

# Nonnegative control of finite-dimensional linear systems

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## Abstract

We consider the controllability problem for finite-dimensional linear autonomous control systems with nonnegative controls. Despite the Kalman condition, the unilateral nonnegativity control constraint may cause a positive minimal controllability time. When this happens, we prove that, if the matrix of the system has a real eigenvalue, then there is a minimal time control in the space of Radon measures, which consists of a finite sum of Dirac impulses. When all eigenvalues are real, this control is unique and the number of impulses is less than half the dimension of the space. We also focus on the control system corresponding to a finite-difference spatial discretization of the one-dimensional heat equation with Dirichlet boundary controls, and we provide numerical simulations.

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## Résumé

Dans cet article, nous considérons la contrôlabilité d'un système linéaire avec des contrôles positifs. Malgré la condition du rang de Kalman, la condition de positivité des contrôles peut conduire à l'existence d'un temps minimal de contrôlabilité strictement positif. Lorsque tel est le cas, nous démontrons que si la matrice du système de contrôle possède une valeur propre réelle, alors il existe dans l'espace des mesures de Radon positives, un contrôle en le temps minimal et ce contrôle est nécessairement une somme finie de masses de Dirac. De plus, lorsque toutes les valeurs propres de la matrice sont réelles, ce contrôle est unique et le nombre de masses de Dirac le constituant est d'au plus la moitié de la dimension de l'espace d'état. Nous particularisons ces résultats sur l'exemple de l'équation de la chaleur unidimensionnelle, avec des contrôles frontières de type Dirichlet, discrétisée en espace et nous proposons quelques simulations numériques.

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**Keywords:** Minimal time; Nonnegative control; Dirac impulse

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## Contents

1. Introduction and main results . . . . .	302
2. Control of the semi-discrete 1D heat equation under a nonnegative control constraint . . . . .	304
3. Numerical approximation of time optimal controls . . . . .	308
4. Further comments and open questions . . . . .	316
5. Proof of the main results . . . . .	320
5.1. Preliminaries . . . . .	320
5.1.1. Accessibility conditions . . . . .	320
5.1.2. Existence of a positive minimal controllability time and minimal time controls . . . . .	322
5.1.3. No gap conditions . . . . .	323
5.2. Controls in time $\mathcal{T}_{\mathcal{M}}$ . . . . .	326
5.2.1. Time rescaling . . . . .	326
5.2.2. Consequences of the Pontryagin maximum principle . . . . .	329
5.2.3. Uniqueness of the minimal time control . . . . .	331
5.3. Approximation of the minimal controllability time $\mathcal{T}_{\mathcal{U}}$ with bang-bang controls . . . . .	332
Acknowledgement . . . . .	335
Appendix A. Proof of Propositions 3 and 7 . . . . .	335
Appendix B. $L^1$ -norm optimal controls . . . . .	336
Appendix C. Technical details of some examples . . . . .	339
C.1. Technical details related to Remarks 5.1.9 and 5.1.10 . . . . .	339
C.1.1. Technical details of the 1st item of Remark 5.1.9 . . . . .	339
C.1.2. Technical details of the 2nd item of Remark 5.1.9 . . . . .	339
C.1.3. Technical details of the 3rd item of Remark 5.1.9 . . . . .	340
C.1.4. Technical details related to Remark 5.1.10 . . . . .	343
C.2. Technical details on the example of Remark 5.3.4 . . . . .	343
References . . . . .	346

## 1. Introduction and main results

Let  $n \in \mathbf{N}^*$ ,  $A$  be an  $n \times n$  real-valued matrix and  $B$  be an  $n \times 1$  real-valued matrix, such that the pair  $(A, B)$  satisfies the Kalman condition. We consider the finite-dimensional linear autonomous control system

$$\dot{y}(t) = Ay(t) + Bu(t) \quad (1.1)$$

where controls  $u$  are real-valued locally integrable functions. Given any initial state  $y^0 \in \mathbf{R}^n$  and any final state  $y^1 \in \mathbf{R}^n$ , the Kalman condition implies that the control system (1.1) can be steered from  $y^0$  to  $y^1$  in any positive time. In other words, the minimal controllability time required to pass from  $y^0$  to  $y^1$  is zero.

Now, we impose the unilateral nonnegativity control constraint

$$u(t) \geq 0 \quad (t > 0 \text{ a.e.}). \quad (1.2)$$

It has been shown in [21] that such a constraint may induce a positive minimal controllability time (this is also the case for unilateral state constraints). Actually, for every  $y^0 \in \mathbf{R}^n$ , there exists a target  $y^1 \in \mathbf{R}^n$  such that the minimal time required to pass from  $y^0$  to  $y^1$  is positive.

The objective of this paper is to study the structure of minimal time controls, which do exist in the class of Radon measures. Actually, we will provide evidence of the importance of the two possible assumptions:

(H.1) The matrix  $A$  has at least one real eigenvalue.

(H.2) All eigenvalues of  $A$  are real.

We will prove that, under assumption (H.1), there exists a minimal time nonnegative control in the class of Radon measures, which consists of a finite number  $N$  of Dirac impulses, and that, under the stronger assumption (H.2), we have  $N \leq \lfloor (n+1)/2 \rfloor$ .

*Application to a discretized 1D heat equation* Nonnegativity control constraints are actually closely related to nonnegativity state constraints (see [20,21]). For example, the comparison principle implies that the control of the heat equation under nonnegativity state constraints by Dirichlet boundary controls is equivalent to the control of the heat equation with nonnegative Dirichlet boundary controls. In this paper we will pay a particular attention to a discretized version of the 1D heat equation

$$\partial_t \psi(t, x) = \partial_x^2 \psi(t, x) \quad (t > 0, x \in (0, 1)), \quad (1.3a)$$

$$\partial_x \psi(t, 0) = 0 \quad (t > 0), \quad (1.3b)$$

$$\psi(t, 1) = u(t) \geq 0 \quad (t > 0), \quad (1.3c)$$

$$\psi(0, x) = \psi^0(x) \quad (x \in (0, 1)). \quad (1.3d)$$

For the continuous version, it has been proved in [20] that for every initial state  $\psi^0 \in L^2(0, 1)$  and every positive constant target  $\psi^1 \neq \psi^0$ , the minimal controllability time is positive, and controllability can be achieved at the minimal time for some nonnegative control in the space of Radon measures. However, uniqueness of this control and its expression as a countable sum of Dirac impulses are open issues.

Here, we consider the finite-difference spatial discretization of (1.3), written as (1.1), where  $n + 1 > 2$  is the number of discretization points and  $y_i(t)$  (the  $i$ th component of  $y(t)$ ) stands for  $\psi(t, (i - 1)/n)$ , with matrices

$$A = n^2 \begin{pmatrix} -2 & 2 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \quad B = n^2 \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (1.4)$$

which do satisfy the Kalman condition. In addition, all eigenvalues of the matrix  $A$  are real and negative: this property is even stronger than (H.2) above. Furthermore, the pair  $(A, B)$  satisfies the comparison principle: if  $y^0 \geq 0$  then the solution  $y$  of (1.1)-(1.2) with initial condition  $y(0) = y^0$  satisfies  $y(t) \geq 0$  for every  $t \geq 0$ . This follows from the fact that  $I_n - \tau A$  is a **M**-matrix (see [2, Chapter 6]) for every  $\tau \geq 0$ , which in turn implies that state and control constraints  $y(t) \geq 0$  and  $u(t) \geq 0$  are equivalent to the sole control constraint  $u(t) \geq 0$  (argument used in [20]).

**Main result** Before stating the main results, we introduce some notations. First, we set  $\mathbf{R}_+ = [0, \infty)$  and  $\mathbf{R}_+^* = (0, \infty)$ . For every  $T > 0$ , we define the set of nonnegative  $L^\infty$  controls by

$$\mathcal{U}_+(T) = \{u \in L^\infty(0, T) \mid u \geq 0\}$$

and the set of nonnegative Radon measure controls by

$$\mathcal{M}_+(T) = \{u \in \mathcal{M}([0, T]) \mid u \geq 0\}$$

where  $\mathcal{M}([0, T])$  is the set of Radon measures on  $[0, T]$ . Note that elements of  $\mathcal{M}([0, T])$  can be identified with those in  $\mathcal{M}(\mathbf{R})$  with support in  $[0, T]$ . In addition, any element  $u \in \mathcal{U}_+(T)$  can be identified with some  $\underline{u} \in \mathcal{M}_+(T)$  by setting  $d\underline{u}(t) = u(t)dt$ . The classical input-to-state mapping  $\Phi_T : L^\infty(0, T) \rightarrow \mathbf{R}^n$  is defined by  $\Phi_T u = \int_0^T e^{(T-t)A} B u(t) dt$  and is extended to  $\Phi_T : \mathcal{M}([0, T]) \rightarrow \mathbf{R}^n$  by  $\Phi_T u = \int_{[0, T]} e^{(T-t)A} B d\underline{u}(t)$ . We define the minimal controllability time required to steer  $y^0$  to  $y^1$ , with nonnegative “classical”  $L^\infty$ -controls, by

$$\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \inf \left\{ T \geq 0 \mid \exists u \in \mathcal{U}_+(T) \text{ s.t. } y^1 = e^{TA} y^0 + \Phi_T u \right\} \quad (1.5)$$

and with nonnegative Radon measure controls, by

$$\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \inf \left\{ T \geq 0 \mid \exists u \in \mathcal{M}_+(T) \text{ s.t. } y^1 = e^{TA} y^0 + \Phi_T u \right\}. \quad (1.6)$$

By convention, we set  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = +\infty$  (respectively  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = +\infty$ ) when  $y^1$  is not accessible from  $y^0$  in any time with nonnegative  $L^\infty$ -controls (respectively Radon measure controls).

Since any element in  $\mathcal{U}_+(T)$  can be identified to an element in  $\mathcal{M}_+(T)$ , we always have

$$\mathcal{T}_{\mathcal{M}}(y^0, y^1) \leq \mathcal{T}_{\mathcal{U}}(y^0, y^1). \quad (1.7)$$

We will see on some examples that this inequality can be strict (see Remark 5.1.9) and we refer to [22] for some *no-gap* conditions. Let us however point out that when  $y^1$  is a *positive steady-state*, i.e.,  $y^1 \in \mathcal{S}_+^*$ , with

$$\mathcal{S}_+^* = \{\bar{y} \in \mathbb{R}^n \mid \exists \bar{u} \in \mathbb{R}_+^* \text{ s.t. } A\bar{y} + B\bar{u} = 0\} \quad (1.8)$$

then  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ .

Recall that, by definition,  $\bar{y} \in \mathbb{R}^n$  is a steady-state for the system (1.1) if there exists  $\bar{u} \in \mathbb{R}$  such that  $A\bar{y} + B\bar{u} = 0$ .

Our main result is the following.

**Theorem 1.** *Let  $y^0 \in \mathbb{R}^n$  and  $y^1 \in \mathbb{R}^n$  be such that  $y^0$  can be steered to  $y^1$  in some positive time with nonnegative  $L^\infty$  controls (i.e.,  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$ ).*

- Under Assumption (H.1), there exists a control  $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}}(y^0, y^1))$  steering the system (1.1) from  $y^0$  to  $y^1$  in time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ , which is a linear combination with nonnegative coefficients of a finite number  $N$  of Dirac impulses.
- Under the stronger assumption (H.2), we have  $N \leq \lfloor (n+1)/2 \rfloor$ . If moreover  $y^1 \in \mathcal{S}_+^*$ , then the minimal time control  $u$  is unique.

The proof of Theorem 1 follows from Propositions 5.1.7, 5.1.11, 5.2.5 and 5.3.1 and Corollary 5.2.2, proved in Section 5. This section actually contains more precise results, most of them being summarized in Table 1.

## Remark 2.

- The result remains true when replacing  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  with  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , except that the additional assumption  $y^1 \in \mathcal{S}_+^*$  is not required to have uniqueness.
- Assumption (H.1) is used in an instrumental way in order to provide the existence of a nonnegative minimal time control in the class of Radon measures (see Proposition 5.1.7). ■

*Organization of the paper* In Section 2, we show on the example of the discretized heat equation (i.e., with the matrices  $A$  and  $B$  given by (1.4)) how the result of Theorem 1 can be used, and we perform some numerical simulations. The proof of the results given in Section 2 is presented in Appendix A. In Section 3 we give some possible strategies to numerically obtain a time optimal control, and in Section 4, we list some open questions. The proof of the main results of this paper (see Table 1) is performed in Section 5. More precisely, in Section 5.1, we recall some results ensuring that the target  $y^1$  can be reached from the initial condition  $y^0$  in some time  $T > 0$  with a nonnegative control (see § 5.1.1), we show that if Assumption (H.1) is satisfied, then there exists a nonnegative Radon measure control at the minimal times  $\mathcal{T}_{\mathcal{U}}$  and  $\mathcal{T}_{\mathcal{M}}$  (see § 5.1.2), and we show that if the target  $y^1$  belongs to  $\mathcal{S}_+^*$  then we have equality in (1.7) (see § 5.1.3). Assuming that Assumption (H.1) is satisfied, we show in Section 5.2 that any nonnegative Radon measure control at the minimal time  $\mathcal{T}_{\mathcal{M}}$  is a finite sum of Dirac impulses. In addition, with the more restrictive assumption (H.2), we bound the number of Dirac impulses and show that this nonnegative Radon control is unique. Section 5.3 gives some results in order to approximate, with bounded  $L^\infty$ -controls, the minimal controllability time  $\mathcal{T}_{\mathcal{U}}$  and the corresponding minimal time control. We provide this section, since we will show in Remark 5.1.9 that a gap phenomena can occur, and in this case, the results given in Section 5.2 are useless for obtaining the minimal time  $\mathcal{T}_{\mathcal{U}}$  and a minimal time control in time  $\mathcal{T}_{\mathcal{U}}$ . In Appendix B, we present some technical results related to the numerical method proposed in Section 3 for the nonnegative controls of minimal  $L^1$ -norm in times greater than  $\mathcal{T}_{\mathcal{M}}$ . Finally, in Appendix C, we give the technical details of some relevant examples presented in Section 5.

## 2. Control of the semi-discrete 1D heat equation under a nonnegative control constraint

We consider the control system (1.1) with matrices  $A$  and  $B$  given by (1.4). We consider an initial state  $y^0$  and a positive steady-state target  $y^1$ , i.e., there exists  $\bar{u}^1 \in \mathbb{R}_+^*$  such that  $y^1 = \bar{u}^1(1, \dots, 1)^\top$ . All results stated in Table 1 apply to this control problem. Moreover, in this case, we can give a more precise result (see Proposition 3) and an a

Table 1

Main results. Note that the assumption  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$  (respectively  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < +\infty$ ) is an implicit assumption on  $y^0$  and  $y^1$  meaning that  $y^1$  is reachable from  $y^0$  in a finite time  $T \in \mathbb{R}_+$  with a control in  $\mathcal{U}_+(T)$  (respectively  $\mathcal{M}_+(T)$ ).

Assumptions on $A$	Assumptions on $y^0$ and $y^1$	Results
No assumption	No assumption $y^1 \in \mathcal{S}_+^*$  $y^0, y^1 \in \mathcal{S}_+^*$	$0 \leq \mathcal{T}_{\mathcal{M}}(y^0, y^1) \leq \mathcal{T}_{\mathcal{U}}(y^0, y^1) \leq +\infty$ . $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ (Proposition 5.1.11, see also Proposition 5.1.12 for a more general result). $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$ (Proposition 5.1.1).
All eigenvalues of $A$ have a negative real part	$y^1 \in \mathcal{S}_+^*$	$\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$ (Proposition 5.1.2).
(H.1) (at least one eigenvalue of $A$ is real)	$\mathcal{T}_{\mathcal{M}}(y^0, y^1) < +\infty$  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$	There exists a control $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{M}}(y^0, y^1))$ steering $y^0$ to $y^1$ in time $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , which is a linear combination with nonnegative coefficients of a finite number of Dirac impulses (Corollary 5.2.3). There exists a control $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}}(y^0, y^1))$ steering $y^0$ to $y^1$ in time $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ (Proposition 5.1.7), which is a linear combination with nonnegative coefficients of a finite number of Dirac impulses (Proposition 5.3.1).
(H.2) (all eigenvalues of $A$ are real)	$\mathcal{T}_{\mathcal{M}}(y^0, y^1) < +\infty$  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$	There exists a unique control $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{M}}(y^0, y^1))$ steering $y^0$ to $y^1$ in time $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , which is a linear combination with nonnegative coefficients of at most $\lfloor (n+1)/2 \rfloor$ Dirac impulses (Proposition 5.2.5). There exists a control $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}}(y^0, y^1))$ steering $y^0$ to $y^1$ in time $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ , which is a linear combination with nonnegative coefficients of at most $\lfloor (n+1)/2 \rfloor$ Dirac impulses (Proposition 5.3.1).

priori lower bound on the minimal time (see Proposition 7). To this end we recall that the eigenvalues  $\lambda_k$  and associated eigenvectors  $\psi_k \in \mathbb{R}^n$  of the matrix  $A$  given by (1.4) are given by

$$\lambda_k = -2n^2 \left( 1 - \cos \frac{(k-1/2)\pi}{n} \right) \quad (k \in \{1, \dots, n\}) \quad (2.1a)$$

and

$$\psi_k = \begin{pmatrix} 1 \\ \cos((k-1/2)\pi/n) \\ \cos(2(k-1/2)\pi/n) \\ \vdots \\ \cos((n-1)(k-1/2)\pi/n) \end{pmatrix} \quad (k \in \{1, \dots, n\}) \quad (2.1b)$$

and the eigenvalues of  $A^\top$  are these  $\lambda_k$  with associated eigenvectors  $\varphi_k \in \mathbb{R}^n$  given by

$$\varphi_k = \begin{pmatrix} 1/2 \\ \cos((k-1/2)\pi/n) \\ \cos(2(k-1/2)\pi/n) \\ \vdots \\ \cos((n-1)(k-1/2)\pi/n) \end{pmatrix} \quad (2.1c)$$

**Proposition 3.** Assume the pair  $(A, B)$  is given by (1.4) and let  $y^0 \in \mathbb{R}^n$  and  $y^1 \in \mathcal{S}_+^*$ , i.e., there exists  $\bar{u}^1 \in \mathbb{R}_+^*$  such that  $y^1 = \bar{u}^1(1, \dots, 1)^\top$ . Then  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  is the minimum of the constraint minimization problem

$$\min \quad T \quad \left| \begin{array}{l} T \geq 0, \\ \exists m_1, \dots, m_N \in \mathbb{R}_+ \text{ and } t_1, \dots, t_N \in [0, T], \text{ s.t.} \\ \frac{\bar{u}^1}{-\lambda_k} - \frac{\langle \varphi_k, y^0 \rangle e^{\lambda_k T}}{\langle \varphi_k, B \rangle} = \sum_{i=1}^N m_i e^{\lambda_k (T-t_i)} \quad (k \in \{1, \dots, n\}) \end{array} \right. \quad (2.2)$$

with  $N = \lfloor (n+1)/2 \rfloor$  and  $\lambda_k$  given by (2.1a).

Furthermore:

- the time control  $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}}(y^0, y^1))$  is unique and given by  $u = \sum_{i=1}^N m_i \delta_{t_i}$ , with  $t_1, \dots, t_N$  and  $m_1, \dots, m_N$  the minimizers of the minimization problem (2.2);
- there exists  $p^1 \in \mathbb{R}^n \setminus \{0\}$  such that the solution  $p$  of the adjoint problem  $\dot{p} = -A^\top p$ , with final condition  $p(\mathcal{T}_{\mathcal{U}}(y^0, y^1)) = p^1$ , satisfies  $B^\top p \geq 0$ , and
 
$$\{t_1, \dots, t_N\} = \{t \in [0, \mathcal{T}_{\mathcal{U}}(y^0, y^1)] \mid B^\top p(t) = 0\}.$$

**Remark 4.** When  $y^0$  is also a steady state, i.e., there exists  $\bar{u}^0 \in \mathbb{R}$  such that  $y^0 = \bar{u}^0(1, \dots, 1)^\top$ , then the minimization problem (2.2) becomes

$$\min \begin{cases} T \\ T \geq 0, \\ \exists m_1, \dots, m_N \in \mathbb{R}_+ \text{ and } t_1, \dots, t_N \in [0, T], \text{ s.t.} \\ \bar{u}^1 - \bar{u}^0 e^{\lambda_k T} = -\lambda_k \sum_{i=1}^N m_i e^{\lambda_k (T-t_i)} \quad (k \in \{1, \dots, n\}). \end{cases} \quad \blacksquare \quad (2.3)$$

**Remark 5.** Given a control  $u = \sum_{i=1}^N m_i \delta_{t_i} \in \mathcal{M}(0, T)$  for some  $m_1, \dots, m_N \in \mathbb{R}$ , some  $T > 0$  and some  $t_1, \dots, t_N \in [0, T]$ , the solution of (1.1) with initial condition  $y^0$  and control  $u$  is given by

$$y(t^-) = e^{tA} y^0 + \sum_{\substack{i=1 \\ t_i < t}}^N e^{(t-t_i)A} B m_i \quad \text{and} \quad y(t^+) = e^{tA} y^0 + \sum_{\substack{i=1 \\ t_i \leq t}}^N e^{(t-t_i)A} B m_i \quad (t \in [0, T]). \quad \blacksquare$$

**Remark 6.** In Proposition 3, we have considered a steady state target  $y^1$ . Recall that, in this case, we have  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ .  $\blacksquare$

**Proposition 7.** With the assumptions and notations introduced in Proposition 3,  $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$  satisfies

$$\sup_{k \in \{1, \dots, n\}} \left( \frac{\bar{u}^1}{-\lambda_k} - e^{\lambda_k \mathcal{T}_{\mathcal{U}}} \frac{\langle \varphi_k, y^0 \rangle}{\langle \varphi_k, B \rangle} \right) \leq \inf_{k \in \{1, \dots, n\}} \left( \frac{\bar{u}^1 e^{-\lambda_k \mathcal{T}_{\mathcal{U}}}}{-\lambda_k} - \frac{\langle \varphi_k, y^0 \rangle}{\langle \varphi_k, B \rangle} \right). \quad (2.4)$$

**Remark 8.** As for Remark 4, when  $y^0$  is a steady state,  $y^0 = \bar{u}^0(1, \dots, 1)^\top$ , the constraint (2.4) becomes

$$\sup_{k \in \{1, \dots, n\}} \frac{1 - e^{\lambda_k \mathcal{T}_{\mathcal{U}}} \bar{u}^0 / \bar{u}^1}{-\lambda_k} \leq \inf_{k \in \{1, \dots, n\}} \frac{e^{-\lambda_k \mathcal{T}_{\mathcal{U}}} - \bar{u}^0 / \bar{u}^1}{-\lambda_k}. \quad \blacksquare \quad (2.5)$$

The Propositions 3 and 7 are proved in Appendix A.

**Numerical simulation** In order to numerically obtain the minimal time control, we numerically solve the minimization problem (2.2), see also the discussion in Section 3 for other possible numerical approaches. In order to numerically solve this constrained optimization problem, we use the interior-point optimization routine IpOpt (see [30]) combined with the automatic differentiation and modeling language AMPL (see [13]).<sup>1</sup> We refer to [3, 28, 29] for a survey on numerical methods in optimal control and how to implement them efficiently according to the context.

<sup>1</sup> See <https://deustotech.github.io/DyCon-Blog/tutorial/wp03/P0002> for some examples of usage of IpOpt and AMPL.

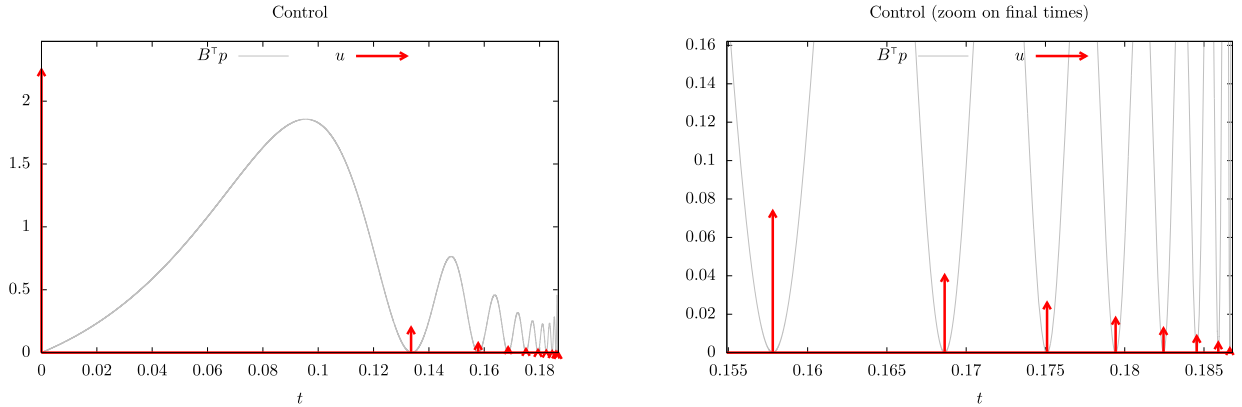


Fig. 1. Minimal time control evolution in order to steer  $y^0 \equiv 1$  to  $y^1 \equiv 5$ . The minimal time computed is  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.186799$ . Dirac impulses are represented by arrows. On this figure, we also plot  $B^T p(t)$ , with  $p(t)$  the adjoint state obtained from `IpOpt`. The corresponding state trajectory is given in Figs. 2 and 3.

In these simulations, we take  $n = 20$ , meaning that we expect that the minimal time control is the sum of at most  $N = 10$  Dirac masses.

Below, we give the numerical results obtained in for  $y^0 \equiv 1$  and  $y^1 \equiv 5$ , for  $y^0 \equiv 5$  and  $y^1 \equiv 1$ , and for  $y^0(x) = 5 \cos(11\pi x/2)/(4x + 1)$  and  $y^1 \equiv 1$ .

In order to make the numerical computation successful, we allow some additional masses, then after the optimal solution has been found, we remove Dirac masses of zero measure and sum the Dirac masses which are located at the same time instant. For the case  $y^0 \equiv 1$  and  $y^1 \equiv 5$ , we allow 35 Dirac masses, and for the other cases, we allow 25 Dirac masses. After having removed the Dirac masses of zero measure and merging the Dirac masses located at the same time instant, we obtain a number of Dirac mass which is coherent with the expected result (10 in the first case, 7 in the second case, and 8 in the last case). Note that if the number of allowed Dirac masses is too small or too large, the numerical algorithm fails to converge, and the proper number of allowed Dirac masses has to be found by hand.

**From  $y^0 \equiv 1$  to  $y^1 \equiv 5$ .** We set  $y^0 = (1, \dots, 1)^T \in \mathbb{R}^n$  and  $y^1 = (5, \dots, 5)^T \in \mathbb{R}^n$  (with  $n = 20$ ). First of all, we numerically evaluate the constraint on the minimal time given in Proposition 7, to obtain  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \geq 0.0924$ .

Computationally, we obtain  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.186799$  which is in accordance with the lower bound obtained from Proposition 7. The control and state trajectories are displayed on Figs. 1 to 3. On Fig. 1, we also plot  $B^T p(t)$ , with  $p(t)$  the adjoint state obtained from `IpOpt` and we observe, as expected from Proposition 3 that the Dirac masses are located at the times  $t$  such that  $B^T p(t) = 0$ . On Fig. 1, we observe that the minimal time control is the sum of 10 Dirac masses.

**From  $y^0 \equiv 5$  to  $y^1 \equiv 1$ .** We set  $y^0 = (5, \dots, 5)^T \in \mathbb{R}^n$  (with  $n = 20$ ) and  $y^1 = (1, \dots, 1)^T \in \mathbb{R}^n$ . First of all, we numerically evaluate the constraint on the minimal time given in Proposition 7, to obtain  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \geq 0.6613$ .

