Local null controllability of the penalized Boussinesq system with a reduced number of controls

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Abstract: In this paper we consider the Boussinesq system with homogeneous Dirichlet boundary conditions, defined in a regular domain $\Omega \subset \mathbb{R}^N$ for $N = 2$ and $N = 3$. The incompressibility condition of the fluid is replaced by its approximation by penalization with a small parameter $\varepsilon > 0$. We prove that our system is locally null controllable using a control with a restricted number of components, localized in an open set ω contained in Ω. We also show that the control cost is bounded uniformly with respect to $\varepsilon \to 0$. The proof is based on a linearization argument. The null controllability of the linearized system is obtained by proving a new Carleman estimate for the adjoint system. This inequality is derived by exploiting the coercivity of some second order differential operator involving crossed derivatives.

Key words: Carleman inequality, controllability, nonlinear system, penalized Stokes system

AMS subject classification: 35K40, 93B05, 93C20

Abbreviated title: null controllability of the penalized Boussinesq system.

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1 Introduction

1.1 Presentation of the system

For a given time $T > 0$, Ω a sufficiently smooth bounded, connected, open set of $\mathbb{R}^N$ $(N = 2, 3)$ and ω a nonempty open set contained in Ω, we consider the controlled penalized Boussinesq system

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \frac{1}{2} (\nabla \cdot y)\, y + \nabla p = \theta e_N + \tilde{v} 1_\omega & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta + \frac{1}{2} (\nabla \cdot y)\, \theta = v_{N+1} 1_\omega & \text{in } Q, \\ \nabla \cdot y = -\varepsilon p & \text{in } Q, \\ y = 0, \ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \ \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where e_N stands for the N-th vector of the canonical basis of $\mathbb{R}^N$, $Q := (0, T) \times \Omega$, $\Sigma := (0, T) \times \partial\Omega$. In the controlled system (1.1), $y = y(t, x)$ represents the velocity of the particles of the fluid, $\theta = \theta(t, x)$ their temperature and $v = (\tilde{v}, v_{N+1}) = (v_1, \ldots, v_N, v_{N+1})(t, x)$ stands for the control, acting on ω.

System (1.1) approximates the classical incompressible Boussinesq system (replacing $\nabla \cdot y = -\varepsilon p$ by $\nabla \cdot y = 0$ that implies that $\frac{1}{2} (\nabla \cdot y)\, y = 0$ and $\frac{1}{2} (\nabla \cdot y)\, \theta = 0$). This system has been introduced by Joseph Boussinesq in 1877 for modelling an incompressible fluid subjected to small variations of temperature. In addition, this way of approximating the incompressibility condition is called the penalty method and was introduced in [Tem68] in the Navier-Stokes case. As explained for instance in the survey [She97] and in the papers [Ber78] and [OJ84], this approximation procedure is widely used for numerical purpose.

We are going to focus on the small-time null controllability of the system (1.1) with a reduced number of controls. In particular, we seek controls v that satisfy:

$$\tilde{v} = 0, \text{ if } N = 2, \qquad v_1 = v_3 = 0, \text{ if } N = 3. \tag{1.2}$$

This choice matches with what is known about the controllability of the classical Boussinesq system when the control satisfies (1.2) (see [Car12]). From a modelling point of view in the two-dimensional case, one wants to control both the velocity and the temperature of the fluid by acting only in the temperature equation. Similarly, in the three-dimensional case, one seeks to control the velocity by acting on both the temperature and the velocity, but with a scalar force acting on the second component of the velocity. One of the main interest in studying the controllability of (1.1) is that the controls of (1.1) converge to the controls of the Boussinesq system as ε goes to 0. Thus, in order to compute numerically the controls of the Boussinesq system, it suffices to compute the controls of (1.1) for $\varepsilon > 0$ small enough.

1.2 Main result

In this part, we precise mathematically the main result of the paper. For this purpose, we introduce three different geometrical assumptions on Ω and ω.

Hypothesis 1.1. For $\Omega \subset \mathbb{R}^2$, let $\sigma^1, \dots, \sigma^k$ be the arc-length parametrizations of the different connected components of $\partial\Omega$. We assume that for any $i \in \{1, \dots, k\}$ and for any $s \in [0, 1]$ such that $(\sigma_1^i)'(s) = 0$ or $(\sigma_2^i)'(s) = 0$, we have that $\kappa^i(s) \neq 0$ where $\kappa^i(s)$ is the curvature of $\partial\Omega$ at the point $\sigma^i(s)$.

Hypothesis 1.2. For $\Omega \subset \mathbb{R}^3$, we assume that there are $\Omega_0 \subset \mathbb{R}^2$ satisfying Hypothesis 1.1, $\delta > 0$ and $H_-, H_+ \in C(\overline{\Omega_0}; [\delta, +\infty))$ such that:

$$\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_0 \text{ and } -H_-(x_1, x_2) < x_3 < H_+(x_1, x_2)\}.$$

Hypothesis 1.3. For $\Omega \subset \mathbb{R}^3$, we assume that there are an interval $I \subset \mathbb{R}$ and a curve $\mathfrak{C} \subset \mathbb{R}^2$ such that $\Gamma := I \times \mathfrak{C}$ is a relative non trivial open set of $\partial\Omega$. In addition, we consider a control domain $\omega \subset \Omega$ such that $\Gamma \cap \partial\omega$ contains a relative non trivial open set.

Example 1.1. We recall that, as proved in [BP20a], Hypothesis 1.1 is satisfied by any strictly convex smooth domain Ω. Moreover, according to [BP20a, Lemma 1.3], for any smooth domain Ω, one can find a rotation (i.e. a linear application of the type $U_\theta(x, y) = ((\cos\theta)x - (\sin\theta)y, (\sin\theta)x + (\cos\theta)y))$ that maps Ω to a domain Ω' satisfying Hypothesis 1.1. Furthermore, we can easily construct regular domains Ω satisfying Hypothesis 1.2, by considering a cylinder and some cupolas. Finally, Hypothesis 1.3 is satisfied by any smooth domain Ω containing a cylindrical part on its boundary.

We now present the main result of the paper:

Theorem 1.2. *Let Ω, ω be such that Hypothesis 1.1, 1.2 or 1.3 holds. Then, there exists $\varepsilon_0 > 0$ such that for every time $T > 0$ there exist $\delta_T > 0$ and $C_T > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and every initial data $(y^0, \theta^0) \in L^2(\Omega)^{N+1}$ satisfying $\|(y^0, \theta^0)\|_{L^2(\Omega)} \leq \delta_T$, there exist a control $v \in L^2(Q_\omega)^{N+1}$ satisfying* (1.2),

$$\|v\|_{L^2(Q_\omega)} \leq C_T, \tag{1.3}$$

and a weak solution (y, θ) of (1.1) *verifying*

$$(y, \theta)(T) = 0.$$

Here and in the whole paper we shorten the notation and denote $\|\cdot\|_V$ the norm $\|\cdot\|_{V^k}$ for V any Banach space and $k \in \mathbb{N}_*$. Indeed, the value of k is always easily deducible from the context.

In order to prove Theorem 1.2 we use the classical approach of proving first the controllability of the linearized system and then using a fixed point theorem. In addition, we prove the controllability of the linearized system by proving a Carleman inequality for the homogeneous adjoint system and then using the approach of [LTT13]. The most difficult and original part is to prove the Carleman inequality and in particular to prove the coercivity of some 2nd order differential operator involving crossed derivatives. Before continuing, we make some comments on Theorem 1.2, its proof and related bibliography:

- For the definition of a weak solution of (1.1), one can adapt [Tem68, Section I.1.]. Remark that the existence of a weak solution is guaranteed by an adaptation of [Tem68, Théorème I.2.] but the uniqueness is only valid in the 2D case.

- Theorem 1.2 is a null controllability result, uniform with respect to the parameter $\varepsilon > 0$ because of the estimate (1.3). So, by letting $\varepsilon \to 0$ in (1.1), we recover the results from [Car12] for less regular initial data but for more restrictive assumptions on Ω, ω.

- We use the geometric hypothesis for proving the coercivity of a 2nd order differential operator involving the crossed derivatives. For $\Omega \subset \mathbb{R}^2$, the assumption we make on the geometry of Ω, i.e. Hypothesis 1.1, has already been introduced in the recent paper [BP20a], dealing with the null controllability of the linear penalized 2-D Stokes system. This is crucial for proving the null controllability of such a system with one scalar control. Indeed, as shown in [Zua96, Theorem 1.2] and [BP20a, Section 2.1] some geometric hypothesis is needed because the linearized system around 0 cannot be controlled due to the fact that some eigenfunctions of the elliptic operator are not observable in a rhombus.

- At the heuristic level, it seems natural to obtain the controllability of the whole Boussinesq system (1.1) by acting only with $N-1$ scalar controls. Indeed, v_{N+1} directly controls the component θ by the last equation of (1.1), then θ acts as an indirect control to control y_N in the N-th equation of $(1.1)_1$. In addition, if $N = 2$ the penalized divergence condition implies that for ε small enough y_2 acts as an indirect control to control y_1. Similarly, if $N = 3$, y_1 is directly controlled by v_1 and y_2 is indirectly controlled by y_1 and y_3 by the penalized divergence condition for ε small enough.

- To prove the Carleman inequality, with the objective of highlighting the main ideas, we prioritize giving a clear proof, even at the expense of not getting the most optimal results. In that spirit, some recurrent operations are stated as lemmas (see Lemmas 2.5 and 2.7 below). These technical results can be useful for proving other Carleman inequalities in a different context.

- The results presented in Theorem 1.2 are original. Indeed, control problems with an approximation by penalization have first been studied in [IPY09, Section 4], where the penalized Stokes system was studied but without restriction on the control v. Next, the null controllability of the penalized Navier-Stokes system has been studied in [Bad11], but again without restriction on the control $\tilde{v}$. Finally, the null controllability of the penalized Stokes system in the 2D-case with a scalar control has been established in [BP20a]. So, Theorem 1.2 is the first null controllability result for a penalized system with a reduced number of controls in the nonlinear setting, see Section 4 for a similar result in the Navier-Stokes case. Moreover, both the linear and nonlinear results in the 3D-case in this paper are new.

- Considering other systems, the study of controllability problems in which the control has a reduced number of components has been an active topic of research recently. In particular, for the Stokes and Navier-Stokes systems we can consult for instance the following papers: [LZ96, FCGIP06, Gue07, CG09, CnG13, CL14, CnGG15, GM18, BP20b]. For more results on the controllability of linear parabolic systems with a reduced number of controls, see the survey [AKBGBdT11] and the references therein.

The rest of the paper is organized as follows:

- In Section 2, we prove the null controllability of the linearized system.

- In Section 3, we prove the local null controllability of (1.1), i.e. we prove Theorem 1.2.

- In Section 4, we make some remarks and present some open problems.

2 Null controllability of the linearized system

In this section we prove the null controllability of the linearized system of (1.1). We divide the proof as follows:

- In Section 2.1 we linearize the system (1.1) around 0, and we recall the equivalence between the null controllability of the linearized system and the observability of the corresponding adjoint system.

