Cost of null controllability for parabolic equations with vanishing viscosity and a transport term

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-Jornada de primavera en EDPs -UAM- 09/04/2021

Talk based on a paper with the same name (see https://hal.archives-ouvertes.fr/hal-02455632)

Introduction to null controllability

The control problem in which we act on the interior

Let $\Omega \subset \mathbb{R}^d$ be an open connected bounded set, $\mathcal{P}$ be a differential operator which acts on the spatial variable, let $\omega \subset \Omega$ be an open set, let $T > 0$ and let $y^0 \in (L^2(\Omega))^N$. We consider the following control problem:

$$\begin{cases} y_t + \mathcal{P}y = f1_\omega & \text{for } t > 0, \\ \text{boundary conditions} \\ y(0) = y^0. \end{cases} \tag{1}$$

Here $f \in L^2((0,T) \times \omega)$ is the control.

Null controllability

We seek to answer the following questions:

- Let $y^0 \in (L^2(\Omega))^N$. Does it exist a force $f \in (L^2((0,T)\times\omega))^N$ such that the solution of the previous system satisfies $y(T,\cdot)=0$?
- If the answer is affirmative for all $y^0 \in (L^2(\Omega))^N$, what is the minimum size of $\|f\|_{(L^2((0,T)\times\omega))^N}$ with respect to $\|y^0\|_{(L^2(\Omega))^N}$?

The answers of both questions, of course, may depend on the final time T.

Acting on the boundary

We may also control the system by acting on the boundary. In particular, we may consider the following control problem:

$$\begin{cases} y_t + \mathcal{P}y = 0 \quad \text{for } t > 0, \\ H(\nabla, n, x)y = f 1_\Gamma, \\ y(0) = y^0, \end{cases} \tag{2}$$

for $\Gamma \subset \partial\Omega$ a subset of the boundary, n the outwards normal vector, $H(\nabla, n, x)$ some generic pseudo-differential operator, and $f \in L^2((0, T) \times \Gamma)$ is the control.

State of the art of parabolic control problems with vanishing viscosity

A toy example: 1D transport equation

Let us consider the following system:

$$\begin{cases} y_t + y_x = 0 & t > 0,\ x \in (0,1), \\ y(t,0) = f(t) & t > 0, \\ y(0,\cdot) = y^0. \end{cases}$$

The solution is given by the Method of Characteristics:

$$y(t,x) = \begin{cases} y^0(x-t), & t \leq x \\ f(t-x), & t > x. \end{cases}$$

Thus, for any $T \geq 1$ we can obtain $y(T,\cdot) = 0$ by considering the null control; i.e. $f = 0$. Also, if $T < 1$, we do not have the null controllability property (consider, for instance, $y^0 = 1$).

1D heat equation with vanishing viscosity

We consider the following problem:

$$\begin{cases} y_t - \varepsilon y_{xx} + y_x = 0 & \text{in } (0,T)\times(0,1), \\ y(t,0) = f(t) & \text{on } (0,T), \\ y(t,1) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y^0 & \text{on } (0,1). \end{cases} \tag{3}$$

The null controllability of (3) is well-known for any time $T > 0$. Thus, the next interesting question is to study the cost of the control when $\varepsilon \to 0$. We recall that the cost is given by:

$$K(T,\varepsilon) := \sup_{y^0\in L^2(0,1)\setminus\{0\}} \inf_{f:y(T,\cdot)=0} \frac{\|f\|_{L^2(0,T)}}{\|y^0\|_{L^2(\Omega)}}.$$

Controllability results for (3)

- The solutions of (3) for a fixed initial value and control converge to the solutions of the transport equation when $\varepsilon \to 0$ in norm $C^0([0, T]; L^2(0,1))$.

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- The transport equation is null controllable if and only if: $T \geq 1$.
- Thus, we may prove by reductio ad absurdum that the cost of the control explodes for $T < 1$. In fact, Coron&Guerrero proved in 2005 that $K(T,\varepsilon) \geq ce^{c\varepsilon^{-1}}$.

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- Thus, we may prove by reductio ad absurdum that the cost of the control explodes for $T < 1$. In fact, Coron&Guerrero proved in 2005 that $K(T,\varepsilon) \geq ce^{c\varepsilon^{-1}}$.
- Moreover, it looks reasonable that the cost decays for $T \geq 1$. This, however, is a conjecture. Coron&Guerrero proved in 2005 that there is $\tilde{T} > 1$ such that for $T \geq \tilde{T}$ we have $K(T,\varepsilon) \leq Ce^{-c\varepsilon^{-1}}$, and Lissy proved in 2012 that this was true at least for $\tilde{T} = 2\sqrt{3}$.