Computationally, we obtain  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.788791$  which is in accordance with the lower bound obtained from Proposition 7. The control and state trajectories are displayed on Figs. 4 to 6. As in the previous example, we also plot, on Fig. 4,  $B^T p(t)$ , with  $p(t)$  the adjoint state obtained from `IpOpt` and similarly, we observe that the minimal time control  $u$  computed by `IpOpt` is supported by the time instants where  $B^T p(t) = 0$ . On Fig. 4, we observe that the minimal time control is the sum of 7 Dirac masses.

**From  $y^0(x) = 5 \cos(11\pi x/2)/(4x + 1)$  to  $y^1 \equiv 1$ .** Let  $f(x) = 5 \cos(11\pi x/2)/(4x + 1)$ , we set  $y^0 = (f(0), f(1/n), \dots, f((n-1)/n))^T \in \mathbb{R}^n$  (with  $n = 20$ ) and  $y^1 = (1, \dots, 1)^T \in \mathbb{R}^n$ . First of all, we numerically evaluate the constraint on the minimal time given in Proposition 7, to obtain  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \geq 0.0939$ .

Computationally, we obtain  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.183000$  which is in accordance with the lower bound obtained from Proposition 7. The control and state trajectories are displayed on Figs. 7 and 8. As in the previous examples, we also plot, on Fig. 7,  $B^T p(t)$ , with  $p(t)$  the adjoint state obtained from `IpOpt` and similarly, we observe that the minimal time control  $u$  computed by `IpOpt` is supported by the time instants where  $B^T p(t) = 0$ . On Fig. 7, we observe that the minimal time control is the sum of 8 Dirac masses.



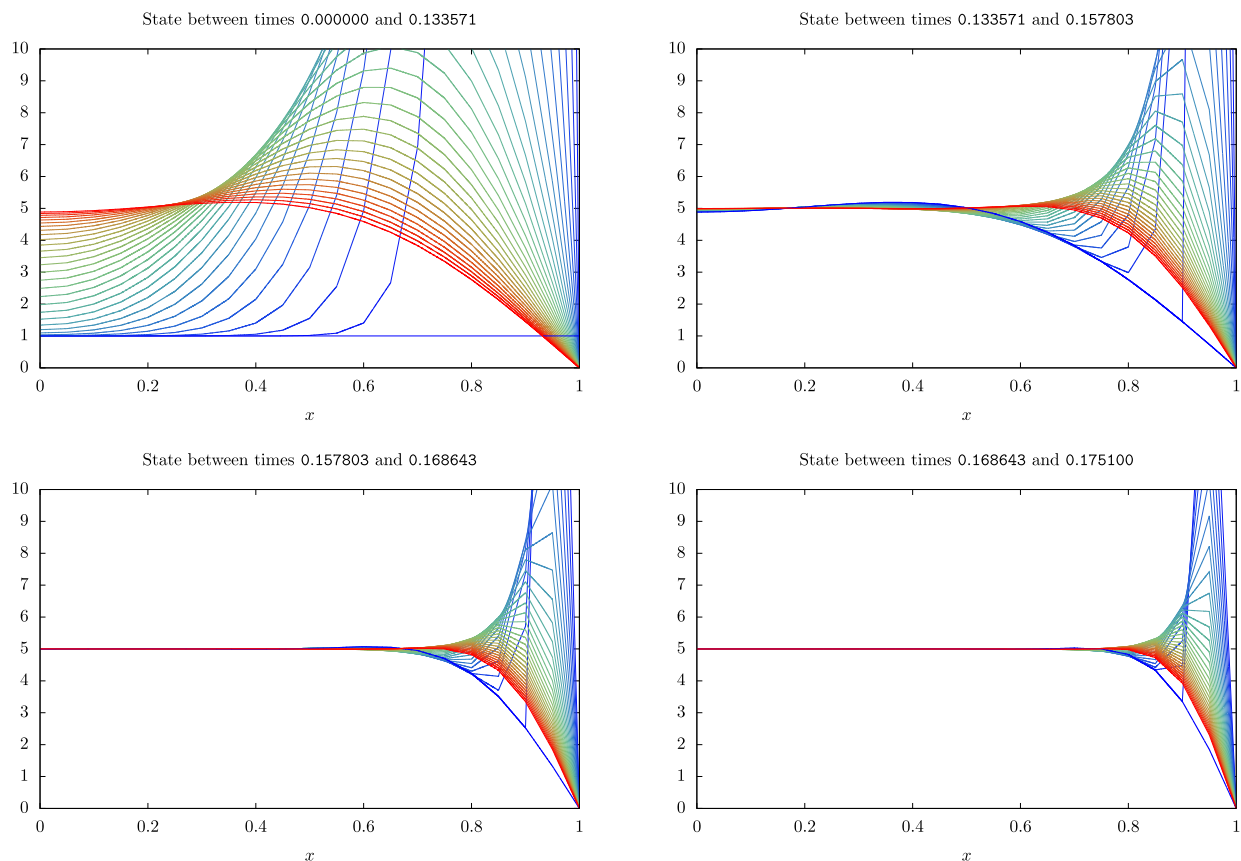


Fig. 2. Evolution of the state between two Dirac impulses. The corresponding control required to steer  $y^0 \equiv 1$  to  $y^1 \equiv 5$  is given in Fig. 1 and the minimal time computed is  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.186799$ . The color of the state goes from blue (for the initial time instant) to red (for the final time instant). (See Fig. 3 for the final times.) (For interpretation of the color(s) in the figure(s), the reader is referred to the web version of this article.)

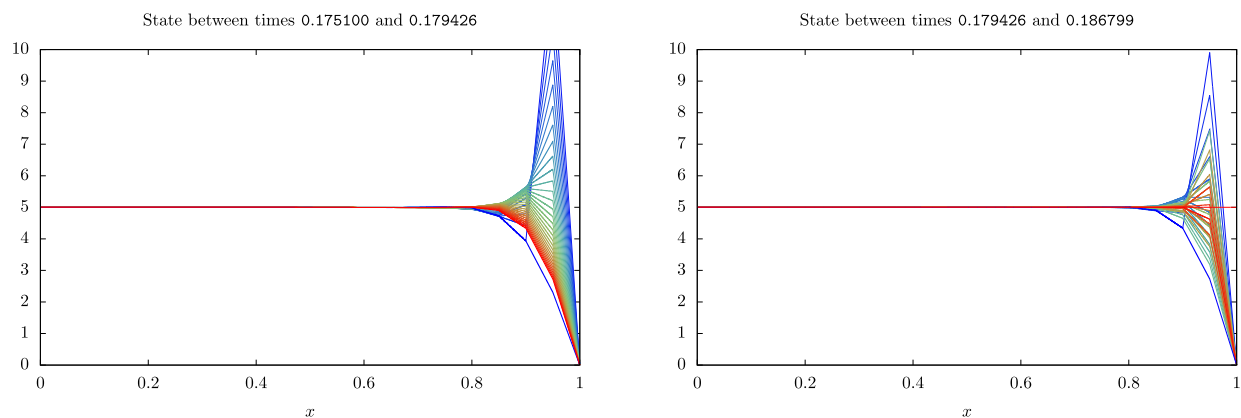


Fig. 3. Fig. 2 continued.

### 3. Numerical approximation of time optimal controls

In Section 2, we present some numerical simulations. The simulations have been performed by solving the minimization problem (2.2) or (2.3). This has been possible because, we know exactly the eigenvalues and eigenvectors of  $A^\top$ . In a general situation, the computation of eigenvalues and eigenvector is in itself a problem. Let us in addition



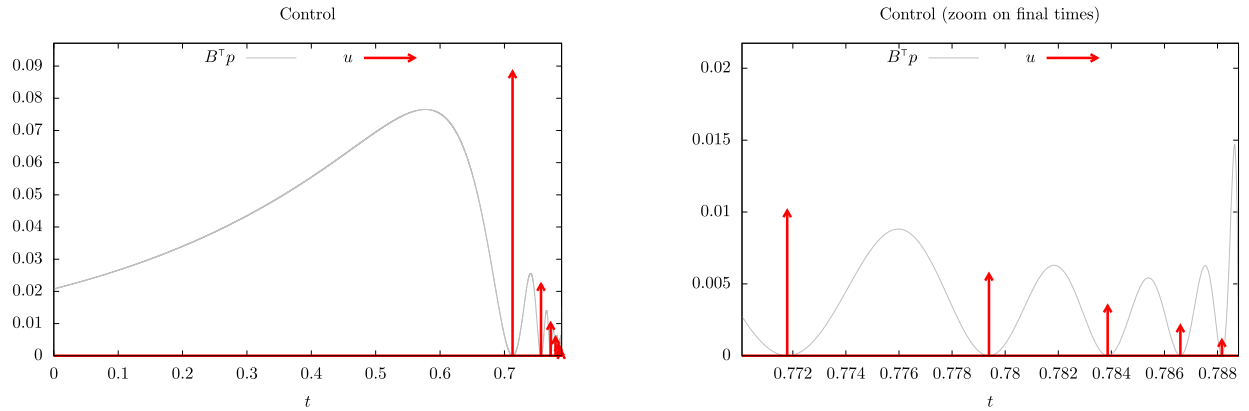


Fig. 4. Minimal time control evolution in order to steer  $y^0 \equiv 5$  to  $y^1 \equiv 1$ . The minimal time computed is  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.788791$ . Dirac impulses are represented by arrows. On this figure, we also plot a multiple of  $B^T p(t)$ , with  $p(t)$  the adjoint state obtained from `IpOpt`. The corresponding state trajectory is given in Figs. 5 and 6.

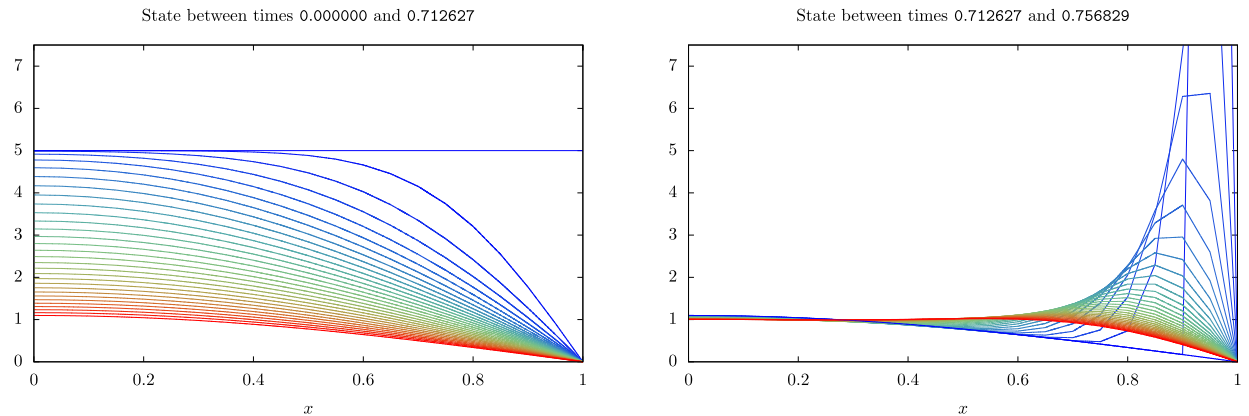


Fig. 5. Evolution of the state between two Dirac impulses. The corresponding control required to steer  $y^0 \equiv 5$  to  $y^1 \equiv 1$  is given in Fig. 4 and the minimal time computed is  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.788791$ . The color of the state goes from blue (for the initial time instant) to red (for the final time instant). (See Fig. 6 for the final times.)

mention that if the dimension of the matrix is large, solving the minimization problem (2.2) directly is hard. This is mainly due to the presence of exponentials in the constraints.

In order to overcome these facts, let us present here some other ways of numerically finding the minimal time and the time optimal control. We have tried all the other approaches proposed below. However, it seems that the method presented in Section 2 is the most efficient, in terms of computational time and result quality, for the discretized heat equation. Having a convergence proof for the numerical methods presented here is pointed in Open problem 9. Note that the construction of an efficient numerical method is also related to a better understanding of the adjoint problem, as pointed in Open problem 8.

Recall that it is possible to have  $\mathcal{T}_{\mathcal{M}} < \mathcal{T}_{\mathcal{U}}$ . We thus present in two different paragraphs the methods which are designed for obtaining the time  $\mathcal{T}_{\mathcal{U}}$  and the one designed for obtaining  $\mathcal{T}_{\mathcal{M}}$ .

### Obtaining the time $\mathcal{T}_{\mathcal{M}}$ .

**Numerical method 1 (Momentum approach).** This approach is based on the expression of optimal control problem in the basis generated by the eigenvectors of  $A$  has been explained in Section 2. Note that this approach is only applicable in the cases where  $A$  is a diagonalizable matrix.

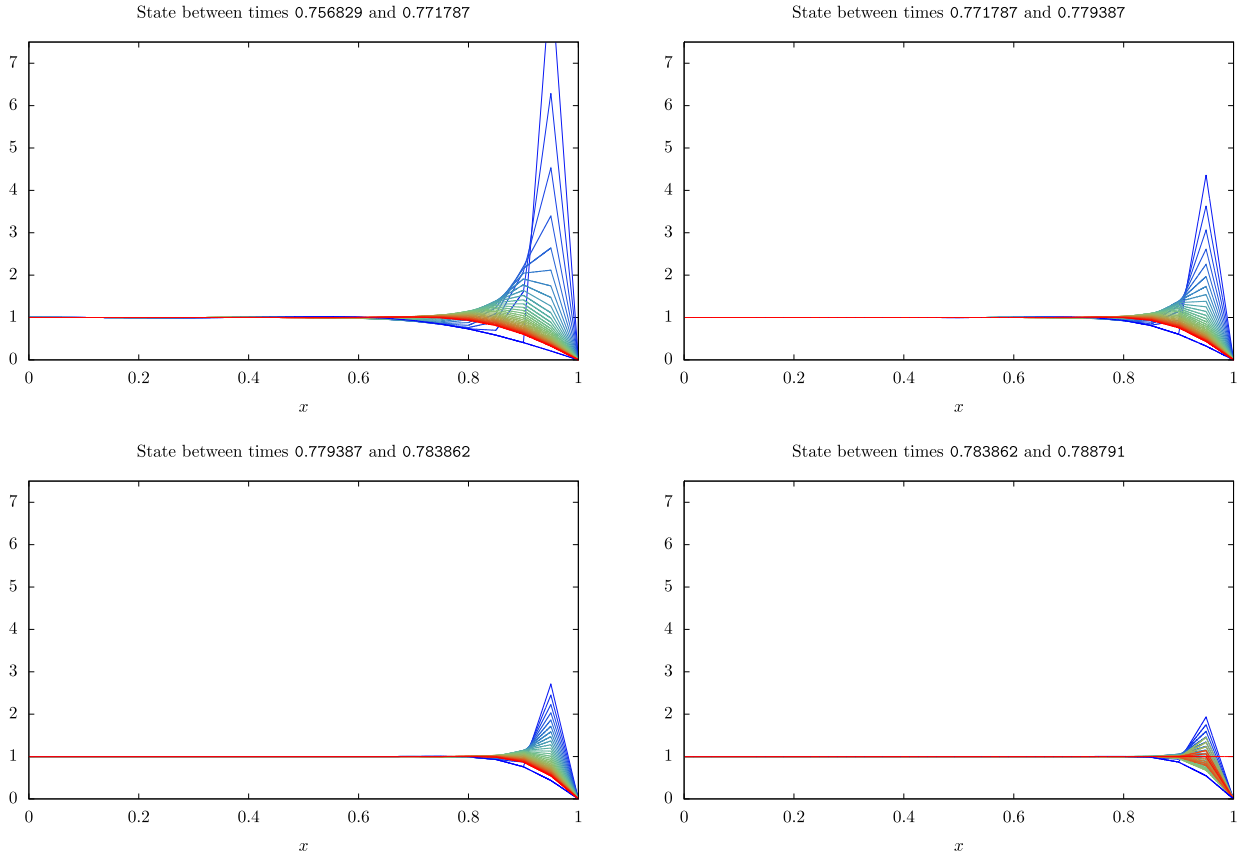


Fig. 6. Fig. 5 continued.

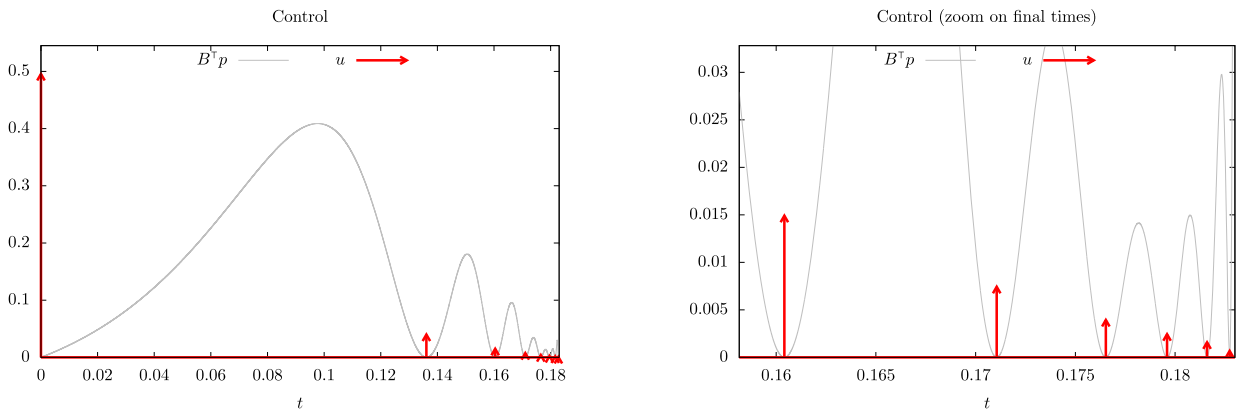


Fig. 7. Minimal time control evolution in order to steer  $y^0(x) = 5 \cos(11\pi x/2)/(4x+1)$  to  $y^1 \equiv 1$ . The minimal time computed is  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.183000$ . Dirac impulses are represented by arrows. On this figure, we also plot a multiple of  $B^\top p(t)$ , with  $p(t)$  the adjoint state obtained from IpOpt. The corresponding state trajectory is given in Fig. 8.

**Numerical method 2 (Total discretization).** This approach is used to find the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  and a control in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . Recall that if a nonnegative measure control exists in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , then it is the sum of at most  $N$  Dirac masses ( $N \leq \lfloor (n+1)/2 \rfloor$ ) when the matrix  $A$  satisfies the Assumption (H.2)). We thus pick  $N > 0$  large enough, and define  $0 = t_0 \leq t_1 \leq \dots \leq t_N \leq T = t_{N+1}$ ,  $t_1, \dots, t_N$  being the time instants where a Dirac impulse can occur. Between times  $t_k$  and  $t_{k+1}$ , the control is 0 and solution is given by  $y(t) = y_k(t - t_k)$ , with  $y_k$  solution of

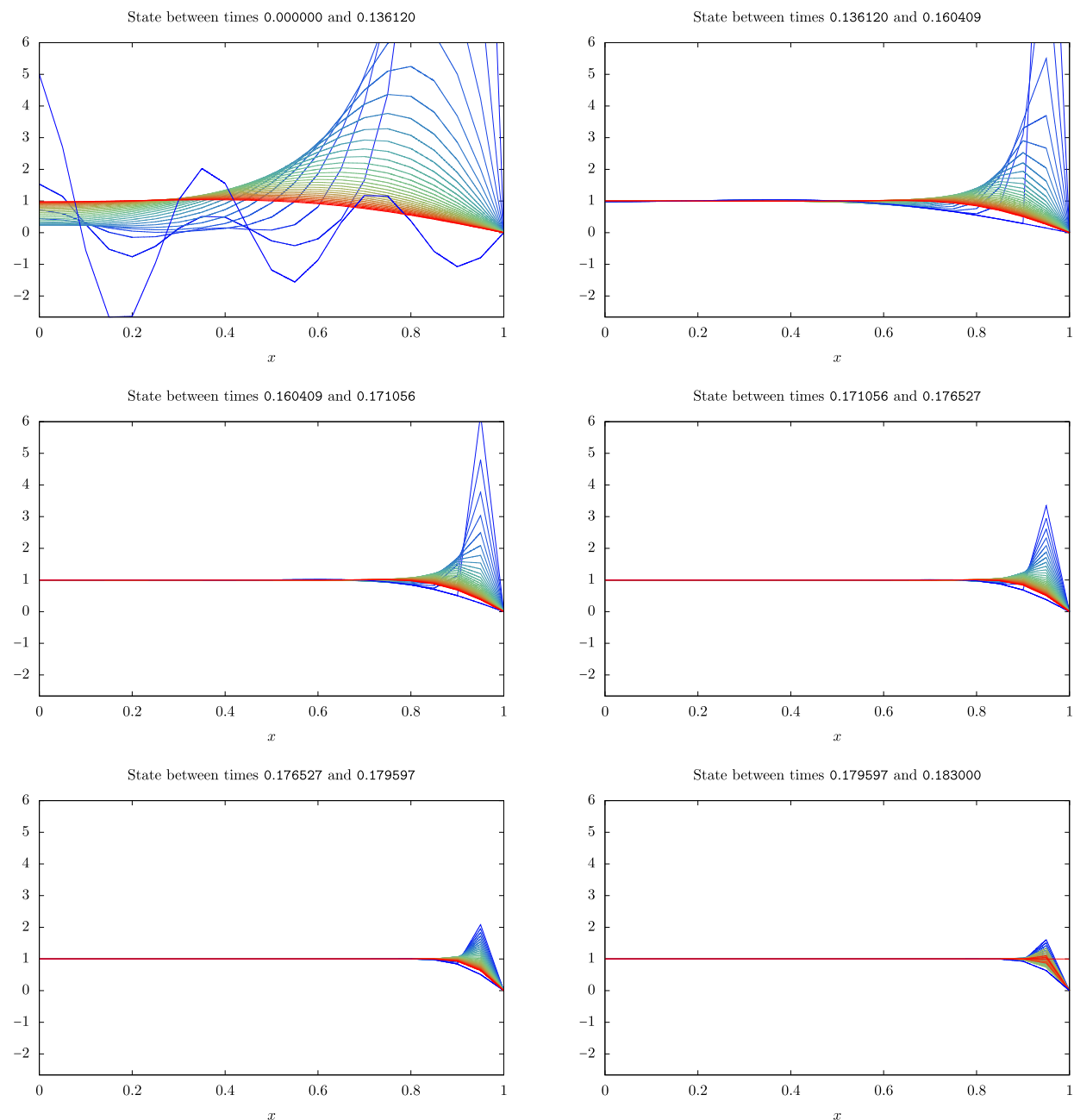


Fig. 8. Evolution of the state between two Dirac impulses. The corresponding control required to steer  $y^0(x) = 5 \cos(11\pi x/2)/(4x+1)$  to  $y^1 \equiv 1$  is given in Fig. 7 and the minimal time computed is  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \simeq 0.183000$ . The color of the state goes from blue (for the initial time instant) to red (for the final time instant).

$$\dot{y}_k(t) = Ay_k(t) \quad (t \in (0, t_{k+1} - t_k)) \quad (3.1)$$

with initial condition given below. For every  $k \in \{1, \dots, N+1\}$  we have  $y_k(0) = y_{k-1}(t_k - t_{k-1}) + \gamma_k B$  for some  $\gamma_k \geq 0$ . At initial and final times we have  $y_0(0) = y^0$  and  $y_N(T - t_N) = y^1$ . Notice that if a Dirac impulse occurs at the initial or final time, we will have  $t_1 = 0$  or  $t_N = T$  respectively.

Let us define  $\tau_k = t_{k+1} - t_k$  for every  $k \in \{0, \dots, N\}$ , we have  $T = \sum_{k=0}^N \tau_k$  and  $y_k(\tau_k) = e^{\tau_k A} y_k(0)$ . We also set  $y_k^0 = y_k(0)$ . The minimization problem is then

$$\min \sum_{k=0}^N \tau_k \quad (3.2a)$$

subject to the constraints:

$$0 \leq \tau_k \text{ and } y_k^0 \in \mathbb{R}^n \quad (k \in \{0, \dots, N\}), \quad (3.2b)$$

$$y_0^0 = y^0 \text{ and } y^1 = e^{\tau_N A} y_N^0, \quad (3.2c)$$

$$P_B \left( y_{k+1}^0 - e^{\tau_k A} y_k^0 \right) \geq 0 \quad \text{and} \quad P_B^\perp \left( y_{k+1}^0 - e^{\tau_k A} y_k^0 \right) = 0 \quad (k \in \{0, \dots, N-1\}), \quad (3.2d)$$

where  $P_B$  (respectively  $P_B^\perp$ ) is an orthogonal projector on  $\text{Ran } B$  (respectively  $(\text{Ran } B)^\perp$ ).

The constraint (3.2c) ensures that the initial condition  $y(0) = y^0$  and the final condition  $y(T) = y^1$  are satisfied, and the constraint (3.2d) ensures the existence of some  $\gamma_k \geq 0$  such that  $y_{k+1}(0) = y_k(\tau_k) + \gamma_k B$ .

In order to perform numerical simulations, one needs to compute  $e^{\tau_k A}$ . To this end, it is possible to perform a time discretization of the ordinary differential equation (3.1).

**Numerical method 3** (Using a time reparametrization). As we will see in § 5.2.1, the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  is obtained through the minimization problem:

$$\min \int_0^S w(s) \, ds \quad (3.3)$$

$$\left| \begin{array}{l} S \geq 0, \\ w(s) \in [0, 1] \quad (s \in [0, S]), \\ z(S) = y^1, \text{ with } z \text{ the solution of} \\ \quad \begin{cases} \dot{z}(s) = w(s)Az(s) + B(1 - w(s)) & (s \in [0, S]), \\ z(0) = y^0. \end{cases} \end{array} \right.$$

We then have  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \int_0^S w(s) \, ds$ .

The interest of this approach is that now, the new control  $w$  is uniformly bounded and any classical method to find the corresponding optimal control problem can be used. As explained in § 5.2.1, under this change of variables, the time instances  $s$  in which  $w(s) = 0$  corresponds to the presence of active Dirac masses while  $w(s) > 0$  corresponds to a bounded  $L^\infty$  control, the limit being  $w(s) = 1$ , corresponding to the absence of control action.

As far as we know, this numerical method is the only one proposed in this article that can be adapted to nonlinear control problems. We refer to [10,22,5,16] for the adaptation of the time reparametrization for nonlinear control systems.

This method can also be adapted to find the minimal time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ , see Numerical method 6.

**Numerical method 4** (Approximation by a sequence of nonnegative controls of minimal  $L^1$  norm). For additional details about this method, we refer to Appendix B.

Assume that  $y^1 \in S_+^*$ , and that  $0 < \mathcal{T}_{\mathcal{M}}(y^0, y^1) < \infty$ , then for every time  $T > \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  (recall that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  when  $y^1 \in S_+^*$ ), there exists a control  $u \in \mathcal{M}_+(T)$  steering  $y^0$  to  $y^1$  in time  $T$ . In particular, there exists a control of minimal measure. Note also that for every nonnegative time  $T < \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , there does not exist a control in  $\mathcal{M}_+(T)$  steering  $y^0$  to  $y^1$  in time  $T$ . The idea is then to find the minimal time  $T$  such that the optimal control problem

$$\inf \left| \begin{array}{l} \|u\|_{\mathcal{M}([0,T])} \\ u \in \mathcal{M}_+(T), \\ y^1 - e^{TA} y^0 = \Phi_T u \end{array} \right. \quad (3.4)$$

admits a solution.