- In Section 2.2 we recall some previous results about parabolic systems, elliptic systems and Carleman inequalities.

- In Sections 2.3 and 2.4 we obtain some Carleman inequalities for the adjoint system, for the 2D case and the 3D case respectively.

- In Section 2.5 we use the source term method to get the null controllability of the linearized system and a source term, exponentially decreasing at $t = T$. This method has recently been used for many other control systems (see, for instance, [DL19, MT18, FCLdM16, BM20, Tak17, LB20, HSLB20, GZ21]).

We also introduce the notations $Q_\omega := (0,T)\times\omega$, δ_{ij} is a constant that is 1 if $i = j$ and 0 otherwise, $f = (\tilde{f}, f_{N+1}) = (f_1, \dots, f_N, f_{N+1})$ denotes the source term, $H^{k_1,k_2}(Q) := H^{k_1}(0,T;L^2(\Omega)) \cap L^2(0,T;H^{k_2}(\Omega))$ and $H^{k_1,k_2}(\Sigma) := H^{k_1}(0,T;L^2(\partial\Omega)) \cap L^2(0,T;H^{k_2}(\partial\Omega))$ (for $k_1, k_2 \in \mathbb{R}^+$).

2.1 Linearization around 0, null controllability results and observability estimates

The linearization of (1.1) around 0 gives

$$\begin{cases} y_t - \Delta y + \nabla p = \theta e_N + \tilde{v} 1_\omega & \text{in } Q, \\ \theta_t - \Delta\theta = v_{N+1} 1_\omega & \text{in } Q, \\ \nabla \cdot y = -\varepsilon p & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

The goal is to obtain the following result:

Proposition 2.1. *Let Ω, ω be such that Hypothesis 1.1, 1.2 or 1.3 holds. Then, there exist $\varepsilon_0 > 0$, $m \geq 1$ and $C > 0$ such that for every $T > 0$, $\varepsilon \in (0, \varepsilon_0)$ and $(y^0, \theta^0) \in L^2(\Omega)^{N+1}$, there exists a control $v \in L^2(Q_\omega)^{N+1}$ satisfying* (1.2),

$$\|v\|_{L^2(Q_\omega)} \leq K(T) \|(y^0, \theta^0)\|_{L^2(\Omega)}, \ \text{where } K(T) := C \exp\left(\frac{C}{T^m}\right), \tag{2.2}$$

and such that the solution (y, θ) of (2.1) *satisfies $(y, \theta)(T) = 0$.*

In order to prove Proposition 2.1, by the Hilbert Uniqueness Method (see, for instance, [Rus78], [Lio88] and [Cor07]), it is equivalent to establish an observability estimate for the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = 0 & \text{in } Q, \\ -\psi_t - \Delta\psi = \varphi_N & \text{in } Q, \\ \nabla \cdot \varphi = -\varepsilon\pi & \text{in } Q, \\ \varphi = 0,\ \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T,\ \psi(T) = \psi^T & \text{in } \Omega. \end{cases} \tag{2.3}$$

Proposition 2.2. *Let Ω, ω be such that Hypothesis 1.1, 1.2 or 1.3 holds. Then, there exist $\varepsilon_0 > 0$, $m \geq 1$ and $C > 0$ such that for every $T > 0$, $\varepsilon \in (0, \varepsilon_0)$ and $(\varphi^T, \psi^T) \in L^2(\Omega)^{N+1}$ the solution (φ, ψ) of (2.3) satisfies:*

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} + \|\psi(0,\cdot)\|^2_{L^2(\Omega)} \leq K(T) \iint_{Q_\omega} \delta_{3,N}|\varphi_1(t,x)|^2 + |\psi(t,x)|^2 dtdx, \tag{2.4}$$

with $K(T)$ as in (2.2).

To obtain the observability estimate (2.4), we will use Carleman estimates. More precisely, Proposition 2.2 will be a direct consequence of the Propositions 2.8, 2.10, 2.12, see below, and a classical dissipation argument.

2.2 Toolbox of elliptic, parabolic estimates and Carleman estimates

In this part, we recall elliptic and parabolic regularity estimates. We also present the Carleman estimates that will be useful in the sequel.

2.2.1 A parabolic and an elliptic result

Let us recall the regularity results of the Cauchy problem of the penalized Stokes system, which is given by the equations:

$$\begin{cases} u_t - \Delta u + \nabla q = \tilde{f} & \text{in } Q, \\ \varepsilon q + \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u^0 & \text{in } \Omega. \end{cases} \tag{2.5}$$

We have the following regularity estimate:

Lemma 2.3. *Let $k \in \mathbb{N}_*$ and $\Omega \subset \mathbb{R}^N$. Then, there are $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $u^0 \in H^{2k-1}(\Omega)^N$ and $\tilde{f} \in H^{k-1,2k-2}(Q)^N$ are such that $(u^0, \tilde{f})$ satisfies the compatibility conditions*

$$g_0 := u^0 \in H^1_0(\Omega),\ g_1 := \tilde{f}(0) - L_\varepsilon g_0 \in H^1_0(\Omega),\ \dots\ ,\ g_{k-1} := \frac{d^{k-2}\tilde{f}}{dt^{k-2}}(0) - L_\varepsilon g_{k-2} \in H^1_0(\Omega),$$

with $L_\varepsilon g = -\Delta g - \varepsilon^{-1}\nabla(\nabla \cdot g)$, we have that the solution (u, q) of (2.5) belongs to $H^{k,2k}(Q)^N \times H^{k-1,2k-1}(Q)$ with the estimate:

$$\|u\|_{H^{k,2k}(Q)} + \|q\|_{H^{k-1,2k-1}(Q)} \leq C\left(\|\tilde{f}\|_{H^{k-1,2k-2}(Q)} + \|u^0\|_{H^{2k-1}(\Omega)}\right). \tag{2.6}$$

As explained in [BP20a, Lemma 2.5] (though it was done for the specific case $u^0 = 0$), the proof of Lemma 2.3 is mainly by induction. The initial case ($k = 1$) can be proved again by the Galerkin method. As for the inductive case, we get the regularity in the time variable by considering that u_t is a solution of (2.5) with $(\tilde{f}, u^0)$ replaced by $(\tilde{f}_t, \Delta u^0 + \varepsilon^{-1}\nabla(\nabla \cdot u^0) + \tilde{f}(0, \cdot))$ and using again the Galerkin method (for instance, as in [Eva10, Theorem 7.1.6]). Moreover, we get the regularity in space by using the estimate for the steady Stokes problem given in [Tem77, Proposition I.2.2].

As for elliptic results, we recall the following result that is proved in [BP20a, Theorem 1.8] (considering the symmetry between the first and second variable):

Proposition 2.4. *Let Ω be a domain such that Hypothesis 1.1 holds. Then, for $a_0 > 0$ small enough, there is $C > 0$ such that for any function $w \in H^4(\Omega) \cap H^1_0(\Omega)$ and for any $a \in [0, a_0]$ we have that:*

$$\|\partial_{x_2} w\|_{C^0(\overline{\Omega})} \leq C(\|\partial_{x_1 x_2} w\|_{H^2(\Omega)} + \|L_a w\|_{H^1(\partial\Omega)}), \tag{2.7}$$

with:

$$L_a w = -\partial_{x_1 x_1} w - a\partial_{x_2 x_2} w. \tag{2.8}$$

Roughly, Proposition 2.4 states that the crossed derivative with some information on the second derivative on the boundary is coercive in $H^4(\Omega) \cap H^1_0(\Omega)$.

2.2.2 Classical Carleman estimates

We consider the following weights defined in Q:

$$\begin{aligned} \alpha(t,x) &:= \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{t^m(T-t)^m}, \qquad \xi(t,x) := \frac{e^{\lambda\eta(x)}}{t^m(T-t)^m}, \\ \alpha^*(t) &:= \sup_{x\in(0,L)} \alpha(t,x), \qquad \xi^*(t) := \inf_{x\in(0,L)} \xi(t,x), \end{aligned} \tag{2.9}$$

for $m \in \mathbb{R}^+$, $\lambda > 0$ that will be fixed later and for $\eta \in C^2(\overline{\Omega}; \mathbb{R})$ satisfying:

$$\eta = 0 \text{ on } \partial\Omega, \quad \eta > 0 \text{ in } \Omega, \quad \inf_{\overline{\Omega}\setminus\omega_0} |\nabla\eta| > 0 \tag{2.10}$$

for some $\omega_0 \subset\subset \omega$. The existence of such a function η is proved in [FI96]. In fact, the weights (2.9) are also taken from [FI96]. We recall that from (2.9) and (2.10) we obtain on Σ that:

$$\alpha(t,x) = \alpha^*(t), \quad \xi(t,x) = \xi^*(t). \tag{2.11}$$

We also recall that for all $m > 0$ and $\delta > 0$ there is $C > 0$ such that for all $s \geq 0$:

$$s(\xi + \xi^* + \alpha + \alpha^*) \leq Ce^{s\delta\alpha}. \tag{2.12}$$

In addition, for all $\delta > 0$ there is $C > 0$ such that if $\lambda \geq C$ we have that:

$$\alpha^* \leq (1+\delta)\alpha. \tag{2.13}$$

The weights (2.9) allow to prove very nice results when working with parabolic equations. For instance, they allow to estimate a function in a quantitative way with its derivatives and with a local term. From [CG09, Lemma 1], we can obtain the following result by an easy induction, recalling that we can deal with the local terms by integrating by parts and leaving just the lower and higher order terms:

Lemma 2.5. *Let $k \in \mathbb{N}$, $m \geq 1$ and $r \in \mathbb{R}$. Then, there are $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\lambda \geq \lambda_0$, $s \geq CT^{2m}$ and $u \in L^2\big(0,T;H^{k+1}(\Omega)\big)$, we have:*

$$\sum_{i=0}^{k} s^{2+r-2i}\lambda^{3+r-2i}\iint_Q e^{-2s\alpha}\xi^{2+r-2i}|\nabla^i u|^2 \leq C\bigg(s^{r-2k}\lambda^{1+r-2k}\iint_Q e^{-2s\alpha}\xi^{r-2k}|\nabla^{k+1}u|^2 + s^{2+r}\lambda^{3+r}\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^{2+r}|u|^2 + s^{2+r-2k}\lambda^{3+r-2k}\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^{2+r-2k}|\nabla^k u|^2\bigg). \quad (2.14)$$

To continue with, we recall the following Carleman estimate for the heat equation with nonhomogeneous boundary conditions.

Lemma 2.6. *There is $C > 0$ such that for all $u \in C^2(\overline{Q})$, $a \in (0,1]$, $m \geq 1$, $r \in \mathbb{R}$, $\lambda \geq C$ and $s \geq Ce^{C\lambda}(T^m + T^{2m})$ we have the inequality:*

$$s^{3+r}\lambda^{4+r}\iint_Q e^{-2s\alpha}\xi^{3+r}|u|^2 + s^{1+r}\lambda^{2+r}\iint_Q e^{-2s\alpha}\xi^{1+r}|\nabla u|^2 \leq Cs^r\lambda^r\bigg(\iint_Q e^{-2s\alpha}\xi^r|g|^2 + s^{3+r}\lambda^{4+r}\iint_{Q_{\omega_0}} \xi^{3+r}|u|^2 + s^{1+r}\lambda^{1+r}\iint_\Sigma e^{-2s\alpha}\xi^{1+r}|\partial_n u|^2\bigg), \quad (2.15)$$

for $g := -au_t - \Delta u$. In addition, if $u \equiv 0$ on Σ, one can drop the trace term in the right-hand side of (2.15).

