(u^T)$. Therefore, for any $y \in \cup_k (z_k + Tf'(S_T^-(u^T)(z_k-)), z_k + Tf'(S_T^-(u^T)(z_k+)))$,

$$\partial_y S_T^+(S_T^-(u^T))(y) = \frac{1}{T}. \tag{23}$$

From (16), we have

$$\begin{aligned} -\int_{\mathbb{R}} (u^T(x) - S_T^+(S_T^-(u^T))(x)) h(x)\, dx = \quad & -\int_{p^{-1}(X(S_T^-(u^T)))} F'(y) h(y)\, dy \\ & -\int_{X(S_T^+ S_T^-(u^T)) \setminus p^{-1}(X(S_T^-(u^T)))} F'(y) h(y)\, dy. \end{aligned} \tag{24}$$

From (21), for every $h \in \mathcal{T}_{S_T^+(S_T^-(u^T))}$,

$$-\int_{p^{-1}(X(S_T^-(u^T)))} F'(y) h(y)\, dy = 0. \tag{25}$$

Since $h \in \mathcal{T}_{S_T^+(S_T^-(u^T))}$ is an admissible perturbation, for every $\epsilon_n > 0$ such that $\epsilon_n \to 0$ when $n \to \infty$ there exists $h_n \in L^2(\mathbb{R})$ such that $\lim_{n\to\infty} h_n = h$ in $L^2(\mathbb{R})$ and $S_T^+(S_T^-(u^T)) + \epsilon_n h_n \in \mathcal{U}_{\text{ad}}^T$. Thus, from (23), $\partial_x h_n \leq 0$. Since $\lim_{n\to\infty} h_n = h$ in $L^2(\mathbb{R})$, h_n tends to h in the sense of distributions and we conclude that for any admissible perturbation $h \in \mathcal{T}_{S_T^+(S_T^-(u^T))}$

$$\partial_x h \leq 0. \tag{26}$$

Applying Theorem A.2 with $u_0 = S_T^-(u^T)$, we have $F(y) \leq 0$ for any $y \in \mathbb{R}$. To simplify the notation, one define the open set $\mathcal{I}_k = (z_k + Tf'(S_T^-(u^T)(z_k-)), z_k + Tf'(S_T^-(u^T)(z_k+)))$. Using a mollifier function ρ_n, there exists $F_n := \rho_n * F \in C_c^\infty(\mathcal{I}_k)$ such that F_n converges to F in $H^1(\mathbb{R})$ and $F_n \leq 0$. From (22) and (26), for every $n \in \mathbb{N}$,

$$-\int_{z_k+Tf'(S_T^-(u^T)(z_k-))}^{z_k+Tf'(S_T^-(u^T)(z_k+))} F_n'(x)h(x)\,dx = \langle F_n, \partial_x h\rangle \geq 0. \tag{27}$$

Since F_n converges to F in $H^1(\mathcal{I}_k)$, F_n' converges to F' in $L^2(\mathcal{I}_k)$. Therefore, by passing to the limit in (27), we conclude that, for any admissible perturbation $h \in \mathcal{T}_{S_T^+(S_T^-(u^T))}$,

$$-\int_{z_k+Tf'(S_T^-(u^T)(z_k-))}^{z_k+Tf'(S_T^-(u^T)(z_k+))} (u^T(x) - S_T^+(S_T^-(u^T))(x))h(x)\,dx \geq 0. \tag{28}$$

From (24), (25), (22) and (28) for any k, the inequality (15) holds. Thus, $S_T^+(S_T^-(u^T))$ is a critical point of (7). Since J_1 is strictly convex, $S_T^+(S_T^-(u^T))$ is the unique optimal solution of (7).

□

Proof of Theorem 2.1: From Lemma 4.2, for every $q \in \mathcal{U}_{\text{ad}}^T$, we have

$$\|u^T - S_T^+(S_T^-(u^T))\|_{L^2(\mathbb{R})} \leq \|u^T - q\|_{L^2(\mathbb{R})}. \tag{29}$$

From Lemma 4.1, $\text{Supp}(S_T^-(u^T)) \subset K_0$. Using the forward maximum principle, $\|S_T^-(u^T)\|_{BV(\mathbb{R})} \leq \|u^T\|_{BV(\mathbb{R})} \leq C$. Therefore, we have $S_T^-(u^T) \in \mathcal{U}_{\text{ad}}^0$ with $\mathcal{U}_{\text{ad}}^0$ defined in (4). Moreover, for every $u_0 \in \mathcal{U}_{\text{ad}}^0$, $S_T^+(u_0) \in \mathcal{U}_{\text{ad}}^T$ with $\mathcal{U}_{\text{ad}}^T$ defined in (8) since

- From Lemma 4.1, $\text{Supp}(S_T^+(u_0)) \subset K_1$.
- From [20, 11.2.2 Theorem] and $\text{Supp}(S_T^+(u_0)) \subset K_1$, we have $S_T^+(u_0) \in L^\infty(\mathbb{R})$ and $\partial_x S_T^+(u_0) \leq \frac{1}{T}$.
- From [20, 6.2.3 Theorem, 6.2.6 Theorem] and $\text{Supp}(S_T^+(u_0)) \subset K_1$, we have $\|S_T^+(u_0)\|_{BV(\mathbb{R})} \leq \|u_0\|_{BV(\mathbb{R})} \leq C$.