Things get tricky when the control acts against the flow

Let us now consider the system:

$$\begin{cases} y_t - \varepsilon y_{xx} + y_x = 0 & \text{in } (0,T)\times(0,1), \\ y(t,0) = 0 & \text{on } (0,T), \\ y(t,1) = f(t) & \text{on } (0,T), \\ y(0,\cdot) = y^0 & \text{on } (0,1). \end{cases}$$

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Similar problem where the higher-order term vanishes

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- Stokes

The Stokes control problem with vanishing viscosity

Let us consider the following control problem:

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_d} y + \nabla q = f 1_\omega & \text{in } (0,T) \times (0,1)^d, \\ \nabla \cdot y = 0 & \text{in } (0,T) \times (0,1)^d, \\ y \cdot n = 0, \quad (Dy \cdot n)_{\text{tg}} = 0 & \text{on } (0,T) \times \partial((0,1)^d), \\ y(0,\cdot) = y^0 & \text{on } (0,1)^d. \end{cases}$$

for $d = 2, 3$. Its null controllability was proved by Guerrero in 2006, so the interest is to study the cost of the control with respect to the diffusivity.

Main results

- In $(0,1)^2$ the cost of the control explodes with ε if the time is small enough and the control domain is compactly included in $(0,1)^2$.
- In $(0,1)^2$ for a sufficiently large time the cost of the control decays exponentially.

Main results

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- In $(0,1)^2$ for a sufficiently large time the cost of the control decays exponentially.
- In $(0,1)^3$ for any time $T > 0$ the cost of the control explodes when $\varepsilon \to 0$ if the control domain is compactly included in $(0,1)^3$.

The control problem of the heat equation with mixed boundary conditions

Formulation of the problem

We are going to study the following control problem:

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } (0,T)\times\Omega, \\ \partial_n y + a^\varepsilon(x) y = 0, & \text{on } (0,T)\times\Gamma, \\ y = 0, & \text{on } (0,T)\times\Gamma^*, \\ y(0,\cdot) = y^0, & \text{on } \Omega. \end{cases}$$

Here, Ω is a C^2 domain of $\mathbb{R}^d$, $\Gamma \subset \partial\Omega$ is relatively open, and $\Gamma^* = \partial\Omega \setminus \Gamma$. We seek to estimate the minimum size of f so that the solution of the previous system satisfies $y(T,\cdot) = 0$, and in particular, the behaviour of the cost when $\varepsilon \to 0$.

The dual observability problem

The cost of the control is estimated with the Hilbert Uniqueness Method. For that, we focus on the observability properties of:

$$\begin{cases} -\varphi_t - \varepsilon\Delta\varphi - \partial_{x_1}\varphi = 0, & \text{in } (0,T)\times\Omega, \\ \varepsilon\partial_n\varphi + (\varepsilon a^\varepsilon + n_1)\varphi = 0, & \text{on } (0,T)\times\Gamma, \\ \varphi = 0, & \text{on } (0,T)\times\Gamma^*, \\ \varphi(T,\cdot) = \varphi^T, & \text{on } \Omega. \end{cases}$$

Indeed, we have that:

$$K(\Omega,\omega,T,\varepsilon) = \sup_{\varphi^T\in L^2(\Omega)\setminus\{0\}} \frac{\|\varphi(0,\cdot)\|_{L^2(\Omega)}}{\|\varphi\|_{L^2((0,T)\times\omega)}}.$$

Main idea

The operator $-\varepsilon\Delta - \partial_{x_1}$ is diagonalizable. Indeed, we can relate the spectral problem associated to the adjoint variable:

$$\begin{cases} -\varepsilon\Delta u - \partial_{x_1} u = \lambda u, & \text{in } \Omega, \\ \varepsilon\partial_n u + (\varepsilon a + n_1)\, u = 0, & \text{on } \Gamma, \\ u = 0, & \text{on } \Gamma^*, \end{cases} \tag{4}$$

and the following spectral problem with a self-adjoint operator:

$$\begin{cases} -\Delta v = \tilde{\lambda} v, & \text{in } \Omega, \\ \varepsilon\partial_n v + \left(\varepsilon a + \frac{n_1}{2}\right) v = 0, & \text{on } \Gamma, \\ v = 0, & \text{on } \Gamma^*. \end{cases} \tag{5}$$

We have that $(v, \tilde{\lambda})$ is a solution of (5) if and only if:

$$\left(v e^{-(2\varepsilon)^{-1} x_1}, \varepsilon\tilde{\lambda} + \frac{1}{4\varepsilon} \right),$$

is a solution of (4). Similarly, (u, λ) is a solution of (4) if and only if:

$$\left(u e^{(2\varepsilon)^{-1} x_1}, \frac{\lambda}{\varepsilon} - \frac{1}{4\varepsilon^2} \right),$$

is a solution of (5).