The dual problem of (3.4) is:

$$\begin{aligned} \inf \quad & \langle e^{TA} y^0 - y^1, p^1 \rangle \\ \mid & p^1 \in \mathbb{R}^n, \\ \mid & B^\top e^{(T-t)A^\top} p^1 \leq 1 \quad (t \in [0, T]). \end{aligned} \quad (3.5)$$

Using weak duality results, one can show that if the infimum of the minimization problem given by (3.5) is  $-\infty$ , then, there does not exist a control in  $\mathcal{M}_+(T)$  steering  $y^0$  to  $y^1$  in time  $T$  (i.e.,  $T < \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ ). By strong duality result, one can also show that for  $T > \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , the minimization problem (3.5) admits a minimum. Reciprocally, using first order optimality conditions, we can prove that if the minimization problem (3.5) admits a minimum, then there exists a control  $u \in \mathcal{M}_+(T)$  steering  $y^0$  to  $y^1$  in time  $T$  and this control is the sum of a finite number of Dirac masses.

It is possible to use these facts to build an algorithm in order to find an approximation of the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  (see Algorithm 1). This algorithm, is based on a dichotomy approach, testing whether the dual problem (3.5) admits a minimizer or not.

---

**Algorithm 1** Approximation of  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ .

---

**Require:**  $\varepsilon > 0$   
**Require:**  $y^0 \in \mathbb{R}^n$ ,  $y^1 \in \mathcal{S}_+^*$  and  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < \infty$   
**Ensure:**  $0 \leq T - \mathcal{T}_{\mathcal{M}}(y^0, y^1) < \varepsilon$   
 {Test if  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = 0$ :}  
 $T_0 \leftarrow 0$   
**if** (3.5) (with  $T = 0$ ) admits a minimizer **then**  
   **return**  $T = 0$   
**else**  
    $T_0 \leftarrow 0$   
   {Find  $T_1 > 0$  such that (3.5) (with  $T = T_1$ ) admits a minimizer:}  
    $T_1 \leftarrow 1$   
   **while** (3.5) (with  $T = T_1$ ) does not admit a minimizer **do**  
      $T_1 \leftarrow T_1 + 1$   
   **end while**  
**end if**  
 {We now have  $T_0 \leq \mathcal{T}_{\mathcal{M}}(y^0, y^1) \leq T_1$ .}  
 {Do the dichotomy procedure:}  
**while**  $T_1 - T_0 \geq \varepsilon$  **do**  
   **if** (3.5) (with  $T = (T_0 + T_1)/2$ ) admits a minimizer **then**  
      $T_1 \leftarrow (T_0 + T_1)/2$   
   **else**  
      $T_0 \leftarrow (T_0 + T_1)/2$   
   **end if**  
**end while**  
**return**  $T = T_1$

---

**Remark 9.** Note that the minimization problem (3.5) is a linear programming problem. To test whether the minimum is achieved or not, one can use the simplex algorithm, see for instance [11]. Furthermore, the linear inequality  $B^\top e^{(T-t)A^\top} p^1 \leq 1$  for every  $t \in [0, T]$ , in (3.5), is numerically tested as  $B^\top e^{(T-t_i)A^\top} p^1 \leq 1$  for every  $i \in \{0, \dots, n_T\}$ , with  $t_i = iT/n_T$  and with  $n_T \in \mathbb{N}^*$  large.

Note also that it is possible to use the numerical strategy proposed in [18,19] in order to find the control of minimal measure. This strategy would avoid the usage of the simplex method. More precisely, it might be possible to adapt the work done in [18,19] to find a nonnegative control of minimal  $L^1$ -norm such that  $y(T)$  (the controlled state at the final time) is at distance  $\varepsilon$  from the target  $y^1$ . The algorithm proposed in [18,19], is based on a greedy algorithm, and might be efficient when  $T > \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . However, there is still some work to do so that for  $T < \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , the algorithm answers that no minimizer exists. ■

**Remark 10.** We have assumed here that  $y^1 \in \mathcal{S}_+^*$ . But, probably, it is sufficient to assume that for every time  $T > \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , there exists a control  $u \in \mathcal{M}_+(T)$  steering  $y^0$  to  $y^1$  in time  $T$ . However, without the assumption  $y^1 \in \mathcal{S}_+^*$ , we are currently unable to prove the strong duality result (see Remark B.3 for more details). ■

**Remark 11.** On one hand, if we are able to pass to the limit  $T \rightarrow \mathcal{T}_{\mathcal{M}}$ , we would obtain the existence of an adjoint state  $p$  such that  $B^\top p \leq 1$  and such that the Dirac masses are located in the set of times  $t$  such that  $B^\top p(t) = 1$ . On the other hand, Corollary 5.2.2 ensures the existence of an adjoint  $p$  such that  $B^\top p \geq 0$  and such that the Dirac masses are located in the set of time  $t$  such that  $B^\top p(t) = 0$ .

This means that at the minimal time, there would exist two different adjoints states leading to a control at the minimal time  $\mathcal{T}_{\mathcal{M}}$ . When the matrix  $A$  satisfies the assumption (H.2), the minimal time control is unique, thus these two adjoints states shall lead to the same control. The relation between these two adjoints is not understood and postponed in Open problem 8. ■

### Obtaining the time $\mathcal{T}_{\mathcal{U}}$ .

**Numerical method 5** (*Approximation by bang-bang controls*). Note that this approach, based on the approximation result given in Proposition 5.3.1, has already been used in [20]. Let us also point out that the minimization problem (2.3) is adapted for finding the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  and in the numerical simulations given in Section 2, we use the fact that  $y^1 \in \mathcal{S}_+^*$ , and hence  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . However, as pointed out in § 5.1.3, there might exist  $y^0$  and  $y^1$  such that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . In this situation, a way to find  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  is to solve, for  $M > 0$ , the minimization problem:

$$\min \quad \begin{cases} T \\ T > 0, \\ 0 \leq u(t) \leq M & (t \in (0, T) \text{ a.e.}), \\ y(0) = y^0 \text{ and } y(T) = y^1, \text{ with } y \text{ solution of (1.1)} \end{cases} \quad (3.6)$$

and let  $M$  go to  $\infty$ . We refer to Section 5.3 for more results of the convergence of the minimizer of this minimization problem as  $M \rightarrow \infty$ . The interest of this approach is that now, the control  $u$  is uniformly bounded and any classical method to find the corresponding optimal control problem can be used.

**Numerical method 6** (*Using a time reparametrization - second version*). The idea used in Numerical method 3 can also be used to design a numerical method aiming to find the minimal time  $\mathcal{T}_{\mathcal{U}}$ . In fact as explained in Numerical method 3,  $w = 0$  correspond to Dirac masses. In order to avoid Dirac masses, we fix  $\varepsilon \in (0, 1)$  and solve the minimization problem:

$$\min \quad \begin{cases} \int_0^S w(s) ds \\ S \geq 0, \\ w(s) \in [\varepsilon, 1] & (s \in [0, S]), \\ z(S) = y^1, \text{ with } z \text{ the solution of} \\ \quad \begin{cases} \dot{z}(s) = w(s)Az(s) + B(1 - w(s)) & (s \in [0, S]), \\ z(0) = y^0. \end{cases} \end{cases} \quad (3.7)$$

In this problem, the constraint  $w(s) \geq \varepsilon$  avoids the presence of Dirac masses and as  $\varepsilon \rightarrow 0$ , we will recover the minimal time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . In fact, the constraint  $w(s) \geq \varepsilon$  ensures that in the original time scale, the control  $u$  is uniformly bounded by  $(1 - \varepsilon)/\varepsilon$ .

### Numerical comparison between the different approaches.

In order to compare the different numerical approaches proposed in the previous paragraphs, we consider the system (1.1) with matrices  $A$  and  $B$  given by (1.4), with  $n = 5$ . For this system, we consider the case  $y^0 \equiv 1$  and  $y^1 \equiv 5$ , and the case  $y^0 \equiv 5$  and  $y^1 \equiv 1$ . Note that we expect to have no more than  $N = \lfloor (n + 1)/2 \rfloor = 3$  Dirac

masses involved in the minimal time control. Since  $y^1 \in \mathcal{S}_+^*$ , we have  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  and the control at the minimal time is unique. Hence, all the numerical methods proposed shall, up to numerical errors, give the same time and optimal control.

Note that for Numerical methods 3, 5 and 6, we end up with an optimal control problem written in its classical form. To computationally solve these problems, we used the *total discretization method* introduced in [28, Part 2, § 9.II.1], combined with the Crank-Nicolson method. The number of time discretization points is precised below.

- For Numerical method 1, we allow  $N = 3$  Dirac masses.
- For Numerical method 2, in order to compute  $e^{\tau_k A} y_k^0$ , appearing in (3.2d), we solve (3.1), with initial condition  $y_k^0$ , using the Crank-Nicolson method with  $n_t = 100$  discretization points.  
Note that this means that on the full time interval, we have 400 discretization points in time. Since we observe that a Dirac mass is located close to the final time, this in fact means that we have, in fact, no more than 300 effective discretization points in time (the last  $\tau_k$  in (3.2) is almost 0). This explains why we use 300 discretization points in time for Numerical methods 4 and 5.
- For Numerical method 3, in the case  $y^0 \equiv 1$  and  $y^1 \equiv 5$  (respectively  $y^0 \equiv 5$  and  $y^1 \equiv 1$ ), we use  $n_t = 900$  (respectively  $n_t = 500$ ) discretization points in time in the Crank-Nicolson method.  
This difference between the number of discretization points, is due to the fact that the system is discretized over  $[0, S]$ , where  $S$  is in fact the sum of the minimal time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  and measure of the optimal control. When  $y^0 \equiv 1$  and  $y^1 \equiv 5$  (respectively  $y^0 \equiv 5$  and  $y^1 \equiv 1$ ), after computation, we obtain  $S \simeq 3.321291$  (respectively  $S \simeq 1.098819$ ).
- For Numerical method 4, we rewrite the constraint  $B^\top e^{(T-t)A} p^1 \leq 1$  (for every  $t \in [0, T]$ ), appearing in (3.5), as  $\langle e^{tA} B, p^1 \rangle \leq 1$  (for every  $t \in [0, T]$ ) and we compute the values of  $e^{tA} B$  using Crank-Nicolson method with  $n_t = 300$  discretization points. Furthermore, the parameter  $\varepsilon$  appearing in Algorithm 1 is fixed to  $10^{-4}$ , and the solution of the linear optimization problem is computed with a simplex algorithm.
- For Numerical method 5, we use  $n_t = 300$  discretization points in the Crank-Nicolson method. Furthermore, the value of  $M$  is fixed to 10 for the case  $y^0 \equiv 1$  and  $y^1 \equiv 5$ , and to 30 for the case  $y^0 \equiv 5$  and  $y^1 \equiv 1$ . We use  $M = 10$  (for the first case) and  $M = 30$  (for the second case) simply for graphical reasons.
- For Numerical method 6, we use here the same number of discretization points as for Numerical method 3 in the Crank-Nicolson method. As for Numerical method 5, we use  $\varepsilon = 1/10$  (respectively  $\varepsilon = 1/30$ ) in the case  $y^0 \equiv 1$  and  $y^1 \equiv 5$  (respectively  $y^0 \equiv 5$  and  $y^1 \equiv 1$ ).

In practice, we use IpOpt and AMPL to numerically solve the optimal problems given in Numerical methods 1 to 3, 5 and 6, and to solve the linear programming problem appearing in Numerical method 4, we use the `linpro` routine (see [6]) of Scilab.

Corresponding results are plotted on Figs. 9 to 14 and the computed minimal times are gathered in Table 2. We also plot on these figures the corresponding adjoint states. We can then see, as expected, that for Numerical methods 1 to 4, there exists an adjoint state  $p$  such that the optimal control is active when  $B^\top p \leq 0$  and null for all the other times. For Numerical method 4, we can also see that there exists an adjoint state  $p$  such that the optimal control is active when  $B^\top p = 1$ . Understanding the relation between these two adjoint states is the goal of Open problem 8 below.

We observe on Table 2 that the times obtained for Numerical methods 5 and 6 are similar and greater than the times obtained for the other numerical methods. This fact was expected, since in this case, we are looking for a bounded control in  $L^\infty$ , hence, the time obtained has to be greater than  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  (the time which shall be obtained with Numerical methods 1 to 4). We also observe that the times obtained for Numerical methods 1 and 4 are similar and lower than the times obtained for Numerical methods 2 and 3. We do not know how to explain the gap between these two times. This can be due to the fact that we are solving a nonlinear minimization problem and that for Numerical methods 2 and 3, we only find a local minimum. Note also that the convergence of the optimization algorithm is really dependent on the initialization point. To compute the above results, we progressively increase the parameters  $n$  and  $n_t$  up to their desired values and between two increments of the parameters, we initialized the optimization algorithm with the previously computed result.



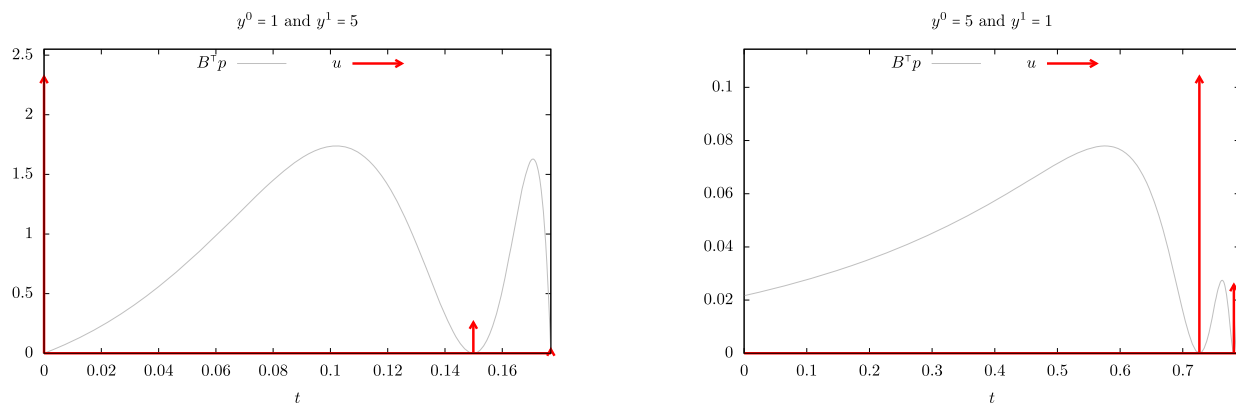


Fig. 9. Results for Numerical method 1. See Table 2 for the corresponding minimal times.

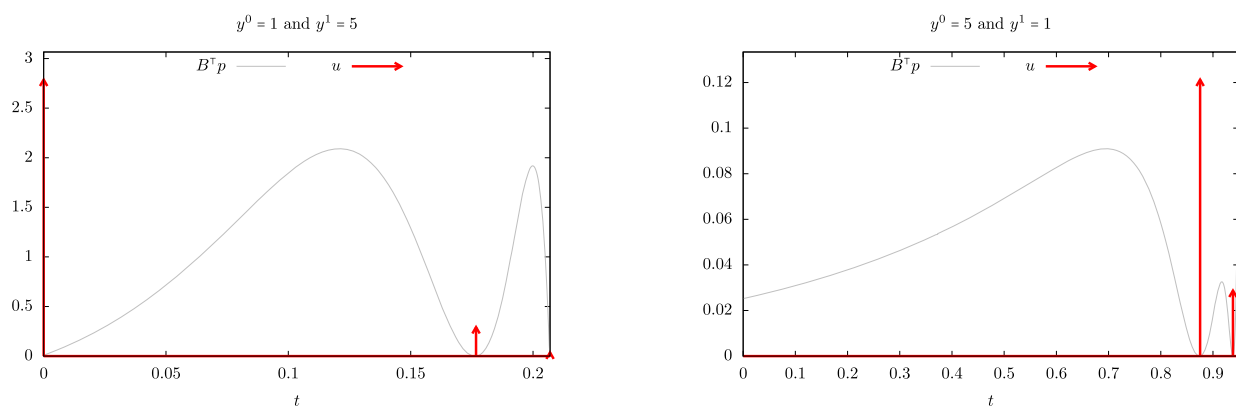
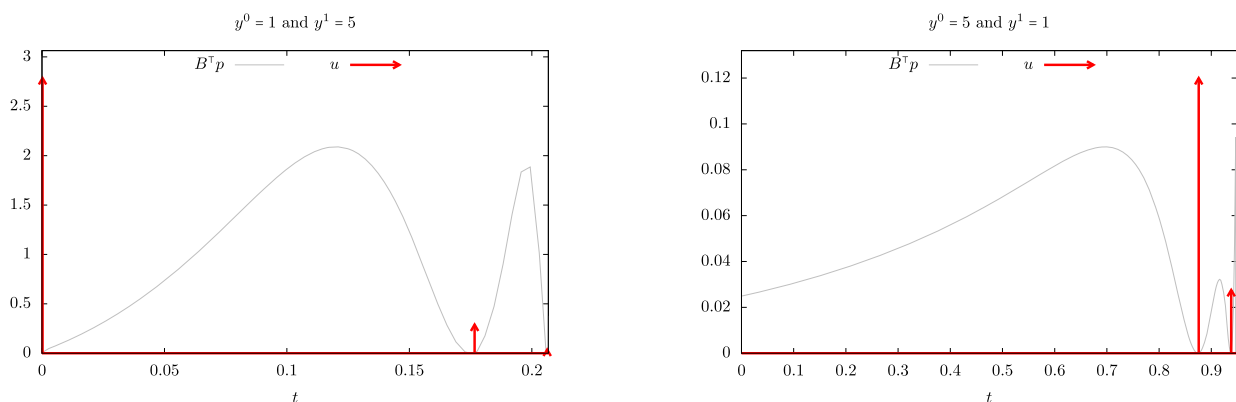


Fig. 10. Results for Numerical method 2. See Table 2 for the corresponding minimal times.

Fig. 11. Results for Numerical method 3. See Table 2 for the corresponding minimal times. Recall that here, the dynamical system is rescaled in time, and the new control  $w$  belongs to  $[0, 1]$ . The figures here are displayed in the original time and a posttreatment of the result has been done to observe Dirac masses.

#### 4. Further comments and open questions

In this paper, we show that controlling with nonnegative controls a finite-dimensional linear autonomous control system  $\dot{y} = Ay + Bu$  satisfying the Kalman condition requires a positive minimal time as soon as the difference

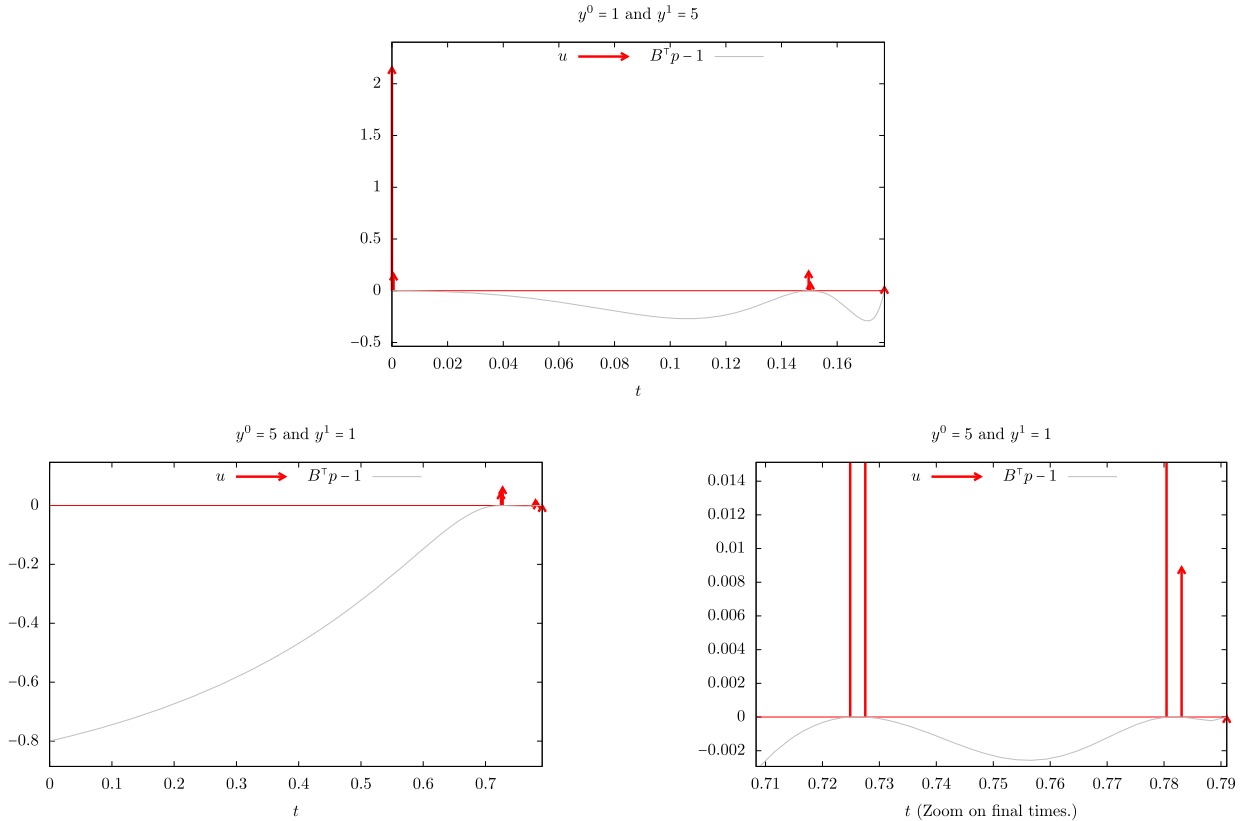


Fig. 12. Results for Numerical method 4. Note that here, instead of plotting  $B^T p$ , we plot  $B^T p - 1$ . See Table 2 for the corresponding minimal times.

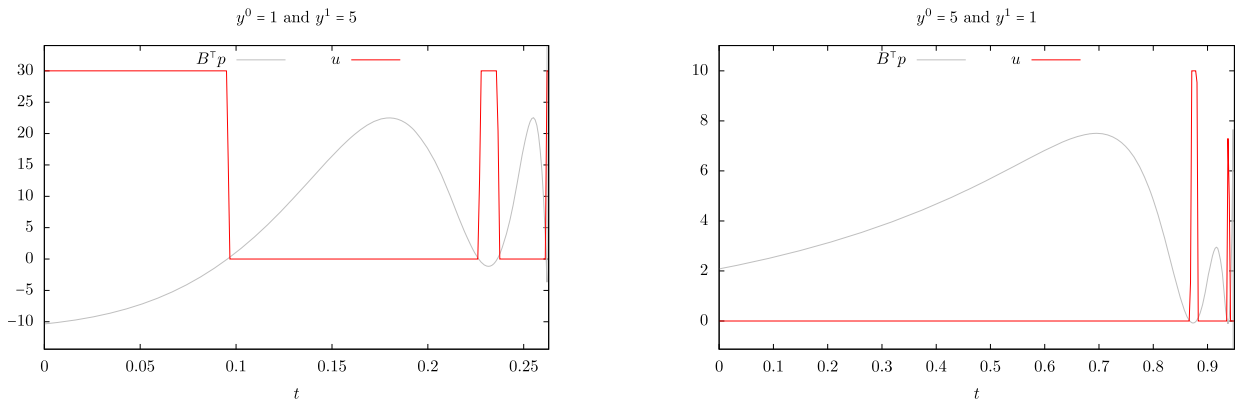


Fig. 13. Results for Numerical method 5. See Table 2 for the corresponding minimal times. Note that in both cases, we expect that the control takes its values in  $\{0, M\}$  for almost every time. But this fact is not observed on the right figure ( $M = 10$  in the case  $y^0 = 5$  and  $y^1 = 1$ ). However, by increasing the number of time discretization points, we will recover this saturation property.

between the initial state and target state does not belong to  $\text{Ran } B$ . When  $A$  admits at least one real eigenvalue (and when the target is reachable with nonnegative Radon measure control, i.e.,  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < \infty$ ), there exists a minimal time nonnegative control in the space of Radon measures and this control is a linear combination with nonnegative coefficients of a finite number of Dirac impulses. Without this spectral assumption the conclusion may fail. In addition,

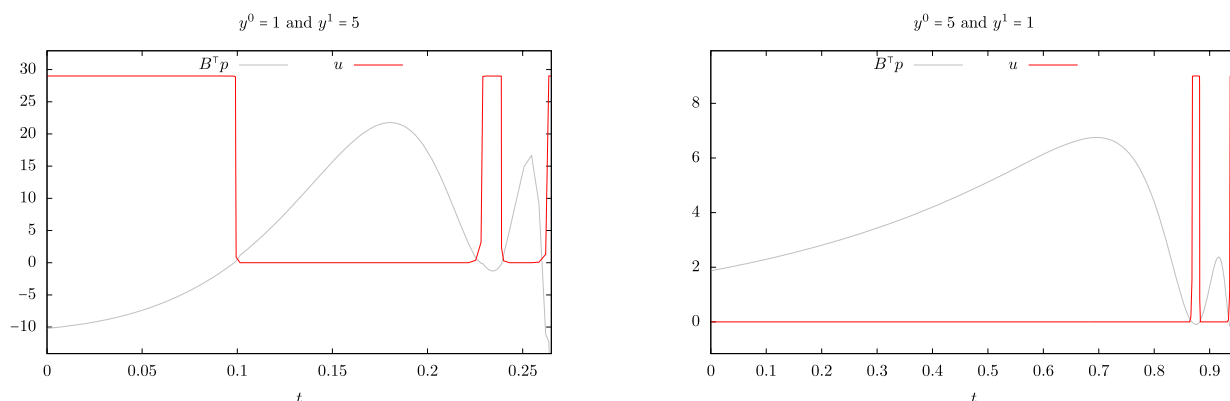


Fig. 14. Results for Numerical method 6. See Table 2 for the corresponding minimal times. Recall that in this case, the control is bounded by  $(1/\varepsilon - 1)/\varepsilon$ , and we had chosen  $\varepsilon = 1/30$  (respectively  $\varepsilon = 1/10$ ) for  $y^0 = 1$  and  $y^1 = 5$  (respectively  $y^0 = 5$  and  $y^1 = 1$ ). Recall also that as for Fig. 11, the dynamical system is rescaled in time, and the new control  $w$  belongs to  $[\varepsilon, 1]$ . The figures here are displayed in the original time scale.

Table 2

Minimal time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  computed with Numerical methods 1 to 6 for the case  $y^0 \equiv 1$  and  $y^1 \equiv 5$  and  $y^0 \equiv 5$  and  $y^1 \equiv 1$ . More details on the parameters used for these numerical methods are given in the last paragraph of Section 3.

Numerical method	1	2	3	4	5	6
$y^0 \equiv 1$ and $y^1 \equiv 5$	0.185024	0.201834	0.206753	0.176994	0.263030	0.265068
$y^0 \equiv 5$ and $y^1 \equiv 1$	0.793984	0.873111	0.948933	0.791016	0.949745	0.949479

when all eigenvalues of  $A$  are real, the number of Dirac masses involved in the time optimal control is no more than half of the space dimension and the time optimal control is unique.

Let us mention several open questions and propose some other numerical strategies aiming to find the minimal controllability time.

**Open problem 1** (*Nonnegative vectorial controls*). In this paper, we only study the case of a nonnegative scalar control. The same question with a control  $u \in \mathbf{R}_+^m$  (with  $m \geq 2$ ) is not presented here. However, it shall be easy to adapt the proof given for the scalar case to the  $m$ -dimensional control case. This extension is, in particular, relevant for discretized versions of higher dimensional heat equation or 1D heat equation with controls at both ends. Note that when controlling the discretized version of a 1D heat equation with Dirichlet controls at both ends, we can use the symmetry properties already discussed in [20] to come back (when the initial state and target states are both symmetric) to the problem present work.

**Open problem 2** (*Do we have  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$* ). We have shown that if the target state is a steady-state then the minimal controllability time for nonnegative Radon measure controls coincides with the one for nonnegative  $L^\infty$ -controls. In general a gap may occur, as exemplified in Remark 5.1.9. But, there is no clear picture for the existence of a gap, and we refer to [22] for further results in this direction.