Lemma 2.6 is well known. We get the case $i = 1$, $m \geq 1$ and $r \in \mathbb{R}$ by repeating all the steps in [FCGBGP06, Theorem 1], where the authors prove the case $(a,r,m) = (1,0,1)$. In addition, we can get the uniformity in the parameter $a \in (0,1]$ following the steps of, for instance, [FI96] or [Bad11, Lemma 4.1]. As for the case with Dirichlet boundary condition, this is the classical Carleman estimate for the heat equation, which is proved in [FI96].

To finish with, we consider the following technical result:

Lemma 2.7. *Let $k \in \mathbb{N}$, $m > 0$, $r \in \mathbb{R}$ and let us consider the weights given in* (2.9)*. Then, there is $C > 0$ such that for all $\lambda \geq 1$ and $s \geq e^{C\lambda}(T^m + T^{2m})$ the following estimate holds:*

$$\|s^r e^{-s\alpha^*}(\xi^*)^r\varphi\|_{H^{k,2k}(Q)} + \|s^r e^{-s\alpha^*}(\xi^*)^r\pi\|_{H^{k-1,2k-1}(Q)} \leq \|s^{r+k+k/m}e^{-s\alpha^*}(\xi^*)^{r+k+k/m}\varphi\|_{L^2(Q)}, \quad (2.16)$$

for φ the solution of:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = 0 & \text{in } Q, \\ \varepsilon\pi + \nabla\cdot\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T & \text{in } \Omega. \end{cases} \tag{2.17}$$

As far as we know, very similar versions of Lemma 2.7 have been used to deal with the trace terms since [Ima01] and it is widely known by the control community of Navier-Stokes like system. However, this result has mainly been used for $k = 1$ or $k = 2$, and as far as we know, the proof of the general case is not written. Thus, for completeness, we give the proof in Appendix A.

2.3 Proof of the observability estimate in 2-D

In this section we prove the following Carleman inequality for the solution of the homogeneous 2-D Boussinesq system (2.3):

Proposition 2.8. *Let Ω be such that Hypothesis 1.1 holds, $\omega \subset \Omega$ be a nonempty open set and $m \geq 8$. Then, there are $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $(\varphi^T, \psi^T) \in L^2(\Omega)^3$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have:*

$$s^{12}\lambda^{13} \iint_Q e^{-5s\alpha^*}(\xi^*)^{15}|\varphi|^2 + s^3\lambda^4 \iint_Q e^{-6s\alpha}\xi^3|\psi|^2 \leq C \iint_{Q_\omega} e^{-4s\alpha}|\psi|^2, \tag{2.18}$$

for the weights defined in (2.9) *and* (φ, ψ) *the solution of* (2.3)*.*

First, we recall from [BP20a, Theorem 1.7] (considering that there is a symmetry between the variable ϕ_1 and ϕ_2 and replacing s by $\frac{5}{2}s$) the following Carleman estimate for the penalized Stokes system with an observation only with the second component.

Proposition 2.9. *Let Ω, ω, m as in Proposition 2.8, $\omega_0 \subset\subset \omega$ be an open set such that $\inf_{\omega\setminus\omega_0} |\nabla\eta| > 0$ and let $m \geq 8$. Then, there are $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\varphi^T \in L^2(\Omega)^2$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have:*

$$s^{15}\lambda^{16} \iint_Q e^{-5s\alpha^*}(\xi^*)^{15}|\varphi|^2 \leq Cs^{34}\lambda^{35} \iint_{Q_{\omega_0}} e^{-5s\alpha}\xi^{34}|\varphi_2|^2,$$

for the weights defined in (2.9) *and φ the solution of* (2.17)*.*

Proof of Proposition 2.8. Considering Proposition 2.9 and Lemma 2.7 for $k = 1$, $r = 6$ and replacing s by $5s/2$, applied to the first equation of (2.3), we obtain the estimate:

$$s^{12}\lambda^{13}\int_0^T e^{-5s\alpha^*}(\xi^*)^{12}\|\varphi\|^2_{H^2(\Omega)} + s^{12}\lambda^{13}\iint_Q e^{-5s\alpha^*}(\xi^*)^{12}|\varphi_t|^2 \leq Cs^{15}\lambda^{16}\iint_Q e^{-5s\alpha^*}(\xi^*)^{15}|\varphi|^2 \leq Cs^{34}\lambda^{35}\iint_{Q_{\omega_0}} e^{-5s\alpha}\xi^{34}|\varphi_2|^2. \quad (2.19)$$

Using the classical Carleman inequality of the heat equation on the ψ variable with homogeneous Dirichlet boundary conditions for the second equation of (2.3), (by Lemma 2.6 with $3s$ instead of s) we find for $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$ that:

$$s^3\lambda^4\iint_Q e^{-6s\alpha}\xi^3|\psi|^2 \leq C\left(\iint_Q e^{-6s\alpha}|\varphi_2|^2 + s^3\lambda^4\iint_{Q_\omega} e^{-6s\alpha}\xi^3|\psi|^2\right). \quad (2.20)$$

We deal with the local term in the right hand side of (2.19) as in [Car12]. We consider a positive function χ supported in ω such that $\chi = 1$ in ω_0. We have for all $\delta > 0$ by integrating by parts, using Lemma A.2, (2.12), (2.13) and Young's inequality the following estimate:

$$\begin{aligned} s^{34}\lambda^{35}\iint_{Q_{\omega_0}} e^{-5s\alpha}\xi^{34}|\varphi_2|^2 &\leq s^{34}\lambda^{35}\iint_{Q_\omega} e^{-5s\alpha}\chi\xi^{34}|\varphi_2|^2 \\ &= -s^{34}\lambda^{35}\iint_{Q_\omega} e^{-5s\alpha}\chi\xi^{34}(\psi_t + \Delta\psi)\varphi_2 = s^{34}\lambda^{35}\iint_{Q_\omega}\psi(\partial_t - \Delta)(e^{-5s\alpha}\chi\xi^{34}\varphi_2) \\ &\leq C_\delta\iint_{Q_\omega} e^{-4s\alpha}|\psi|^2 + \delta s^{12}\lambda^{13}\left(\int_0^T e^{-5s\alpha^*}(\xi^*)^{12}\|\varphi\|^2_{H^2(\Omega)} + \iint_Q e^{-5s\alpha^*}(\xi^*)^{12}|\varphi_t|^2\right). \quad (2.21)\end{aligned}$$

Thus, if we combine (2.19)-(2.21), by choosing δ small enough we obtain (2.18). □

2.4 Proof of the observability estimate in 3-D

In this section we prove the following Carleman inequality, for the solution of the homogeneous 3-D Boussinesq system assuming Hypothesis 1.2 or 1.3.

2.4.1 A Carleman inequality assuming Hypothesis 1.2

In this section we prove the following result:

Proposition 2.10. *Let Ω be such that Hypothesis 1.2 holds, $\omega \subset \Omega$ be a nonempty open set and let $m \geq 2$. Then, there are $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $(\varphi^T, \psi^T) \in L^2(\Omega)^4$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have:*

$$s^{51}\lambda^{52}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2 + s^3\lambda^4\iint_Q e^{-3s\alpha}\xi^3|\psi|^2 \leq C\iint_{Q_\omega} e^{-s\alpha}(|\psi|^2 + |\varphi_2|^2), \quad (2.22)$$

for the weights defined in (2.9) and (φ, ψ) the solution of (2.3).

A geometrical result. First of all we prove the following geometrical result:

Proposition 2.11. *Let Ω be such that Hypothesis 1.2 holds. Then, for $a_0 > 0$ small enough, there is $C > 0$ such that for any function $w \in H^4(\Omega) \cap H^1_0(\Omega)$ and for any $a \in [0, a_0]$ we have that:*

$$\|\partial_{x_1} w\|_{L^2(\Omega)} \leq C \left(\|\partial_{x_1 x_2} w\|_{H^2(\Omega)} + \|\partial_{x_1 x_3} w\|_{H^2(\Omega)} + \|L_a w\|_{H^1(\partial\Omega)} \right), \tag{2.23}$$

with

$$L_a w = -\partial_{x_1 x_1} w - a\partial_{x_2 x_2} w - a\partial_{x_3 x_3} w. \tag{2.24}$$

Proof. We recall that by Hypothesis 1.2 there are a domain $\Omega_0 \subset \mathbb{R}^2$ and $H_-, H_+ \in C^0(\overline{\Omega_0}, [\delta, +\infty))$ such that $\Omega = \{x : -H_-(x_1, x_2) < x_3 < H_+(x_1, x_2)\}$. Using Proposition 2.4 we obtain that there are $C > 0$ and $a_0 > 0$ such that for all $a \in [0, a_0)$ and $s \in (-\delta, \delta)$ we have that:

$$\|\partial_{x_1} w\|_{C^0(\overline{\Omega}_0 \times \{s\})} \leq C \left(\|\partial_{x_1 x_2} w\|_{H^2(\Omega_0 \times \{s\})} + \| - \partial_{x_1 x_1} w - a\partial_{x_2 x_2} w\|_{H^1(\partial\Omega_0 \times \{s\})} \right). \tag{2.25}$$

Considering that $\partial_{x_3 x_3} w = 0$ on $\partial\Omega_0 \times (-\delta, \delta)$ (because of Dirichlet boundary conditions) we have that:

$$L_a w = -\partial_{x_1 x_1} w - a\partial_{x_2 x_2} w \quad \text{ on } \Omega_0 \times (-\delta, \delta). \tag{2.26}$$

Thus, if we integrate (2.25) squared on $(-\delta, \delta)$ and consider (2.26), we obtain that:

$$\|\partial_{x_1} w\|^2_{L^2(\Omega_0 \times (-\delta,\delta))} \leq C \left(\|\partial_{x_1 x_2} w\|^2_{H^2(\Omega_0 \times (-\delta,\delta))} + \|L_a w\|^2_{H^1(\partial\Omega_0 \times (-\delta,\delta))} \right). \tag{2.27}$$

Finally, considering that for all $x = (x_1, x_2, x_3) \in \Omega$ and $s \in (-\delta, \delta)$ we have that:

$$\partial_{x_1} w(x_1, x_2, x_3) = \partial_{x_1} w(x_1, x_2, s) + \int_s^{x_3} \partial_{x_1 x_3} w(x_1, x_2, s')ds'. \tag{2.28}$$

By integrating (2.28), using (2.27) and remarking that $\Omega_0 \times (-\delta, \delta) \subset \Omega$, we easily obtain (2.23). □

Proof of Proposition 2.10. Because the proof is quite technical, we divide it into twelve steps.