The first eigenvalue of the symmetrized system

We recall that by Rayleigh principle we have the equality:

$$\tilde{\lambda}_0^\varepsilon = \min \left\{ \int_\Omega |\nabla v|^2 dx + \int_\Gamma \left(a^\varepsilon + \frac{n_1}{2\varepsilon} \right) |v|^2 dx : \right.$$
$$\left. v \in H^1(\Omega), \|v\|_{L^2(\Omega)} = 1, v = 0 \text{ on } \Gamma^* \right\}.$$

This quantity will be key for studying the behaviour of the cost of null controllability.

Systems for which the cost of the controllability decays for a sufficiently large time

Conditions for the decay of the cost

Theorem

Let Ω be a C^2 domain, $\omega \subset \Omega$ be a subdomain and assume that (Γ, a^ε) satisfies that $a^\varepsilon \in L^\infty(\Gamma)$ and:

$$(a^\varepsilon + (2\varepsilon)^{-1} n_1) 1_\Gamma \geq 0 \tag{6}$$

for ε small enough. Then, there are $T_0, c, C > 0$ depending only on ω and Ω such that for ε small enough and all $T \geq T_0$ we have that:

$$K(\Omega, \omega, T, \varepsilon) \leq Ce^{-c\varepsilon^{-1}}.$$

Examples

- Dirichlet boundary conditions ($\Gamma = \emptyset$).
- Segments in which we have Dirichlet boundary conditions on the left end and Neumann boundary conditions on the right end.
- Any system in which $a^{\varepsilon} \geq 0$ and $n_1 1_{\Gamma} \geq 0$; that is, in which we have Dirichlet boundary conditions on the part of the boundary in which the flux of the transport enters and either Dirichlet or Robin with a positive coefficient on the other part of the boundary.
- Any system in which we almost have Dirichlet boundary conditions on the part of the boundary in which the flux of the transport enters and either Dirichlet or Robin with a coefficient whose negative part is not too large on the other part of the boundary.

Step 1: decay of the solutions

Under the geometric condition (6) we can see that the first eigenvalue of $-\varepsilon\Delta - \partial_{x_1}$ is greater than $\frac{1}{4\varepsilon}$. Thus, we obtain for a large T that:

$$\|\varphi(0,\cdot)\|_{L^2(\Omega)} \leq Ce^{-c\varepsilon^{-1}T}\|\varphi(T-1,\cdot)\|_{L^2(\Omega)}.$$

The dual meaning of this is that the solution of the control problem can be taken approximately to 0 with the null control for a sufficiently large time.

Step 2: exact null observability

With a Carleman inequality for the adjoint system we can obtain the estimate:

$$\|\varphi(T-1,\cdot)\|_{L^2(\Omega)} \leq Ce^{C\varepsilon^{-1}}\|\varphi\|_{L^2((T-1,T)\times\omega)}.$$

This observability inequality has the dual meaning that we can take the solution to 0 with a cost proportional to the initial value, but exponentially increasing with ε^{-1}.

Remark

The hypothesis $(a^\varepsilon + (2\varepsilon)^{-1}n_1)1_\Gamma \geq 0$ is also needed in this step.

Step 3: combining both inequalities

Combining both inequalities we obtain that:

$$\|\varphi(0,\cdot)\|_{L^2(\Omega)} \leq Ce^{-c\varepsilon^{-1}T}\|\varphi(T-1,\cdot)\|_{L^2(\Omega)} \\ \leq Ce^{(C-cT)\varepsilon^{-1}}\|\varphi\|_{L^2((T-1,T)\times\omega)},$$

so for a sufficiently large time the cost decays exponentially.

Systems for which the cost of the controllability explodes for any time $T > 0$

Examples of the explosion of the cost for any $T > 0$ (i)

We consider the system:

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_1} y = 1_\omega f, & \text{in } (0,T) \times \Omega, \\ \partial_n y = 0, & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y^0, & \text{on } \Omega. \end{cases} \tag{7}$$

Theorem

Let $h > 0$, Ω be a domain, and $\omega \subset \Omega$ be an open subset such that:

$$\pi_1(\omega) \subset (\inf \pi_1(\Omega) + h, \sup \pi_1(\Omega)).$$

Then, for all $T > 0$ there is $c > 0$ depending on h and T such that for all $\varepsilon > 0$:

$$K(\Omega, \omega, T, \varepsilon) \geq c e^{c\varepsilon^{-1}},$$

for K the cost of the null controllability of (7).