**Open problem 3** (*Existence of a minimal time control*). When no eigenvalue of  $A$  is real, we are not able to show that a minimal time control in the space of nonnegative Radon measure exists. As shown in Remark 5.1.8, the answer to this question might be negative in some situations. The difficulty encountered in order to solve this problem is due to the lack of uniform bound on the norm of the controls in times greater than the minimal time.

**Open problem 4** (*Number of Dirac impulses in the minimal time control*). We show in Corollary 5.2.2 that if a minimal time control exists, then this control is a linear combination with nonnegative coefficients of a finite number of Dirac impulses. But we do not have an estimate on the number of impulses. We also show in Proposition 5.2.5 that under the stronger assumption (H.2) (all eigenvalues of  $A$  are real), we have an explicit bound on the number of Dirac impulses.

Is it possible to obtain an upper bound on the number of Dirac masses involved in the time optimal control without the assumption (H.2)?

**Open problem 5** (*Uniqueness of the minimal time control*). The uniqueness of the minimal time control (if it exists) remains open when at least one of the eigenvalues of  $A$  is not real. If we aim to follow the proof of Proposition 5.2.5, we need to show that any minimal time control consists of at most  $\lfloor (n+1)/2 \rfloor$  Dirac impulses. Consequently, this question might also be related to the Open problem 4.

**Open problem 6** (*Localization of the Dirac masses*). We only know under the assumption (H.1) that a finite number of Dirac masses are involved in any time optimal control. Finding their location passes through an optimization algorithm, and we do not know a priori repartition of the Dirac masses. In particular, for the discretized heat equation, it seems that the localization of the Dirac masses and their amplitude is rather organized (see Figs. 1, 4 and 7).

**Open problem 7** (*Convergence speed of the minimal controllability time with bounded  $L^\infty$ -controls to the one with unbounded controls*). We have proved that the minimal time for the minimal time control problem under the additional control constraint  $0 \leq u(t) \leq M$  converges to the minimal time as  $M \rightarrow +\infty$ . Obtaining convergence rates is an interesting problem and the answer to this question would be helpful for numerical simulations. Some key arguments may be found in [25–27].

**Open problem 8** (*Obtaining the minimal time control through the adjoint*). In classical optimal control problems, the optimal control is given in function of the adjoint. This means that there exists a function  $f$  such that  $u(t) = f(B^\top p(t))$ , where  $p$  is solution of  $\dot{p} = -A^\top p$  with the terminal condition  $p(T) = p^1 \in \mathbb{R}^n$ . At this level, this structure is badly understood.

In Appendix B, we have tried, without success, to obtain the optimal control by computing the control of  $L^1$ -minimal norm in time  $T > \mathcal{T}_M$  and passing to the limit as  $T \rightarrow \mathcal{T}_M$ . For  $T > \mathcal{T}_M$ , the control of minimal  $L^1$ -norm is a sum of Dirac impulses and these Dirac impulses are located on a level set of the adjoint observations (more precisely on the set of time  $t \in [0, T]$  such that  $B^\top p(t) = 1$ ). However, we have shown that the terminal condition  $p^1$  minimizes a linear cost and is subject to a unilateral linear constraint. This fact is not enough to obtain compactness on  $p^1$  and does not allow us to pass to the limit as  $T \rightarrow \mathcal{T}_M$ . In addition, once the control is characterized by the adjoint state, an optimal control is given by an adjoint minimizing some functional. The existence of such minimum is usually related to an observability inequality. In Appendix B, for  $T > \mathcal{T}_M$ , we do not know how to interpret the existence of a minimum of (B.3) for the dual problem in terms of an observability inequality.

Finally, in the proof of Proposition 5.2.5, we use the adjoint state related to the minimization problem (5.6). Similarly, this leads to an adjoint state  $p(t)$  such that  $B^\top p(t)$  is of constant sign and such that the control is active only at the time instants such that  $B^\top p(t) = 0$ .

The understanding of the relations between these two adjoint states remains open.

**Open problem 9** (*Numerical approximation of the time optimal control*). Nothing ensures the convergence of the numerical method proposed in Sections 2 and 3, except for the Numerical method 4, which is based on Algorithm 1 (see also Appendix B), where we only consider the controllability to a positive steady state target.

This lack of convergence proof is mainly due to the fact that we are solving a nonlinear control problem. Having some efficient and general numerical method ensuring that the computed control is at some distance  $\varepsilon$  from the real control is as far as we know an open problem. This question is also related to the previous one (*Obtaining the minimal time control through the adjoint*), since it is usually more efficient to minimize a cost function related to the adjoint variable, than looking directly for an optimal control. Note also that the main question is the time location of the Dirac masses. In fact, once these locations are found, the amplitude of the Dirac masses is obtained by solving a linear system.

**Open problem 10** (*Limit as  $n \rightarrow +\infty$  of the discretized heat equation*). One of the issues of this paper concerns the study of the controllability of the discretized heat equation with nonnegative Dirichlet controls. An open issue is the convergence of the obtained results as  $n \rightarrow +\infty$ . In particular, we would expect that for the heat system described by (1.3) the minimal time nonnegative control is a linear combination with nonnegative coefficients of a countable

number of Dirac impulses. If this is true, we would also aim to know how these Dirac impulses are distributed over the time interval. The answer to this question may require a better understanding of the adjoint system.

## 5. Proof of the main results

### 5.1. Preliminaries

#### 5.1.1. Accessibility conditions

In this paragraph, we do not aim to give an exhaustive description of the accessible points from some  $y^0$ . For this question, we refer to [14, 15]. Here we only recall that a steady-state connectedness assumption ensures controllability in large enough time.

Since  $B$  is a vector of  $\mathbf{R}^n$  and since the pair  $(A, B)$  satisfies the Kalman rank condition, the set of steady-states is a subspace of  $\mathbf{R}^n$  of dimension one. Let us point out that  $S_+^*$  is either a half-line or the empty set. In fact, it is easy to see that  $S_+^*$  is empty when  $B \notin \text{Ran } A$  and a half-line when  $B \in \text{Ran } A$ .

The next result can be obtained with the quasi-static strategy and easily follows from small-time local controllability combined with a compactness argument. We refer to [8, 9, 23] for more details.

**Proposition 5.1.1.** *Assume  $S_+^* \neq \emptyset$ . Let  $\bar{y}^0, \bar{y}^1 \in S_+^*$ , let  $\bar{u}^0, \bar{u}^1 \in \mathbf{R}_+^*$  their associated steady-state controls (i.e.,  $A\bar{y}^i + B\bar{u}^i = 0$  for  $i \in \{0, 1\}$ ) and let  $\mu = \min(\bar{u}^0, \bar{u}^1) > 0$ . Then there exist  $\rho = \rho(\mu) > 0$  and a positive time  $T \leq (3 + |\bar{y}^1 - \bar{y}^0|/2\rho)\mu$  such that for every  $y^0 \in B(\bar{y}^0, \rho)$  and every  $y^1 \in B(\bar{y}^1, \rho)$ , there exists a control  $u \in \mathcal{U}_+(T)$  such that the solution of (1.1) with initial condition  $y^0$  and control  $u$  reaches  $y^1$  at time  $T$ .*

**Proof.** Let us recall that a linear control system is small-time locally controllable around any steady-state  $(\bar{y}, \bar{u}) \in \mathbf{R}^n \times \mathbf{R}$ , i.e. (see [7, Definition 3.2 p. 125]), for every  $\varepsilon > 0$ , there exists  $\rho(\varepsilon) > 0$  such that for every  $y^0$  and every  $y^1$  in  $B(\bar{y}, \rho(\varepsilon))$ , there exists a measurable function  $u : [0, \varepsilon] \rightarrow \mathbf{R}$  such that  $|u(t) - \bar{u}| \leq \varepsilon$  for every  $t \in [0, \varepsilon]$  and the solution of (1.1) starting from  $y^0$  reaches  $y^1$  at time  $\varepsilon$ . Note that for linear control systems,  $\rho$  can be chosen independent of  $\bar{y}$ .

In particular, choosing  $\varepsilon = \mu$  (and  $\rho = \rho(\mu)$ ), for every steady-state  $(\bar{y}, \bar{u})$ , with  $\bar{u} > \mu$ , and for every  $y^0$  and  $y^1$  in  $B(\bar{y}, \rho)$ , there exists a control  $u \in \mathcal{U}_+(\mu)$  such that the solution of (1.1) starting from  $y^0$  reaches  $y^1$  at time  $\mu$ .

To prove the statement of Proposition 5.1.1, we consider the sequence of points

$$\tilde{y}^0 = y^0, \tilde{y}^1 = \bar{y}^0 + (\bar{y}^1 - \bar{y}^0)\alpha, \dots, \tilde{y}^N = \bar{y}^0 + (2N - 1)(\bar{y}^1 - \bar{y}^0)\alpha, \tilde{y}^{N+1} = y^1,$$

where  $\alpha \in \mathbf{R}_+^*$  and  $N \in \mathbf{N}$  are designed so that

- $\tilde{y}^k$  and  $\tilde{y}^{k+1}$  belong to the ball of radius  $\rho$  centered on the steady-state point  $(\tilde{y}^k + \tilde{y}^{k+1})/2$  for every  $k \in \{1, \dots, N - 1\}$ ;
- $\tilde{y}^1$  belong to the ball of radius  $\rho$  centered on the steady-state point  $\bar{y}^0$ ;
- $\tilde{y}^N$  belong to the ball of radius  $\rho$  centered on the steady-state point  $\bar{y}^1$ .

These conditions lead to  $\alpha < \frac{\rho}{|\bar{y}^1 - \bar{y}^0|}$  and  $\frac{1}{2} \left(1 + \frac{1}{\alpha} - \frac{\rho}{\alpha|\bar{y}^1 - \bar{y}^0|}\right) < N < \frac{1}{2} \left(1 + \frac{1}{\alpha} + \frac{\rho}{\alpha|\bar{y}^1 - \bar{y}^0|}\right)$ .

By construction, it is then easy to build a control in  $\mathcal{U}_+(\mu)$  steering  $y^k$  to  $y^{k+1}$  in time  $\mu$ . Thus, by concatenation of these controls, we have built a control steering  $y^0$  to  $y^1$  in a time  $T$  lower than  $\frac{\mu}{2} \left(1 + \frac{1}{\alpha} + \frac{\rho}{\alpha|\bar{y}^1 - \bar{y}^0|}\right) + 2\mu$ . For the sake of readability, we illustrate this construction on Fig. 15.

Taking the limit  $\alpha \rightarrow \rho/|\bar{y}^1 - \bar{y}^0|$ , we obtain the upper bound estimation on  $T$ .  $\square$

**Proposition 5.1.2.** *Assume that all the eigenvalues of  $A$  have a negative real part. Let  $\bar{y}^1 \in S_+^*$  and let  $\bar{u}^1 \in \mathbf{R}_+^*$  its associated steady-state control (i.e.,  $A\bar{y}^1 + B\bar{u}^1 = 0$ ). Then there exists  $\rho = \rho(\bar{u}^1) > 0$  such that for every  $y^0 \in \mathbf{R}^n$  and  $y^1 \in B(\bar{y}^1, \rho)$  there exists a positive time  $T \leq \inf\{t > 0 \mid C(t) < \rho/|\bar{y}^1 - y^0|\} + \bar{u}^1$ , with  $C(t) = \sup\{|e^{tA}z| \mid z \in \mathbf{R}^n, |z| \leq 1\}$ , and a control  $u \in \mathcal{U}_+(T)$  such that the solution of (1.1) with initial condition  $y^0$  and control  $u$  reaches  $y^1$  at time  $T$ .*

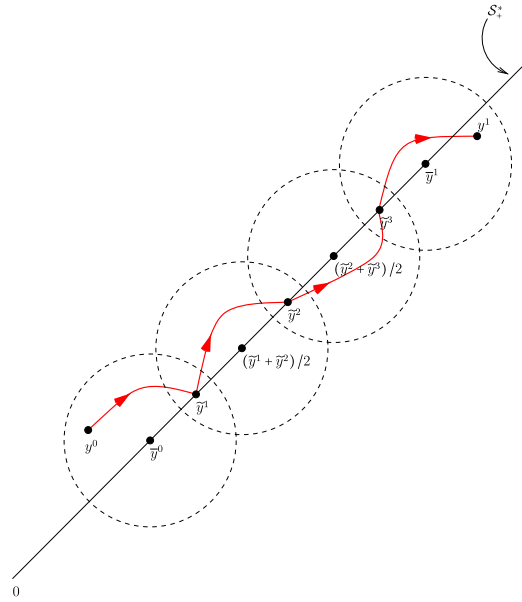


Fig. 15. State trajectory for the control built in the proof of Proposition 5.1.1.

**Proof.** We use the dissipativity of the system. More precisely, taking the constant control  $u(t) = \bar{u}^1$ , the solution of (1.1) with initial condition  $y^0$  (and control  $u$ ) exponentially converges to  $\bar{y}^1$ . More precisely, we have  $|\bar{y}^1 - y(t)| \leq C(t)|\bar{y}^1 - y^0|$ .

We then use the small-time local controllability around the steady-state  $(\bar{y}^1, \bar{u}^1)$ . This means that there exists  $\rho = \rho(\bar{u}^1) > 0$  such that for every  $y^0$  and  $y^1$  in  $B(\bar{y}^1, \rho)$ , there exists a control  $u \in \mathcal{U}_+(\bar{u}^1)$  such that the solution of (1.1) starting from  $y^0$  reaches  $y^1$  at time  $\bar{u}^1$ .

The upper bound on the reachability time easily follows.  $\square$

**Remark 5.1.3.** In addition to the reachability condition, the Propositions 5.1.1 and 5.1.2 also give an upper bound on the reachability time. However, to explicitly know this bound, one needs to know the parameter  $\rho$ , which is not explicit.  $\blacksquare$

**Lemma 5.1.4.** Assume  $\mathcal{S}_+^* \neq \emptyset$ . Let  $y^0, y^1 \in \mathbb{R}^n$  and assume that  $y^0$  or  $y^1$  belongs to  $\mathcal{S}_+^*$ . Assume, in addition, the existence of  $T > 0$  such that  $y^1$  is reachable from  $y^0$  in time  $T$  with a control in  $\mathcal{M}_+(T)$ . Then for every  $\tau \geq 0$ ,  $y^1$  is reachable from  $y^0$  in time  $T + \tau$ .

The proof of this lemma is straightforward: a control in time  $T + \tau$  is obtained by concatenation of a control in time  $T$  with the constant control  $\bar{u}$  associated to the steady-state  $y^0$  or  $y^1$ .

**Remark 5.1.5.** As for Propositions 5.1.1 and 5.1.2, the condition  $y^0 \in \mathcal{S}_+^*$  or  $y^1 \in \mathcal{S}_+^*$  (in the statement of Lemma 5.1.4) can be relaxed to  $y^i \in B(\bar{y}^i, \rho)$ , with  $\bar{y}^i \in \mathcal{S}_+^*$  (for  $i = 0$  or  $i = 1$ ) and  $\rho > 0$  small enough (depending on  $\tau$  and  $\bar{y}^0$  or  $\bar{y}^1$ ). To this end, we use small time controllability around the steady state  $\bar{y}^0$  or  $\bar{y}^1$ .  $\blacksquare$

**Remark 5.1.6.** The result of Lemma 5.1.4 can be trivially extended to the problem of controllability to trajectories. In fact, set  $\bar{y}$  is a solution of (1.1) with a nonnegative control  $\bar{u} \in L^\infty(\mathbb{R}_+)$ , and set  $y^0 \in \mathbb{R}^n$ . If there exists a time  $T$  and a control  $u \in \mathcal{U}_+(T)$  such that the solution  $y$  of (1.1), with initial condition  $y^0$  and control  $u$ , satisfies  $y(T) = \bar{y}(T)$ , then for every  $\tau > 0$ , there exists a control  $u_\tau \in \mathcal{U}_+(T + \tau)$  such that the solution  $y$  of (1.1), with initial condition  $y^0$  and control  $u_\tau$  satisfies  $y(T + \tau) = \bar{y}(T + \tau)$ . To this end, we only take,  $u_\tau(t) = u(t)$  for  $t \in (0, T)$ , and  $u_\tau(t) = \bar{u}(t)$  for  $t \in (T, T + \tau)$ .  $\blacksquare$

### 5.1.2. Existence of a positive minimal controllability time and minimal time controls

An important notion to define the minimal time is the accessible set with nonnegative controls

$$\text{Acc}_+(T) = \{\Phi_T u, u \in \mathcal{U}_+(T)\}.$$

The minimal controllability time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  defined by (1.5) is then

$$\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \inf \left\{ T > 0 \mid y^1 - e^{TA} y^0 \in \text{Acc}_+(T) \right\} \quad (y^0, y^1 \in \mathbb{R}^n),$$

and by convention,  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = +\infty$  when  $y^1$  is not accessible from  $y^0$  in any time, i.e.,  $y^1 - e^{TA} y^0 \notin \text{Acc}_+(T)$  for every  $T > 0$ .

As explained in [21], for  $T > 0$  small enough,  $\text{Acc}_+(T)$  is isomorphic to the positive orthant of  $\mathbb{R}^n$ . This ensures that whatever  $y^0 \in \mathbb{R}^n$  is, there always exists  $y^1 \in \mathbb{R}^n$  such that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) > 0$ . The problem is to characterize this minimal control time and to determine whether there exists a control at the minimal time. Similarly to [20], it can be checked that the existence of a minimal controllability time is ensured in the set of Radon measures.

**Proposition 5.1.7.** *Let  $y^0$  and  $y^1$  be two points of  $\mathbb{R}^n$  such that  $0 \leq \mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$ , i.e.,  $y^1$  is accessible from  $y^0$ . Under Assumption (H.1), there exists a control  $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}}(y^0, y^1))$  steering the system (1.1) from  $y^0$  to  $y^1$  in time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ .*

*The same result holds with  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  replaced with  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ .*

**Proof.** The argument is similar to the one used in [20]. We prove here this result in the case of  $L^\infty$  nonnegative controls. The same proof can be made for nonnegative Radon measure controls.

Let us denote  $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . There exists a non-increasing sequence  $(T_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} T_n = \mathcal{T}_{\mathcal{U}}$  and for every  $n \in \mathbb{N}$ , there exists  $u_n \in \mathcal{U}_+(T_n)$  such that the system (1.1) is steered from  $y^0$  to  $y^1$  in time  $T_n$ , i.e.

$$y^1 - e^{T_n A} y^0 = \int_0^{T_n} e^{(T_n-t)A} B u_n(t) dt \quad (5.1)$$

Since the pair  $(A, B)$  satisfies the Kalman rank condition, for every eigenvector  $\varphi$  of  $A^\top$ , we have  $\langle \varphi, B \rangle \neq 0$ . Let us denote by  $\lambda$  the associated eigenvalue ( $A^\top \varphi = \lambda \varphi$ ). Since  $A$  satisfies the assumption (H.1), we can choose the eigenvector  $\varphi \in \mathbb{R}^n$  so that it is associated to a real eigenvalue  $\lambda$ .

We define  $Y^0 = \langle \varphi, y^0 \rangle$  and  $Y^1 = \langle \varphi, y^1 \rangle$ . Then from (5.1), we deduce that

$$\frac{Y^1 - e^{\lambda T_n} Y^0}{\langle \varphi, B \rangle} = \int_0^{T_n} e^{\lambda(T_n-t)} u_n(t) dt.$$

Since  $u_n$  is nonnegative and  $t \mapsto e^{\lambda(T_n-t)}$  is also nonnegative, we have

$$\int_0^{T_n} e^{\lambda(T_n-t)} u_n(t) dt \geq e^{-|\lambda|T_n} \int_0^{T_n} u_n(t) dt$$

and hence

$$\|u_n\|_{L^1(0, T_n)} \leq e^{|\lambda|T_n} \frac{|Y^1 - e^{\lambda T_n} Y^0|}{|\langle \varphi, B \rangle|}$$

and considering that  $(T_n)_{n \in \mathbb{N}}$  is non-increasing, we have,

$$\|u_n\|_{L^1(0, T_n)} \leq e^{|\lambda|T_0} \frac{|Y^1| + e^{|\lambda|T_0} |Y^0|}{|\langle \varphi, B \rangle|} \quad (n \in \mathbb{N}).$$

Extending  $u_n$  on  $(0, T_0)$  by 0 on  $(T_n, T_0)$ , we obtain that the sequence  $(u_n)_n$  is uniformly bounded in  $L^1(0, T_0)$  and hence, up to a subsequence, it converges in vague topology of measures to some  $\underline{u} \in \mathcal{M}(0, T_0)$  (see e.g.,



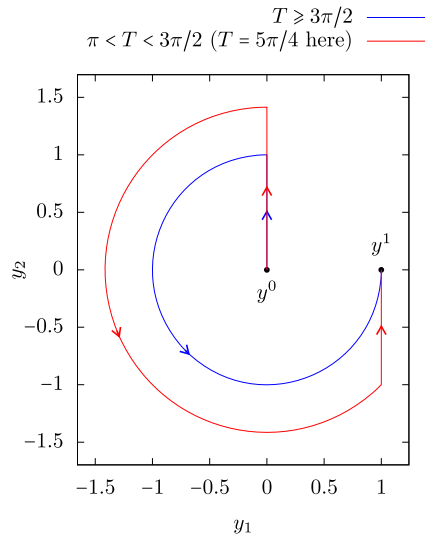


Fig. 16. State trajectories related to the impulsive control introduced in Remark 5.1.8.

[1, Corollary 31.3 p. 206]). In addition, since  $u_n$  is nonnegative and since  $u_n$  has its support in  $[0, T_n]$ , we obtain that  $\underline{u}$  is a nonnegative measure which has its support contained in  $[0, \mathcal{T}_{\mathcal{U}}]$ . Finally, taking the limit  $n \rightarrow +\infty$  in (5.1), we obtain

$$y^1 - e^{\mathcal{T}_{\mathcal{U}}A} y^0 = \int_{[0, \mathcal{T}_{\mathcal{U}}]} e^{(\mathcal{T}_{\mathcal{U}}-t)A} B d\underline{u}(t),$$

i.e.,  $\underline{u}$  satisfies the control requirement.  $\square$

**Remark 5.1.8.** Assumption (H.1) is instrumental here. However, without this assumption, it is possible that no nonnegative measure control exists at the minimal time (even if the target  $y^1$  is reachable from the initial condition  $y^0$  for every time  $T > \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ ). As example, consider

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad y^0 = 0, \quad y^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is easy to see that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \pi$  and no nonnegative Radon measure control exists in time  $\pi$ . In fact, we have:

- for  $T \geq 3\pi/2$ , the impulsive control  $u = \delta_{T-3\pi/2}$  steers 0 to  $y^1$  (the corresponding state trajectory is plotted on Fig. 16);
- for  $T \in (\pi, 3\pi/2)$ , the impulsive control  $u = \frac{-1}{\sin T} \delta_0 + \frac{\cos T}{\sin T} \delta_T$  steers 0 to  $y^1$  (the corresponding state trajectory is plotted on Fig. 16);
- for  $T \in [0, \pi]$ , for every nonnegative Radon measure control  $u$ , the first component of  $\Phi_T u$  is nonpositive. Consequently, there does not exist any control in time  $T \leq \pi$ .

This example shows that (without restrictive conditions) even if there exists a nonnegative control for every  $T > \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  steering  $y^0$  to  $y^1$ , nothing ensures the existence of such a control at the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . Having a better understanding of the conditions ensuring a control in the minimal time is the point of Open problem 3. ■

### 5.1.3. No gap conditions

As proved in Proposition 5.1.7, there exists (under some sufficient assumption) a Radon measure control at the minimal time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . It is then natural to wonder if  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . It turns out that this is not the case in general. In fact, one can see in the examples provided in Remark 5.1.9 some trivial situations where a gap occurs.

The examples provided in the 1st and 2nd items of Remark 5.1.9, deal with situations where the target state can be reached in zero time with a Dirac impulse, but cannot be reached in arbitrarily small time with classical  $L^\infty$  controls. A less trivial situation, where  $0 < \mathcal{T}_M < \mathcal{T}_U < \infty$  is given in the 3rd item of Remark 5.1.9. All these examples show that the inequality (1.7) can be strict.

**Remark 5.1.9.** The technical details related to the examples given in this remark are postponed in Appendix C.1.

1. Consider  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , with  $y^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $y^1 = \begin{pmatrix} 0 \\ 1 + \varepsilon \end{pmatrix}$  (for some  $\varepsilon > 0$ ). Then we have  $\mathcal{T}_U(y^0, y^1) = +\infty$ , i.e.,  $y^1$  is not accessible from  $y^0$ , but since  $y^1 \in \{y^0\} + \mathbf{R}_+ B$ , it is trivial that  $\mathcal{T}_M(y^0, y^1) = 0$ . The technical details of this example are provided in Appendix C.1.1.
2. Consider now the matrix  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $B$ ,  $y^0$  and  $y^1$  being the same as in the previous item. Then, one can show that  $\mathcal{T}_U(y^0, y^1) = \pi$ , but here again,  $\mathcal{T}_M(y^0, y^1) = 0$  (see Appendix C.1.2 for the technical details related to this example).
3. Consider  $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ , with  $y^0 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$  and  $y^1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ . Then we have  $\pi/2 = \mathcal{T}_M(y^0, y^1) < \mathcal{T}_U(y^0, y^1) < \infty$ . The technical details of this example are provided in Appendix C.1.3. On Fig. 17, we have plotted the corresponding trajectories and controls. More precisely, on Fig. 17, one of the plots corresponds to the minimal time control with measures (red) and the other one is obtained through a numerical simulation and gives the minimal time control satisfying the additional constraint  $0 \leq u(t) \leq M$ , with  $M = 5$  (blue).

The examples given in the 1st and 2nd items also show that we cannot expect some nonsingularity of  $(y^0, y^1) \mapsto \mathcal{T}_U(y^0, y^1)$ . However, due to small time local controllability, we also know that if  $y^1 \in S_+^*$  then  $y^0 \mapsto \mathcal{T}_U(y^0, y^1)$  is continuous at  $y^1$ . ■

**Remark 5.1.10.** It is natural to ask whether a gap can occur also for reachability of trajectories. More precisely, given  $\bar{y}$ , a solution of (1.1) for some initial condition  $\bar{y}^0$  and some nonnegative control  $\bar{u} \in L^\infty(\mathbf{R}_+)$ , we can define similarly the minimal time to reach this trajectory from an initial condition  $y^0$  with nonnegative  $L^\infty$  control,

$$\mathcal{T}_U^{\rightarrow \bar{y}}(y^0) = \inf \left\{ T \geq 0 \mid \exists u \in \mathcal{U}_+(T), e^{TA} y^0 + \Phi_T u = \bar{y}(T) \right\}$$

or with nonnegative Radon measure control,

$$\mathcal{T}_M^{\rightarrow \bar{y}}(y^0) = \inf \left\{ T \geq 0 \mid \exists u \in \mathcal{M}_+(T), e^{TA} y^0 + \Phi_T u = \bar{y}(T) \right\}.$$

But in fact, on the example given in the 3rd item of Remark 5.1.9, it appears that a gap can occur. To this end, consider the trajectory  $\bar{y}$  solution of (1.1) with initial condition  $\bar{y}^0 = (0, 1, 0)^\top$  and null control. With the initial condition  $y^0$  given in the 3rd item of Remark 5.1.9, it is obvious that  $\mathcal{T}_M^{\rightarrow \bar{y}}(y^0) = 0$ . But we have  $\mathcal{T}_U^{\rightarrow \bar{y}}(y^0) \geq \pi/4$  (see Appendix C.1.4 for computational details).