- **Step 1:** We first use the coercivity estimate given in Proposition 2.11 for φ_1. We transform this coercivity estimate into a weighted estimate of φ_1.

- **Step 2:** We add global terms involving φ_2 and φ_3 in both sides of the previous estimate.

- **Step 3:** We get rid of the terms involving φ_1 in the right-hand side of the previous estimate thanks to the equations satisfied by φ_2 and φ_3.

- **Step 4:** We choose from this step forward to treat only one term in the right-hand side of the estimate to simplify, i.e. $\partial_{x_2 x_3} \varphi_2$, all the other terms can be treated in a similar way. In this step we use Lemma 2.5.

- **Step 5:** We begin with a Carleman estimate for the derivatives of $\partial_{x_2x_3}\varphi_2$, this Carleman estimate comes from Lemma 2.6.

- **Step 6:** The previous inequality leads to a global term involving the pressure π. This is why we also use a Carleman estimate for the derivatives of $\partial_{x_2x_3x_2}\pi$.

- **Step 7:** We then get rid of the trace terms containing $\partial_{x_2x_3}\varphi_2$ and $\partial_{x_2x_3x_2}\pi$ by using the regularity result stated in Lemma 2.7.

- **Step 8:** We eliminate the local term involving the pressure i.e. $\partial_{x_2x_3x_2}\pi$ by the second equation of the system.

- **Step 9:** We gather Steps 4, 5, 6, 7 and 8.

- **Step 10:** We first estimate the local term involving the derivatives of $\partial_{x_2x_3}\varphi_2$ by a local term involving $\partial_{x_2x_3}\varphi_2$ by using standard integration by parts. We then estimate the local term of $\partial_{x_2x_3}\varphi_2$ in function of a local term of φ_2.

- **Step 11:** We apply the standard Carleman estimate for the heat equation satisfied by ψ.

- **Step 12:** We estimate the local term of φ_3 by the equation satisfied by ψ.

First, let us fix three nonempty open sets $\omega_1, \omega_2, \omega_3$ such that $\omega_0 \subset\subset \omega_1 \subset\subset \omega_2 \subset\subset \omega_3 \subset\subset \omega$.

Step 1: From (2.3), we readily see that the equation of φ_1 on the boundary is given by

$$-\partial_{x_1x_1}\varphi_1 - \frac{\varepsilon}{1+\varepsilon}\partial_{x_2x_2}\varphi_1 - \frac{\varepsilon}{1+\varepsilon}\partial_{x_3x_3}\varphi_1 = \frac{1}{1+\varepsilon}\left(\partial_{x_1x_2}\varphi_2 + \partial_{x_1x_3}\varphi_3\right) \text{ on } \Sigma. \tag{2.29}$$

Thus, using Proposition 2.11 and Poincaré inequality we have that for every $t \in [0,T)$:

$$\|\varphi_1(t,\cdot)\|_{L^2(\Omega)} \leq C\Bigg(\|\partial_{x_1x_2}\varphi_1(t,\cdot)\|_{H^2(\Omega)} + \|\partial_{x_1x_3}\varphi_1(t,\cdot)\|_{H^2(\Omega)} + \|\partial_{x_1x_2}\varphi_2(t,\cdot)\|_{H^2(\Omega)} + \|\partial_{x_1x_3}\varphi_3(t,\cdot)\|_{H^2(\Omega)}\Bigg). \tag{2.30}$$

Therefore, we deduce that:

$$s^{51}\lambda^{52}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi_1|^2 \leq C\Bigg(s^{51}\lambda^{52}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}(|\nabla^i\partial_{x_1x_2}\varphi_1|^2 + |\nabla^i\partial_{x_1x_3}\varphi_1|^2) + s^{51}\lambda^{52}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}(|\nabla^i\partial_{x_1x_2}\varphi_2|^2 + |\nabla^i\partial_{x_1x_3}\varphi_3|^2\Bigg), \tag{2.31}$$

for α^* and ξ^* defined in (2.9). Note that the previous exponents: 51, 52 are chosen sufficiently big to absorb the trace terms, see Step 7. They are not optimal.

Step 2: We deduce from (2.31) the following estimate

$$s^{51}\lambda^{52}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2 \le C\Bigg(s^{51}\lambda^{52}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}(|\nabla^i\partial_{x_1x_2}\varphi_1|^2+|\nabla^i\partial_{x_1x_3}\varphi_1|^2)$$
$$+ s^{51}\lambda^{52}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}(|\nabla^i\partial_{x_1x_2}\varphi_2|^2+|\nabla^i\partial_{x_1x_3}\varphi_3|^2+|\varphi_2|^2+|\varphi_3|^2\Bigg). \quad (2.32)$$

1 **Step 3:** From (2.3), the equations satisfied by φ_2 and φ_3 are

$$\begin{cases} -\frac{\varepsilon}{1+\varepsilon}\partial_t\varphi_2 - \frac{\varepsilon}{1+\varepsilon}\partial_{x_1x_1}\varphi_2 - \partial_{x_2x_2}\varphi_2 - \frac{\varepsilon}{1+\varepsilon}\partial_{x_3x_3}\varphi_2 - \frac{1}{1+\varepsilon}\left(\partial_{x_1x_2}\varphi_1+\partial_{x_2x_3}\varphi_3\right) = 0 \text{ in } Q, \\ -\frac{\varepsilon}{1+\varepsilon}\partial_t\varphi_3 - \frac{\varepsilon}{1+\varepsilon}\partial_{x_1x_1}\varphi_3 - \frac{\varepsilon}{1+\varepsilon}\partial_{x_2x_2}\varphi_3 - \partial_{x_3x_3}\varphi_3 - \frac{1}{1+\varepsilon}\left(\partial_{x_1x_3}\varphi_1+\partial_{x_2x_3}\varphi_2\right) = 0 \text{ in } Q. \end{cases} \quad (2.33)$$

So we easily deduce that

$$s^{51}\lambda^{52}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}(|\nabla^i\partial_{x_1x_2}\varphi_1|^2+|\nabla^i\partial_{x_1x_3}\varphi_1|^2)$$
$$\le Cs^{51}\lambda^{52}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}\Bigg(|\nabla^i\partial_{x_2x_3}\varphi_3|^2+|\nabla^i\partial_t\varphi_2|^2+|\nabla^i\partial_{x_1x_1}\varphi_2|^2+|\nabla^i\partial_{x_2x_2}\varphi_2|^2+|\nabla^i\partial_{x_3x_3}\varphi_2|^2$$
$$+|\nabla^i\partial_{x_2x_3}\varphi_2|^2+|\nabla^i\partial_t\varphi_3|^2+|\nabla^i\partial_{x_1x_1}\varphi_3|^2+|\nabla^i\partial_{x_2x_2}\varphi_3|^2+|\nabla^i\partial_{x_3x_3}\varphi_3|^2\Bigg). \quad (2.34)$$

2 From (2.3), we also have that φ_2, φ_3 satisfy

$$\forall i\in\{2,3\},\ -\partial_t\varphi_i-\Delta\varphi_i = -\partial_{x_i}\pi \text{ in } Q,\ \varphi_i = 0 \text{ on } \Sigma. \quad (2.35)$$

Therefore, for every $i=2,3$, we have that

$$|\partial_t\varphi_i|^2 \le C(|\Delta\varphi_i|^2+|\partial_{x_i}\pi|^2) \le C(|\partial_{x_1x_1}\varphi_i|^2+|\partial_{x_2x_2}\varphi_i|^2+|\partial_{x_3x_3}\varphi_i|^2+|\partial_{x_i}\pi|^2),$$

so from (2.34), we obtain

$$s^{51}\lambda^{52}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}(|\nabla^i\partial_{x_1x_2}\varphi_1|^2+|\nabla^i\partial_{x_1x_3}\varphi_1|^2)$$
$$\le Cs^{51}\lambda^{52}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}\Bigg(|\nabla^i\partial_{x_2x_3}\varphi_3|^2+|\nabla^i\partial_{x_2}\pi|^2+|\nabla^i\partial_{x_1x_1}\varphi_2|^2+|\nabla^i\partial_{x_2x_2}\varphi_2|^2+|\nabla^i\partial_{x_3x_3}\varphi_2|^2$$
$$+|\nabla^i\partial_{x_2x_3}\varphi_2|^2+|\nabla^i\partial_{x_3}\pi|^2+|\nabla^i\partial_{x_1x_1}\varphi_3|^2+|\nabla^i\partial_{x_2x_2}\varphi_3|^2+|\nabla^i\partial_{x_3x_3}\varphi_3|^2\Bigg). \quad (2.36)$$

We now gather (2.36) with (2.31) to deduce that

$$s^{51}\lambda^{52}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2 \le C\Bigg(s^{51}\lambda^{52}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}\Bigg(|\varphi_2|^2+|\varphi_3|^2+|\nabla^i\partial_{x_2}\pi|^2+|\nabla^i\partial_{x_3}\pi|^2$$
$$+\sum_{i=0}^{2}\sum_{j=2,3}|\nabla^i\partial_{x_1x_2}\varphi_j|^2+|\nabla^i\partial_{x_2x_3}\varphi_j|^2+|\nabla^i\partial_{x_1x_1}\varphi_j|^2+|\nabla^i\partial_{x_2x_2}\varphi_j|^2+|\nabla^i\partial_{x_3x_3}\varphi_j|^2\Bigg)\Bigg). \quad (2.37)$$

Step 4: We focus on the estimation of $\partial_{x_2x_3}\varphi_2$. We apply Lemma 2.5 to $\partial_{x_2x_3}\varphi_2$ with $k = 24$, $i = 53$, this leads to

$$\begin{aligned} s^{55}\lambda^{56}\iint_Q e^{-2s\alpha}\xi^{55}|\partial_{x_2x_3}\varphi_2|^2 + s^{53}\lambda^{54}\iint_Q e^{-2s\alpha}\xi^{53}|\nabla\partial_{x_2x_3}\varphi_2|^2 + s^{51}\lambda^{52}\iint_Q e^{-2s\alpha}\xi^{51}|\nabla^2\partial_{x_2x_3}\varphi_2|^2 \\ + \cdots + s^7\lambda^8\iint_Q e^{-2s\alpha}\xi^7|\nabla^{24}\partial_{x_2x_3}\varphi_2|^2 \\ \leq C\Big(s^5\lambda^6\iint_Q e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 + s^{55}\lambda^{56}\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^{55}|\partial_{x_2x_3}\varphi_2|^2 \\ + s^7\lambda^8\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^7|\nabla^{24}\partial_{x_2x_3}\varphi_2|^2\Big). \quad (2.38) \end{aligned}$$

Step 5: We now apply the Carleman estimate given by Lemma 2.6 to the equation satisfied by $\nabla^{25}\partial_{x_2x_3}\varphi_2$, i.e.

