

CONTROL OF HYPERBOLIC AND PARABOLIC EQUATIONS ON NETWORKS AND SINGULAR LIMITS

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ABSTRACT. We study the controllability properties of the transport equation and of parabolic equations posed on a tree. Using a control localized on the exterior nodes, we prove that the hyperbolic and the parabolic systems are null-controllable. The hyperbolic proof relies on the method of characteristics, the parabolic one on duality arguments and Carleman inequalities. We also show that the parabolic system may not be controllable if we do not act on all exterior vertices because of symmetries. Moreover, we estimate the cost of the null-controllability of transport-diffusion equations with diffusivity $\varepsilon > 0$ and study its asymptotic behavior when $\varepsilon \rightarrow 0^+$. We prove that the cost of the controllability decays for a time sufficiently large and explodes for short times. This is done by duality arguments allowing to reduce the problem to obtain observability estimates which depend on the viscosity parameter. These are derived by using Agmon and Carleman inequalities.

1. INTRODUCTION

In the past few decades, models based on partial differential equations have been very effective in tackling many problems dealing with flows on networks (e.g. irrigation channels, gas pipelines, blood circulation, vehicular traffic, supply chains, air traffic management – see [11] for a survey of the topic).

In particular, linear advection-diffusion equations on graphs have been employed to describe the flow of a fluid with a dissolved contaminant through a network of one-dimensional cracks (see [70]). Following [31], we represent the network by a finite, directed, and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V} = \{v_1, \dots, v_n\}$ and edges $\mathcal{E} = \{e_1, \dots, e_m\} \subset \mathcal{V} \times \mathcal{V}$. For every edge $e \in \mathcal{E}$, we define an incidence vector $(n^e)_{v \in \mathcal{V}}$ by

$$n^e(v) = -1 \text{ if } e = (v, \cdot), \quad n^e(v) = 1 \text{ if } e = (\cdot, v), \quad \text{and} \quad n^e(v) = 0 \text{ otherwise.}$$

Here, n^e plays the same role as the normal vector for problems in multi-dimensional domains. For any $v \in \mathcal{V}$, we define the set of incident edges $\mathcal{E}(v) := \{e \in \mathcal{E} : n^e(v) \neq 0\}$; we also distinguish between inner vertices $\mathcal{V}_0 := \{v \in \mathcal{V} : |\mathcal{E}(v)| \geq 2\}$ and boundary or external vertices $\mathcal{V}_\partial := \mathcal{V} \setminus \mathcal{V}_0$. We suppose that for all $v \in \mathcal{V}_0$ there are $e_1, e_2 \in \mathcal{E}(v)$ such that $n^{e_1}(v) = -1$ and $n^{e_2}(v) = 1$. We may model the proposed problem with the system

$$(1.1) \quad \begin{cases} a^e \partial_t y_\varepsilon^e + b^e \partial_x y_\varepsilon^e - \varepsilon \partial_{xx}^2 y_\varepsilon^e = 0, & \text{in } (0, T) \times e, \quad e \in \mathcal{E}, \\ y_\varepsilon^e(t, v) = u_\varepsilon^v(t), & \text{on } (0, T) \times \mathcal{V}_\partial, \\ y_\varepsilon^{e_1}(t, v) = y_\varepsilon^{e_2}(t, v), & t \in (0, T), \quad v \in \mathcal{V}, \quad \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} (b^e y_\varepsilon^e - \varepsilon \partial_x y_\varepsilon^e) n^e = 0, & \text{on } (0, T) \times \mathcal{V}_0, \\ y_\varepsilon(0, \cdot) = y_0, & \text{on } \mathcal{E}, \end{cases}$$

with $a^e, b^e, \varepsilon > 0$ and for a fixed time-horizon $T > 0$. Each edge is identified with a closed interval $[x_e, \tilde{x}_e]$, and the left end (resp. right end) is identified with the vertex v such that $n^e(v) = -1$ (resp. $n^e(v) = 1$). We shall also use the notation $y_\varepsilon := (y_\varepsilon^{e_1}, \dots, y_\varepsilon^{e_m})$ and $u = (u_\varepsilon^{v_1}, \dots, u_\varepsilon^{v_k})$, where $k = |\mathcal{V}_\partial|$ and $m = |\mathcal{E}|$. The boundary condition $(1.1)_3$ is a continuity condition in the internal nodes and $(1.1)_4$ implies that the flux of the mass is null. Here, y_ε^e denotes the concentration of the contaminant and b^e the flow rate in each edge of the graph and u_ε^e is the boundary datum on each boundary vertex of the graph, which we may use as control.

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We additionally assume

$$(1.2) \quad \sum_{e \in \mathcal{E}(v)} b^e n^e(v) = 0, \quad v \in \mathcal{V}_0,$$

which is a balance relation for the flow across a junction. In addition to being an assumption used in the parabolic energy estimate to annihilate the terms at junctions, condition (1.2) is motivated by the corresponding hyperbolic problem, for which it ensures that energy does not increase at junctions. Such condition is also pivotal in proving the vanishing viscosity approximation result in [32].

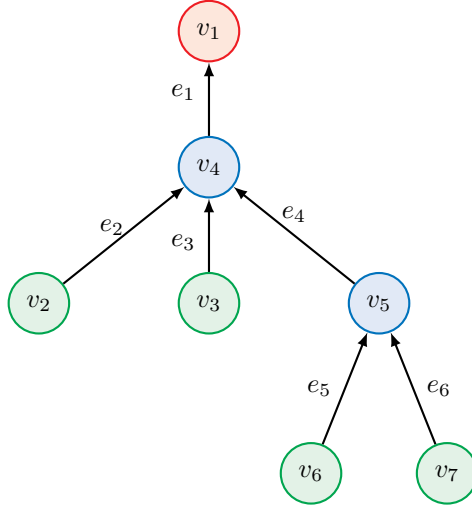


FIGURE 1. An example of tree with edges $e_1 = (v_4, v_1)$, $e_2 = (v_2, v_4)$, $e_3 = (v_3, v_4)$, and $e_4 = (v_5, v_4)$, $e_5 = (v_6, v_5)$, $e_6 = (v_7, v_5)$; inner vertices $\mathcal{V}_0 = \{v_4, v_5\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_2, v_3, v_6, v_7\}$. We split the set of boundary vertices into inflow and outflow vertices: $\mathcal{V}_\partial^{in} = \{v_2, v_3, v_6, v_7\}$ (green) and $\mathcal{V}_\partial^{out} = \{v_1\}$ (red), respectively. The set $\mathcal{E}(v_4) = \{e_1, e_2, e_3, e_4\}$ denotes the edges adjacent to the junction v_4 . In a similar manner, we can split the set $\mathcal{E}(v_4)$ by $\mathcal{E}^{in}(v_4) = \{e_2, e_3, e_4\}$ and $\mathcal{E}^{out}(v_4) = \{e_1\}$ into edges that go into or out of the vertex v_4 . The arrows illustrate the flow direction. The non-zero incidence vectors are $n^{e_1}(v_1) = n^{e_2}(v_4) = n^{e_3}(v_4) = n^{e_4}(v_4) = n^{e_5}(v_5) = n^{e_6}(v_5) = 1$ and $n^{e_1}(v_4) = n^{e_2}(v_2) = n^{e_3}(v_3) = n^{e_4}(v_5) = n^{e_5}(v_6) = n^{e_6}(v_7) = -1$.