Of course, studying gap condition for the control to trajectory require much more work, and this is not developed in this paper. ■

In Remark 5.1.9, we have given examples where  $\mathcal{T}_M(y^0, y^1) < \mathcal{T}_U(y^0, y^1)$ . We are now going to give some conditions ensuring that  $\mathcal{T}_M(y^0, y^1) = \mathcal{T}_U(y^0, y^1)$ . However, the results below do not solve all the possible situations and general condition ensuring that  $\mathcal{T}_M(y^0, y^1) = \mathcal{T}_U(y^0, y^1)$  is the goal of Open problem 2.

**Proposition 5.1.11.** *Given any  $y^0 \in \mathbf{R}^n$  and  $y^1 \in S_+^*$ , we have  $\mathcal{T}_U(y^0, y^1) = \mathcal{T}_M(y^0, y^1)$ .*

**Proof.** Let  $T = \mathcal{T}_M(y^0, y^1)$ . Let us first note that if  $T = \infty$ , then, obviously,  $\mathcal{T}_U(y^0, y^1) = T = \infty$ . We thus assume that  $T \in \mathbf{R}_+$ .

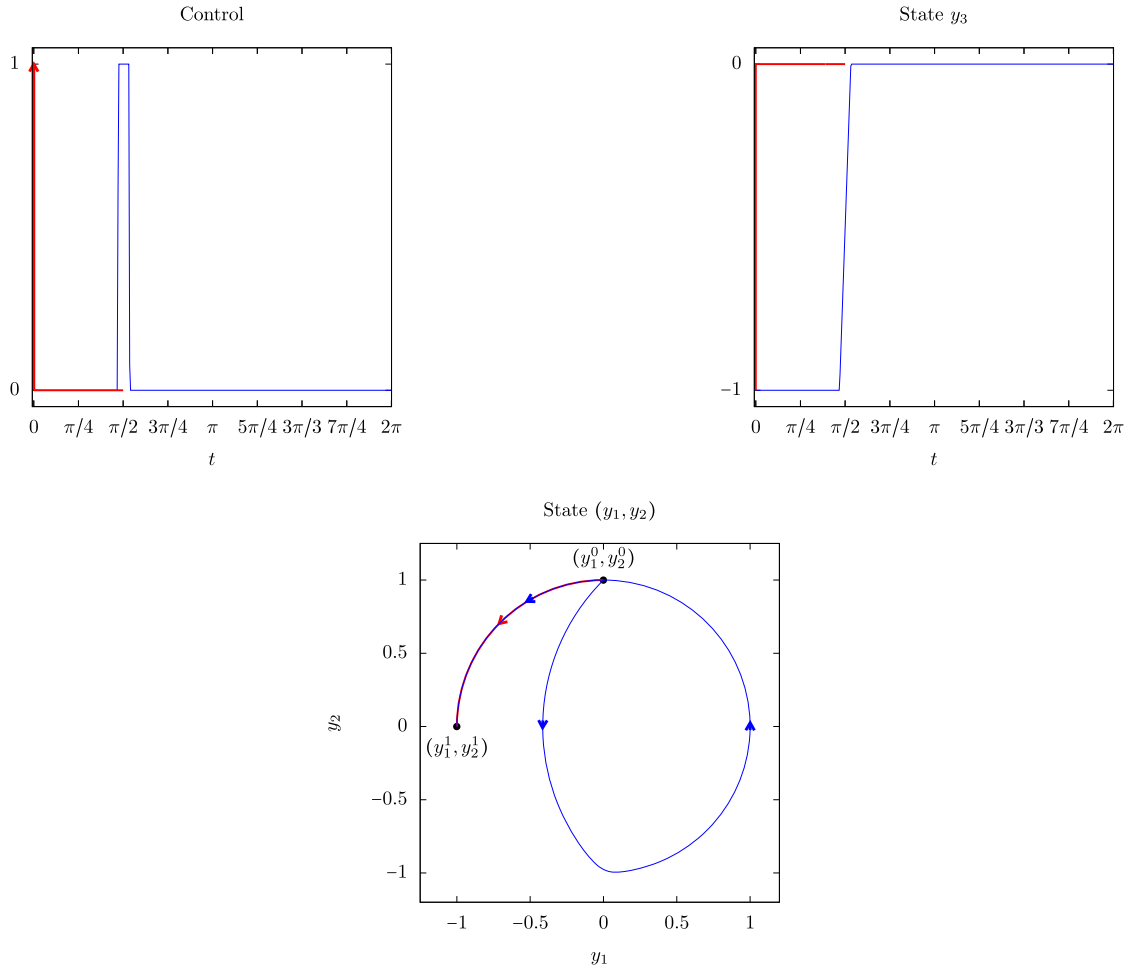


Fig. 17. Control and state trajectories for the 3rd item of Remark 5.1.9. The impulsive control and its corresponding state trajectories are plotted in red (arrow stand for Dirac mass) and the minimal time control and its associated state trajectories, with the additional control constraint  $u(t) \leq 5$ , are plotted in blue. (On the “Control figure”, we plot in blue  $t \mapsto u(t)/5$ .)

Let  $\underline{u} \in \mathcal{M}_+(T)$  be a control steering  $y^0$  to  $y^1$  in time  $T$ . For every  $\varepsilon > 0$ , there exist  $\tau_1 = \tau_1(\varepsilon)$  and  $\tilde{u} \in L^\infty(-\tau, T + \tau)$  such that  $\left| \int_{-\tau_1}^{T+\tau_1} f(t) \tilde{u}(t) dt - \int_{[-\tau_1, T+\tau_1]} f(t) d\underline{u}(t) \right| \leq \varepsilon$  for every  $f \in C([- \tau_1, T + \tau_1])$ . Here  $\underline{u}$  has been extended by 0 to a Radon measure on  $[-\tau_1, T - \tau_1]$ . Let us then define  $u^1 \in \mathcal{U}_+(T + 2\tau_1)$  by  $u^1(t) = \tilde{u}(t - \tau_1)$ . We set

$$\tilde{y}^1 = e^{(T+2\tau_1)A} y^0 + \int_0^{T+2\tau_1} e^{(T+2\tau_1-t)A} B u^1(t) dt = e^{2\tau_1 A} e^{TA} y^0 + e^{\tau_1 A} \int_{-\tau_1}^{T+\tau_1} e^{(T-t)A} B \tilde{u}(t) dt.$$

Then there exists  $\eta = \eta(\varepsilon, \tau_1) > 0$  such that  $|y^1 - \tilde{y}^1| \leq \eta$ . Recall that  $y^1 = e^{TA} y^0 + \int_{[0, T]} e^{(T-t)A} B d\underline{u}(t)$ . Noticing that  $\tau_1$  and  $\eta$  can be chosen so that  $\tau_1 \rightarrow 0$  and  $\eta \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we conclude that  $\tilde{y}^1$  can be arbitrarily close to  $y^1$ . Now, for  $\eta > 0$  small enough, we use the small time local controllability around the positive steady state  $y^1$ , *i.e.*, there exists a time  $\tau_2 = \tau_2(\eta)$  such that  $\tilde{y}^1$  can be steered to  $y^1$  with a control  $u^2 \in \mathcal{U}_+(\tau_2)$ , and we have  $\tau_2 \rightarrow 0$  as  $\eta \rightarrow 0$ .

All in all, we have designed a control  $u \in \mathcal{U}_+(T + 2\tau_1 + \tau_2)$  steering  $y^0$  to  $y^1$  in time  $T + 2\tau_1 + \tau_2$ . The conclusion follows, since  $\tau_1$  and  $\tau_2$  can be arbitrarily small.  $\square$

Adapting the regularization argument used in the proof of Proposition 5.1.11, it is possible to slightly extend the result of Proposition 5.1.11.

**Proposition 5.1.12.** *Given any  $y^0 \in \mathbb{R}^n$  and  $y^1 \in \mathbb{R}^n$  such that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < \infty$ . If there exist a sequence of time  $(T_n)_{n \geq 0}$  and a sequence of controls  $(u_n)_{n \geq 0}$  such that  $u_n \in \mathcal{M}_+(T_n)$  and*

1.  $T_n \rightarrow \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  as  $n \rightarrow \infty$ ;
2. for every  $n \geq 0$ ,  $u_n$  steers  $y^0$  to  $y^1$  in time  $T_n$ ;
3. for every  $n \geq 0$ , there exists  $\bar{t}_n \in [0, T_n]$  such that  $[y_n(\bar{t}_n^-), y_n(\bar{t}_n^+)] \cap \mathcal{S}_+^* \neq \emptyset$ , with  $y_n$ , the solution of (1.1) with control  $u_n$  and initial condition  $y^0$ ,

then we have  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ .

**Proof.** To prove this result, we will show that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \leq T_n$  for every  $n$ .

For every  $n$ , let us set  $\bar{y}_n \in [y^n(\bar{t}_n^-), y^n(\bar{t}_n^+)] \cap \mathcal{S}_+^*$ . With the regularization argument used in the proof of Proposition 5.1.11, we can show that  $\mathcal{T}_{\mathcal{U}}(y^0, \bar{y}_n) \leq \bar{t}_n$ , and by performing the change of variables  $t \mapsto T_n - t$ , we can use again the same regularization argument to show that  $\mathcal{T}_{\mathcal{U}}(\bar{y}_n, y^1) \leq T_n - \bar{t}_n$ . Noticing that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \leq \mathcal{T}_{\mathcal{U}}(y^0, \bar{y}_n) + \mathcal{T}_{\mathcal{U}}(\bar{y}_n, y^1)$ , we easily conclude.  $\square$

**Remark 5.1.13.** The result of Proposition 5.1.12 covers the case where no nonnegative Radon measure control exists at time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . If there exists a control  $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{M}}(y^0, y^1))$  steering  $y^0$  to  $y^1$  in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , it is possible to try the assumptions of Proposition 5.1.12 with  $T_n = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  and  $u_n = u$  for every  $n$ .  $\blacksquare$

**Remark 5.1.14.** When  $y^1$  does not belong to  $\mathcal{S}_+^*$ , one could expect to obtain that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  by regularization, as in the proof of Proposition 5.1.11. But this does not seem easy.

In fact, let us pick a nonnegative Radon measure control,  $u$  steering  $y^0$  to  $y^1$  in some time  $T \geq \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . By smoothing the control  $u$ , for every  $\varepsilon > 0$ , there exists a nonnegative control  $\tilde{u} \in L^\infty(0, T)$  steering  $y^0$  to  $\tilde{y}^1$  in time  $T$ , with  $|\tilde{y}^1 - y^1| < \varepsilon$ . The difficulty is then to show that  $\tilde{y}^1$  can be steered to  $y^1$  with a nonnegative  $L^\infty$  control in a time  $\tau(\varepsilon)$ , with  $\tau(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . But as illustrated by the example given in the 1st and 2nd items of Remark 5.1.9, the closeness between  $y^1$  and  $\tilde{y}^1$  is not enough to ensure the reachability of  $y^1$  from  $\tilde{y}^1$  in small time.  $\blacksquare$

**Remark 5.1.15.** As example of application of Proposition 5.1.12, note that if  $y^0$  and  $y^1$  in  $\mathbb{R}^n$  are such that there exists  $\bar{y} \in \mathcal{S}_+^*$  such that  $y^0 = \bar{y} - \gamma^0 B$  and  $y^1 = \bar{y} + \gamma^1 B$  for some  $\gamma^0 \geq 0$  and  $\gamma^1 \geq 0$ . Then we have  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \mathcal{T}_{\mathcal{U}}(y^0, y^1) = 0$ .

Note also that given  $y^0$  and  $y^1$  in  $\mathbb{R}^n$ , we have  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = 0$  if and only if  $y^1 \in \{y^0\} + \mathbb{R}_+ B$ .  $\blacksquare$

## 5.2. Controls in time $\mathcal{T}_{\mathcal{M}}$

Recall that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \geq \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  and that if  $y^1 \in \mathcal{S}_+^*$  then these two times coincide (see Proposition 5.1.11). Let us also mention that at the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , a nonnegative measure control still exists as soon as the matrix  $A$  satisfies Assumption (H.1). We are now going to analyze in this section the Radon measure controls at the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  defined by (1.6). To this end, we use a time reparametrization to obtain an optimal control problem with controls bounded in  $[0, 1]$ . This allows us to use the Pontryagin maximum principle and by application of this principle we will see that for the original control problem, the control in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  is a finite sum of Dirac impulses. Then under the additional assumption (H.2), we will show that the control in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  is unique and is the sum of at most  $\lfloor (n+1)/2 \rfloor$  Dirac impulses.

### 5.2.1. Time rescaling

Following [5,10,16,22] (see also [24] for initial work on Pontryagin maximum principle with Radon measure controls), we redefine the solution of (1.1) with initial condition  $y(0) = y^0$  and with measure inputs. This definition is based on the time reparametrization of (1.1) recalled hereafter.

Given a nonnegative control  $u$ , we are going to define a change of variable  $s = \sigma(t)$  and a control  $s \mapsto w(s)$  (both  $\sigma$  and  $w$  will be function of  $u$ ), and we will define a new control system

$$\dot{z}(s) = w(s)Az(s) + B(1 - w(s)) \quad (s \in [0, \sigma(T)]), \quad (5.2)$$

so that, for every  $y^0 \in \mathbb{R}^n$ , the solution of (5.2) with initial condition  $y^0$  satisfies,

$$y(t) = z(\sigma(t)) \quad (t \geq 0 \text{ a.e.}),$$

where  $y$  is the solution of (1.1), with initial condition  $y^0$  and control  $u$ . The interest of this change of variables is that the new control will be bounded,

$$0 \leq w(s) \leq 1 \quad (s \geq 0 \text{ a.e.}), \quad (5.3)$$

and the time  $t$  can be recovered from  $w$  by  $t = \int_0^{\sigma(t)} w(s) ds$ . All in all, in order to find the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , we finally aim to find the minimum of

$$(S, w) \in \mathbb{R}_+ \times L^\infty(0, \infty) \mapsto \int_0^S w(s) ds$$

subject to the constraints  $w(s) \in [0, 1]$  for almost every  $s \in \mathbb{R}_+$  and the solution of (5.2) with initial condition  $y^0$  and control  $w$  satisfies  $z(S) = y^1$ . Before presenting the change of variables in the general case, we first present it in two particular cases:

- Case  $u \in \mathcal{U}_+(T)$ : we define the new time variable

$$s = \sigma(t) = t + \int_0^t u(\tau) d\tau \quad (t \in [0, T])$$

and we set  $S = \sigma(T)$ . Since  $u$  is nonnegative, we have that  $\sigma : [0, T] \rightarrow [0, S]$  is a continuous and increasing function, and hence  $\sigma^{-1}$  is well-defined. We then set  $z(\sigma(t)) = y(t)$  and have, for  $y$  solution of (1.1),

$$\dot{y}(t) = \dot{\sigma}(t)\dot{z}(\sigma(t)) = (1 + u(t))\dot{z}(\sigma(t)) \quad \text{and} \quad \dot{y}(t) = Ay(t) + Bu(u) = Az(\sigma(t)) + Bu(t).$$

Setting  $s = \sigma(t)$ , this leads to:

$$\dot{z}(s) = \frac{1}{1 + u(\sigma^{-1}(s))} Az(s) + B \left( 1 - \frac{1}{1 + u(\sigma^{-1}(s))} \right).$$

Then, setting  $w = 1/(1 + u \circ \sigma^{-1})$ , we obtain that  $w$  satisfies (5.3) and  $z$  is solution of (5.2).

Note also that  $w(s) = 1$  when  $u(\sigma^{-1}(s)) = 0$ , and that  $w(s) \rightarrow 0$  as  $u(\sigma^{-1}(s)) \rightarrow +\infty$ . Roughly speaking, this means that the new control  $w$  is 0 at the time corresponding to impulses of  $u$ . In addition, we have  $T = \int_0^T dt = \int_0^{\sigma(T)} \frac{1}{\sigma'(s)} ds = \int_0^S w(s) ds$ .

- Case  $u = m\delta_\tau$ , with  $\tau \in (0, T)$ , and  $m > 0$ :

For the sake of simplicity, we have assumed that  $\tau \in (0, T)$ . We define

$$\sigma(t) = t + \int_{[0,t]} du = \begin{cases} t & \text{if } 0 \leq t \leq \tau, \\ t + m & \text{if } \tau < t \leq T \end{cases}$$

and we set  $S = \sigma(T) = T + m$ . Here  $\sigma$  is not a valid change of variables, since it is not continuous anymore. However, one can see that the solution of (1.1) with initial condition  $y^0$  is

$$y(t) = \begin{cases} e^{tA} y^0 & \text{if } 0 \leq t < \tau, \\ e^{(t-\tau)A} (e^{\tau A} y^0 + mB) & \text{if } \tau < t \leq T. \end{cases}$$

Let us also define a control  $w$  by

$$w(s) = \begin{cases} 1 & \text{if } 0 \leq s < \tau, \\ 0 & \text{if } \tau < s < \tau + m, \\ 1 & \text{if } \tau + m < s \leq T + m. \end{cases}$$

Obviously,  $w$  satisfies (5.3). Let  $z$  be the solution of (5.2) with this control  $w$  and with the initial condition  $z(0) = y^0$ . Then we have

$$z(s) = \begin{cases} e^{sA} y^0 & \text{if } 0 \leq s \leq \tau, \\ e^{\tau A} y^0 + B(s - \tau) & \text{if } \tau \leq s \leq \tau + m, \\ e^{(s-(\tau+m))A} (e^{\tau A} y^0 + Bm) & \text{if } \tau + m \leq s \leq T + m, \end{cases}$$

and it is easy to check that we have  $y(t) = z(\sigma(t))$  for almost every  $t \in [0, T]$  and that  $\int_0^S w(s) ds = T$ . In other words the impulse at time  $\tau$  is replaced by a linear increase of the solution in the direction of  $B$  during on the interval  $(\tau, \tau + m)$ .

Based on these ideas, we build a time reparametrization for every  $u \in \mathcal{M}_+(T)$  (following [5,10,16,22,24]): given any nonnegative Radon measure  $u \in \mathcal{M}_+(T)$ , we define

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(t) = t + v(t) \quad (t \in (0, T]),$$

with  $v(t) = u([0, t])$ . Since  $v$  is a function of bounded variation,  $\sigma$  is left continuous and can only have a countable number of jumps. We set  $\mathcal{T}$  the set of jump times. For every  $t \in [0, T]$ ,

- if  $s = \sigma(t^-)$ , with  $t \in [0, T] \setminus \mathcal{T}$ , we set  $\tau(s) = t$  and  $\gamma(s) = v(t)$ ;
- if  $s \in [\sigma(t^-), \sigma(t^+)]$ , with  $t \in \mathcal{T}$ , we set  $\tau(s) = t$  and  $\gamma(s) = v(t^-) + \frac{v(t^+) - v(t^-)}{\sigma(t^+) - \sigma(t^-)}(s - \sigma(t^-)) = s - t^-$ .

It is easy to see that for every  $s \in [0, \sigma(T))$  and every  $\eta \in [0, \sigma(T) - s]$ , we have  $0 \leq \tau(s + \eta) - \tau(s) \leq \eta$ . Consequently,  $\tau \in W^{1,\infty}(0, \sigma(T))$  and  $0 \leq \tau'(s) \leq 1$  for almost every  $s \in [0, \sigma(T)]$ . In addition, when  $s \in [\sigma(t^-), \sigma(t^+)]$ , for some  $t \in \mathcal{T}$ , we have  $\gamma'(s) = 1$  and when  $s = \sigma(t^-)$ , for some  $t \notin \mathcal{T}$ , we have  $s = \tau(s) + \gamma(s)$  leading to  $\gamma \in W^{1,\infty}(0, \sigma(T))$  and  $\gamma'(s) = 1 - \tau'(s)$  for almost every  $s \in [0, \sigma(T)]$ .

The path  $\gamma$  leads to a reparametrization of the graph of  $v$ , and to the reparametrized dynamical system

$$\dot{z}(s) = \tau'(s)Az(s) + B(1 - \tau'(s)) \quad (s \in (0, \sigma(T))), \quad (5.4a)$$

with initial condition

$$z(0) = y^0. \quad (5.4b)$$

Given  $u \in \mathcal{M}_+(T)$ , the solution  $y$  of (1.1) with initial state  $y^0$  and control  $u$  is given by  $y(t) = z(\sigma(t))$  with  $z$  solution of the Cauchy problem (5.4). Setting  $w(s) = \tau'(s) \in [0, 1]$ , as new control, we end up with the system (5.2) and  $T$  is given by  $T = \int_0^{\sigma(T)} w(s) ds$ .

On Fig. 18, we show on an example, how the new control  $w$  is related to the control  $u \in \mathcal{M}_+(T)$ .

All in all, the minimal time control problem is written as

$$\begin{aligned} \min \quad & \int_0^S w(s) ds \\ \text{subject to} \quad & \begin{cases} S \geq 0, \\ w(s) \in [0, 1] & (s \in [0, S]), \\ z(S) = y^1, \text{ with } z \text{ the solution of (5.2) with initial condition } z(0) = y^0. \end{cases} \end{aligned} \quad (5.5)$$

The interest of this formulation is that now the control is bounded in  $[0, 1]$  and we can apply the classical Pontryagin maximum principle (see [17, Theorem 1 p. 310]).

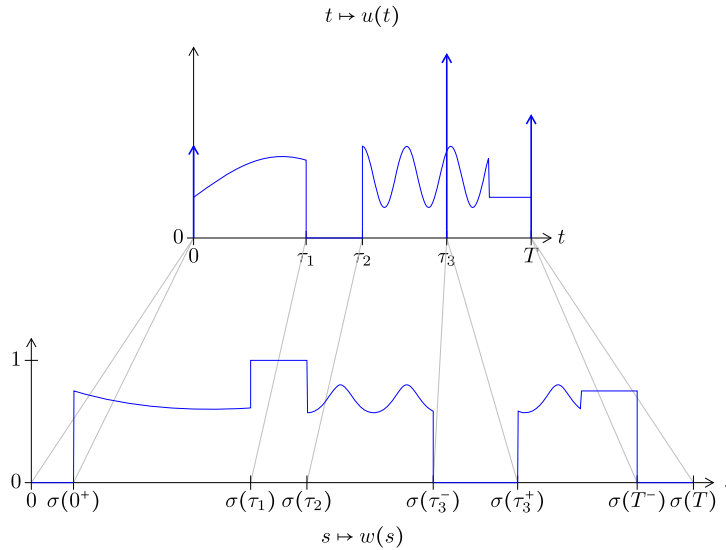


Fig. 18. Relation between the control  $u$  and the control  $w$ . For the plot of the control  $u$ , arrows stand for Dirac impulses.

### 5.2.2. Consequences of the Pontryagin maximum principle

Applying the Pontryagin maximum principle to (5.5), we obtain the following result.

**Proposition 5.2.1.** *Let  $y^0$  and  $y^1$  be two points of  $\mathbb{R}^n$  such that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < +\infty$ , i.e.,  $y^1$  is accessible from  $y^0$  with a nonnegative Radon measure control, and assume in addition that there exists a control  $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{M}}(y^0, y^1))$  steering  $y^0$  to  $y^1$  at the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . Then for any pair  $(S, w)$  minimizing the optimal control problem (5.5), we have*

$$w(s) = \begin{cases} 1 & \text{if } s \notin \bigcup_{i=1}^N I_k, \\ 0 & \text{if } s \in \bigcup_{i=1}^N I_k \end{cases} \quad (s \in [0, S] \text{ a.e.}),$$

where  $N \in \mathbb{N}$  and  $I_1, \dots, I_N$  are two-by-two disjoint intervals of  $[0, S]$ .

Furthermore,

- if  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = 0$ ,  $w \equiv 0$ ;
- if  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) > 0$ , there exists a nontrivial solution  $q$  of  $\dot{q} = -wA^\top q$  such that  $B^\top q$  has a constant sign, and  $\bigcup_{i=1}^N I_k = \{s \in [0, S] \mid B^\top q(s) = 0\}$ .

**Corollary 5.2.2.** *Let  $y^0$  and  $y^1$  be two points of  $\mathbb{R}^n$  such that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < +\infty$ , i.e.,  $y^1$  is accessible from  $y^0$  with a nonnegative Radon measure control and assume in addition that there exists a control  $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{M}}(y^0, y^1))$  steering  $y^0$  to  $y^1$  at the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ .*

*Then any control in  $\mathcal{M}_+(\mathcal{T}_{\mathcal{M}}(y^0, y^1))$  steering the solution of (1.1) from  $y^0$  to  $y^1$  at the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  is a linear combination with nonnegative coefficients of a finite number of Dirac impulses. That is to say that there exist  $N \in \mathbb{N}^*$ ,  $\tau_1, \dots, \tau_N \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)]$  and  $m_1, \dots, m_N > 0$  such that  $u = \sum_{i=1}^N m_i \delta_{\tau_i}$ .*

*In addition, then there exists a nontrivial solution  $p$  of  $\dot{p} = -A^\top p$  such that  $B^\top p$  has a constant sign, and  $\{\tau_1, \dots, \tau_N\} = \{t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)] \mid B^\top p(t) = 0\}$ .*

In order to use Corollary 5.2.2, one needs to prove that a minimal time control exists. This is ensured by Assumption (H.1) (or by the stronger assumption (H.2)), see Proposition 5.1.7). This leads to the following corollary. Note that without the assumption (H.1), it may happen that no control exists at the minimal time (see the example of Remark 5.1.8).



**Corollary 5.2.3.** *Let  $y^0 \in \mathbb{R}^n$  and  $y^1 \in \mathbb{R}^n$  be such that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < +\infty$ . Under Assumption (H.1), there exists a minimal time nonnegative control in the space of Radon measures steering  $y^0$  to  $y^1$  in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  and this control is a linear combination with nonnegative coefficients of a finite number of Dirac impulses. Furthermore, as in Corollary 5.2.2, the time localization of these Dirac masses is given by the zero set of  $B^\top p$  with  $p$  a nontrivial solution of the adjoint system such that  $B^\top p$  has a constant sign.*

**Proof of Proposition 5.2.1.** Let us first notice that if there exists a nonnegative Radon measure control steering  $y^0$  to  $y^1$  in zero time, then this control is impulsive (see Remark 5.1.15), meaning that  $w \equiv 0$ .

Assume now that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) > 0$ . To prove the result, we are going to apply the Pontryagin maximum principle. The Hamiltonian associated with the optimal control problem (5.5) is

$$H(z, w, p_0, q) = -p_0 w + \langle q, wAz + (1 - w)B \rangle \quad (z \in \mathbb{R}^n, w \in [0, 1], p_0 \in \mathbb{R}_+, q \in \mathbb{R}^n).$$

Optimality conditions gives  $\dot{q} = -\partial H / \partial z$ , i.e.,  $\dot{q} = -wA^\top q$ . Let us also define  $\varphi(s) = -p_0 + \langle q(s), Az(s) - B \rangle$ , then due to the maximization condition, the optimal control  $w$  satisfies

$$w(s) = \begin{cases} 0 & \text{if } \varphi(s) < 0, \\ 1 & \text{if } \varphi(s) > 0. \end{cases}$$

At this step,  $w(s)$  is undetermined when  $\varphi(s) = 0$ . In addition, since the final time is free and since the system is autonomous, for any optimal solution, we have  $H(z(s), w(s), p_0, q(s)) = 0$  for  $s \in [0, S]$ . In particular, we have  $\varphi(s) \leq 0$  if and only if  $B^\top q(s) = 0$ , and  $\varphi(s) > 0$  if and only if  $B^\top q(s) < 0$ . This in particular ensures that  $B^\top q$  has a constant sign, the set  $E_0 = \{s \in [0, S] \mid w(s) = 0\}$  is contained in the set  $\{s \in [0, S] \mid B^\top q(s) = 0\}$ , and the set  $E_1 = \{s \in [0, S] \mid w(s) = 1\}$  contains the set  $\{s \in [0, S] \mid B^\top q(s) \neq 0\}$ .