$$-\partial_t(\nabla^{25}\partial_{x_2x_3}\varphi_2) - \Delta(\nabla^{25}\partial_{x_2x_3}\varphi_2) = -\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi \text{ in } Q, \quad (2.39)$$

we obtain the following inequality

$$\begin{aligned} s^5\lambda^6\iint_Q e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 + s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 \\ \leq C\Big(s^2\lambda^2\iint_Q e^{-2s\alpha}|\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2 + s^5\lambda^6\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 \\ + s^3\lambda^3\iint_\Sigma e^{-2s\alpha}\xi^3|\partial_n\nabla^{25}\partial_{x_2x_3}\varphi_2|^2\Big). \quad (2.40) \end{aligned}$$

Step 6: In order to estimate the global term involving the pressure in the right-hand side of (2.40), we also employ a Carleman estimate for the pressure. Indeed, by taking the divergence in $(2.3)_1$ we can easily see that:

$$-\varepsilon\partial_t\pi - (1-\varepsilon)\Delta\pi = 0 \text{ in } Q,$$

so in particular

$$\forall i \in \{2,3\},\ -\frac{\varepsilon}{1-\varepsilon}\partial_t(\partial_{x_i}\pi) - \Delta(\partial_{x_i}\pi) = 0 \text{ in } Q.$$

Consequently, from Lemma 2.6 we obtain that:

$$\begin{aligned} s^2\lambda^3\iint_Q e^{-2s\alpha}\xi^2|\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2 + \lambda\iint_Q e^{-2s\alpha}|\nabla^{26}\partial_{x_2x_3}\partial_{x_2}\pi|^2 \\ \leq C\Big(s^2\lambda^3\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^2|\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2 + s^{-1}\lambda^{-1}\iint_\Sigma e^{-2s\alpha}\xi^{-1}|\partial_n\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2\Big). \quad (2.41) \end{aligned}$$

Step 7: We now gather the estimates (2.40) and (2.41) and take λ, s sufficiently large to get

$$
\begin{aligned}
s^5\lambda^6 \iint_Q e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\nabla^{26}\partial_{x_2x_3}\varphi_2|^2 \\
s^2\lambda^3 \iint_Q e^{-2s\alpha}\xi^2|\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2 + \lambda \iint_Q e^{-2s\alpha}|\nabla^{26}\partial_{x_2x_3}\partial_{x_2}\pi|^2 \\
\leq C\Big(s^5\lambda^6 \iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 + s^3\lambda^3 \iint_\Sigma e^{-2s\alpha}\xi^3|\partial_n\nabla^{26}\partial_{x_2x_3}\varphi_2|^2 \\
+ s^2\lambda^3 \iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^2|\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2 + s^{-1}\lambda^{-1} \iint_\Sigma e^{-2s\alpha}\xi^{-1}|\partial_n\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2\Big). \quad (2.42)
\end{aligned}
$$

From (2.11) and the regularity result stated in Lemma 2.7 with $k = 16$ and $r = 3/2$, we deduce that for $m \geq 2$,

$$
\begin{aligned}
\|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}\varphi\|_{H^{16,32}(Q)} + \|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}\pi\|_{H^{15,31}(Q)} \\
\leq \|s^{3/2+16+16/m}e^{-s\alpha^*}(\xi^*)^{3/2+16+16/m}\varphi\|_{L^2(Q)} \leq \|s^{51/2}e^{-s\alpha^*}(\xi^*)^{51/2}\varphi\|_{L^2(Q)}. \quad (2.43)
\end{aligned}
$$

From (2.11) and (2.43), we then obtain that

$$
\begin{aligned}
s^3\lambda^3 \iint_\Sigma e^{-2s\alpha}\xi^3|\partial_n\nabla^{26}\partial_{x_2x_3}\varphi_2|^2 + s^{-1}\lambda^{-1} \iint_\Sigma e^{-2s\alpha}\xi^{-1}|\partial_n\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2 \\
= s^3\lambda^3 \iint_\Sigma e^{-2s\alpha^*}(\xi^*)^3|\partial_n\nabla^{26}\partial_{x_2x_3}\varphi_2|^2 + s^{-1}\lambda^{-1} \iint_\Sigma e^{-2s\alpha^*}(\xi^*)^{-1}|\partial_n\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2 \\
\leq \|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}\varphi\|^2_{H^{16,32}(Q)} + \|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}\pi\|^2_{H^{15,31}(Q)} \\
\leq \lambda^3 s^{51} \iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2. \quad (2.44)
\end{aligned}
$$

So by taking λ sufficiently large, the trace terms of the right hand side of (2.42) can be absorbed by the left-hand side of (2.32).

Step 8: From (2.35) and (2.43), by integration by parts and cut-off arguments, we have that there exist $r, r' > 0$ such that

$$
\begin{aligned}
s^2\lambda^3 \iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^2|\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi|^2 = -s^2\lambda^3 \iint_{Q_{\omega_0}} e^{-2s\alpha}[\nabla^{25}\partial_{x_2x_3}\partial_{x_2}\pi][(\partial_t - \Delta)\nabla^{25}\partial_{x_2x_3}\varphi_2] \\
\leq \|s^{1/2}e^{-s\alpha^*}(\xi^*)^{1/2}\pi\|^2_{H^{15,31}(Q)} + s^r\lambda^{r'} \iint_{Q_{\omega_1}} e^{-4s\alpha+2s\alpha^*}\xi^r|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 \\
\leq s^{51} \iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2 + Cs^r\lambda^{r'} \iint_{Q_{\omega_1}} e^{-4s\alpha+2s\alpha^*}\xi^r|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2. \quad (2.45)
\end{aligned}
$$

So for λ sufficiently large, the local term of the pressure in the right hand side of (2.42) can be absorbed by the left-hand side of (2.32).

Step 9: By using (2.38), (2.42), (2.44) and (2.45), we have

$$
\begin{aligned}
s^{55}\lambda^{56}\iint_Q e^{-2s\alpha}\xi^{55}|\partial_{x_2x_3}\varphi_2|^2+s^{53}\lambda^{54}\iint_Q e^{-2s\alpha}\xi^{53}|\nabla\partial_{x_2x_3}\varphi_2|^2+s^{51}\lambda^{52}\iint_Q e^{-2s\alpha}\xi^{51}|\nabla^2\partial_{x_2x_3}\varphi_2|^2 \\
+\cdots+s^5\lambda^6\iint_Q e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2+s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\nabla^{26}\partial_{x_2x_3}\varphi_2|^2 \\
\leq C\bigg(\lambda^3 s^{51}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2+s^r\lambda^{r'}\iint_{Q_{\omega_1}} e^{-4s\alpha+2s\alpha^*}\xi^r|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 \\
+s^{55}\lambda^{56}\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^{55}|\partial_{x_2x_3}\varphi_2|^2+s^7\lambda^8\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^7|\nabla^{24}\partial_{x_2x_3}\varphi_2|^2 \\
+s^5\lambda^6\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2\bigg). \quad (2.46)
\end{aligned}
$$

Step 10: We now estimate the local term in φ_2, we proceed by standard integrations by parts, as in [BP20a, Section 5, Step 3], to obtain

$$
\begin{aligned}
s^r\lambda^{r'}\iint_{Q_{\omega_1}} e^{-4s\alpha+2s\alpha^*}\xi^r|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2+s^7\lambda^8\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^7|\nabla^{24}\partial_{x_2x_3}\varphi_2|^2 \\
+s^5\lambda^6\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^3|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2 \\
\leq \delta\bigg(s^{55}\lambda^{56}\iint_Q e^{-2s\alpha}\xi^{55}|\partial_{x_2x_3}\varphi_2|^2+s^{53}\lambda^{54}\iint_Q e^{-2s\alpha}\xi^{53}|\nabla\partial_{x_2x_3}\varphi_2|^2+s^{51}\lambda^{52}\iint_Q e^{-2s\alpha}\xi^{51}|\nabla^2\partial_{x_2x_3}\varphi_2|^2 \\
+\cdots+s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\nabla^{26}\partial_{x_2x_3}\varphi_2|^2\bigg) \\
+C_\delta s^r\lambda^{r'}\iint_{Q_{\omega_2}} e^{-4s\alpha+2s\alpha^*}\xi^r|\partial_{x_2x_3}\varphi_2|^2, \quad (2.47)
\end{aligned}
$$

for some other parameters $r, r' > 0$. By taking δ sufficiently small in (2.47), we deduce from (2.46) the following estimate

$$
\begin{aligned}
s^{55}\lambda^{56}\iint_Q e^{-2s\alpha}\xi^{55}|\partial_{x_2x_3}\varphi_2|^2+s^{53}\lambda^{54}\iint_Q e^{-2s\alpha}\xi^{53}|\nabla\partial_{x_2x_3}\varphi_2|^2+s^{51}\lambda^{52}\iint_Q e^{-2s\alpha}\xi^{51}|\nabla^2\partial_{x_2x_3}\varphi_2|^2 \\
+\cdots+s^5\lambda^6\iint_Q e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2+s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\nabla^{26}\partial_{x_2x_3}\varphi_2|^2 \\
\leq C\lambda^3 s^{51}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2+Cs^r\lambda^{r'}\iint_{Q_{\omega_2}} e^{-4s\alpha+2s\alpha^*}\xi^r|\partial_{x_2x_3}\varphi_2|^2. \quad (2.48)
\end{aligned}
$$

By integration by parts in the same spirit as before, we have

$$
\begin{aligned}
s^{55}\lambda^{56}\iint_Q e^{-2s\alpha}\xi^{55}|\partial_{x_2x_3}\varphi_2|^2+s^{53}\lambda^{54}\iint_Q e^{-2s\alpha}\xi^{53}|\nabla\partial_{x_2x_3}\varphi_2|^2+s^{51}\lambda^{52}\iint_Q e^{-2s\alpha}\xi^{51}|\nabla^2\partial_{x_2x_3}\varphi_2|^2 \\
+\cdots+s^5\lambda^6\iint_Q e^{-2s\alpha}\xi^5|\nabla^{25}\partial_{x_2x_3}\varphi_2|^2+s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\nabla^{26}\partial_{x_2x_3}\varphi_2|^2 \\
\leq C\lambda^3 s^{51}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2+Cs^r\lambda^{r'}\iint_{Q_{\omega_3}} e^{-4s\alpha+2s\alpha^*}\xi^r|\varphi_2|^2, \quad (2.49)
\end{aligned}
$$

for other parameters $r, r' > 0$.