In this paper, we study the appropriate boundary data (*boundary control*) in order to lead the solution of the parabolic problem (1.1) to rest; and, in particular, its behavior as the diffusivity parameter ε vanishes. In other words, we are interested in controlling the solution of (1.1) across the network and in studying the cost of controllability as $\varepsilon \rightarrow 0^+$, i.e. as the solution of (1.1) approaches the one of the hyperbolic problem

$$(1.3) \quad \begin{cases} a^e \partial_t y^e + b^e \partial_x y^e = 0, & \text{in } (0, T) \times e, \quad e \in \mathcal{E}, \\ y^e(t, v) = u^v(t), & \text{on } (0, T) \times \mathcal{V}_\partial^{in}, \\ y^{e_1}(t, v) = y^{e_2}(t, v), & t \in (0, T), \quad v \in \mathcal{V}, \quad \forall e_1, e_2 \in \mathcal{E}^{out}(v), \\ \sum_{e \in \mathcal{E}^{in}(v)} b^e y^e = y^{e_1}(t, v) \sum_{e \in \mathcal{E}^{out}(v)} b^e, & \text{on } (0, T) \times \mathcal{V}_0, \quad e_1 \in \mathcal{E}^{out}(v), \\ y(0, \cdot) = y_0, & \text{on } \mathcal{E}, \end{cases}$$

where the contaminant does not undergo diffusion and is only driven by the (constant) velocity of the liquid flow. Here, $\mathcal{V}_\partial^{in}$ are the vertices from which the flow is coming into the network (i.e. with just outgoing edges), $\mathcal{V}_\partial^{out}$ are the vertices where the flow goes out of the network (i.e. with just incoming edges), $\mathcal{E}^{in}(v)$ are the edges incoming into v , and $\mathcal{E}^{out}(v)$ are the edges outgoing from v (see Figure 1). The junction condition (1.3)₃ expresses continuity of the concentrations exiting a junction, and (1.3)₄ gives the conservation of the mass. Throughout this paper, we will mainly consider tree-shaped networks (that is, networks without loops), an example of which is given in Figure 1.

We remark that, in order to properly approximate the hyperbolic problem, at the parabolic level, we needed to choose suitable complementary junction conditions (which amounts to picking a realization of the Laplacian on graphs; see [69]). In particular, for the hyperbolic problem, the number of coupling conditions at each junction $v \in \mathcal{V}_0$ is $|\mathcal{E}^{out}(v)|$, which only suffices to guarantee conservation of mass at the junction and to prescribe the concentrations at the outflow edges; on the other hand, for the parabolic problem, the number of coupling conditions at each junction $v \in \mathcal{V}_0$ is $|\mathcal{E}(v)|$, which allows to guarantee continuity of the solution and conservation of mass at the junction.

In order to obtain and understand the limiting behaviour of the controls for (1.1) we study two additional problems that have interest on their own.

- The first problem is the controllability of (1.3) without the assumption (1.2). We show that system (1.3) is controllable for a sufficiently large times, and not controllable for small times. In the process, we show that determining the time from which the system is null-controllable is not a simple task (compared to determining it in segments) since it depends on the geometry and metric of the network. Moreover, in networks with loops, applying a null control may not be enough to take the solution to rest. This problem has interest on its own since (1.3) models an inviscid flow on the network.
- The second problem is the study of null-controllability of parabolic systems in network. In particular, we study the controllability of

$$(1.4) \quad \begin{cases} a^e \partial_t y^e - \mu^e \partial_{xx}^2 y^e + b^e(t, x) \partial_x y^e + c^e(t, x) y^e = 0, & \text{in } (0, T) \times e, \quad e \in \mathcal{E}, \\ y^e(t, v) = u^v(t), & \text{on } (0, T) \times \mathcal{V}_\partial, \\ y^{e_1}(t, v) = y^{e_2}(t, v), & t \in (0, T), \quad v \in \mathcal{V}, \quad \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} \mu^e \partial_x y^e n^e = \gamma(t, v) y(t, v), & \text{on } (0, T) \times \mathcal{V}_0, \\ y(0, \cdot) = y_0, & \text{on } \mathcal{E}, \end{cases}$$

which generalizes (1.1). Here, we assume $\inf a, \inf \mu > 0$. We show that (1.4) is null controllable by acting on all the external vertices (except at most one); more surprisingly, it may not be null-controllable if we just act on a smaller subset of the boundary vertices. This already allows to determine the controllability of (1.1), and it has interest on its own since (1.4) models diffusive processes in networks.

Our main novel contributions can be then summarized as follows:

- By the method of characteristics, we control the hyperbolic problem (1.3) to zero (for sufficiently long times) by acting on the incoming vertices and also discuss about the optimal time for which (1.3) is null-controllable, which is not as trivial in graphs as in segments.
- By H.U.M. (*Hilbert Uniqueness Method*, introduced in [61, 62]) and Carleman inequalities, we control the parabolic problem (1.4) to zero by acting on all the boundary vertices in \mathcal{V}_∂ *except at most one* – here, as explained later in Proposition 3.2, we cannot steer the system to zero by just acting on fewer vertices. We do it with a Carleman inequality, with the purpose that our results may also be useful for researchers interested in bang-bang controls, inverse problems, the controllability of the semilinear heat-equations, insensitizing problems, etc. (see [76]). Analogously, we also show a controllability result by acting in the interior of a sufficient number of edges.
- We estimate on the cost of controllability for the parabolic problem (1.1), which depends on the time-horizon: for small times, we prove the blow-up of the cost of controllability; for sufficiently long time horizon, we prove its decay.

1.1. Flows on networks, control and singular limits. For conservation laws on \mathbb{R} , the vanishing viscosity limit as $\varepsilon \rightarrow 0^+$ of the uniformly parabolic viscous regularization

$$\partial_t y_\varepsilon + \partial_x f(y_\varepsilon) = \varepsilon \partial_{xx}^2 y_\varepsilon$$

is pivotal in Kruřkov's well-posedness theory for entropy solutions (see [28, 46] for a modern exposition). Vanishing viscosity limits of scalar conservation laws with degenerate (second-order) diffusion have also been extensively studied (see [49, 47, 9, 48] and references therein). For zero diffusion-dispersion limits approximating entropy solutions, we refer to the large body of literature starting with the papers [74, 58].