From (29) and for every $u_0 \in \mathcal{U}_{\text{ad}}^0$, $S_T^+(u_0) \in \mathcal{U}_{\text{ad}}^T$, we conclude that, for every $u_0 \in \mathcal{U}_{\text{ad}}^0$,

$$\|u^T - S_T^+(S_T^-(u^T))\|_{L^2(\mathbb{R})} \leq \|u^T - S_T^+(u_0)\|_{L^2(\mathbb{R})}, \tag{30}$$

which concludes the proof of Theorem 2.1.

A Characterization of the set of initial data evolving to the same target u^T at time T

Fix $u^T \in L^\infty(\mathbb{R})$, we introduce the set

$$\mathcal{I}^+(u^T) = \{u_0 \in L^\infty(\mathbb{R};\mathbb{R}) : S_T^+(u_0) = u^T\}. \tag{31}$$

From [15, Corollary 3.2], $\mathcal{I}^+(u^T) \neq \emptyset$ if and only if a suitable representative of u^T satisfies the Oleinik condition [8, 24, 36, 20], i.e for every $x \in \mathbb{R}$ and $y \in \mathbb{R}^+\backslash\{0\}$,

$$f'(u^T(x+y) - f'(u^T(x))) \leq \frac{y}{T}. \tag{32}$$

Theorem A.1 gives a full characterization of the set of initial data $u_0 \in L^\infty(\mathbb{R})$ such that $S_T^+(u_0) = u^T$.

Theorem A.1 *Let $T > 0$ and a suitable representation of $u^T \in L^\infty(\mathbb{R})$ satisfies the Oleinik condition* (2). *Then the initial data $u_0 \in L^\infty(\mathbb{R})$ verifies $S_T^+(u_0) = u^T$ if and only if the following statements holds. For any $(x, y) \in X(u^T) \times \mathbb{R}$*

$$\int_{x-Tf'(u^T(x))}^{y} S_T^-(u^T)(s)\, ds \leq \int_{x-Tf'(u^T(x))}^{y} u_0(s)\, ds, \tag{33}$$

For any $(x, y) \in X(u^T)^2$,

$$\int_{x-Tf'(u^T(x))}^{y-Tf'(u^T(y))} S_T^-(u^T)(s)\, ds = \int_{x-Tf'(u^T(x))}^{y-Tf'(u^T(y))} u_0(s)\, ds, \tag{34}$$

where $X(u^T) = \{x \in \mathbb{R}, u^T(x-) = u^T(x+)\}$ and $S_T^-(u^T)$ is defined in (3).

Remark 6 *When $u^T \in L^\infty(\mathbb{R})$ satisfies the Oleinik condition* (2)*, then $u^T \in BV_{loc}(\mathbb{R})$. Thus, $X(u^T)$ is well-defined.*

Theorem A.1 points out the richness and the diversity of initial data evolving to the same target at time T.

- There exists $u_0 \in L^\infty(\mathbb{R})$ such that $S_T^+(u_0) = u^T$ with $\min_{x\in\mathbb{R}} u_0(x) < \min_{x\in\mathbb{R}} u^T(x)$ and/or $\max_{x\in\mathbb{R}} u^T(x) < \max_{x\in\mathbb{R}} u_0(x)$, see Figure 10.
- The set $\mathcal{I}^+(u^T)$ defined in (31) is a convex cone having as unique extremal point at its vertex the map $S_T^-(u^T)$, see [15, Proposition 5.2]; for any $u_0 \in \mathcal{I}^+(u^T)$, for every $\eta > 0$, $u_0^\eta = u_0^* + \eta(u_0 - u_0^*) \in \mathcal{I}^+(u^T)$.

The proof is given in [21, 32]. Fix $u_0 \in L^\infty(\mathbb{R})$, we introduce the set

$$\mathcal{I}^-(u_0) = \{u^T \in L^\infty(\mathbb{R};\mathbb{R}) : S_T^-(u^T) = u_0\}. \tag{35}$$

From (3) and [15, Corollary 3.2], $\mathcal{I}^-(u_0) \neq \emptyset$ if and only if for every $x \in \mathbb{R}$ and $y \in \mathbb{R}^+\backslash\{0\}$,

$$f'(u_0(x+y) - f'(u_0(x))) \geq -\frac{y}{T}. \tag{36}$$

Theorem A.2 gives a full characterization of the set of function $u^T \in L^\infty(\mathbb{R})$ such that $S_T^-(u^T) = u_0$.

Theorem A.2 *Let $T > 0$ and a suitable representation of $u_0 \in L^\infty(\mathbb{R})$ satisfies* (36). *Then a map $u^T \in L^\infty(\mathbb{R})$ satisfies $S_T^-(u^T) = u_0$ if and only if the following statements holds. For any $(x, y) \in X(u_0), \times\mathbb{R}$*

$$\int_{x+Tf'(u_0(x))}^{y} S_T^+(u_0)(s)\, ds \geq \int_{x+Tf'(u_0(x))}^{y} u^T(s)\, ds, \tag{37}$$

For any $(x, y) \in X(u_0)^2$,

$$\int_{x+Tf'(u_0(x))}^{y+Tf'(u_0(y))} S_T^+(u_0)(s)\, ds = \int_{x+Tf'(u_0(x))}^{y+Tf'(u_0(y))} u^T(s)\, ds, \tag{38}$$

where $X(u_0) = \{x \in \mathbb{R}, u_0(x-) = u_0(x+)\}$.

Theorem A.2 is a consequence of Theorem A.1 noticing that $S_T^-(u^T) : x \to S_T^+(x \to u^T(-x))(-x)$.

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