By using (2.37), (2.43), (2.49) and performing the same strategy from Step 5 to Step 10 for other terms appearing in the right-hand side of (2.37), we obtain that there exist $r, r' > 0$ such that

$$s^{51}\lambda^{52}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2 \leq C s^r \lambda^{r'} \iint_{Q_{\omega_3}} e^{-4s\alpha+2s\alpha^*}\xi^r(|\varphi_2|^2+|\varphi_3|^2). \tag{2.50}$$

Step 11: From (2.3), we apply the classical Carleman estimate for the heat equation with homogenous Dirichlet boundary condition satisfied by ψ, i.e. we use Lemma 2.6,

$$s^3\lambda^4\iint_Q e^{-3s\alpha}\xi^3|\psi|^2 + s\lambda^2\iint_Q e^{-3s\alpha}\xi|\nabla\psi|^2 \leq C\Big(\iint_Q e^{-3s\alpha}|\varphi_3|^2 + s^3\lambda^4\iint_{Q_{\omega_0}} e^{-3s\alpha}\xi^3|\psi|^2\Big). \tag{2.51}$$

By using (2.13), we sum (2.50) and (2.51) to get

$$s^{51}\lambda^{52}\iint_Q e^{-2s\alpha^*}(\xi^*)^{51}|\varphi|^2 + s^3\lambda^4\iint_Q e^{-3s\alpha}\xi^3|\psi|^2 + s\lambda^2\iint_Q e^{-3s\alpha}\xi|\nabla\psi|^2 \leq C\left(s^r\lambda^{r'}\iint_{Q_{\omega_3}} e^{-4s\alpha+2s\alpha^*}\xi^r(|\varphi_2|^2+|\varphi_3|^2) + s^3\lambda^4\iint_{Q_{\omega_0}} e^{-3s\alpha}\xi^3|\psi|^2\right). \tag{2.52}$$

Step 12: To eliminate the local term in φ_3 in the right-hand side of (2.52), we proceed exactly as in the proof of Proposition 2.8 in Section 2.3. Therefore, we obtain the desired estimate (2.22). □

2.4.2 A Carleman inequality assuming Hypothesis 1.3

The goal of this part is to prove the following result:

Proposition 2.12. *Let Ω, ω be such that Hypothesis 1.3 holds and let $m \geq 12$. Then, there are $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $(\varphi^T, \psi^T) \in (L^2(\Omega))^4$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have:*

$$s^{16}\lambda^{17}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi|^2 + s^3\lambda^4\iint_Q e^{-\frac{201}{100}s\alpha}\xi^3|\psi|^2 \leq C\iint_{Q_\omega} e^{-s\alpha^*}(|\psi|^2+|\varphi_2|^2), \tag{2.53}$$

for the weights defined in (2.9) *and for* (φ, ψ) *the solution of* (2.3).

To prove Proposition 2.12, we follow the strategy of [FCGIP06], which consists in getting a Carleman estimate with some local terms and then using that the observation domain touches the boundary. The fact that the coupling of the system is of order 2 is an additional difficulty with respect to the systems treated in [FCGIP06].

Proof of Proposition 2.12. First of all, we consider that, by taking a smaller control domain if necessary, we can suppose that there are $\gamma \in \mathrm{Span}(e_2, e_3)$ and $\tilde{\Gamma} \subset \Gamma$ a relative open set such that $\omega = \{x + \lambda\gamma : x \in \tilde{\Gamma},\ \lambda \in (0,1)\}$. In addition, we consider

$$\omega_1 := \{x + \lambda\gamma : x \in \hat{\Gamma},\ \lambda \in (0, 1/2)\}, \tag{2.54}$$

for $\hat{\Gamma}$ a relative open set contained in $\tilde{\Gamma}$. Finally, we consider ω_0 a non empty open set compactly contained in ω_1.

In order to prove the Carleman inequality (2.53), using (2.3), we first remark that $\partial_{x_1}\varphi_1$ satisfies the equation:

$$\left(-\frac{\varepsilon}{1+\varepsilon}\partial_t - \Delta\right)\partial_{x_1}\varphi_1 = \frac{1}{1+\varepsilon}\left(\partial_{x_1x_1x_2}\varphi_2 + \partial_{x_1x_1x_3}\varphi_3 + \partial_{x_1x_2x_2}\varphi_1 + \partial_{x_1x_3x_3}\varphi_1\right). \tag{2.55}$$

To continue with, using Poincaré's inequality for φ_1 and Lemma 2.5 for $\partial_{x_1}\varphi_1$ with $k = 1$, we obtain that:

$$\begin{aligned} s^{16}\lambda^{17}\iint_Q e^{-2s\alpha^*}(\xi^*)^{16}|\varphi_1|^2 + s^{16}\lambda^{17}\iint_Q e^{-2s\alpha}\xi^{16}|\partial_{x_1}\varphi_1|^2 + s^{14}\lambda^{15}\iint_Q e^{-2s\alpha}\xi^{14}|\nabla\partial_{x_1}\varphi_1|^2 \\ \le C\Bigg(s^{12}\lambda^{13}\iint_Q e^{-2s\alpha}\xi^{12}|\nabla^2\partial_{x_1}\varphi_1|^2 + s^{16}\lambda^{17}\iint_{Q_{\omega_0}} \xi^{16}|\partial_{x_1}\varphi_1|^2 \\ + s^{14}\lambda^{15}\iint_{Q_{\omega_0}} \xi^{14}|\nabla\partial_{x_1}\varphi_1|^2\Bigg). \end{aligned} \tag{2.56}$$

In particular, by using (2.55) and the Carleman estimate coming from Lemma 2.6 for the terms of $\nabla^2\partial_{x_1}\varphi_1$ we obtain that:

$$\begin{aligned} s^{12}\lambda^{13}\iint_Q e^{-2s\alpha}\xi^{12}|\nabla^2\partial_{x_1}\varphi_1|^2 \le C\Bigg(s^{9}\lambda^{9}\iint_Q e^{-2s\alpha}\xi^{9}|\nabla^2 g|^2 + s^{12}\lambda^{13}\iint_{Q_{\omega_0}} \xi^{12}|\nabla^2\partial_{x_1}\varphi_1|^2 \\ + s^{10}\lambda^{10}\iint_{\Sigma} e^{-2s\alpha}\xi^{10}|\partial_n\nabla^2\partial_{x_1}\varphi_1|^2\Bigg), \end{aligned} \tag{2.57}$$

for $g := \partial_{x_1x_1x_2}\varphi_2 + \partial_{x_1x_1x_3}\varphi_3 + \partial_{x_1x_2x_2}\varphi_1 + \partial_{x_1x_3x_3}\varphi_1$.

To continue with, we find from Poincaré's inequality and Lemma 2.5 with $k = 0$ that:

$$\begin{aligned} s^{17}\lambda^{18}\iint_Q e^{-2s\alpha^*}(\xi^*)^{17}(|\varphi_2|^2 + |\varphi_3|^2) + s^{17}\lambda^{18}\iint_Q e^{-2s\alpha}\xi^{17}(|\partial_{x_1}\varphi_2|^2 + |\partial_{x_1}\varphi_3|^2) \\ \le C\Bigg(s^{15}\lambda^{16}\iint_Q e^{-2s\alpha}\xi^{15}(|\nabla\partial_{x_1}\varphi_2|^2 + |\nabla\partial_{x_1}\varphi_3|^2) \\ + s^{17}\lambda^{18}\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^{17}(|\partial_{x_1}\varphi_2|^2 + |\partial_{x_1}\varphi_3|^2)\Bigg). \end{aligned} \tag{2.58}$$

Next, we remark that we can follow the steps 3-11 of Section 2.4.1 and by using the estimates on the weights (2.12), (2.13), Lemmas 2.6 and 2.7 and (2.56), (2.57), (2.58), we deduce that:

$$
\begin{aligned}
s^{16}\lambda^{17}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi|^2 + s^3\lambda^4 \iint_Q e^{-\frac{201}{100}s\alpha}\xi^3|\psi|^2 \leq C\Big(\iint_{Q_{\omega_1}} e^{-\frac{31}{16}s\alpha}(|\psi|^2+|\varphi_2|^2)\\
\iint_{Q_{\omega_1}} e^{-\frac{31}{16}s\alpha^*}|\partial_{x_1}\varphi_1|^2 + s^{10}\lambda^{10}\iint_\Sigma e^{-2s\alpha}\xi^{10}|\partial_n\nabla^2\partial_{x_1}\varphi_1|^2\Big). \quad (2.59)
\end{aligned}
$$

Remark 2.13. It helps us to have only derivatives of $\partial_{x_1}\varphi_2$ and $\partial_{x_1}\varphi_3$ in the right-hand side of (2.58) because we eventually do a Carleman inequality for $\nabla^i\partial_{x_1}\varphi$ as in [BP20a, Section 5], so the local term that we get is of the form $\nabla^i\partial_{x_1}\varphi$. If instead of having $\partial_{x_1}\varphi_2$ and $\partial_{x_1}\varphi_3$ we had $\nabla\varphi_2$ and $\nabla\varphi_3$, after applying the Carleman estimate to $\nabla^i\varphi$ we would get a local term which has $\partial^i_{x_2}\varphi_1$ and $\partial^i_{x_3}\varphi_1$ terms from which we do not know how to get a Carleman inequality with a local term that only depends on φ_2 and ψ.

Let us now treat the boundary term of φ_1 as in Section 2.4.1. By interpolation and Lemma 2.7 we have:

$$
\begin{aligned}
&s^{10}\lambda^{10}\iint_\Sigma e^{-2s\alpha}\xi^{10}|\partial_n\nabla^2\partial_{x_1}\varphi_1|^2\\
&\quad \leq C\|s^{23/4}\lambda^{23/4}e^{-s\alpha^*}(\xi^*)^{23/4}\varphi\|^{1/2}_{L^2(0,T;H^4(\Omega))}\|s^{19/4}\lambda^{19/4}e^{-s\alpha^*}(\xi^*)^{19/4}\varphi\|^{3/2}_{L^2(0,T;H^6(\Omega))}\\
&\qquad \leq C\|s^{31/4+2/m}\lambda^{23/4}e^{-s\alpha^*}(\xi^*)^{23/4}\varphi\|^{1/2}_{L^2(Q)}\|s^{31/4+3/m}\lambda^{19/4}e^{-s\alpha^*}(\xi^*)^{31/4+3/m}\varphi\|^{3/2}_{L^2(Q)}.
\end{aligned}
$$

So, if $m \geq 12$, we have that the trace term in the right-hand side of (2.59) can be absorbed by the left-hand side of (2.59), then we obtain:

$$
\begin{aligned}
s^{16}\lambda^{17}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi|^2 + s^3\lambda^4 \iint_Q e^{-\frac{201}{100}s\alpha}\xi^3|\psi|^2 \leq C\Big(\iint_{Q_{\omega_1}} e^{-\frac{31}{16}s\alpha}(|\psi|^2+|\varphi_2|^2)\\
+\iint_{Q_{\omega_1}} e^{-\frac{31}{16}s\alpha^*}|\partial_{x_1}\varphi_1|^2\Big). \quad (2.60)
\end{aligned}
$$

Let us now estimate the third right-hand side term of (2.60). From (2.54), recalling that e_1 is tangent to every point of $\tilde{\Gamma}$ by Hypothesis 1.3, $\partial_{x_1}\varphi_1 \equiv 0$ in $\partial\omega_1 \cap \partial\Omega$, and from Poincaré's inequality we obtain that:

$$
\iint_{Q_{\omega_1}} e^{-\frac{31}{16}s\alpha^*}|\partial_{x_1}\varphi_1|^2 \leq C\iint_{Q_{\omega_1}} e^{-\frac{31}{16}s\alpha^*}(|\partial_{x_2x_1}\varphi_1|^2+|\partial_{x_3x_1}\varphi_1|^2). \quad (2.61)
$$

Consequently, by using the equations (2.33), we can estimate the local terms in the right-hand side of (2.61) in function of local terms of φ_2 and φ_3. By putting this in (2.60), we bound the local term in $\partial_{x_1}\varphi_1$ by local terms in φ_2 and φ_3. In addition, we can deal with the local term of φ_3 by using the equation satisfied by ψ as before. Hence, by straightforward computations, we obtain the expected estimate (2.53). □

2.5 Source term method

In this section we adapt the source term method of [LTT13] to our case. It is worth mentioning that we use the uniform analyticity of the penalized semigroup, that come from uniform maximal regularity estimates of the linearized Boussinesq system (2.1).