Here, $\pi_1(x) := x_1$.

Study of the cost when we have Neumann b. c.

In this situation we have that $K \geq ce^{c/\varepsilon}$. Indeed, $e^{-x_1/\varepsilon}$ is a static solution of the adjoint system for any domain Ω.

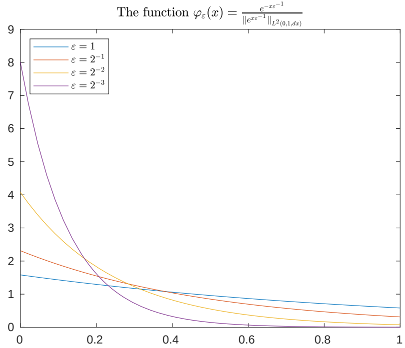

Figure: Normalized eigenfunctions of the adjoint of Neumann b. c.

Examples of the explosion of the cost for any $T > 0$ (ii)

We consider the control problem:

$$\begin{cases} y_t - \varepsilon\partial_{xx}y + \partial_x y = f1_\omega, & \text{in } (0,T)\times(-L,0), \\ \partial_x y(\cdot,-L) = 0, & \text{on } (0,T), \\ y(\cdot,0) = 0, & \text{on } (0,T), \\ y(0,\cdot) = y^0, & \text{on } (-L,0). \end{cases} \tag{8}$$

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$$\begin{cases} y_t - \varepsilon \partial_{xx} y + \partial_x y = f 1_\omega, & \text{in } (0,T) \times (-L,0), \\ \partial_x y(\cdot, -L) = 0, & \text{on } (0,T), \\ y(\cdot, 0) = 0, & \text{on } (0,T), \\ y(0,\cdot) = y^0, & \text{on } (-L,0). \end{cases} \tag{8}$$

Theorem

Let $h > 0$ and $\omega \subset (-L+h, 0)$ be an open subset. Then, for all $T > 0$ there are $c, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have the estimate:

$$K(\Omega, \omega, T, \varepsilon) \geq c e^{c\varepsilon^{-1}},$$

for K the cost of the control problem (8).

Proof of a the case with mixed b.c.

We prove the previous theorem by computing the first eigenfunction of the adjoint operator. Indeed, we can see that it is given by:

$$(u^{\varepsilon}(x), \lambda_0^{\varepsilon}) := \left(\sinh(-r_{\varepsilon}x)e^{-(2\varepsilon)^{-1}x}, -\varepsilon r_{\varepsilon}^2 + \frac{1}{4\varepsilon}\right),$$

for $r_{\varepsilon} \in (0, 1/(2\varepsilon))$ a value such that:

$$\frac{1}{2\varepsilon} - r_{\varepsilon} \leq \frac{1}{\varepsilon e^{L(2\varepsilon)^{-1}}},$$

which implies that:

$$-\varepsilon r_{\varepsilon}^2 + \frac{1}{4\varepsilon} \to 0.$$

Proof of a the case with mixed b.c. (illustration)

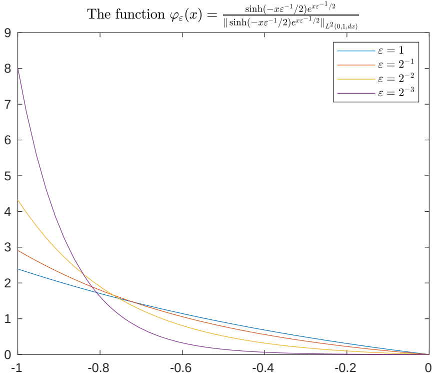

Figure: Approximation of the normalized eigenfunctions in $(0, 1)$ with mixed b. c.

Perspectives and open problems

Some open problems

- Determine if with any boundary condition the cost of the control is at most $Ce^{C\varepsilon^{-1}}$ for all $a \in L^\infty(\Gamma \times (0, \varepsilon_0))$, which is what can be expected considering the work of Lebeau&Guerrero in 2007.

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- Determine what happens when a depends of the time variable. The difficulty is that the spectral decomposition that we use does not work if a depends on the time variable.
- Studying the cost of approximate controllability on the heat equation with vanishing viscosity.

Thank you for your attention!

Is there any question?