The problem of singular limits related to scalar conservation laws modeling traffic flow on networks has been addressed more recently in [22, 3, 20, 21]. The authors considered the vanishing viscosity approximation of the traffic model (with constant or degenerate diffusivity) on a network composed by a single junction with

n incoming and m outgoing edges. They proved that, under suitable coupling conditions in the junction, the solution of the parabolic approximation exists and, as the viscosity vanishes, it converges to a solution of the original problem (which is entropy admissible – in the sense of [3, 34]). In the context of Hamilton-Jacobi equations, a convergence result for the vanishing viscosity approximation was established in [12] under Kirchhoff-type conditions.

For linear transport equations, further results are available: in [32], suitable coupling condition that guarantee conservation of mass, energy dissipation, and continuity are imposed and a vanishing viscosity convergence result is established; on the other hand, in [40], the equations are coupled by transmission conditions set at the inner node, which do not impose continuity on the unknown and, as the diffusion coefficient vanishes, the family of solutions converges to the unique solution of the first order equations and satisfies suitable transmission conditions at the inner node, which are determined by the parameters appearing in the parabolic transmission conditions. In the present paper, we adopted the coupling conditions of [32].

The study of uniform controllability problems for singular perturbations of partial differential equations started with the pioneering works [62, Chapter 3], [59, 60, 67, 66]. In the context of linear advection-diffusion equations in the vanishing viscosity limit, the first result was obtained by Coron and Guerrero in [27], where they made a conjecture on the minimal time needed to achieve uniform controllability. Then, the estimates on this minimal time are improved in [36] with a complex analytic method. The result of [27] was also generalized in several space dimensions and for non-constant transport speed in [41].

For nonlinear transport terms, the only available results have been obtained by Glass and Guerrero (see [37]) for the Burgers equation in the vanishing viscosity limit and later generalized by Léautaud for more general flux functions in [57]. As for other systems, several results have been recently obtained: for the Stokes system (see [6]), for an artificial advection-diffusion problem (see [24, 25]), and for fourth-order parabolic equations (see [15, 68, 50]).

In the network setting, much fewer controllability results are available. The null-controllability of the heat equation in trees with coefficients that depends on the space variable was established in [4]. In [29], the controllability of several classes of PDEs on networks is considered (wave, Schrödinger, heat, beam, etc.): in particular, in [29, Chapter 8.1] the heat equation with Kirchhoff-type junction conditions is controlled to zero by the action of a controller under suitable topological assumptions on the graph, which are needed as their proof is based on the transmutation method. For the well-posedness, controllability, and stabilization of several hyperbolic problems, we also point to [53, 52, 51, 54] and references therein (for models of thermoelastic beams, linked plates, and plate-beam systems); and to [7, Chapter 1.15], [43, 45, 75, 30, 42, 44] (for models arising in water flow, gas transport, etc.).

However, results on the controllability in the singular limit on networks are, to the best of our knowledge, still open. With the present paper, we aim to address this gap in the literature.

We remark that, in the case of zero dispersion limit, the uniform control properties of the linearized Korteweg-de Vries equation were studied in [38]; subsequently, in [39], the authors addressed the case of zero diffusion-dispersion. Recent work on the KdV equation with a vanishing parameter in the diffusive term include [13, 14]. On a star-graph, well-posedness and controllability results for KdV are also available in the literature (see [16, 19, 18, 2]), but the uniform controllability problem has not been addressed yet.

1.2. Outline of the paper. The paper is organized as follows.

In Section 2, we introduce the preliminary information on the function spaces used throughout the work and present the known results on the well-posedness of problems (1.3) and (1.4) and on the convergence of (1.1) to (1.3) (which have been obtained in [32]).

In Section 3, we state our main theorems: a controllability result for the hyperbolic problem (1.3) on a tree (for sufficiently long times); a controllability result for the parabolic problem (1.4); and an estimate on the cost of controllability for the parabolic problem (1.1) (which depends on the time-horizon) as the diffusivity parameter vanishes. Moreover, we present some pathological cases to illustrate the sharpness of our results.

In Section 4, we prove the controllability result for the transport equation on a tree by relying on the classical method of characteristics: thanks to the flux conservation condition in (1.3), we are able to argue analogously to the case of a bounded interval, where it suffices to take a null boundary control. We also show that, using non-null boundary controls, we can control the system to zero in a shorter time-span.

In Section 5, we use a duality argument (H.U.M.) and Carleman inequalities to prove the null-controllability for the parabolic problem (1.4) with controls in the exterior vertices. We also prove that a similar result can be obtained using controls localized in the interior of the edges of the network.

Sections 6 and 7 are dedicated to the singular limit problem. In the first one, we prove the blow-up of the cost of controllability for the parabolic problem (1.1) and, in the second one, we prove the decay (for sufficiently long time-horizon). Our strategy is based on the H.U.M. and, in particular, on the ideas of [27, 41]. To prove the blow-up, we rely on an Agmon-type inequality. For the proof of the decay, we need to establish a decay property for the L^2 -mass of the adjoint system and a Carleman-type inequality.

Section 8 concludes the paper with some open problems for future consideration.

2. PRELIMINARIES

2.1. Function spaces on a network and parametrization of the edges. Each edge $e \in \mathcal{E}$ has a positive length l^e and we identify it with the interval $(0, l^e)$. More specifically, we state the following lemma.

Lemma 2.1 (Edge parametrization in a tree). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed tree. Then, each edge can be identified with an interval $[x^e, x^e + l^e]$ such that for all $v \in \mathcal{V}$ there is $C^v \in \mathbb{R}$ such that*

$$(2.1) \quad \begin{cases} x^e + l^e = C^v & \forall e \in \mathcal{E}^{in}(v), \\ x^e = C^v & \forall e \in \mathcal{E}^{out}(v), \end{cases}$$

Proof. Lemma 2.1 can be proved with an induction on the number of edges. The base case, a tree with one edge, is trivial. Let us then assume that the property is true for a tree of N vertices and let us prove that it holds for a tree of $N + 1$ vertices. It is well-known that there is at least one vertex u with degree 1 and with an edge e incident in some vertex $v \in \mathcal{V} \setminus \{u\}$. The graph $(\mathcal{V} \setminus \{u\}, \mathcal{E} \setminus \{e\})$ satisfies the inductive hypothesis, and thus we can identify those segments in a way that they satisfy (2.1). Thus, we just have to identify the edge e . If $e \in \mathcal{E}^{in}(v)$, we might identify e with $[C^v - l^e, C^v]$. Otherwise, if $e \in \mathcal{E}^{out}(v)$ we might identify e with $[C^v, C^v + l^e]$. \square

Remark 2.1 (Parametrization of star-graphs). *For a star-graph, we identify each edge $e \in \mathcal{E}^{in}(v)$ as $[-l^e, 0]$ and each edge $e \in \mathcal{E}^{out}(v)$ as $[0, l^e]$.*

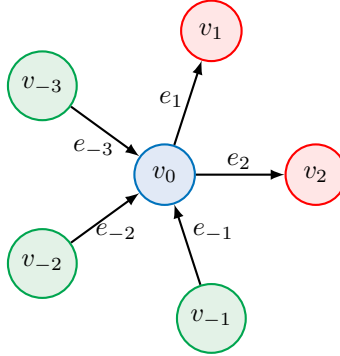


FIGURE 2. An example of a star-graph (particular case of tree).