Let us also notice that  $q(s) = \exp\left(\int_s^S w(\sigma) d\sigma A^\top\right) q(S)$  consequently, if  $q(s) = 0$  for some  $s \in [0, S]$ , then  $q \equiv 0$ . Assume that  $q \equiv 0$ , we thus have  $-p_0 w \equiv H(z, w, p_0, 0) = 0$ , and hence, since  $(p_0, q)$  is not trivial, we necessarily have  $w \equiv 0$ . This ensures that  $E_0 = [0, S]$  and  $E_1 = \emptyset$  (when  $q \equiv 0$ ). This in particular implies that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = 0$  and this particular case has already been treated.

Let us now assume that  $q \not\equiv 0$ , i.e., for every  $s \in [0, S]$ ,  $q(s) \neq 0$ . Defining  $\tau(s) = \int_0^s w(\sigma) d\sigma$ , we have  $q(s) = p(\tau(s))$ , with  $p(t) = \exp((\tau(S) - t)A^\top) q(S)$  for every  $t \in [0, \tau(S)]$ . Since  $B^\top p$  is analytic and not identically equal to 0, there exists a finite number of times  $t_1, \dots, t_N$ , such that this function reaches 0. Consequently,  $E_0 \subset \tau^{-1}(\{t_1, \dots, t_N\})$ . Using the fact that  $\tau$  is continuous and nondecreasing, we have that  $E_0$  is included in a finite union of closed intervals.

We now aim to show that  $E_0$  contains the interior of this finite union. To this end, we define the set  $E = \{s \in [0, S] \mid w(s) > 0\} \cap \{s \in [0, S] \mid B^\top q(s) = 0\}$  and we assume by contradiction that  $|E| > 0$ . Let us also define  $E^d$ , the set of density points of  $E$ . Since  $|E| > 0$ , almost every point of  $E$  is a density point of  $E$ , i.e.,  $|E^d| = |E|$ .

For every  $s_0 \in E^d$ , there exists a sequence  $(s_n)_{n \in \mathbb{N}^*} \in (E^d)^{\mathbb{N}^*}$  such that  $\lim_{n \rightarrow +\infty} s_n = s_0$  and  $|E^d \cap [s_0, s_n]| > 0$  for every  $n \in \mathbb{N}^*$ . Since  $B^\top q \in W^{1,\infty}(0, S)$  and since  $B^\top q(s_0) = B^\top q(s_n) = 0$ , we have

$$0 = \int_{s_0}^{s_n} B^\top \dot{q}(\sigma) d\sigma = - \int_{s_0}^{s_n} w(s) B^\top A^\top q(\sigma) d\sigma.$$

But  $w \geq 0$  and  $w > 0$  on a set of positive measure of  $[s_0, s_n]$ , hence  $B^\top A^\top q(s)$  is either constant and equal to zero or has a sign change in  $[s_0, s_n]$ . In any case, by continuity of  $s \mapsto B^\top A^\top q(s)$  and by the intermediate value Theorem, there exists  $\sigma_n \in (s_0, s_n)$  such that  $B^\top A^\top q(\sigma_n) = 0$ . Taking the limit  $n \rightarrow +\infty$ , we have  $\sigma_n \rightarrow s_0$  and by continuity of  $B^\top A^\top q$ ,  $B^\top A^\top q(s_0) = 0$ . In other words, for every  $s \in E^d$ , we have  $B^\top q(s) = B^\top A^\top q(s) = 0$ . By repeating the procedure, we obtain  $B^\top (A^\top)^k q(s) = 0$  for  $s \in E^d$ ,  $k \in \{0, \dots, n-1\}$ , i.e., using the Kalman condition,  $q(s) = 0$  for every  $s \in E^d$ . This leads to a contradiction with  $q \not\equiv 0$ .

All in all, we have shown that  $w(s) = 0$  (resp.,  $w(s) = 1$ ) for almost every  $s \in [0, S]$  such that  $B^\top q(s) = 0$  (resp.,  $B^\top q(s) \neq 0$ ) and the set  $\{s \in [0, S] \mid B^\top q(s) = 0\}$  is a finite union of intervals. This ends the proof.  $\square$

**Remark 5.2.4.** From [22, Theorem 3.1], if we aim to show that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ , then one needs to show that the minimizer of (5.5) is a normal extremal, i.e., we need to show in the above proof that  $p_0 \neq 0$ . This question is open and is addressed in Open problem 2. ■

In Corollary 5.2.2, we have shown that any minimal time control with nonnegative Radon measure controls is a finite sum of Dirac impulses, i.e., it takes the form  $u = \sum_{i=1}^N m_i \delta_{\tau_i}$ , for some  $m_i \geq 0$ ,  $\tau_i \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)]$  and  $N \in \mathbf{N}$ . Furthermore, these parameters ( $N$ ,  $T = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ ,  $\tau_1, \dots, \tau_N$  and  $m_1, \dots, m_N$ ) satisfy the constrained minimization problem

$$\min \quad T \quad \left| \begin{array}{l} 0 \leq \tau_1 \leq \dots \leq \tau_N \leq T, \\ 0 \leq m_i \quad (i \in \{1, \dots, N\}), \\ y^1 - e^{TA} y^0 = \sum_{i=1}^N e^{(T-\tau_i)A} B m_i. \end{array} \right. \quad (5.6)$$

Due to the results of Corollary 5.2.2, if Assumption (H.1) is satisfied then this minimization problem has a solution for some large enough  $N$ .

### 5.2.3. Uniqueness of the minimal time control

**Proposition 5.2.5.** Let  $y^0$  and  $y^1$  be two points of  $\mathbf{R}^n$  such that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < +\infty$ , i.e.,  $y^1$  is accessible from  $y^0$ . Under Assumption (H.2), the minimal time control (given by Proposition 5.1.7) is unique and is a linear combination with nonnegative coefficients of at most  $\lfloor (n+1)/2 \rfloor$  Dirac impulses. Furthermore, as in Corollary 5.2.2, the time localization of these Dirac masses is given by the zero set of  $B^\top p$  with  $p$  a nontrivial solution of the adjoint system such that  $B^\top p$  has a constant sign.

**Remark 5.2.6.** Without the Assumption (H.2), we are not able to prove the uniqueness of the time optimal control. This is due to the fact that we do not have a sharp upper bound on the number of Dirac masses involved in the time optimal control. Proving the uniqueness of the time optimal control in general situation is the goal of Open problem 5. ■

The proof of the uniqueness of the minimal time control relies on the following lemma.

**Lemma 5.2.7.** Let  $k \leq n$ . Under Assumption (H.2), for every  $0 \leq \tau_1 < \tau_2 < \dots < \tau_k$ , the family  $\{e^{\tau_1 A} B, \dots, e^{\tau_k A} B\}$  is free in  $\mathbf{R}^n$ .

**Proof.** It suffices to prove this result for  $k = n$ , i.e., to prove that for every  $0 \leq \tau_1 < \tau_2 < \dots < \tau_n$ , the family  $\{e^{\tau_1 A} B, \dots, e^{\tau_n A} B\}$  is a basis of  $\mathbf{R}^n$ . Equivalently, we have to show that  $\text{rank } M = n$  with  $M = (e^{\tau_1 A} B, \dots, e^{\tau_n A} B) \in M_n(\mathbf{R})$ ; equivalently,  $\text{Ker } M^\top = \{0\}$ . Let  $p \in \text{Ker } M^\top$ , i.e.,  $B^\top e^{\tau_k A} p = 0$  for every  $k \in \{1, \dots, n\}$ , i.e., the function  $t \mapsto B^\top e^{tA} p$  vanishes  $n$  times. But since all eigenvalues of  $A$  are real, according to [17, Theorem 20 p. 143], either the function  $t \mapsto B^\top e^{tA} p$  is identically 0 or vanishes at most  $n-1$  times. This ensures that  $p = 0$ . □

**Proof of Proposition 5.2.5.** From Corollary 5.2.3, we already know that any minimal time control is a sum of at most  $N$  nonnegative Dirac impulses, for some large enough  $N \in \mathbf{N}$ . That is to say that any optimal control takes the form  $\sum_{i=1}^N m_i \delta_{\tau_i}$ , with  $m_1, \dots, m_N > 0$  and  $0 \leq \tau_1 < \dots < \tau_N \leq \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . Furthermore, Corollary 5.2.3 also ensures that there exists a nontrivial solution  $p$  of  $\dot{p} = -A^\top p$  such that  $B^\top p$  has a constant sign and  $\{\tau_1, \dots, \tau_N\} = \{t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)] \mid B^\top p(t) = 0\}$ .

Using the  $C^1$ -regularity of  $B^\top p$ , we deduce that,

1. if  $\tau_1 > 0$  and  $\tau_N < \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , then  $B^\top p$  admits at least  $2N$  zeros, counted with their multiplicity;
2. if  $\tau_1 = 0$  and  $\tau_N < \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , or if  $\tau_1 > 0$  and  $\tau_N = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , then  $B^\top p$  admits at least  $2N - 1$  zeros, counted with their multiplicity;

3. if  $\tau_1 = 0$  and  $\tau_N = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , then  $B^\top p$  admits at least  $2N - 2$  zeros, counted with their multiplicity.

Under Assumption (H.2), we have from [17, Exercise 13 p. 154] that if  $p \neq 0$ , then  $s \mapsto B^\top p(s)$  admits at most  $n - 1$  zeros (counted with their multiplicity). Consequently, in order to have a non-degenerate solution, we shall have  $2N - 2 \leq n - 1$ , i.e.,  $N \leq \lfloor (n + 1)/2 \rfloor$ .

Let us now consider two controls (consisting of a finite sum of nonnegative Dirac impulses)  $u_1$  and  $u_2$  steering the system from  $y^0$  to  $y^1$  in minimal time. The controls  $u_1$  and  $u_2$  are in one of the cases described by 1-3. By examining the various possible situations, we conclude that  $u_1 - u_2$  consists of at most  $n$  Dirac impulses (note that when  $n$  is even, this fact is direct, and when  $n$  is odd, we have to consider the possible presence of Dirac masses at time 0 and time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ ). Finally, using Lemma 5.2.7, we conclude that  $u_1 = u_2$ .  $\square$

### 5.3. Approximation of the minimal controllability time $\mathcal{T}_{\mathcal{U}}$ with bang-bang controls

The aim of this section is to give the structure of optimal controls in time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . Recall that we have  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) \leq \mathcal{T}_{\mathcal{U}}(y^0, y^1)$  and it is not proved that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ , except when the conditions of Proposition 5.1.12 are fulfilled. To obtain this structure, we are going to approach the nonnegative control in time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$  with nonnegative minimal time controls bounded in  $L^\infty$ -norm by some constant  $M$  which will then be taken larger and larger. For every  $M > 0$ , we define

$$\mathcal{T}_{\mathcal{U}}^M(y^0, y^1) = \inf \left\{ T > 0 \mid y^1 - e^{TA} y^0 \in \text{Acc}_+^M(T) \right\}$$

with

$$\text{Acc}_+^M(T) = \left\{ \Phi_T u, u \in \mathcal{U}_+^M(T) \right\}$$

where  $\mathcal{U}_+^M(T) = \{u \in \mathcal{U}_+(T) \mid u(\cdot) \leq M\}$ . Here,  $\mathcal{T}_{\mathcal{U}}^M(y^0, y^1) = +\infty$  if  $y^1$  is not reachable from  $y^0$  with controls  $u$  such that  $0 \leq u(\cdot) \leq M$ . It can be easily checked that  $\mathcal{T}_{\mathcal{U}}^M(y^0, y^1) \geq \mathcal{T}_{\mathcal{U}}(y^0, y^1) \geq \mathcal{T}_{\mathcal{M}}(y^0, y^1)$  and

$$\lim_{M \rightarrow +\infty} \mathcal{T}_{\mathcal{U}}^M(y^0, y^1) = \mathcal{T}_{\mathcal{U}}(y^0, y^1).$$

In this section, we are going to extract as  $M \rightarrow +\infty$  a limit control which consists of a finite sum of Dirac impulses. More precisely, we obtain the following result.

**Proposition 5.3.1.** Assume that  $A$  satisfies Assumption (H.1). Let  $y^0$  and  $y^1$  be points of  $\mathbb{R}^n$  such that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$ .

At time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ , there exists a control  $\underline{u} \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}}(y^0, y^1))$  which is a linear combination with nonnegative coefficients of a finite number  $N$  of Dirac impulses. If  $A$  satisfies Assumption (H.2) then  $N \leq \lfloor (n + 1)/2 \rfloor$ .

Furthermore, by defining  $u^M \in \mathcal{U}_+^M(\mathcal{T}_{\mathcal{U}}^M(y^0, y^1))$  for every  $M > 0$  large enough, there exists an increasing sequence  $(M_n) \in (\mathbb{R}_+^*)^{\mathbb{N}}$  such that  $(u^{M_n})_{n \in \mathbb{N}}$  converges vaguely to  $\underline{u}$ .

**Remark 5.3.2.** Since  $A$  satisfies the assumption (H.1), from Proposition 5.1.7, there exists a control  $\underline{u} \in \mathcal{M}_+(\mathcal{T}_{\mathcal{M}}(y^0, y^1))$  steering  $y^0$  to  $y^1$  in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . Furthermore, due to Corollary 5.2.2, we also know that this control is a linear combination with nonnegative coefficients of a finite number of Dirac impulses. This result shows that this is also the case in time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . Recall that,  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$  is not true in general.  $\blacksquare$

Before proving Proposition 5.3.1, let us first establish an auxiliary lemma ensuring that the number of zeros of  $t \in [0, T] \mapsto B^\top e^{(T-t)A^\top} p^1$  is uniformly bounded by some constant independent of  $p^1 \in \mathbb{R}^n \setminus \{0\}$ .

**Lemma 5.3.3.** For  $T > 0$  and  $p^1 \in \mathbb{R}^n \setminus \{0\}$ , we define  $Z(p^1) = \left\{ t \in [0, T] \mid B^\top e^{(T-t)A^\top} p^1 = 0 \right\}$ . Then there exists a constant  $N(T)$  independent of  $p^1$  such that  $\#Z(p^1) \leq N(T)$  and  $T \mapsto N(T)$  is non-decreasing. In addition, under Assumption (H.2), we have  $N(T) \leq n - 1$  for every  $T > 0$ .

**Proof.** When  $A$  satisfies the assumption (H.2) this result can be directly obtained from [17, Exercise 13 p. 154]. Let us then prove this result without this assumption.

Assume by contradiction that  $\sup_{p^1 \in \mathbb{R}^n \setminus \{0\}} \#Z(p^1) = +\infty$ , then there exists a sequence  $(p_k^1)_{k \in \mathbb{N}} \in (\mathbb{R}^n \setminus \{0\})^{\mathbb{N}}$  such that  $\#Z(p_k^1) \rightarrow +\infty$  as  $k \rightarrow +\infty$ . By linearity, and since  $p_k^1 \neq 0$ , we have  $Z(p_k^1) = Z(p_k^1/|p_k^1|)$  for every  $k \in \mathbb{N}$  and hence, we can assume that  $p_k^1 \in \mathbb{S}^{n-1}$  for every  $k \in \mathbb{N}$ . Consequently, up to a subsequence,  $p_k^1$  converges to some  $p_\infty^1 \in \mathbb{S}^{n-1}$  and by continuity of  $p^1 \mapsto \left(t \mapsto B^\top e^{(T-t)A^\top} p^1\right) \in C^0([0, T], \mathbb{R}^n)$ , we obtain that  $\#Z(p_\infty^1) = +\infty$ . But this is impossible since  $t \mapsto B^\top e^{(T-t)A^\top} p_\infty^1$  is analytic,  $p_\infty^1 \neq 0$  and the pair  $(A, B)$  satisfies the Kalman condition. Finally, it is obvious that  $T \mapsto N(T)$  is non-decreasing.  $\square$

**Proof of Proposition 5.3.1.** Let us write  $\mathcal{T}_U$  (resp.  $\mathcal{T}_U^M$ ) instead of  $\mathcal{T}_U(y^0, y^1)$  (resp.  $\mathcal{T}_U^M(y^0, y^1)$ ). Let us also define  $M_0 > 0$  large enough such that  $\mathcal{T}_U^{M_0} < +\infty$ .

For every  $M \geq M_0$ , there exists a minimal time control  $u^M \in \mathcal{U}_+^M(\mathcal{T}_U^M)$  steering  $y^0$  to  $y^1$  in time  $\mathcal{T}_U^M$ . According to [17, Corollary 2 p. 135]  $u^M$  is unique, takes its values in  $\{0, M\}$  and has a finite number of switches. More precisely, there exists  $p^M \in \mathbb{R}^n \setminus \{0\}$  such that

$$u^M(t) = \frac{M}{2} \left(1 + \operatorname{sign} \left(B^\top e^{tA^\top} p^M\right)\right) \quad (t \in [0, \mathcal{T}_U^M] \text{ a.e.}). \quad (5.7)$$

According to Lemma 5.3.3 and since  $M \mapsto \mathcal{T}_U^M$  is non-increasing, this control has at most  $N_0 = N(\mathcal{T}_U^{M_0})$  switches, where  $N(T)$  is defined by Lemma 5.3.3.

We define  $\Theta^M = \{t \in [0, \mathcal{T}_U^M] \mid u^M(t) = M\}$ . Since  $u^M$  has at most  $N_0$  switches, we have  $\Theta^M = \bigcup_{k=1}^{K^M} (t_k^M - \varepsilon_k^M, t_k^M + \varepsilon_k^M)$  with

$$\begin{aligned} 2K^M &\leq N_0 + 2, & 0 < t_1^M < \dots < t_k^M < t_{k+1}^M < \dots < t_{K^M}^M < \mathcal{T}_U^M, & \varepsilon_k^M > 0, \\ & & 0 \leq t_1^M - \varepsilon_1^M, & t_{K^M}^M + \varepsilon_{K^M}^M \leq \mathcal{T}_U^M, & t_k^M + \varepsilon_k^M < t_{k+1}^M - \varepsilon_{k+1}^M. \end{aligned}$$

Let us first check that  $|\Theta^M| = O\left(\frac{1}{M}\right)$  as  $M \rightarrow +\infty$ . Since  $u^M$  satisfies the control requirement, we have

$$y^1 - e^{\mathcal{T}_U^M A} y^0 = \int_0^{\mathcal{T}_U^M} e^{(\mathcal{T}_U^M - t)A} B u^M(t) dt = M \int_{\Theta^M} e^{(\mathcal{T}_U^M - t)A} B dt.$$

Let  $\varphi \in \mathbb{R}^n$  be a real eigenvector of  $A^\top$  and  $\lambda \in \mathbb{R}$  its associated eigenvalue (this real eigenvalue exists due to assumption (H.1)). Multiplying by  $\varphi$  the above equality, we obtain

$$\langle y^1 - e^{\mathcal{T}_U^M A} y^0, \varphi \rangle = M \langle \varphi, B \rangle \int_{\Theta^M} e^{(\mathcal{T}_U^M - t)\lambda} dt$$

and hence, since  $\langle \varphi, B \rangle \neq 0$  (the pair  $(A, B)$  satisfies the Kalman rank condition), and since  $\int_{\Theta^M} e^{(\mathcal{T}_U^M - t)\lambda} dt \geq e^{-|\lambda| \mathcal{T}_U^M} |\Theta^M|$ , we have

$$M |\Theta^M| \leq \frac{e^{|\lambda| \mathcal{T}_U^M} \langle e^{(\mathcal{T}_U - \mathcal{T}_U^M)A} y^1 - e^{\mathcal{T}_U A} y^0, \varphi \rangle}{\langle \varphi, B \rangle}.$$

This fact, together with  $\mathcal{T}_U^M \rightarrow \mathcal{T}_U$  as  $M \rightarrow +\infty$ , ensures that  $|\Theta^M| = O_{M \rightarrow +\infty}(1/M)$ .

We are now in a position to prove that, exactly in time  $\mathcal{T}_U$ , there exists a control realizing the controllability problem, which is composed of a sum of nonnegative Dirac impulses. Since  $K^M \in \mathbb{N}$  and  $2K^M \leq N_0 + 2$ , we have  $K = \liminf_{M \rightarrow +\infty} K^M \in \mathbb{N}$  and  $2K \leq N_0 + 2$ , and there exists an increasing sequence  $(M_n)_{n \in \mathbb{N}}$  such that  $M_n \rightarrow +\infty$  and  $K = K^{M_n}$  for every  $n \in \mathbb{N}$ . Since  $u^{M_n}$  satisfies the control requirement, we have for every  $n \in \mathbb{N}$ ,

$$y^1 - e^{\mathcal{T}_U^{M_n} A} y^0 = \sum_{k=1}^K M_n \int_{t_k^{M_n} - \varepsilon_k^{M_n}}^{t_k^{M_n} + \varepsilon_k^{M_n}} e^{(\mathcal{T}_U^M - t)A} B \, dt$$

or equivalently,

$$e^{(\mathcal{T}_U - \mathcal{T}_U^{M_n})A} \left( y^1 - e^{\mathcal{T}_U^{M_n} A} y^0 \right) = \sum_{k=1}^K M_n \int_{t_k^{M_n} - \varepsilon_k^{M_n}}^{t_k^{M_n} + \varepsilon_k^{M_n}} e^{(\mathcal{T}_U - t)A} B \, dt. \quad (5.8)$$

Now, by taking sub-sequences if necessary, for every  $k \in \{1, \dots, K\}$ ,  $t_k^{M_n}$  converges as  $n \rightarrow +\infty$  to some  $t_k \in [0, \mathcal{T}_U]$ . Since  $|\Theta^M| = O\left(\frac{1}{M}\right)$  as  $M \rightarrow +\infty$ , we also have  $\varepsilon_k^{M_n} = O\left(\frac{1}{M_n}\right)$  as  $n \rightarrow +\infty$ . We then define, for every  $k \in \{1, \dots, K\}$ ,  $m_k = \lim_{n \rightarrow +\infty} 2M_n \varepsilon_k^{M_n} \in \mathbf{R}_+$ . By continuity of  $t \mapsto e^{(\mathcal{T}_U - t)A}$ , we have

$$m_k e^{(\mathcal{T}_U - t_k)A} B = \lim_{n \rightarrow +\infty} M_n \int_{t_k^{M_n} - \varepsilon_k^{M_n}}^{t_k^{M_n} + \varepsilon_k^{M_n}} e^{(\mathcal{T}_U - t)A} B \, dt$$

and, taking the limit  $n \rightarrow +\infty$  in (5.8), we obtain

$$y^1 - e^{\mathcal{T}_U A} y^0 = \sum_{k=1}^K m_k e^{(\mathcal{T}_U - t_k)A} B.$$

This means that the nonnegative control  $\underline{u} = \sum_{k=1}^K m_k \delta_{t_k}$  steers  $y^0$  to  $y^1$  in time  $\mathcal{T}_U$ . In addition, this control consists of at most  $\lfloor N_0/2 \rfloor + 1$  Dirac impulses, and it is easy to see that  $u^{M_n}$  converges vaguely to  $\underline{u}$  as  $n \rightarrow \infty$ .

Finally, when  $A$  satisfies the assumption (H.2), we have  $N_0 \leq n - 1$ , and we obtain that the limit control is composed of at most  $\lfloor (n + 1)/2 \rfloor$  Dirac impulses.  $\square$

**Remark 5.3.4.** As in Remark 5.1.8, if Assumption (H.1) is not satisfied then the result of Proposition 5.3.1 may fail. In fact, in the above proof, it is not clear that  $|\Theta^M| = O_{M \rightarrow +\infty}(1/M)$ . Considering the example given in Remark 5.1.8, with  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $y^0 = 0$  and  $y^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , we see that the sequence of minimal time controls  $u^M$  (with  $0 \leq u^M(t) \leq M$ ) does not converge to a Radon measure as  $M$  goes to  $+\infty$ . Note that this fact was expected due to the discussion made in Remark 5.1.8. The computation details of this example are given in Appendix C.2.  $\blacksquare$

**Remark 5.3.5.** When Assumption (H.2) is satisfied, we have seen in Proposition 5.2.5 and in Proposition 5.3.1 that there exists a minimal time control at time  $\mathcal{T}_M(y^0, y^1)$  and a minimal time control in time  $\mathcal{T}_U(y^0, y^1)$ . These two controls are nonnegative Radon measures and sums of at most  $\lfloor (n + 1)/2 \rfloor$  Dirac impulses. This seems to indicate that these two controls are equal and the two minimal times are equal. However, this fact remains open in general.

Consequently, when  $y^1 \notin \mathcal{S}_+^*$  or when the assumptions of Proposition 5.1.12 (see also Remark 5.1.13), the minimization problem (5.6) cannot be used to determine  $\mathcal{T}_U(y^0, y^1)$ . Hence, a numerical strategy to approximate the minimal time  $\mathcal{T}_U(y^0, y^1)$  and corresponding minimal time is based on Proposition 5.3.1 (i.e., compute  $\mathcal{T}_U^M(y^0, y^1)$  and let  $M \rightarrow +\infty$ , see Numerical method 5).

Note that another numerical strategy to numerically find  $\mathcal{T}_U$  is proposed in Section 3. This is the Numerical method 6, which is based on the time rescaling presented in § 5.2.1.  $\blacksquare$

## Declaration of competing interest

The authors declare no competing interest.

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## Appendix A. Proof of Propositions 3 and 7

*Preliminaries* Let us first observe that  $A$  satisfies the Assumption (H.2). Let us denote  $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$  (the fact that  $\mathcal{T}_{\mathcal{U}} < \infty$  is ensured by Proposition 5.1.1 or Proposition 5.1.2) and  $\underline{u}$  the associated nonnegative minimal time control (the existence and uniqueness of  $\underline{u} \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}})$  is ensured by Theorem 1). We have  $y^1 - e^{\mathcal{T}_{\mathcal{U}}A}y^0 = \int_0^{\mathcal{T}_{\mathcal{U}}} e^{(\mathcal{T}_{\mathcal{U}}-t)A} B \, d\underline{u}(t)$ , i.e., since  $A$  is diagonalizable,

$$\langle \varphi_k, y^1 \rangle - e^{\lambda_k \mathcal{T}_{\mathcal{U}}} \langle \varphi_k, y^0 \rangle = \langle \varphi_k, B \rangle \int_0^{\mathcal{T}_{\mathcal{U}}} e^{\lambda_k(\mathcal{T}_{\mathcal{U}}-t)} \, d\underline{u}(t) \quad (k \in \{1, \dots, n\}),$$

with  $\lambda_k$  and  $\varphi_k$  given by (2.1). Since the pair  $(A, B)$  satisfies the Kalman rank condition, we have  $\langle \varphi_k, B \rangle \neq 0$  for every  $k \in \{1, \dots, n\}$ . Hence, we have

$$\frac{\langle \varphi_k, y^1 \rangle - e^{\lambda_k \mathcal{T}_{\mathcal{U}}} \langle \varphi_k, y^0 \rangle}{\langle \varphi_k, B \rangle} = \int_0^{\mathcal{T}_{\mathcal{U}}} e^{\lambda_k(\mathcal{T}_{\mathcal{U}}-t)} \, d\underline{u}(t) \quad (k \in \{1, \dots, n\}). \quad (\text{A.1})$$

This estimate is quite general. Taking in account the fact that  $y^1 = \bar{u}^1(1, \dots, 1)^\top$  and the explicit values of  $B$  and  $\varphi_k$  (see (2.1c)). We obtain

$$\langle \varphi_k, B \rangle = (-1)^{k+1} (n+1)^2 \sin((k-1/2)\pi/n)$$

and, after some computations,

$$\langle \varphi_k, y^1 \rangle = \bar{u}^1 \left( \sum_{j=0}^{n-1} \cos(j(k-1/2)\pi/n) - 1/2 \right) = \frac{\bar{u}^1}{2} \frac{(-1)^{k+1} \sin((k-1/2)\pi/n)}{1 - \cos((k-1/2)\pi/n)},$$

and finally, we have

$$\frac{\langle \varphi_k, y^1 \rangle}{\langle \varphi_k, B \rangle} = \frac{\bar{u}^1}{2n^2} \frac{1}{1 - \cos((k-1/2)\pi/n)} = \frac{-\bar{u}^1}{\lambda_k}. \quad (\text{A.2})$$

*Proof of Proposition 3* Since  $y^1 \in \mathcal{S}_+^*$ , we have, using Proposition 5.1.11, that  $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{M}}(y^0, y^1)$ . In addition, since all the eigenvalues of  $A$  have a negative real part, Proposition 5.1.2 ensures that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < \infty$ . Noticing that  $A$  satisfies the Assumption (H.2), we have, using Proposition 5.2.5, that the time optimal control  $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}})$  is unique

and is the sum of at most  $N = \lfloor (n+1)/2 \rfloor$  Dirac masses, that is to say that  $u = \sum_{i=1}^N m_i \delta_{t_i}$  for some  $m_1, \dots, m_N \in \mathbf{R}_+$

and some  $t_1, \dots, t_N \in [0, \mathcal{T}_{\mathcal{U}}]$ . Proposition 5.2.5 also ensures the existence of a nontrivial solution  $p$  of the adjoint problem such that  $B^\top p \geq 0$  and  $\{t_1, \dots, t_N\} = \{t \in [0, \mathcal{T}_{\mathcal{U}}] \mid B^\top p(t) = 0\}$ .