From Proposition 2.1 we have an estimate for the control cost in L^2 of system (2.1). We now fix $M>0$ such that $K(T) \leq Me^{M/T^m}$, with $K(T)$ defined as in (2.2). In addition, we fix the values:

$$q \in (1, \sqrt[2m]{2}), \qquad p > q^{2m}/(2-q^{2m}), \tag{2.62}$$

and the weights:

$$\rho_0(t) := M^{-p} \exp\left(-\frac{Mp}{(q-1)^m (T-t)^m}\right), \tag{2.63}$$

$$\rho_{\mathcal{F}}(t) := M^{-1-p} \exp\left(-\frac{(1+p)q^{2m} M}{(q-1)^m (T-t)^m}\right). \tag{2.64}$$

We introduce the linearized Boussinesq system, with source term $f = (\tilde{f}, f_{N+1}) \in L^2(Q)^{N+1}$,

$$\begin{cases} y_t - \Delta y + \nabla p = \theta e_N + \tilde{f} + \tilde{v} 1_\omega & \text{in } Q, \\ \theta_t - \Delta \theta = f_{N+1} + v_{N+1} 1_\omega & \text{in } Q, \\ \nabla \cdot y = -\varepsilon p & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega. \end{cases} \tag{2.65}$$

In addition, we define associated spaces for the source term, the state and the control, respectively:

$$\mathcal{F} := \left\{ f \in L^2(Q)^{N+1} \ : \ \frac{f}{\rho_{\mathcal{F}}} \in L^2(Q)^{N+1} \right\}, \tag{2.66}$$

$$\mathcal{Y} := \left\{ (y,\theta) \in L^2(Q)^{N+1} \ : \ \frac{(y,\theta)}{\rho_0} \in L^2(Q)^{N+1} \right\}, \tag{2.67}$$

$$\mathcal{V} := \left\{ v \in L^2(Q_\omega)^{N+1} \ : \ \frac{v}{\rho_0} \in L^2(Q_\omega)^{N+1} \text{ and } v \text{ satisfies (1.2)} \right\}. \tag{2.68}$$

From an adaptation of the proof of [LTT13, Proposition 2.3], since the linearized Boussinesq semigroup is analytic, we deduce the null controllability of the system (2.65) for source terms f that exponentially decrease at $t = T$.

Proposition 2.14. *Let Ω, ω be such that Hypothesis 1.1, 1.2 or 1.3 holds. Then, there exist $\varepsilon_0 > 0$, $m \geq 1$ and $C > 0$ such that for every $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $f \in \mathcal{F}$, $(y^0, \theta^0) \in L^2(\Omega)^{N+1}$, there exists a control $v \in \mathcal{V}$ such that the solution (y, θ) of (2.65) belongs to $\mathcal{Y}$ and we have the following estimate:*

$$\|(y,\theta)/\rho_0\|_{C([0,T];L^2(\Omega))} + \|v/\rho_0\|_{L^2(Q_\omega)} \leq C \left(\|(y^0,\theta^0)\|_{L^2(\Omega)} + \|f/\rho_{\mathcal{F}}\|_{L^2(Q)} \right). \tag{2.69}$$

In particular, $(y,\theta)(T)=0$.

The next proposition gives more information on the regularity of the controlled trajectory obtained in Proposition 2.14. We consider the weight:

$$\rho(t)=\exp\left(-\frac{M\beta}{(q-1)^m(T-t)^m}\right),\ \text{with}\ \frac{(1+p)q^{2m}}{2}<\beta<p.$$

We remark that $\rho(T)=0$ and that ρ satisfies the inequalities:

$$\rho_0\le C\rho,\ \rho_{\mathcal{F}}\le C\rho,\ |\rho'|\rho_0\le C\rho^2,\ \rho^2\le C\rho_{\mathcal{F}}. \tag{2.70}$$

Proposition 2.15. *Let Ω, ω be such that Hypothesis 1.1, 1.2 or 1.3 holds. Then, there exist $\varepsilon_0>0$, $m\ge 1$ and $C>0$ such that for every $T>0$, $\varepsilon\in(0,\varepsilon_0)$, $f\in\mathcal{F}$, $(y^0,\theta^0)\in H_0^1(\Omega)^{N+1}$, there exists a control $v\in\mathcal{V}$ such that the solution (y,θ) of* (2.65) *satisfies:*

$$\|(y,\theta)/\rho\|_{L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))}+\|v/\rho_0\|_{L^2(Q_\omega)}\le C\left(\|(y^0,\theta^0)\|_{H_0^1(\Omega)}+\|f/\rho_{\mathcal{F}}\|_{L^2(Q)}\right). \tag{2.71}$$

In particular, $(y,\theta)(T)=0$.

The proof of Proposition 2.15 is a straightforward adaptation of the proof of [LTT13, Proposition 2.8] by using uniform maximal regularity estimates of the penalized Stokes system (see Lemma 2.3) and maximal regularity estimate of the heat equation.

Remark 2.16. For each $f\in\mathcal{F}$ and $(y^0,\theta^0)\in H_0^1(\Omega)^{N+1}$, by classical arguments (see [LTT13, Proposition 2.9]), we can fix a control $v\in\mathcal{V}$ such that (y,θ) and v satisfy (2.71) by choosing among those the unique minimizer of the functional $v\mapsto\|(y,\theta)/\rho\|^2_{L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))}+\|v/\rho_0\|^2_{L^2(Q_\omega)}$.

3 Local null controllability of the nonlinear Boussinesq system

In this section we conclude the proof of Theorem 1.2. It is organized as follows:

- In Section 3.1 we use a fixed point argument to get the local null controllability of the nonlinear system (1.1) for initial data in $H_0^1(\Omega)^{N+1}$.
- In Section 3.2 we use a regularization argument to deduce the local null controllability of (1.1) for initial data in $L^2(\Omega)^{N+1}$.

3.1 Fixed point argument

In this section, we give the proof of Theorem 1.2 for initial data in $H_0^1(\Omega)^{N+1}$. In this part, C denotes a positive constant that depends on Ω, ω and T but independent of ε and varying from line to line. In addition, we denote:

$$\mathcal{F}_r := \{f \in \mathcal{F} \ : \ \|f/\rho_{\mathcal{F}}\|_{L^2(Q)} \leq r\}, \tag{3.1}$$

for $r > 0$ a small enough parameter independent of ε that will be determined later.

We now fix $(y^0, \theta^0) \in H_0^1(\Omega)^{N+1}$ such that:

$$\|(y^0, \theta^0)\|_{H_0^1(\Omega)} \leq r. \tag{3.2}$$

It follows that we can define an operator $\mathcal{N}$ acting on $\mathcal{F}_r$ by:

$$\mathcal{N}(f) := \left(-(y \cdot \nabla)y - \frac{1}{2}(\nabla \cdot y)y, -y \cdot \nabla\theta - \frac{1}{2}(\nabla \cdot y)\theta \right), \tag{3.3}$$

where (y, θ) is the corresponding trajectory of (2.65) associated to the initial values (y^0, θ^0), the force f and the control v given by Proposition 2.15 and Remark 2.16.

To conclude the proof of Theorem 1.2 for regular initial data it suffices to check that, for $r > 0$ small enough not depending on ε, $\mathcal{N}$ is a contractive mapping from $\mathcal{F}_r$ into itself and then apply the Banach fixed point theorem.

Step 1: $\mathcal{F}_r$ is invariant for $\mathcal{N}$ provided that r is small enough. By using $(2.70)_4$ and the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$ because $N \leq 3$, we have for almost every $t \in (0, T)$:

$$\begin{aligned}
\left\| \frac{\mathcal{N}(f)}{\rho_{\mathcal{F}}}(t) \right\|_{L^2(\Omega)} &\leq C \left| \frac{\rho^2}{\rho_{\mathcal{F}}}(t) \right| \left(\left\| \frac{(y \cdot \nabla)y}{\rho^2}(t) \right\|_{L^2(\Omega)} + \left\| \frac{(\nabla \cdot y)y}{\rho^2}(t) \right\|_{L^2(\Omega)} \right. \\
&\quad \left. + \left\| \frac{y \cdot \nabla\theta}{\rho^2}(t) \right\|_{L^2(\Omega)} + \left\| \frac{(\nabla \cdot y)\theta}{\rho^2}(t) \right\|_{L^2(\Omega)} \right) \\
&\leq C \left(\left\| \frac{y}{\rho}(t) \right\|_{L^4(\Omega)} \left\| \frac{\nabla y}{\rho}(t) \right\|_{L^4(\Omega)} + \left\| \frac{y}{\rho}(t) \right\|_{L^4(\Omega)} \left\| \frac{\nabla\theta}{\rho}(t) \right\|_{L^4(\Omega)} \right. \\
&\quad \left. + \left\| \frac{\nabla y}{\rho}(t) \right\|_{L^4(\Omega)} \left\| \frac{\theta}{\rho}(t) \right\|_{L^4(\Omega)} \right) \\
&\leq C \left(\left\| \frac{y}{\rho}(t) \right\|_{H^1(\Omega)} \left\| \frac{y}{\rho}(t) \right\|_{H^2(\Omega)} + \left\| \frac{y}{\rho}(t) \right\|_{H^1(\Omega)} \left\| \frac{\theta}{\rho}(t) \right\|_{H^2(\Omega)} \right. \\
&\quad \left. + \left\| \frac{y}{\rho}(t) \right\|_{H^2(\Omega)} \left\| \frac{\theta}{\rho}(t) \right\|_{H^1(\Omega)} \right).
\end{aligned}$$

Then, by integrating in the time interval $t \in (0,T)$ and by using (2.71), (3.2) and (3.1), we have:

$$\begin{aligned}\left\|\frac{\mathcal{N}(f)}{\rho_{\mathcal{F}}}\right\|_{L^2(Q)} &\leq C\left(\left\|\frac{y}{\rho}\right\|_{C([0,T];H_0^1(\Omega))}\left(\left\|\frac{y}{\rho}\right\|_{L^2(0,T;H^2(\Omega))}+\left\|\frac{\theta}{\rho}\right\|_{L^2(0,T;H^2(\Omega))}\right)\right)\\ &\quad + C\left\|\frac{\theta}{\rho}\right\|_{C([0,T];H_0^1(\Omega))}\left\|\frac{y}{\rho}\right\|_{L^2(0,T;H^2(\Omega))}\\ &\leq C\left(\|(y^0,\theta^0)\|^2_{H_0^1(\Omega)}+\|f/\rho_{\mathcal{F}}\|^2_{L^2(Q)}\right)\\ &\leq Cr^2.\end{aligned}$$

So, for $r>0$ small enough, $\mathcal{N}$ stabilizes $\mathcal{F}_r$.