Throughout the paper, as in [32], we use the following notation for the space of square-integrable functions:

$$L^2(\mathcal{E}) := L^2(e_1) \times \cdots \times L^2(e_n) = \{w : w^e \in L^2(e), e \in \mathcal{E}\},$$

with the norm and scalar product

$$\|w\|_{L^2(\mathcal{E})}^2 := \sum_{e \in \mathcal{E}} \|w^e\|_{L^2(e)}^2 \quad \text{and} \quad (w_1, w_2)_{L^2(\mathcal{E})} := \sum_{e \in \mathcal{E}} (w_1^e, w_2^e)_{L^2(e)}.$$

Sometimes, we write $\int_{\mathcal{E}} w \, dx := \sum_{e \in \mathcal{E}} \int_e w^e \, dx$. We also use the (piecewise) Sobolev space

$$H_{pw}^s(\mathcal{E}) = \{w \in L^2(\mathcal{E}) : w^e \in H^s(e), e \in \mathcal{E}\},$$

with

$$\|w\|_{H_{pw}^s(\mathcal{E})}^2 := \sum_{e \in \mathcal{E}} \|w^e\|_{H^s(e)}^2 \quad \text{and} \quad (w_1, w_2)_{H_{pw}^s(\mathcal{E})} := \sum_{e \in \mathcal{E}} (w_1^e, w_2^e)_{H^s(e)}.$$

Similarly, we define the spaces of (piecewise) k -times differentiable functions $C_{pw}^k(\mathcal{E})$. For $s > \frac{1}{2}$, we note that the functions in $H_{pw}^s(\mathcal{E})$ are continuous on $e \in \mathcal{E}$, but may be discontinuous across the junction. For $s > \frac{1}{2}$, we denote by $H^s(\mathcal{E})$, the subspace of functions belonging to $H_{pw}^s(\mathcal{E})$ which are also continuous across the junction. We remark that every $w \in H^1(\mathcal{E})$ has a unique value $w(v)$ at every vertex $v \in \mathcal{V}$ and we use $\ell^2(\mathcal{V})$ to denote the set of possible vertex values.

Also, we define distance between vertices and layers of a tree as follows.

Definition 2.1 (Distance and layers on a graph). *We define the distance $d(v_1, v_2)$ between two vertices v_1, v_2 in the graph \mathcal{G} as the minimum number of vertices contained in a path joining them (if any, otherwise, $d(v_1, v_2) = \infty$). In addition, for a tree \mathcal{G} with root v , we define the i -th layer of the tree as the vertices at distance i from v .*

Throughout the paper, we use c or C to denote positive constants which may change from line to line and that may depend on the network (namely, on parameters of the type a^e , b^e , c^e or l^e).

2.2. Well-posedness of the parabolic and hyperbolic problems. We recall a well-posedness result for the parabolic problem (1.4) (see [32, Theorem 3]).

Theorem 2.1 (Well-posedness for the parabolic problem). *For $a^e > 0$, $\mu^e > 0$, $b^e \in L^\infty(0, T; W_{pw}^{1,\infty}(\mathcal{E}) \cap C^0(\bar{\mathcal{E}})) \cap W^{1,\infty}(0, T; L^\infty(\mathcal{E}))$, $\gamma \in W^{1,\infty}((0, T) \times \mathcal{V}_0)$, $c^e \in L^\infty((0, T) \times \mathcal{E})$, $y_0 \in H^1(\mathcal{E}) \cap H_{pw}^2(\mathcal{E})$ and $u_\varepsilon \in C^2([0, T]; \ell_2(\mathcal{V}_\partial))$, the parabolic problem (1.4) has a unique classical solution*

$$y_\varepsilon \in C^1([0, T]; L^2(\mathcal{E})) \cap C^0([0, T]; H^1(\mathcal{E}) \cap H_{pw}^2(\mathcal{E})).$$

Similarly, in [32, Theorem 6], a well-posedness result for the transport problem was obtained.

Theorem 2.2 (Well-posedness for the hyperbolic problem). *For any $y_0 \in H_{pw}^1(\mathcal{E})$ and $u \in C^2([0, T]; \ell_2(\mathcal{V}_\partial^{in}))$, the hyperbolic problem (1.3) has a unique classical solution*

$$y \in C^1([0, T]; L^2(\mathcal{E})) \cap C^0([0, T]; H_{pw}^1(\mathcal{E})).$$

Finally, the following convergence result holds (see [32, Theorem 10]).

Theorem 2.3 (Vanishing viscosity error estimate). *For any $y_0 \in H^1(\mathcal{E}) \cap H_{pw}^2(\mathcal{E})$ and $u_\varepsilon \in C^2([0, T]; \ell_2(\mathcal{V}_\partial))$. Let*

$$y_\varepsilon \in C^1([0, T]; L^2(\mathcal{E})) \cap C^0([0, T]; H^1(\mathcal{E}) \cap H_{pw}^2(\mathcal{E})).$$

be the solution of problem (1.1) and

$$y \in C^1([0, T]; L^2(\mathcal{E})) \cap C^0([0, T]; H_{pw}^1(\mathcal{E})).$$

be the solution of the corresponding limit problem (1.3). Then,

$$(2.2) \quad \|y_\varepsilon - y\|_{L^\infty(0, T; L^2(\mathcal{E}))} \leq C\sqrt{\varepsilon},$$

with a constant C that depends on the time-horizon T but is independent of the diffusion parameter $0 < \varepsilon \leq 1$.