Using the expression of the optimal control as a sum of Dirac masses in (A.1), we obtain

$$\frac{\langle \varphi_k, y^1 \rangle - e^{\lambda_k \mathcal{T}_U} \langle \varphi_k, y^0 \rangle}{\langle \varphi_k, B \rangle} = \sum_{i=1}^N m_i e^{\lambda_k (\mathcal{T}_U - t)} \quad (k \in \{1, \dots, n\}).$$

Taking into account the explicit expression given by (A.2) and the fact that  $\mathcal{T}_U$  shall be minimal, we obtain the result of Proposition 3 and the claim of Remark 4.

*Proof of Proposition 7* In order to prove a priori estimates on the minimal time we follow the sketch of the proof of Proposition 5.1.7.

Since  $\lambda_k \leq 0$  and  $\underline{u} \geq 0$ , for every  $k \in \{1, \dots, n\}$ , we have

$$e^{\lambda_k \mathcal{T}_U} \int_0^{\mathcal{T}_U} d\underline{u}(t) \leq \int_0^{\mathcal{T}_U} e^{\lambda_k (\mathcal{T}_U - t)} d\underline{u}(t) \leq \int_0^{\mathcal{T}_U} d\underline{u}(t)$$

and hence, using (A.1), we deduce,

$$\frac{\langle \varphi_k, y^1 \rangle - e^{\lambda_k \mathcal{T}_U} \langle \varphi_k, y^0 \rangle}{\langle \varphi_k, B \rangle} \leq \int_0^{\mathcal{T}_U} d\underline{u}(t) \leq \frac{e^{-\lambda_k \mathcal{T}_U} \langle \varphi_k, y^1 \rangle - \langle \varphi_k, y^0 \rangle}{\langle \varphi_k, B \rangle}.$$

Consequently,  $\mathcal{T}_U$  satisfies

$$\sup_{k \in \{1, \dots, n\}} \frac{\langle \varphi_k, y^1 \rangle - e^{\lambda_k \mathcal{T}_U} \langle \varphi_k, y^0 \rangle}{\langle \varphi_k, B \rangle} \leq \inf_{k \in \{1, \dots, n\}} \frac{e^{-\lambda_k \mathcal{T}_U} \langle \varphi_k, y^1 \rangle - \langle \varphi_k, y^0 \rangle}{\langle \varphi_k, B \rangle}.$$

The above inequality, together with the explicit expression (A.2), ensure that the minimal time  $\mathcal{T}_U$  shall satisfy (2.4), and (2.5), when  $y^0$  is a steady state.

## Appendix B. $L^1$ -norm optimal controls

Since according to Proposition 5.1.7, we expect to obtain a Radon measure control at the minimal time, given some time  $T > \mathcal{T}_M(y^0, y^1)$ , we consider the norm optimal control problem

$$\begin{cases} \inf & \|u\|_{\mathcal{M}(0,T)} \\ & u \in \mathcal{M}_+(T), \\ & y^1 - e^{TA} y^0 = \Phi_T u \end{cases} \quad (\text{B.1})$$

and we expect that the infimum of times for which the above optimization problem admits a solution will be  $\mathcal{T}_M(y^0, y^1)$ . Based on Proposition 5.1.11, if we assume in addition that  $y^1 \in \mathcal{S}_+^*$ , then we obtain that the minimization problem (B.1) admits a solution  $u \in \mathcal{M}_+(T)$  for every  $T > \mathcal{T}_M(y^0, y^1)$ , and, in addition, we have  $\mathcal{T}_M(y^0, y^1) = \mathcal{T}_U(y^0, y^1)$ . If, in addition, the matrix  $A$  satisfies the assumption (H.1), then the minimization problem (B.1) admits a solution for every  $T \geq \mathcal{T}_U(y^0, y^1) = \mathcal{T}_M(y^0, y^1)$ . In particular, we have the following result.

**Proposition B.1.** Assume that  $y^1 \in \mathcal{S}_+^*$  and let  $y^0 \in \mathbb{R}^n$ , then  $\mathcal{T}_U(y^0, y^1) = \mathcal{T}_M(y^0, y^1)$  is the infimum of times  $T > 0$  such that the minimization problem (B.1) admits a solution.

Furthermore, if the matrix  $A$  satisfies Assumption (H.1) then this minimal time is achieved, i.e., the minimization problem (B.1) admits a solution.

Consequently, in this paragraph, we are going to assume that  $y^1 \in \mathcal{S}_+^*$  (and  $\mathcal{T}_U(y^0, y^1) < +\infty$ ).

Since we are looking for nonnegative controls, the optimal control problem (B.1) can also be expressed as

$$\begin{cases} \inf & \int_{[0,T]} du(t) \\ & u \in \mathcal{M}_+(T), \\ & y^1 - e^{TA} y^0 = \Phi_T u \end{cases}$$



or, equivalently, as

$$\inf_{u \in \mathcal{M}(0,T)} f_T(u) + g_T(\Phi_T u), \quad (\text{B.2})$$

with,

$$g_T(Y) = \begin{cases} 0 & \text{if } Y = y^1 - e^{TA} y^0, \\ \infty & \text{otherwise} \end{cases} \quad (Y \in \mathbb{R}^n),$$

$$f_T(u) = \int_{[0,T]} du(t) + \mathcal{I} \left( \int_{[0,T]} du^-(t) \right) \quad (u \in \mathcal{M}(0,T)),$$

where  $u^-$  is the negative part of  $u$ , and where  $\mathcal{I}(U) = \begin{cases} 0 & \text{if } U = 0, \\ \infty & \text{otherwise} \end{cases}$ , for every  $U \in \mathbb{R}$ .

Our aim is now to use Fenchel-Rockafellar duality. To this end, we first compute the adjoint of  $\Phi_T$  and the convex conjugates of  $g_T$  and  $f_T$ . We obtain, for every  $p^1 \in \mathbb{R}^n$ ,

$$\Phi_T^* p^1 = \left( t \mapsto B^\top e^{(T-t)A^\top} p^1 \right) \quad \text{and} \quad g_T^*(p^1) = \sup_{Y \in \mathbb{R}^n} (\langle p^1, Y \rangle - g(Y)) = \langle y^1 - e^{TA} y^0, p^1 \rangle.$$

Let us now compute the convex conjugate of  $f_T$ . We have, for every  $v \in C([0, T])$ ,

$$\begin{aligned} f_T^*(v) &= \sup_{u \in \mathcal{M}(0,T)} \left( \int_{[0,T]} v du - \int_{[0,T]} du - \mathcal{I} \left( \int_{[0,T]} du^- \right) \right) \\ &= \sup_{\substack{u \in \mathcal{M}(0,T) \\ u(\cdot) \geq 0}} \int_{[0,T]} (v(t) - 1) du^+(t) = \begin{cases} 0 & \text{if } \forall t \in [0, T], v(t) \leq 1, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Then the dual problem of (B.2) is

$$\inf_{p^1 \in \mathbb{R}^n} f_T^*(\Phi_T^* p^1) + g_T^*(-p^1).$$

Let us notice that for  $p^1 = 0$ , we have  $f_T^*(\Phi_T^* p^1) + g_T^*(-p^1) = 0$ , consequently, we have  $\inf_{p^1 \in \mathbb{R}^n} f_T^*(\Phi_T^* p^1) + g_T^*(-p^1) \leq 0$  and hence, this optimization problem can be expressed as

$$\begin{aligned} &\inf \quad \langle e^{TA} y^0 - y^1, p^1 \rangle \\ &\quad \left| \begin{array}{l} p^1 \in \mathbb{R}^n, \\ B^\top e^{(T-t)A^\top} p^1 \leq 1 \quad (t \in [0, T]). \end{array} \right. \end{aligned} \quad (\text{B.3})$$

By weak duality (see [4, Theorem 4.4.2 p. 135] or [12]), we always have

$$\inf_{u \in \mathcal{M}(0,T)} f_T(u) + g_T(\Phi_T u) \geq - \inf_{p^1 \in \mathbb{R}^n} f_T^*(\Phi_T^* p^1) + g_T^*(-p^1) \quad (\text{B.4})$$

In addition, using the strong duality result of [4, Theorem 4.4.3 p. 136], we obtain the equality in (B.4), for  $T > \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ .

**Lemma B.2.** *Let  $y^0 \in \mathbb{R}^n$  and  $y^1 \in \mathcal{S}_+^*$  be such that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$ . For every  $T > \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ , we have*

$$\inf_{u \in \mathcal{M}(0,T)} f_T(u) + g_T(\Phi_T u) = - \inf_{p^1 \in \mathbb{R}^n} f_T^*(\Phi_T^* p^1) + g_T^*(-p^1) = -f_T^*(\Phi_T^* p_T^1) - g_T^*(-p_T^1)$$

for some  $p_T^1 \in \mathbb{R}^n$ .

**Proof.** Let us first set  $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . It is easy to show that  $f_T$  and  $g_T$  are lower semi-continuous functions. In order to prove the strong duality result, we apply [4, Theorem 4.4.3 p. 136]. To this end, we show that if  $T > \mathcal{T}_{\mathcal{U}}(y^0, y^1)$  then  $0 \in \text{core}(\text{dom } g_T - \Phi_T \text{dom } f_T)$  where  $\text{core } A$  is the algebraic interior of  $A$  and  $\text{dom } h$  is the set of points for which  $h$  takes finite values. Since  $\text{dom } g_T = \{y^1 - e^{TA}y^0\}$ , we have to show that  $y^1 - e^{TA}y^0 \in \text{core}(\Phi_T \text{dom } f_T)$ . Firstly, since  $y^1 \in \mathcal{S}_+^*$ , for every  $\tau > 0$ , there exists a control  $u \in \mathcal{M}_+(\mathcal{T}_{\mathcal{U}} + \tau)$  such that  $y^1 - e^{(\mathcal{T}_{\mathcal{U}} + \tau)A}y^0 = \Phi_{\mathcal{T}_{\mathcal{U}} + \tau}u$  (see Lemma 5.1.4). Let us pick  $\tau = (T - \mathcal{T}_{\mathcal{U}})/2$  (so that  $T = \mathcal{T}_{\mathcal{U}} + 2\tau$ ). Secondly,  $y^1 \in \mathcal{S}_+^*$ , we use the small time local controllability, to show that there exists  $\varepsilon_0 > 0$  (depending on  $\tau$  and  $y^1$ ) such that  $B(y^1, \varepsilon_0) \subset \{\Phi_{\tau}v \mid v \in \mathcal{M}_+(\tau)\}$ . Consequently, for every  $\tilde{y} \in \mathbb{R}^n$  and every  $0 \leq \varepsilon < \varepsilon_0$ , there exists a control  $u_{\varepsilon} \in \mathcal{M}_+(T)$  such that  $y^1 + \varepsilon\tilde{y} - e^{TA}y^0 = \Phi_T u_{\varepsilon}$ . This ends the proof.  $\square$

**Remark B.3.** When  $T = \mathcal{T}_{\mathcal{U}}$ , this result is not clear. In fact, to mimic the above proof, we have to show that for every  $\tilde{y}$  small enough, there exists a perturbation  $\tilde{u}$  of the minimal time control  $u \in \mathcal{M}_+(T)$  such that  $u + \tilde{u}$  steers  $y^0$  to  $y^1 + \tilde{y}$  in time  $T$  and  $u + \tilde{u} \in \mathcal{M}_+(T)$ .  $\blacksquare$

**Lemma B.4.** Let  $y^0$  and  $y^1$  be two points of  $\mathbb{R}^n$  and let  $T \geq 0$ . If  $T \geq 0$  is such that the minimization problem (B.3) admits a minimizer, then there exists a control  $u \in \mathcal{M}_+(T)$  steering  $y^0$  to  $y^1$  in time  $T$ , which is a linear combination with nonnegative coefficients of a finite number of Dirac impulses.

**Proof.** Assume there exists a minimizer  $p^1 \in \mathbb{R}^n$  of (B.3). Let us write the optimality condition of this problem. Its Lagrangian is

$$\mathcal{L}(p^1, u) = \langle e^{TA}y^0 - y^1, p^1 \rangle + \int_{[0, T]} \left( B^{\top} e^{(T-t)A^{\top}} p^1 - 1 \right) du(t)$$

for  $p^1 \in \mathbb{R}^n$  and  $u \in \mathcal{M}([0, T])$ . The first-order optimality conditions give  $u \in \mathcal{M}_+(T)$  and  $\text{supp } u \subset \{t \in [0, T] \mid B^{\top} e^{(T-t)A^{\top}} p^1 = 1\}$  and  $0 = \frac{\partial \mathcal{L}}{\partial p^1}(p^1, u) = e^{TA}y^0 - y^1 + \int_{[0, T]} e^{(T-t)A} B du(t)$ . This means that there exists a non-negative Radon measure control  $u$  steering  $y^0$  to  $y^1$  in time  $T$ . In addition, since the pair  $(A, B)$  satisfies the Kalman rank condition, the set  $\{t \in [0, T] \mid B^{\top} e^{(T-t)A^{\top}} p^1 = 1\}$  is a finite union of singletons and hence the corresponding control is a finite sum of Dirac impulses.  $\square$

**Remark B.5.** Note that the existence of a minimum to the adjoint problem (here (B.3)) is usually related to an observability inequality. Here, even if the existence of a minimum to (B.3) leads to a control for the direct problem, we have not found any observability inequality.  $\blacksquare$

**Remark B.6.** Proposition B.1 and Lemmas B.2 and B.4 allows us to build the Algorithm 1 which aim is to find an approximation of the minimal time  $\mathcal{T}_{\mathcal{U}}(y^0, y^1)$ .  $\blacksquare$

Let us also provide a  $\Gamma$ -convergence result.

**Lemma B.7.** Let  $y^0 \in \mathbb{R}^n$  and  $y^1 \in \mathcal{S}_+^*$  be such that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < +\infty$  and set  $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . Let us define  $(T_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$  a sequence converging to  $\mathcal{T}_{\mathcal{U}}$  such that  $T_n > \mathcal{T}_{\mathcal{U}}$  for every  $n \in \mathbb{N}$ , and let us also define

$$J_n(p^1) = f_{T_n}^*(\Phi_{T_n}^* p^1) + g_{T_n}^*(-p^1) \quad (p^1 \in \mathbb{R}^n, n \in \mathbb{N})$$

$$\text{and} \quad J(p^1) = f_{\mathcal{T}_{\mathcal{U}}}^*(\Phi_{\mathcal{T}_{\mathcal{U}}}^* p^1) + g_{\mathcal{T}_{\mathcal{U}}}^*(-p^1) \quad (p^1 \in \mathbb{R}^n).$$

Then the sequence  $(J_n)_{n \in \mathbb{N}}$   $\Gamma$ -converges to  $J$ .

**Proof.** The proof of this result follows from the two following facts:

1. For every sequence  $(p_n)_n \in (\mathbb{R}^n)^{\mathbb{N}}$  converging to  $p \in \mathbb{R}^n$ , we have  $J(p) \leq \liminf_{n \rightarrow +\infty} J_n(p_n)$ .  
 If  $\liminf_{n \rightarrow +\infty} J_n(p_n) = +\infty$ , this fact is obvious. Let us then assume that  $J_n(p_n) < +\infty$  for every  $n \in \mathbb{N}$ . By continuity of  $(T, p) \mapsto \langle e^{TA}y^0 - y^1, p \rangle$ , we have  $\lim_{n \rightarrow +\infty} g_{T_n}^*(-p_n) = g_{\mathcal{T}_{\mathcal{U}}}^*(-p)$ . It is also obvious that  $f_{\mathcal{T}_{\mathcal{U}}}^*(\Phi_{\mathcal{T}_{\mathcal{U}}}^* p) \leq$

$f_{T_n}^*(\Phi_{T_n}^* p)$ . Consequently, for every  $n \in \mathbb{N}$ , we have  $f_{T_U}^*(\Phi_{T_U}^* p_n) \leq f_{T_n}^*(\Phi_{T_n}^* p_n)$ . Finally, by continuity of  $p \mapsto B^\top e^{(\mathcal{T}_U - t)A^\top} p$ , we obtain  $\lim_{n \rightarrow +\infty} f_{T_U}^*(\Phi_{T_U}^* p_n) = f_{T_U}^*(\Phi_{T_U}^* p)$ .

2. For every  $p \in \mathbb{R}^n$ , there exists a sequence  $(p_n)_{n \in \mathbb{N}} \in (\mathbb{R}^n)^{\mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} p_n = p$  and  $J(p) \geq \limsup_{n \rightarrow +\infty} J_n(p_n)$ .

If  $J(p) = +\infty$ , we also have  $J_n(p) = +\infty$  and hence, we simply take  $p_n \equiv p$ . If  $J(p) < +\infty$ , then  $f_{T_U}^*(\Phi_{T_U}^* p) = 0$  and there exists  $\lambda_n$  with  $\lambda_n \rightarrow 1$  as  $n \rightarrow +\infty$  such that  $f_{T_n}^*(\lambda_n \Phi_{T_n}^* p) = 0$ . It is also clear that  $\lim_{n \rightarrow +\infty} g_{T_n}^*(-\lambda_n p) = g_{T_U}^*(-p)$ . This ensures that  $\lim_{n \rightarrow +\infty} J_n(\lambda_n p) = J(p)$ .  $\square$

**Remark B.8.** Let us first note that with the assumptions of Lemma B.7, there exists  $p_n^1 \in \mathbb{R}^n$  a minimizer of  $J_n$  for every  $n \in \mathbb{N}$ . As a consequence of the  $\Gamma$ -convergence result, if the sequence  $(p_n^1)_n$  admits a cluster point, then this cluster point is a minimizer of  $J$  and hence, from Lemma B.4, this would ensure the existence of a Radon measure control at the minimal time  $\mathcal{T}_U$ . The difficulty is then to show that the sequence  $(p_n^1)_n$  admits a cluster point. In fact, the constraints in the minimization problem (B.3) are not enough to ensure some compactness.  $\blacksquare$

## Appendix C. Technical details of some examples

### C.1. Technical details related to Remarks 5.1.9 and 5.1.10

#### C.1.1. Technical details of the 1st item of Remark 5.1.9

In this example, we have considered the system (1.1), with matrices  $A$  and  $B$  given by  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , with the initial condition  $y^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and target  $y^1 = \begin{pmatrix} 0 \\ 1 + \varepsilon \end{pmatrix}$  (for some  $\varepsilon > 0$ ). Since  $y^1 \in \{y^0\} + \mathbb{R}_+ B$ , it is clear that  $\mathcal{T}_M(y^0, y^1) = 0$ . Let us show that  $\mathcal{T}_U(y^0, y^1) = +\infty$ , i.e.,  $y^1$  is not accessible from  $y^0$  with nonnegative  $L^\infty$  controls.

In fact, it is easy to see that  $e^{tA} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ , and hence, the solution  $y$  of (1.1) with initial state  $y^0$  and control  $u$  is given by

$$y(t) = \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} + \int_0^t \begin{pmatrix} \sinh(t - \tau) \\ \cosh(t - \tau) \end{pmatrix} u(\tau) d\tau \quad (t \in (0, T)).$$

In particular, at time  $t = T$ , the first component of  $y$  is given by

$$y_1(T) = \sinh T + \int_0^T \sinh(T - t) u(t) dt.$$

Note that whatever the time  $T > 0$  and the control  $u \geq 0$  is, we have  $y_1(T) > 0$ . This ensures that the target  $y^1$  can be reached only at time  $t = 0$ . But  $y^0 \neq y^1$  and hence, there does not exist a time  $T \geq 0$  and a control  $u \in L^\infty(0, T)$  such that  $y(T) = y^1$ , i.e.,  $\mathcal{T}_U(y^0, y^1) = +\infty$ .

#### C.1.2. Technical details of the 2nd item of Remark 5.1.9

In this example, we have considered the system (1.1), with matrices  $A$  and  $B$  given by  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , with the initial condition  $y^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and target  $y^1 = \begin{pmatrix} 0 \\ 1 + \varepsilon \end{pmatrix}$  (for some  $\varepsilon > 0$ ). Since  $y^1 \in \{y^0\} + \mathbb{R}_+ B$ , it is clear that  $\mathcal{T}_M(y^0, y^1) = 0$ . Let us show that  $\mathcal{T}_U(y^0, y^1) = \pi$ , ensuring that  $y^1$  is accessible from  $y^0$ , but not in arbitrarily small time.

In fact, it is easy to see that  $e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ , and hence, the solution  $y$  of (1.1) with initial state  $y^0$  and control  $u$  is given by

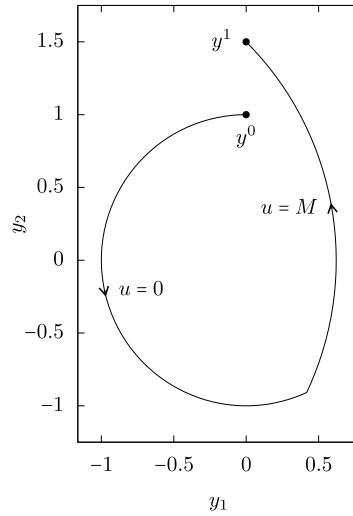


Fig. 19. State trajectory for the example given in the 2nd item of Remark 5.1.9, with the control given by (C.1), with  $M = 3/2$  and  $\varepsilon = 1/2$ .

$$y(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + \int_0^t \begin{pmatrix} -\sin(t-\tau) \\ \cos(t-\tau) \end{pmatrix} u(\tau) d\tau \quad (t \in (0, T)).$$

In particular, at time  $t = T$ , the first component of  $y$  is given by

$$y_1(T) = -\sin T - \int_0^T \sin(T-t)u(t) dt.$$

As for the 1st item of Remark 5.1.9, by considering the sign of  $y_1(T)$ , we easily obtain that  $\pi \leq \mathcal{T}_{\mathcal{U}}(y^0, y^1)$ . Let us now show that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \pi$ . To this end, we consider the control  $u$  given by

$$u(t) = \begin{cases} 0 & \text{if } t \in (0, \pi + \tau_0), \\ M & \text{if } t \in (\pi + \tau_0, \pi + \tau_0 + \tau_1) \end{cases} \quad (t \in (0, \pi + \tau)), \quad (\text{C.1})$$

where  $\tau_0, \tau_1 > 0$  and  $M > 0$  will be adjusted later so that  $u$  steers the system from  $y^0$  to  $y^1 = y^0 + \varepsilon B$  in time  $T = \pi + \tau_0 + \tau_1$ . With this control, the state follows a circle centered on 0 during the time interval  $[0, \pi + \tau_0]$ , and then follows a circle centered on  $(-M, 0)^\top$  during the time interval  $[\pi + \tau_0, \pi + \tau_0 + \tau_1]$  (see Fig. 19 for a graphical example of this trajectory). Using these geometrical considerations, we deduce that the control  $u$  given by (C.1) steers  $y^0$  to  $y^1$  for  $M > \varepsilon(2 + \varepsilon)/2$ , and we have

$$\tau_0 = \arcsin \frac{\varepsilon(2 + \varepsilon)}{2M} \quad \text{and} \quad \tau_1 = \arcsin \frac{M \cos \tau_0 + (1 + \varepsilon)(M + \sin \tau_0)}{M^2 + (1 + \varepsilon)^2}.$$

One can check that  $\tau_0$  and  $\tau_1$  go to 0 as  $M \rightarrow \infty$ , ensuring that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \pi$  (for every  $\varepsilon > 0$ ).

### C.1.3. Technical details of the 3rd item of Remark 5.1.9

In this example, we have considered the system (1.1), with matrices  $A$  and  $B$  given by

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For this system, we consider the initial condition  $y^0 = (0, 1, -1)^\top$  and the target  $y^1 = (-1, 0, 0)^\top$ .

It is easy to see that the pair  $(A, B)$  satisfies the Kalman rank condition and that the matrix  $A$  satisfies the assumption (H.1).

*Proof of  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \pi/2$*  It is easy to see that the impulse control  $u = \delta_0$  steers the solution of (1.1) from  $y^0$  to  $y^1$  in time  $\pi/2$  (see Fig. 17 for an illustration of this control and the associated state trajectory), consequently, we have  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) \leq \pi/2$ .

As shown (see Corollary 5.2.3), the measure control in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  is a linear combination Dirac masses which are located on the set of time  $t$  such that  $B^\top p(t) = 0$ , where  $p$  is a non-trivial solution of the adjoint system and is such that  $B^\top p(t) \geq 0$  for every  $t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)]$ . Consequently, we are going to study the sign of  $B^\top p$ .

First of all, given  $p^0 = (p_1^0, p_2^0, p_3^0)^\top \in \mathbb{R}^3 \setminus \{0\}$ , we have,

$$B^\top p(t) = p_1^0 \cos t - p_2^0 \sin t + p_3^0 - p_1^0,$$

where  $p$  is the solution of  $\dot{p} = -A^\top p$ , with initial condition  $p(0) = p^0$ .

Let us first show that  $p_1^0$  and  $p_2^0$  cannot be simultaneously equal to 0. To this end, we assume by contradiction that  $p_1^0 = p_2^0 = 0$ , then from the non-triviality condition, we have  $p_3^0 \neq 0$ , and hence  $B^\top p(t) = p_3^0$  is a non-zero constant. Since  $B^\top p$  has to be non-negative (and  $p_3^0 \neq 0$ ), we necessarily have  $B^\top p(t) > 0$  for every  $t \geq 0$ . This means that the control in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  would be constant equal to 0. But, it can be easily checked that with such a control the target is never reached, and this leads to a contradiction.

Consequently, we have  $(p_1^0, p_2^0) \neq 0$ , and since the controls are defined up to a multiplicative constant, we can assume that  $p_1^0 = \cos \theta$  and  $p_2^0 = \sin \theta$  for some  $\theta \in \mathbb{R}$ . Let us also denote by  $\alpha$  the value of  $p_3^0 - p_1^0$ . Thus, we have to study, for  $\theta, \alpha \in \mathbb{R}$ , the sign of

$$\varphi_{\theta, \alpha}(t) = \cos(t + \theta) + \alpha.$$

In other words, at the minimal time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , there exist  $\theta \in \mathbb{R}$  and  $\alpha \in \mathbb{R}$  such that  $\varphi_{\theta, \alpha}(t) \geq 0$  for every  $t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)]$ , and a control in time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$  is of the form  $u = \sum_{i=1}^N m_i \delta_{t_i}$ , for some  $N \in \mathbb{N}$ , some  $m_i > 0$  and some  $t_i \in \{t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)] \mid \varphi_{\theta, \alpha}(t) = 0\}$ .