Step 2: $\mathcal{N}$ is contracting provided that r is small enough. Using the same kind of arguments, it is not difficult to obtain that:

$$\left\|\frac{\mathcal{N}(f^1)-\mathcal{N}(f^2)}{\rho_{\mathcal{F}}}\right\|_{L^2(Q)} \leq Cr\left\|\frac{f^1-f^2}{\rho_{\mathcal{F}}}\right\|_{L^2(Q)}.$$

Consequently, by taking r sufficiently small, $\mathcal{N}$ is a contracting mapping on the closed ball $\mathcal{F}_r$. Therefore, by the Banach fixed point theorem, $\mathcal{N}$ has a unique fixed point f. By denoting (y,θ,v) the associated trajectory to f, we find that (y,θ,v) satisfies the system (1.1) and $(y,\theta)(T)=0$. Remark that r does not depend on ε, so the control is bounded uniformly when ε goes to 0. This concludes the proof.

3.2 Smoothing effect of the nonlinear Boussinesq system

In this section, we give the proof of Theorem 1.2 for initial data in $L^2(\Omega)^{N+1}$. This type of arguments have already been used in [CMS20] and [CSFCB+20]. The key remark is the following regularity lemma.

Lemma 3.1. *Let $T>0$. There exists a positive constant $C_T>0$ such that for every $\varepsilon>0$, any $(y_0,\theta_0)\in L^2(\Omega)^{N+1}$ and any weak solution of (1.1) with control $v\equiv 0$,*

$$\exists t_0\in[0,T],\ \ \|(y,\theta)(t_0,\cdot)\|_{H^1(\Omega)} \leq C_T\|(y_0,\theta_0)\|_{L^2(\Omega)}.$$

Proof. We multiply $(1.1)_1$ by y and $(1.1)_2$ by θ, then integrate in Ω, we find

$$\frac{d}{dt}\left(\int_\Omega y(t,\cdot)^2\right)+\int_\Omega|\nabla y(t,\cdot)|^2+\frac{1}{\varepsilon}\int_\Omega(\nabla\cdot y(t,\cdot))^2=\int_\Omega\theta(t,\cdot)y_N(t,\cdot),$$

$$\frac{d}{dt}\left(\int_\Omega\theta(t,\cdot)^2\right)+\int_\Omega|\nabla\theta(t,\cdot)|^2=0.$$

We sum, use Young's inequality and Gronwall's lemma to obtain that there exists $C > 0$ such that for all $t \in [0, T]$:

$$\|(y,\theta)(t,\cdot)\|_{L^2(\Omega)}^2 + \int_0^t \|(y,\theta)(s,\cdot)\|_{H^1(\Omega)}^2 \, ds \leq C \,\|(y_0,\theta_0)\|_{L^2(\Omega)}^2 .$$

In particular, this gives us the existence of a time $t_0 \in [0, T]$ such that

$$\|(y,\theta)(t_0,\cdot)\|_{H^1(\Omega)} \leq \sqrt{\frac{C}{T}} \,\|(y_0,\theta_0)\|_{L^2(\Omega)} .$$

This concludes the proof with $C_T = \sqrt{C/T}$. □

We now have all the tools to end the proof of Theorem 1.2.

Proof of Theorem 1.2. We divide the control strategy into three steps.

Step 1: regularization of the data. By setting $v \equiv 0$ and by using Lemma 3.1, we deduce that there exists $t_0 \in (0, T/2)$ such that $\|(y,\theta)(t_0,\cdot)\|_{H^1(\Omega)} \leq C_{T/2} \,\|(y_0,\theta_0)\|_{L^2(\Omega)}$.

Step 2: local null controllability result in time $T/2$**.** By taking $(y_0, \theta_0) \in L^2(\Omega)^{N+1}$ such that $C_{T/2} \,\|(y_0,\theta_0)\|_{L^2(\Omega)} \leq \delta_{T/2}$, where $\delta_{T/2}$ is the radius of local null controllability of (1.1) for initial data in $H_0^1(\Omega)^{N+1}$, we obtain that there exists a control $v \in L^2((t_0, t_0 + T/2) \times \omega)^{N+1}$ satisfying (1.2) such that $(y,\theta)(t_0 + T/2, \cdot) = 0$.

Step 3: do nothing at the end of the time interval. We set $v \equiv 0$ in $(t_0 + T/2, T) \times \omega$, so $(y,\theta)(T,\cdot) = 0$. □

4 Remarks and open problems

In this section, we make some remarks and formulate some open problems concerning the null controllability of the penalized Boussinesq system (1.1).

- If $N = 3$ by symmetry and by adapting Hypotheses 1.2 or 1.3, we can construct controls which satisfy:

  $$\lambda_1 v_1 + \lambda_2 v_2 = v_3 = 0$$

  for any $\lambda_1, \lambda_2 \in \mathbb{R}$. Indeed, all the proofs in this paper can be adapted to these situations by a simple change of coordinates.

- We can prove analogue controllability results assuming Hypothesis 1.1, 1.2 or 1.3 for the penalized Navier-Stokes system, which we recall is given by:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \frac{1}{2} (\nabla \cdot y)\, y + \nabla p = v 1_\omega & \text{in } Q, \\ \nabla \cdot y = -\varepsilon p & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(0) = y^0, & \text{in } \Omega. \end{cases} \tag{4.1}$$

  Indeed, by following the strategy of this paper and omitting the steps related to the heat equation satisfied by θ, we can prove that (4.1) is locally null controllable uniformly on ε with a control v with one null component. Obtaining the local null controllability of (4.1) with a control with two null-components in 3-D is an interesting open problem. A good strategy seems to employ the return method in the spirit of [CL14].

- The exponents of s, λ and ξ and the constant m stated in Propositions 2.8, 2.10 and 2.12 are a bit arbitrary. In particular, by combining Lemmas 2.5 and 2.7 a sufficient number of times we can get an analogous result for any $m > 1$ by choosing the exponent of s, λ and ξ large enough. This implies that the cost of the null controllability of the linearized system without a source term is less than $Ce^{CT^{-m}}$ for all $m > 1$.

- Removing the geometrical hypothesis on Ω is an interesting open problem.

- There are still plenty of systems with penalizations which approximate the incompressibility condition (see [She97]) whose null controllability properties have not been studied yet.

A Proof of Lemma 2.7

Let us prove Lemma 2.7. The proof consists in two steps: first we use parabolic regularity estimates and then we estimate the weights.

As a first step, we prove the following result:

Lemma A.1. *Let $k \in \mathbb{N}$ and $h \in H_0^k(0,T)$. Then, there is $C > 0$ such that for all $\varphi^T \in L^2(\Omega)$:*

$$\|h\varphi\|_{H^{k,2k}(Q)} + \|h\pi\|_{H^{k-1,2k-1}(Q)} 1_{k \geq 1} \leq C \|h^{(k)} \varphi\|_{L^2(Q)}, \tag{A.1}$$

for (φ, π) the solution of (2.17).

Proof of Lemma A.1. The proof of Lemma A.1 is done by induction. The base case, $k = 0$, is trivial. Let us now prove the inductive case. By hypothesis we have (A.1) for any function $h \in H_0^k(0,T)$ and we have to prove the estimate:

$$\|h\varphi\|_{H^{k+1,2(k+1)}(Q)} + \|h\pi\|_{H^{k,2k+1}(Q)} \le C\|h^{(k+1)}\varphi\|_{L^2(Q)}, \tag{A.2}$$

for any function $h \in H_0^{k+1}(0,T)$. We have that $(h\varphi, h\pi)$ satisfies:

$$\begin{cases} -(h\varphi)_t - \Delta(h\varphi) + \nabla(h\pi) = -h'\varphi & \text{in } Q, \\ \varepsilon h\pi + \nabla\cdot(h\varphi) = 0 & \text{in } Q, \\ h\varphi = 0 & \text{on } \Sigma, \\ (h\varphi)(T,\cdot) = 0 & \text{in } \Omega. \end{cases}$$

Thus, since h and its derivatives vanish at $t = T$, Lemma 2.3 implies that:

$$\|h\varphi\|_{H^{k+1,2(k+1)}(Q)} + \|h\pi\|_{H^{k,2k+1}(Q)} \le C\|h'\varphi\|_{H^{k,2k}(Q)}. \tag{A.3}$$

As $h' \in H_0^k(0,T)$ we can now use the inductive hypothesis (A.1) and obtain (A.2) from (A.3). $\square$

Using Lemma A.1 with $h = s^r e^{-s\alpha^*}(\xi^*)^r$ we find that:

$$\|s^r e^{-s\alpha^*}(\xi^*)^r\varphi\|_{H^{k,2k}(Q)} + \|s^r e^{-s\alpha^*}(\xi^*)^r\pi\|_{H^{k-1,2k-1}(Q)} \le \|\partial_t^k(s^r e^{-s\alpha^*}(\xi^*)^r)\varphi\|_{L^2(Q)}. \tag{A.4}$$

To conclude, we have to estimate the time derivative of the weights. We first recall the following result, which is classical, whose proof is sketched for completeness:

Lemma A.2. *Let $k \in \mathbb{N}_*$. Then, for all $s \ge e^{C\lambda}T^m$ we have the following estimate:*

$$|\partial_t^k \xi^*| + |\partial_t^k \alpha^*| \le s^{k/m}(\xi^*)^{1+k/m}. \tag{A.5}$$

Sketch of the proof of Lemma A.2 . It is easy to prove by induction that:

$$\partial_t^k \xi^* = \frac{q(t,T)}{(t(T-t))^{m+k}}, \qquad \partial_t^k \alpha^* = \frac{q(t,T)(e^{2\lambda\|\eta\|_\infty} - 1)}{(t(T-t))^{m+k}}. \tag{A.6}$$

for q a homogeneous polynomial of degree k. Thus, considering that $t \le T \le e^{-C\lambda}s^{1/m}$ as $s \ge e^{C\lambda}T^m$, we easily obtain (A.5) from (A.6). $\square$

Finally, (A.5) and $s\xi^* \geq 1$ imply the estimate:

$$\partial_t^k (s^r e^{-s\alpha^*} (\xi^*)^r) \leq C s^r \sum_{i_1+\cdots+i_{k+1}=k} \left(\prod_{j=1,\ldots,k: i_j \neq 0} s \partial_t^{i_j} \alpha^* \right) e^{-s\alpha^*} \partial_t^{i_{k+1}} [(\xi^*)^r]$$
$$\leq s^{r+k+k/m} (\xi^*)^{r+k+k/m} e^{-s\alpha^*}. \quad \text{(A.7)}$$

Indeed, we get the maximum exponent for s and ξ^* in (A.7) by picking $j_1, \ldots, j_k = 1$ and $j_{k+1} = 0$. Consequently, we obtain the desired estimate (2.16) from (A.4) and (A.7).

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