Example 2.1 (Explicit computations for the hyperbolic problem and continuity at junctions). *Let us consider the graph in Figure 3 made of the vertices v_1, v_2, v_3, v_4 and edges $e_1 = [v_1, v_3] \simeq [0, 1]$, $e_2 = [v_2, v_3] \simeq [0, 1]$, and $e_3 = [v_3, v_4] \simeq [0, 1]$. We consider the equation (1.3) with $a^{e_1} = a^{e_2} = a^{e_3} = b^{e_1} = b^{e_2} = 1$ and $b^{e_3} = 2$. Let us consider as initial datum $y_0 = 0$ and as boundary datum $u_1 = 1$ and $u_2 = 2$. Then the solution can be computed as follows: $y^{e_1} = \mathbf{1}_{t>x}$, $y^{e_2} = 2 \cdot \mathbf{1}_{t>x}$, $y^{e_3} = \frac{3}{2} \cdot \mathbf{1}_{t>1+\frac{x}{2}}$: this satisfies the flux conservation condition $b^{e_1}y^{e_1} + b^{e_2}y^{e_2} = b^{e_3}y^{e_3}$ at the junction expressed in (1.3)₄. However, we note that, in this example, the continuity across the junction is indeed lost from the parabolic system with vanishing viscosity even if the controls are regular (only the weaker condition (1.3)₃ is preserved).*

As an additional example, let us consider the graph in Figure 4 made of the vertices v_1, v_2, v_3, v_4 and the edges $e_1 = [v_1, v_2] \simeq [0, 1]$, $e_2 = [v_2, v_3] \simeq [0, 1]$, and $e_3 = [v_2, v_4] \simeq [0, 1]$. We take $a^{e_1} = a^{e_2} = a^{e_3} = b^{e_1} = 1$, $b^{e_2} = 1/3$, $b^{e_3} = 2/3$. Let us consider as initial datum $y_0 = 0$ and as boundary datum $u_1 = 1$. Then the solution can be computed as follows: $y^{e_1} = \mathbf{1}_{t>x}$, $y^{e_2} = \mathbf{1}_{t>1+\frac{x}{3}}$, $y^{e_3} = \mathbf{1}_{t>1+\frac{2x}{3}}$: this satisfies the flux

conservation condition $b^{e_1}y^{e_1} = b^{e_2}y^{e_2} + b^{e_3}y^{e_3}$ at the junction expressed in (1.3)₃. In this example, the continuity is kept at the junction.

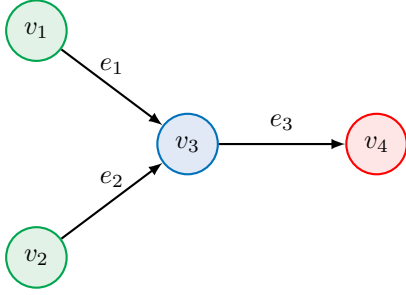


FIGURE 3. Star-graph with edges $e_1 = (v_1, v_3)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$; inner vertices $\mathcal{V}_0 = \{v_3\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_2, v_4\}$. We split the set of boundary vertices into inflow and outflow vertices: $\mathcal{V}_\partial^{in} = \{v_1, v_2\}$ (green) and $\mathcal{V}_\partial^{out} = \{v_4\}$ (red), respectively.

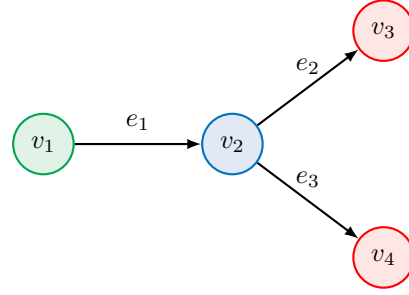


FIGURE 4. Star-graph with edges $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_2, v_4)$; inner vertices $\mathcal{V}_0 = \{v_2\}$ (blue), and boundary vertices $\mathcal{V}_\partial = \{v_1, v_3, v_4\}$. We split the set of boundary vertices into inflow and outflow vertices: $\mathcal{V}_\partial^{in} = \{v_1\}$ (green) and $\mathcal{V}_\partial^{out} = \{v_3, v_4\}$ (red), respectively.

3. MAIN RESULTS

3.1. Control of the hyperbolic problem. Our first result concerns the null-controllability of problem (1.3) on a tree. For the sake of simplicity, we consider coefficients that are constant in each edge; however, as explained in Remark 4.2, the results are also valid for strictly positive space-dependent coefficients. In particular, we are interested in controlling the flow across the network by means of controls placed on the inflow vertices $\mathcal{V}_\partial^{in}$.

To this end, we define an upper bound and a lower bound on the time in which the information propagates across the network (i.e., the maximal and minimal travel time of the characteristics across the network).

Definition 3.1 (Propagation time on a network). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree. We define the functions $\hat{T}, \tilde{T} : \mathcal{V} \rightarrow \mathbb{R}_+$ as follows:*

$$\begin{aligned} \hat{T}(v) &= 0, & \tilde{T}(v) &= 0 & \text{if } v \in \mathcal{V}_\partial^{in}, \\ \hat{T}(v) &= \sup_{e=(v^e, v) \in \mathcal{E}^{in}(v)} \left(\hat{T}(v^e) + \frac{a^e l^e}{b^e} \right) & & & \text{if } v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{out}, \\ \tilde{T}(v) &= \inf_{e=(v^e, v) \in \mathcal{E}^{in}(v)} \left(\tilde{T}(v^e) + \frac{a^e l^e}{b^e} \right) & & & \text{if } v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{out}. \end{aligned}$$

The times $\hat{T}(v)$ and $\tilde{T}(v)$ are respectively the maximal and minimal propagation time required for information to reach $v \in \mathcal{V}$ from a node in $\mathcal{V}_\partial^{in}$.

Since the network \mathcal{G} has no loops (being a tree), we can prove inductively that \hat{T} and \tilde{T} are well-defined.

Example 3.1 (Propagation times). *Let consider the graph in Figure 3 with vertices v_1, v_2, v_3, v_4 , and edges $e_1 = [v_1, v_3] \simeq [0, 2]$, $e_2 = [v_2, v_3] \simeq [0, 1]$, and $e_3 = [v_3, v_4] \simeq [0, 2]$. We consider the equation (1.3) with $a^{e_1} = a^{e_2} = a^{e_3} = b^{e_1} = b^{e_2} = 1$ and $b^{e_3} = 2$. We can compute the maximal travel time to reach $v \in \mathcal{V}$ as follows:*

$$\hat{T}(v_1) = \hat{T}(v_2) = 0, \quad \hat{T}(v_3) = \sup_{i=1,2} (\hat{T}(v_i) + l^i) = \max\{1, 2\} = 2, \quad \hat{T}(v_4) = \hat{T}(v_3) + \frac{l^3}{b^3} = 2 + 1 = 3.$$

Moreover, we compute the minimal travel time to reach $v \in \mathcal{V}$ as follows:

$$\tilde{T}(v_1) = \tilde{T}(v_2) = 0, \quad \tilde{T}(v_3) = \inf_{i=1,2} (\tilde{T}(v_i) + l^i) = \min\{1, 2\} = 1, \quad \tilde{T}(v_4) = \tilde{T}(v_3) + \frac{l^3}{b^3} = 1 + 1 = 2.$$

By relying on this notion of propagation time and on the classical method of characteristics, we can prove the following controllability result.

Theorem 3.1 (Null-controllability for the hyperbolic problem). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree and let y be the solution of (1.3) for $u = 0$. Then, $y = 0$ for all $T \geq \max_{v \in \mathcal{V}_\partial^{out}} \hat{T}(v)$.*

The proof of Theorem 3.1 is given in Section 4.