We also know that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) \leq \pi/2$ . It is easy to see that for every  $T \leq \pi/2$ , the condition  $\varphi_{\theta, \alpha}(t) \geq 0$  for every  $t \in [0, T]$  implies that the set  $\{t \in [0, T] \mid \varphi_{\theta, \alpha}(t) = 0\}$  is the empty set, a singleton, or the set  $\{0, T\}$ . Let us consider the three possibilities.

1.  $\{t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)] \mid \varphi_{\theta, \alpha}(t) = 0\} = \emptyset$ :

In this case, the optimal control is the null control, and as already explained, this control does not steer the initial state to the target state.

2.  $\{t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)] \mid \varphi_{\theta, \alpha}(t) = 0\} = \{\tau\}$ :

In this case, any optimal control is of the form  $u = m\delta_\tau$  for some  $\tau \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)]$  and some  $m \geq 0$ .

Since the target condition has to be fulfilled, we necessarily have  $m = 1$  (in order to have  $y_3(T) = 0$ ). Assume by contradiction that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) < \pi/2$ . Then we have,

- if  $\tau > 0$ , since  $\tau \leq \mathcal{T}_{\mathcal{M}}(y^0, y^1) < \pi/2$ , we have  $y_1(t)^2 + y_2(t)^2 = y_1(\tau)^2 + y_2(\tau)^2 < 1$  for every  $t \geq \tau$ . Consequently, the target cannot be reached;
- if  $\tau = 0$ , then we have  $y_2(t) = \cos(t)$ , and hence, for every  $t \in [0, \pi/2)$  we have  $y_2(\mathcal{T}_{\mathcal{M}}(y^0, y^1)) > 0$ . Consequently, the target cannot be reached in a time lower than  $\pi/2$ .

3.  $\{t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)] \mid \varphi_{\theta, \alpha}(t) = 0\} = \{0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)\}$ : In this case, any optimal control is of the form  $u = m_0\delta_0 + m_1\delta_{\mathcal{T}_{\mathcal{M}}(y^0, y^1)}$ , with  $m_0, m_1 \in \mathbb{R}_+$ .

Since the target condition has to be fulfilled, we necessarily have  $m_0 + m_1 = 1$  (in order to have  $y_3(T) = 0$ ). Note that the case  $m_0 = 0$  and the case  $m_0 = 1$  are already covered by the previous item. Consequently, we assume that  $m_0 \in (0, 1)$ . Then for every  $t \in [0, \mathcal{T}_{\mathcal{M}}(y^0, y^1)]$ , we have,

$$(y_1(t) + m_0 - 1)^2 + y_2(t)^2 = (m_0 - 1)^2 + 1.$$

In particular, at time  $\mathcal{T}_{\mathcal{M}}(y^0, y^1)$ , the target shall be reached. Thus,  $m_0$  shall satisfy  $(m_0 - 2)^2 = (m_0 - 1)^2 - 1$ , i.e.,  $m_0 = 2$ , this leads to a contradiction with the fact that  $m_0 \in (0, 1)$ .

In conclusion, we have shown that  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \pi/2$  and that an optimal control in this time is  $u = \delta_0$ .

*Proof of  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) < \infty$*  To this end, given some  $\mu > 0$ , we consider the control  $u$  given by:

$$u(t) = \begin{cases} 0 & \text{if } t \in [0, \tau), \\ 1/\mu & \text{if } t \in (\tau, \tau + \mu), \\ 0 & \text{if } t \in (\tau + \mu, T], \end{cases}$$

where  $\tau \geq 0$  and  $T \geq \tau + \mu$  has to be chosen so that  $y^0$  is steered to  $y^1$  in time  $T$  (see Fig. 17 for an illustration of this type of control and its associated state trajectory). Let us denote by  $y$  the solution of (1.1) with this control  $u$ . First of all, we have,

$$y_3(t) = \begin{cases} -1 & \text{if } t \in [0, \tau), \\ -1 + (t - \tau)/\mu & \text{if } t \in (\tau, \tau + \mu), \\ 0 & \text{if } t \in (\tau + \mu, T]. \end{cases}$$

From this expression, we deduce after some computation that,

$$\begin{aligned} y_1(T) &= -\cos T - \sin T + \cos(T - \tau) \frac{\sin \mu}{\mu} - \sin(T - \tau) \frac{\cos \mu - 1}{\mu}, \\ y_2(T) &= \cos T - \sin T + \sin(T - \tau) \frac{\sin \mu}{\mu} + \cos(T - \tau) \frac{\cos \mu - 1}{\mu}. \end{aligned}$$

We now aim to find  $T$  and  $\tau$  such that the terminal conditions  $y_1(T) = -1$  and  $y_2(T) = 0$  are satisfied. This leads to the equations  $F(\mu, T, \tau) = 0$ , where we have set:

$$F(\mu, T, \tau) = \begin{pmatrix} 1 - \cos T - \sin T + \cos(T - \tau) \sin(\mu)/\mu - \sin(T - \tau)(\cos \mu - 1)/\mu \\ \cos T - \sin T + \sin(T - \tau) \sin(\mu)\mu + \cos(T - \tau)(\cos \mu - 1)/\mu \end{pmatrix}.$$

Note that  $F$  is  $C^\infty$  smooth on  $\mathbb{R}^3$  and for  $\mu = 0$ , we have,

$$F(0, T, \tau) = \begin{pmatrix} 1 - \cos T - \sin T + \cos(T - \tau) \\ \cos T - \sin T + \sin(T - \tau) \end{pmatrix}.$$

It is clear that  $(T, \tau) = (\pi/2, 0)$  is solution of  $F(0, T, \tau) = 0$ . But, in order to prove the existence of a gap, we look for another solution, and we observe that  $F(0, 2\pi, \pi/2) = 0$ . In order to prove that for every small enough  $\mu > 0$ , there exist  $T(\mu)$  and  $\tau(\mu)$  such that  $F(\mu, T(\mu), \tau(\mu)) = 0$ , we are going to use the implicit function theorem. To this end, we only need to check that  $\det(\partial_T F(0, 2\pi, \pi/2), \partial_\tau F(0, 2\pi, \pi/2)) \neq 0$ . In fact, we have,

$$(\partial_T F(0, T, \tau), \partial_\tau F(0, T, \tau)) = \begin{pmatrix} \sin T - \cos T - \sin(T - \tau) & \sin(T - \tau) \\ -\sin T - \cos T + \cos(T - \tau) & -\cos(T - \tau) \end{pmatrix},$$

and hence,

$$\det(\partial_T F(0, 2\pi, \pi/2), \partial_\tau F(0, 2\pi, \pi/2)) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -1 \neq 0.$$

This ensures, using the implicit function Theorem, that for small enough  $\mu > 0$ , there exist  $T(\mu)$  and  $\tau(\mu)$  such that  $F(\mu, T(\mu), \tau(\mu)) = 0$ , and  $(T(\mu), \tau(\mu))$  converges to  $(2\pi, \pi/2)$  as  $\mu$  goes to 0. Hence, we conclude that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \leq 2\pi < \infty$ .

*Proof of  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) > \pi/2$*  We already know that  $2\pi \geq \mathcal{T}_{\mathcal{U}}(y^0, y^1) \geq \mathcal{T}_{\mathcal{M}}(y^0, y^1) = \pi/2$ . Let us assume by contradiction that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \mathcal{T}_{\mathcal{M}}(y^0, y^1) = \pi/2$ . This would ensure the existence of a time  $T \in [\pi/2, \pi)$  and of a nonnegative control  $u$  in  $L^\infty(0, T)$  which steers  $y^0$  to  $y^1$  in time  $T$ . We define  $y$  the solution of (1.1) with this control. Note first that  $y_3$  is continuous, nondecreasing,  $y_3(0) = -1$  and  $y_3(T) = 0$ . Note also that we have

$$y_1(T) = -\sin(T) - \int_0^T \sin(T - s) y_3(s) \, ds.$$

Taking into account that we shall have  $y_1(T) = -1$ , the above equation is

$$1 - \sin(T) = \int_0^T \sin(T-s)y_3(s) \, ds.$$

Since  $y_3$  is continuous, non-positive and non-constant to zero, and since  $s \in [0, T] \mapsto \sin(T-s)$  is non-negative (recall that  $\pi/2 \leq T \leq \pi$ ), we have  $\int_0^T \sin(T-s)y_3(s) \, ds < 0$ . This obviously leads to a contradiction with  $T < \pi$  (since  $1 - \sin(T) \geq 0$ ). In fact, we just proved that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) \geq \pi$ .

*Conclusion* In this example, we provided a situation where  $0 < \mathcal{T}_{\mathcal{M}}(y^0, y^1) < \mathcal{T}_{\mathcal{U}}(y^0, y^1) < \infty$ . On Fig. 17, we have plotted the corresponding trajectories and controls. In this figure, one of the plots corresponds to the minimal time control with measures (red) and the other one is obtained through a numerical simulation and gives the minimal time control satisfying the additional constraint  $0 \leq u(t) \leq M$ , with  $M = 5$  (blue).

#### C.1.4. Technical details related to Remark 5.1.10

Recall that we consider here the initial condition  $y^0$  and the matrices  $A$  and  $B$  defined in the 3rd item of Remark 5.1.9 (see also Appendix C.1.3). We also consider the trajectory  $\bar{y}$ , solution of (1.1) with initial condition  $\bar{y}^0 = (0, 1, 0)^\top$  and null control. We thus have

$$\bar{y}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \quad (t \geq 0).$$

Since  $\bar{y}^0 \in \{y^0\} + \mathbf{R}_+ B$ , it is obvious that  $\mathcal{T}_{\mathcal{M}}^{\rightarrow \bar{y}}(y^0) = 0$  (recall that  $\mathcal{T}_{\mathcal{M}}^{\rightarrow \bar{y}}(y^0)$  and  $\mathcal{T}_{\mathcal{U}}^{\rightarrow \bar{y}}(y^0)$  are defined in Remark 5.1.10).

In order to prove that  $\mathcal{T}_{\mathcal{U}}^{\rightarrow \bar{y}}(y^0) \geq \pi/4$  we are going to proceed as in Appendix C.1.3. Recall that for every  $T > 0$  such that there exists a control  $u \in \mathcal{U}_+(T)$  steering  $y^0$  to  $\bar{y}(T)$  in time  $T$ , the third component of  $y$  is continuous and nondecreasing from  $-1$  to  $0$ . Note also that we have

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + \int_0^t \begin{pmatrix} -\sin \tau \\ \cos \tau \end{pmatrix} y_3(t-\tau) \, d\tau$$

and

$$\frac{1}{2} \frac{d}{dt} (y_1(t)^2 + y_2(t)^2) = y_2(t)y_3(t).$$

From the above equality and sign consideration, we have,

$$y_2(t) \geq \cos t - \sin t \quad (t \in [0, \pi/2]).$$

In particular, we deduce that  $\frac{d}{dt} (y_1(t)^2 + y_2(t)^2) < 0$  for every  $t \in (0, \pi/4)$ . But, since  $(y_1^0)^2 + (y_2^0)^2 = \bar{y}_1(T)^2 + \bar{y}_2(T)^2 = 1$ , we easily deduce that  $\bar{y}(T)$  cannot be reached in a time  $T$  lower than  $\pi/4$ .

#### C.2. Technical details on the example of Remark 5.3.4

As in Remark 5.1.8, if Assumption (H.1) is not satisfied then the result of Proposition 5.3.1 may fail. In fact, in the proof of Proposition 5.3.1, it is not clear that  $|\Theta^M| = O_{M \rightarrow +\infty}(1/M)$ . Considering the example given in Remark 5.1.8, with  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $y^0 = 0$  and  $y^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , we see that the sequence of minimal time controls  $u^M$  (with  $0 \leq u^M(t) \leq M$ ) does not converge to a Radon measure as  $M$  goes to  $+\infty$ . More precisely:

- From the discussion made in Remark 5.1.8, we have  $\mathcal{T}_{\mathcal{M}}(y^0, y^1) = \pi$ . Using the family of controls introduced in Remark 5.1.8, we can see that the assumptions of Proposition 5.1.12 are fulfilled (see in particular Fig. 16 and

note that the set of positive steady state is  $\mathcal{S}_+^* = \{0\} \times \mathbf{R}_+^*$ . This ensures that  $\mathcal{T}_{\mathcal{U}}(y^0, y^1) = \pi$  and hence, for  $M$  large enough, we can assume that  $\mathcal{T}_{\mathcal{U}}^M(y^0, y^1) \leq 3\pi/2$ .

• Using the bang-bang principle, we know that for every  $M > 0$ ,  $u^M(t) \in \{0, M\}$  for almost every  $t \in [0, T^M]$ . Consequently, the state trajectory lies on circles centered on 0 (when the control is null) and on circles centered on  $(-M, 0)^\top$  (when the control is equal to  $M$ ).

• By application of the Pontryagin maximum principle, we know that for every  $M > 0$  there exists  $p^M \in \mathbf{R}^2$  such that

$$u^M(t) = \begin{cases} M & \text{if } B^\top e^{tA^\top} p^M > 0, \\ 0 & \text{if } B^\top e^{tA^\top} p^M < 0. \end{cases}$$

Setting  $p^M = \alpha^M (\cos \theta^M, \sin \theta^M)^\top$ , for some  $\alpha^M \in \mathbf{R}$  and  $\theta^M \in \mathbf{R}$ , the above relation leads to

$$u^M(t) = \begin{cases} M & \text{if } \alpha^M \sin(\theta^M - t) > 0, \\ 0 & \text{if } \alpha^M \sin(\theta^M - t) < 0. \end{cases}$$

This together with the fact that  $T^M(y^0, y^1) \leq 3\pi/2$  for large enough values of  $M$ , ensures that the minimal time control  $u^M$  admits at most two jumps. It is also clear that there does not exist  $\varepsilon > 0$  such that  $u^M(t) = 0$  for every  $t \in [0, \varepsilon]$ . Consequently, we conclude that we have, for  $M > 0$  large enough,

$$u^M(t) = \begin{cases} M & \text{if } t \in (0, \tau_0), \\ 0 & \text{if } t \in (\tau_0, \tau_0 + \tau_1), \\ M & \text{if } t \in (\tau_0 + \tau_1, \tau_0 + \tau_1 + \tau_2), \end{cases} \quad (\text{C.2})$$

with  $\tau_0 = \tau_0(M) > 0$ ,  $\tau_1 = \tau_1(M) > 0$  and  $\tau_2 = \tau_2(M) \geq 0$ , and the minimal time is  $\mathcal{T}_{\mathcal{U}}^M(y^0, y^1) = \tau_0 + \tau_1 + \tau_2$ .

• We now compute the minimal time control  $u^M$  for large values of  $M$ , i.e., we give the explicit expression of the parameters  $\tau_0$ ,  $\tau_1$  and  $\tau_2$  in the expression (C.2). First of all, let us define

$$R(t) = e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad (t \in \mathbf{R}).$$

From the expression of  $u^M$ , it follows that the state trajectory follows a circle centered on  $(-M, 0)^\top$ , then a circle centered on 0 and finally a circle centered on  $(-M, 0)^\top$ . Since the final condition shall be satisfied, we have the relation between  $\tau_0$ ,  $\tau_1$ ,  $\tau_2$  and  $M$ :

$$y^1 = R(\tau_2) \left( R(\tau_1) \left( R(\tau_0) \left( y^0 + M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) - M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) - M \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

i.e., taking in account that  $y^0 = 0$  and  $y^1 = (1, 0)^\top$ ,

$$\begin{aligned} \frac{1}{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= (R(\tau_2) (R(\tau_1) (R(\tau_0) - I_2) + I_2) - I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (R(\tau_0 + \tau_1 + \tau_2) - R(\tau_1 + \tau_2) + R(\tau_2) - I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned}$$

that is to say,

$$1/M = \cos(\tau_0 + \tau_1 + \tau_2) - \cos(\tau_1 + \tau_2) + \cos(\tau_2) - 1, \quad (\text{C.3a})$$

$$0 = \sin(\tau_0 + \tau_1 + \tau_2) - \sin(\tau_1 + \tau_2) + \sin(\tau_2). \quad (\text{C.3b})$$

• Let us now find the minimum of  $\tau_0 + \tau_1 + \tau_2$  (for  $\tau_0, \tau_1, \tau_2 \in \mathbf{R}_+^*$ ) under the constraint (C.3). The Lagrangian  $L : \mathbf{R}^5 \rightarrow \mathbf{R}$  of this minimization problem is given by

$$\begin{aligned} L(\tau_0, \tau_1, \tau_2, \lambda_c, \lambda_s) &= \tau_0 + \tau_1 + \tau_2 \\ &\quad + \lambda_c (\cos(\tau_0 + \tau_1 + \tau_2) - \cos(\tau_1 + \tau_2) + \cos(\tau_2) - 1 - 1/M) \\ &\quad + \lambda_s (\sin(\tau_0 + \tau_1 + \tau_2) - \sin(\tau_1 + \tau_2) + \sin(\tau_2)). \end{aligned}$$



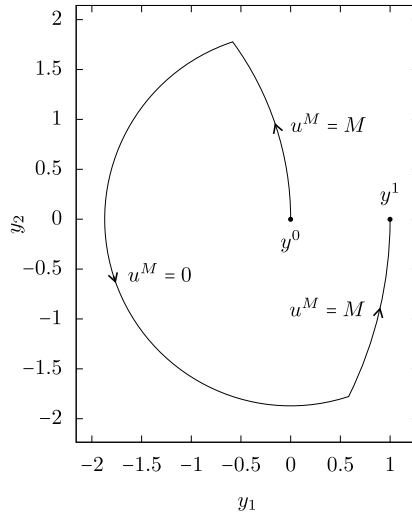


Fig. 20. Associated state trajectory for the minimal time control  $u^M$  computed in Remark 5.3.4. We chose  $M = 3$  and obtain  $T^M(y^0, y^1) \simeq 4.2362699$ .

The first order optimality conditions lead to

$$0 = \partial_{\tau_0} L = 1 - \lambda_c \sin(\tau_0 + \tau_1 + \tau_2) + \lambda_s \cos(\tau_0 + \tau_1 + \tau_2), \quad (\text{C.4a})$$

$$0 = \partial_{\tau_1} L = 1 - \lambda_c (\sin(\tau_0 + \tau_1 + \tau_2) - \sin(\tau_1 + \tau_2)) + \lambda_s (\cos(\tau_0 + \tau_1 + \tau_2) - \cos(\tau_1 + \tau_2)), \quad (\text{C.4b})$$

$$0 = \partial_{\tau_2} L = 1 - \lambda_c (\sin(\tau_0 + \tau_1 + \tau_2) - \sin(\tau_1 + \tau_2) + \sin(\tau_2)) + \lambda_s (\cos(\tau_0 + \tau_1 + \tau_2) - \cos(\tau_1 + \tau_2) + \cos(\tau_2)). \quad (\text{C.4c})$$

From (C.4c) and (C.3), we easily obtain  $\lambda_s = -M/(M+1)$ .

By subtracting (C.4a) to (C.4b) and (C.4b) to (C.4c), we obtain that  $\lambda_s$  and  $\lambda_c$  satisfy

$$\begin{pmatrix} \sin(\tau_1 + \tau_2) & -\cos(\tau_1 + \tau_2) \\ -\sin(\tau_2) & \cos(\tau_2) \end{pmatrix} \begin{pmatrix} \lambda_c \\ \lambda_s \end{pmatrix} = 0.$$

Since  $\lambda_s = -M/(M+1) \neq 0$ , the determinant of the above squared matrix shall be null, that is to say  $\sin \tau_1 = 0$ . Since we necessarily have  $\tau_1 > 0$  and since the minimal time shall not exceed  $3\pi/2$  (for  $M$  large enough), we necessarily have  $\tau_1 = \pi$  (for  $M$  large enough). Consequently, the constraints given by (C.3) are

$$1/M = -\cos(\tau_0 + \tau_2) + 2\cos(\tau_2) - 1, \quad (\text{C.5a})$$

$$0 = -\sin(\tau_0 + \tau_2) + 2\sin(\tau_2). \quad (\text{C.5b})$$

Since we just see that  $\tau_1 = \pi$ , and since  $\mathcal{T}_{\mathcal{U}}^M(y^0, y^1) \leq 3\pi/2$  (for  $M$  large enough), we have  $\tau_0 + \tau_2 \leq \pi/2$  (for  $M$  large enough). This ensures that  $\cos(\tau_0 + \tau_2) = \sqrt{1 - \sin^2(\tau_0 + \tau_2)}$ , and from (C.5b) we obtain  $\cos(\tau_0 + \tau_2) = \sqrt{1 - 4\sin^2 \tau_2} = \sqrt{4\cos^2 \tau_2 - 3}$ . This implies that  $\tau_2 \in (0, \pi/6]$ . From (C.5a), we obtain  $2\cos \tau_2 - \frac{M+1}{M} = \sqrt{4\cos^2 \tau_2 - 3}$ . This implies that  $\cos \tau_2 = \frac{M+1}{4M} + \frac{3M}{4(M+1)}$ . Finally, we deduce that

$$\tau_0 = \arccos\left(\frac{3M}{2(M+1)} - \frac{M+1}{2M}\right) - \arccos\left(\frac{M+1}{4M} + \frac{3M}{4(M+1)}\right), \quad (\text{C.6a})$$

$$\tau_1 = \pi, \quad (\text{C.6b})$$

$$\tau_2 = \arccos\left(\frac{M+1}{4M} + \frac{3M}{4(M+1)}\right), \quad (\text{C.6c})$$

Note that  $\tau_2 \rightarrow 0$  and  $\tau_0 \rightarrow 0$  as  $M \rightarrow \infty$ , and we recover the fact that  $\mathcal{T}_u^M(y^0, y^1) = \tau_0 + \tau_1 + \tau_2 \rightarrow \pi$  as  $M \rightarrow \infty$ . For  $M = 3$ , we plot on Fig. 20 the state trajectory associated to the minimal time control,  $u^M$ , given by (C.2), with parameters  $\tau_0$ ,  $\tau_1$  and  $\tau_2$  given by (C.6).

**Conclusion** In order to prove the claim of this remark (i.e., the optimal control  $u^M$  does not converge to a Radon measure), let us show that  $M\tau_2$  goes to  $+\infty$  when  $M$  goes to  $+\infty$ . In fact, we have

$$\tau_2 = \arccos\left(\frac{M+1}{4M} + \frac{3M}{4(M+1)}\right) = \arccos\left(1 - \frac{2M-1}{4M(M+1)}\right),$$

from which we conclude that  $\tau_2 \sim \sqrt{\frac{1}{2M}}$ , and hence  $\lim_{M \rightarrow +\infty} M\tau_2 = \infty$ .

## References

- [1] H. Bauer, *Measure and Integration Theory*, Translated from the German by Robert B. Burckel, De Gruyter Studies in Mathematics, vol. 26, Walter de Gruyter & Co., Berlin, 2001, xvi+230 pp.
- [2] A. Berman, R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, PA, 1994.
- [3] J.T. Betts, *Practical Methods for Optimal Control and Estimation Using Nonlinear Programming*, 2nd ed., Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2010.
- [4] J.M. Borwein, Q.J. Zhu, *Techniques of Variational Analysis*, Springer, New York, NY, 2005.
- [5] A. Bressan, F. Rampazzo, On differential systems with vector-valued impulsive controls, *Boll. Unione Mat. Ital.*, B (7) 2 (3) (1988) 641–656.
- [6] E. Casas, C. Pola, An algorithm for indefinite quadratic programming based on a partial Cholesky factorization, *RAIRO. Rech. Opér.* 27 (4) (1993) 401–426.
- [7] J.-M. Coron, *Control and Nonlinearity*, American Mathematical Society (AMS), Providence, RI, 2007.
- [8] J.-M. Coron, E. Trélat, Global steady-state controllability of one-dimensional semilinear heat equations, *SIAM J. Control Optim.* 43 (2) (2004) 549–569.
- [9] J.-M. Coron, E. Trélat, Global steady-state stabilization and controllability of 1D semilinear wave equations, *Commun. Contemp. Math.* 8 (4) (2006) 535–567.
- [10] G. Dal Maso, F. Rampazzo, On systems of ordinary differential equations with measures as controls, *Differ. Integral Equ.* 4 (4) (1991) 739–765.
- [11] G.B. Dantzig, A. Orden, P. Wolfe, The generalized simplex method for minimizing a linear form under linear inequality restraints, *Pac. J. Math.* 5 (1955) 183–195.
- [12] I. Ekeland, R. Temam, *Convex Analysis and Variational Problems*, Translated by Minerva Translations, Ltd., London, Studies in Mathematics and Its Applications, vol. 1, North-Holland Publishing Company, American Elsevier Publishing Company, Inc. IX, Amsterdam, Oxford, New York, 1976, 402 pp.
- [13] R. Fourer, D.M. Gay, B.W. Kernighan, A modeling language for mathematical programming, *Manag. Sci.* 36 (5) (1990) 519–554.
- [14] W. Heemels, M. Camlibel, Controllability of linear systems with input and state constraints, in: *Proc. IEEE CDC'07*, 2007.
- [15] M.I. Krastanov, On the constrained small-time controllability of linear systems, *Automatica* 44 (9) (2008) 2370–2374.
- [16] K. Kunisch, Z. Rao, Minimal time problem with impulsive controls, *Appl. Math. Optim.* 75 (1) (2017) 75–97.
- [17] E. Lee, L. Markus, *Foundations of Optimal Control Theory*, The SIAM Series in Applied Mathematics, vol. XII, John Wiley and Sons, Inc., New York, London, Sydney, 1967, 576 pp.
- [18] Y. Li, S. Osher, Coordinate descent optimization for  $\ell^1$  minimization with application to compressed sensing, a greedy algorithm, *Inverse Probl. Imaging* 3 (3) (2009) 487–503.
- [19] Y. Li, S. Osher, R. Tsai, Heat source identification based on  $\ell_1$  constrained minimization, *Inverse Probl. Imaging* 8 (1) (2014) 199–221.
- [20] J. Lohéac, E. Trélat, E. Zuazua, Minimal controllability time for the heat equation under unilateral state or control constraints, *Math. Models Methods Appl. Sci.* 27 (09) (2017) 1587–1644.
- [21] J. Lohéac, E. Trélat, E. Zuazua, Minimal controllability time for finite-dimensional control systems under state constraints, *Automatica* 96 (2018) 380–392.
- [22] M. Motta, F. Rampazzo, R. Vinter, Normality and gap phenomena in optimal unbounded control, *ESAIM Control Optim. Calc. Var.* 24 (4) (2018) 1645–1673.
- [23] D. Pighin, E. Zuazua, Controllability under positivity constraints of semilinear heat equations, *Math. Control Relat. Fields* 8 (3–4) (2018) 935–964.
- [24] R. Rishel, An extended Pontryagin principle for control systems whose control laws contain measures, *J. Soc. Ind. Appl. Math., A, on Control* 3 (1965) 191–205.
- [25] T.I. Seidman, How violent are fast controls?, *Math. Control Signals Syst.* 1 (1) (1988) 89–95.
- [26] T.I. Seidman, J. Yong, How violent are fast controls? II, *Math. Control Signals Syst.* 9 (4) (1996) 327–340.
- [27] T.I. Seidman, How violent are fast controls. III, *J. Math. Anal. Appl.* 339 (1) (2008) 461–468.
- [28] E. Trélat, *Contrôle optimal. Théorie et applications*, Vuibert, Paris, 2005.
- [29] E. Trélat, Optimal control and applications to aerospace: some results and challenges, *J. Optim. Theory Appl.* 154 (3) (2012) 713–758.
- [30] A. Wächter, L.T. Biegler, On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming, *Math. Program.* 106 (1(A)) (2006) 25–57.