Remark 3.1 (Generalization of the hyperbolic result). *We can actually prove a stronger result: $y^e(T, x) = 0$ for all $x \in e = [v_1, v_2] \simeq [0, l^e]$ and $T \geq \hat{T}(v_1) + \frac{a^e x}{b^e}$.*

Moreover, the following result regarding the minimal propagation time holds.

Proposition 3.1 (Minimal time for the null-controllability for the hyperbolic problem). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree. Then, system (1.3) is not null-controllable for $T < \max_{v \in \mathcal{V}_\partial^{out}} \hat{T}(v)$.*

Proposition 3.1 is also proved in Section 4. We will also observe (in Remark 4.3) that the time given in Theorem 3.1 may not be optimal if we consider non-null controls. In addition, as we see below in Remark 4.4, the optimal time for null-controllability can be almost any in $[\max_{v \in \mathcal{V}_\partial^{out}} \tilde{T}(v), \max_{v \in \mathcal{V}_\partial^{out}} \hat{T}(v)]$ (by possibly using non-zero controls) by considering the right network, so the upper and lower bound are optimal if we do not add any extra hypothesis on the structure of the network.

3.2. Control of the parabolic problem. Our next theorem provides the controllability of the parabolic system (1.4) on tree-shaped networks by acting on the external vertices (except at most one) without any further geometric constraint.

Theorem 3.2 (Controllability of parabolic systems of networks). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree, $a^e > 0$, $\mu^e > 0$, $b^e \in L^\infty(0, T; W_{pw}^{1,\infty}(\mathcal{E}) \cap C^0(\bar{\mathcal{E}})) \cap W^{1,\infty}(0, T; L^\infty(\mathcal{E}))$, $\gamma \in W^{1,\infty}((0, T) \times \mathcal{V}_0)$, $c^e \in L^\infty((0, T) \times \mathcal{E})$, and $y_0 \in L^2(\mathcal{E})$. Then, there exists $u \in L^2((0, T) \times \mathcal{V}_\partial)$ such that the solution of (1.4) satisfies $y(T, \cdot) = 0$.*

The proof of Theorem 3.2 is given in Section 5 and is based on the H.U.M. method (see, for example, [61]) – i.e., on the proof of an observability result for the adjoint system, which is given by

$$(3.1) \quad \begin{cases} -a^e \partial_t \varphi^e - \mu^e \partial_{xx}^2 \varphi^e - \partial_x b^e(t, x) \partial_x \varphi^e - b^e(t, x) \partial_x \varphi^e + c^e(t, x) \varphi^e = 0, & \text{in } (0, T) \times e, e \in \mathcal{E}, \\ \varphi^e(v) = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ \varphi^{e_1}(t, v) = \varphi^{e_2}(t, v), & t \in (0, T), v \in \mathcal{V}, \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} \mu^e \partial_{n^e} \varphi^e(t, v) = (-\sum_{e \in \mathcal{E}(v)} n^e b^e(t, v) + \gamma(t, v)) \varphi^e, & \text{on } (0, T) \times \mathcal{V}_0, \\ \varphi(T, \cdot) = \varphi_T, & \text{on } \mathcal{E}. \end{cases}$$

Generally, in the parabolic setting, it does not suffice to act on $\mathcal{V}_\partial^{in}$ to drive the system to zero (see Proposition 3.3). Instead, we can establish a null-controllability result by considering all the boundary vertices \mathcal{V}_∂ , except at most one (as highlighted in the proof of the Carleman inequality established in Proposition 5.1). We remark that such Carleman inequality could also be used to obtain bang-bang controls, to study inverse problems, to study the controllability of the semilinear heat-equation, insensitizing problems, etc. (see [76]).

Remark 3.2 (Acting on all but one boundary vertices in \mathcal{V}_∂). *More precisely, we will see that the proof is valid if we act in all the boundary vertices except at most one (the one chosen as a root of the tree), i.e. if we consider $\mathcal{V}_\partial \setminus \{\tilde{v}\}$. As we will establish in Proposition 3.2 below, this result is sharp in some graphs: acting on fewer vertices may not suffice.*

3.3. Cost of controllability in the singular limit. Our final main theorem provides estimates on the cost of controllability as $\varepsilon \rightarrow 0^+$ for the parabolic problem (1.1). As in the hyperbolic case, for the sake of simplicity, we consider coefficients that are constant in each edge; but, as explained in Remark 7.2, the results are also valid for strictly positive space-dependent coefficients.

We consider the following quantity which measures the cost of the null-controllability of (1.1):

$$(3.2) \quad K(\varepsilon, T, a^e, b^e, \mathcal{G}, \tilde{\mathcal{V}}_\partial) := \sup_{y_0 \in L^2(\mathcal{E}) \setminus \{0\}} \inf_{u \in U} \frac{\|u\|_{L^2(0, T)}}{\|y_0\|_{L^2(\mathcal{E})}},$$

where $U := U(y_0, \varepsilon, T, a^e, b^e, \tilde{\mathcal{V}})$ denotes the subset of $\{u : \ell^2(\mathcal{V}_\partial) : u(v) = 0 \ \forall v \in \mathcal{V}_\partial \setminus \tilde{\mathcal{V}}_\partial\}$ such that the solution of (1.1) satisfies $y(T, \cdot) = 0$.

Regarding the cost of controllability, from the hyperbolic result in Theorem 3.1, we expect the following behavior: for small times, $K \rightarrow +\infty$ as $\varepsilon \rightarrow 0^+$; on the other hand, for T sufficiently large, $K \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Our main result concerns a quantitative estimate on these limits.

Theorem 3.3 (Estimates on the cost of controllability). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree and assume that (1.2) holds.*

- (1) *There exists $c > 0$ such that, for ε small enough and all $T < \max_{v \in \mathcal{V}_\partial^{\text{out}}} \tilde{T}(v)$, the following lower bound holds:*

$$(3.3) \quad K(\varepsilon, T, a^e, b^e, \mathcal{G}, \mathcal{V}_\partial^{\text{in}}) \geq c\varepsilon^{c/\varepsilon}.$$

- (2) *There exist $\bar{T}, c > 0$ such that, for ε small enough and all $T < \bar{T}$, the following lower bound holds:*

$$(3.4) \quad K(\varepsilon, T, a^e, b^e, \mathcal{G}, \mathcal{V}_\partial) \geq c\varepsilon^{c/\varepsilon}.$$

- (3) *There exist $T_0, c, C > 0$ such that, for ε small enough and all $T \geq T_0$, the following upper bound holds:*

$$(3.5) \quad K(\varepsilon, T, a^e, b^e, \mathcal{G}, \mathcal{V}_\partial) \leq C\varepsilon^{-c/\varepsilon}.$$

Remark 3.3 (Acting on all but one boundary vertices in \mathcal{V}_∂). *As in the previous section, we will see that the proof of Item (3) is valid if we act in all the boundary vertices except at most one, i.e. if we consider $\mathcal{V}_\partial \setminus \{\tilde{v}\}$ in formula (3.5). As we will establish in Proposition 3.3 below, this result is sharp in some graphs: acting on fewer vertices (even if we act on every vertex in \mathcal{V}^{in}) may not suffice.*

The proof of Theorem 3.3 is given in Section 6 (Claims (1) and (2)) and Section 7 (Claim (3)) and is based on the H.U.M. (see, for example, [61]) – that is, on study the cost of observation of the adjoint variable. The adjoint system is given by

$$(3.6) \quad \begin{cases} -a^e \partial_t \varphi_\varepsilon^e - b^e \partial_x \varphi_\varepsilon^e - \varepsilon \partial_{xx}^2 \varphi_\varepsilon^e = 0, & \text{in } (0, T) \times e, e \in \mathcal{E}, \\ \varphi_\varepsilon^e(v) = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ \varphi_\varepsilon^{e_1}(t, v) = \varphi_\varepsilon^{e_2}(t, v), & t \in (0, T), v \in \mathcal{V}, \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} \varepsilon \partial_{n^e} \varphi_\varepsilon^e(t, v) = 0, & \text{on } (0, T) \times \mathcal{V}_0, \\ \varphi_\varepsilon(T, \cdot) = \varphi_T, & \text{on } \mathcal{E}, \end{cases}$$

where we have used (1.2) to obtain (3.6)₄. We recall that, by using the H.U.M., the cost of the control defined in (3.2) is equivalently given by

$$(3.7) \quad K(\varepsilon, T, a^e, b^e, \mathcal{G}, \tilde{\mathcal{V}}_\partial) = \sup_{\varphi^T \in L^2(\mathcal{E}) \setminus \{0\}} \frac{\|\varphi_\varepsilon(0, \cdot)\|_{L^2(\mathcal{E})}}{\left(\iint_{(0, T) \times \tilde{\mathcal{V}}_\partial} |\partial_\nu \varphi_\varepsilon|^2 dt \right)^{1/2}}.$$

In order to do some computations, we will often use the following symmetrized system as in [5]: with the identification of the edges given by Lemma 2.1, we define the function

$$(3.8) \quad z_\varepsilon := \varphi_\varepsilon e^{(xb^e)/2\varepsilon},$$

which satisfies

$$(3.9) \quad \begin{cases} -a^e \partial_t z_\varepsilon^e - \varepsilon \partial_{xx}^2 z_\varepsilon^e + \frac{|b^e|^2}{4\varepsilon} z_\varepsilon^e = 0, & \text{in } (0, T) \times e, e \in \mathcal{E}, \\ z_\varepsilon^e(v) = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ z_\varepsilon^{e_1}(t, v) = z_\varepsilon^{e_2}(t, v), & t \in (0, T), v \in \mathcal{V}, \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} \varepsilon \partial_{n^e} z_\varepsilon^e = 0, & \text{on } (0, T) \times \mathcal{V}_0, \\ z_\varepsilon(T, \cdot) = z_T, & \text{on } \mathcal{E}. \end{cases}$$

Here, we relied on (1.2) to obtain (3.9)₄.

Roughly speaking, the strategy of our proofs is as follows:

- for Claims (1) and (2), we obtain that the cost explodes for small times by considering data for the adjoint problem supported far away from the observation domains and computing its observability cost – this explodes as $\varepsilon \rightarrow 0^+$ because for small times the mass remains in the domain, but the mass reaching the observation domain is of order $\exp(C\varepsilon^{-1})$;
- for Claim (3), we show that the cost decays for large times by using first the decay of the free solutions, and then, when the mass of the state is almost null, by exactly observing the mass.

3.4. Pathological examples and further remarks.

3.4.1. *Hyperbolic problem.* For particular graphs and choices of the coefficients in (1.1) and (1.3), we can build several pathological examples to illustrate the scope of our controllability results.

Remark 3.4 (Counterexample to exact controllability to any target state $y(T, \cdot) \in C_{pw}^0(\mathcal{E})$). *While we are able to prove null-controllability, and thus controllability to trajectories because of linearity, we may not have exact controllability to any $y \in C_{pw}^0(\mathcal{E})$ – namely, when $|\mathcal{V}_\partial^{out}| > |\mathcal{V}_\partial^{in}|$. In fact, by the method of characteristics, we get an over-determined system on $\mathcal{V}_\partial^{out}$. For example, let us consider the graph in Figure 4, made of the vertices v_1, v_2, v_3, v_4 and the edges $e_1 = [v_1, v_2] \simeq [0, 1]$, $e_2 = [v_2, v_3] \simeq [0, 1]$, and $e_3 = [v_2, v_4] \simeq [0, 1]$. In (1.3), we take $a^{e_1} = a^{e_2} = a^{e_3} = b^{e_1} = 1$, $b^{e_2} = 1/2$, $b^{e_3} = 1/2$. Then, for any $y_0 \in L^2(\mathcal{E})$ and $u \in C^2([0, T]; \ell^2(v_1))$, we have that the solution y of (1.3) satisfies the equality $y^{e_2}(t, x) = y^{e_3}(t, x)$ for $t > 1$.*

Next, we illustrate some issues arising from networks with loops.

Remark 3.5 (Networks with loops and controls). *Generalizing Theorem 2.2 to networks with loops is not a straightforward problem, but a challenging open one.*

Let us consider a graph with $\mathcal{V} := \{v_1, v_2, v_3, v_4\}$ and with edges $e_1 = [v_1, v_2]$, $e_2 = [v_2, v_3]$, $e_3 = [v_3, v_4]$, $e_4 = [v_4, v_2]$ (see the left-side picture in Figure 5). In that case, in the free system (i.e., with Dirichlet boundary condition $u_1 \equiv 0$ at v_1) we can prove that the total mass is constant (which can be done by showing that $\frac{d}{dt} \int_{\mathcal{E}} u \, dx = 0$ with energy methods or by the method of characteristics). And yet, we can use the method of characteristics to prove that the hyperbolic system is null-controllable with a control that is non-null.

A similar example consists of the same graph with an additional output vertex v_5 and $e_5 = [v_3, v_5]$ (see the right-side picture in Figure 5). In that case, the mass is not conserved, but the null control do not take the system to equilibrium (the mass on the loop decreases exponentially, though). As in the previous example, we can use the method of characteristics to prove that the hyperbolic system is null controllable, with a control that is non-null.

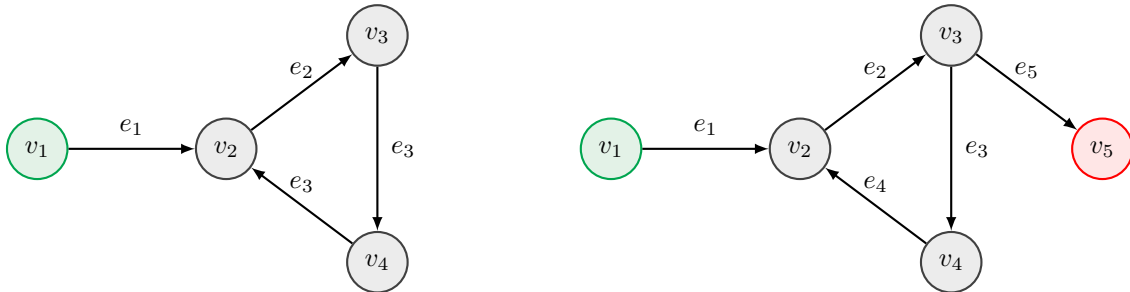


FIGURE 5. Graphs with loops used in Remark 3.5. **Left:** graph with one incoming vertex v_1 (green) and loop made of vertices v_2, v_3, v_4 (gray). **Right:** graph with incoming vertex v_1 (green), outgoing vertex v_5 (red) and loop made of vertices v_2, v_3, v_4 (gray).

3.4.2. *Parabolic problem.* At the parabolic level, it might look surprising to the reader that we cannot also just consider controls acting on $\mathcal{V}_\partial^{in}$. However, system (1.4) may not always be null-controllable depending on the metric of the graph, as shown in the following proposition.

Proposition 3.2 (Lack of null-controllability for general graphs). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph in Figure 4, made of the vertices v_1, v_2, v_3, v_4 and the edges $e_1 = [v_1, v_2] \simeq [0, 1]$, $e_2 = [v_2, v_3] \simeq [0, 1]$, and $e_3 = [v_2, v_4] \simeq [0, 1]$. Then, system (1.4), with coefficients $a = \mu = 1$ and $b = c = \gamma = 0$, is not approximately controllable by acting only on v_1 (i.e., if $u^{v_3} = u^{v_4} = 0$) for any $\varepsilon > 0$.*

Heuristically, the motivation for such result is that, by symmetry, the effect of the control on e_2 and e_3 is identical, so we cannot control both y^{e_2} and y^{e_3} simultaneously. The proof is made rigorous by a duality argument.

Proof. By duality, it suffices to show that there are non-null solutions of (3.1) satisfying $\partial_{n^\varepsilon} \varphi(\cdot, v_1) = 0$. Considering (3.8), this is equivalent to showing that there are non-null solutions of (3.9) satisfying $\partial_{n^\varepsilon} z(\cdot, v_1) = 0$. Considering the spectral decomposition of the Laplacian on the graph, we can construct such a solution as follows:

$$\varphi^{e_1} = 0, \quad \varphi^{e_2} = e^{-\pi^2 t} \sin(\pi x), \quad z^{e_3} = -e^{-\pi^2 t} \sin(\pi x).$$

□

3.4.3. Singular limit problem. As in Proposition 3.2, we note that system (1.1) cannot be controlled for any $\varepsilon > 0$ by acting on fewer boundary vertices.

Proposition 3.3 (Lack of null-controllability for general graphs). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph in Figure 4, made of the vertices v_1, v_2, v_3, v_4 and the edges $e_1 = [v_1, v_2] \simeq [0, 1]$, $e_2 = [v_2, v_3] \simeq [0, 1]$, and $e_3 = [v_2, v_4] \simeq [0, 1]$. Then, system (1.1) (with coefficients $b^{e_1} = a^{e_1} = a^{e_2} = a^{e_3} = 1$ and $b^{e_2} = b^{e_3} = \frac{1}{2}$) is not approximately controllable by acting only on v_1 (i.e. if $u^{v_3} = u^{v_4} = 0$) for any $\varepsilon > 0$.*

Proof. By duality, it suffices to show that there are non-null solutions of (3.6) satisfying $\partial_{n^\varepsilon} \varphi(\cdot, v_1) = 0$. Considering (3.8), this is equivalent to showing that there are non-null solutions of (3.9) satisfying $\partial_{n^\varepsilon} z(\cdot, v_1) = 0$. Again, considering the spectral decomposition of the Laplacian in the graph we can construct such a solution as follows:

$$z^{e_1} = 0, \quad z^{e_2} = \exp \left[- \left(\varepsilon \pi^2 + \frac{1}{16\varepsilon} \right) t \right] \sin(\pi x), \quad z^{e_3} = - \exp \left[- \left(\varepsilon \pi^2 + \frac{1}{16\varepsilon} \right) t \right] \sin(\pi x).$$

□

Finally, we note the relevance of (1.2) for our results.

Remark 3.6 (On the balance relation (1.2)). *The balance relation (1.2) is needed to ensure that the symmetrized system is dissipative. Indeed, without (1.2), we must replace (3.8)₄ by*

$$(3.10) \quad \sum_{e \in \mathcal{E}(v)} \varepsilon \partial_{n^\varepsilon} z_\varepsilon^e = - \sum_{e \in \mathcal{E}(v)} \frac{n^e b^e}{2}.$$

Also, without condition (1.2), system (1.3) may not be dissipative at junctions. Indeed, from (1.3), we compute

$$(3.11) \quad \frac{d}{dt} \int_{\mathcal{E}} (y^e)^2 dx = - \sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} n^e b^e (y^e)^2(v),$$

which is not non-positive in general. For example, let us consider a simple 1-to-1 junction modeled as follows:

$$(3.12) \quad \begin{cases} \partial_t y^- + b^- \partial_x y^- = 0, & x \in (-1, 0), \\ \partial_t y^+ + b^+ \partial_x y^+ = 0, & x \in (0, 1), \\ y^-(t, -1) = 0, & t > 0, \\ b^- y^-(t, 0) = b^+ y^+(t, 0), & t > 0, \\ y^-(0, x) = y_0^-, & x \in (-1, 0), \\ y^+(0, x) = y_0^+, & x \in (0, 1). \end{cases}$$

Here, the term in (3.11) is given by

$$\frac{|y^-|^2}{2} (b^-) \left(1 - \frac{b^-}{b^+} \right),$$

so if $b^+ > b^-$ energy is added on junctions; thus the problem is not dissipative.

We refer to Section 8 (Item 2) for further comments on this matter.

4. CONTROL OF THE TRANSPORT PROBLEM

4.1. Main control result. We prove Theorem 3.1 by means of the method of characteristics, following [26, Chapter 2.1.2., p. 30] (see Figure 6 for a sample illustration in case of 1-to-1 junctions). More specifically, we use an induction argument to show that, for any tree \mathcal{G} , the system (1.3) with $u \equiv 0$ is at rest for any time larger than $\max_{v \in \mathcal{V}} \hat{T}(v)$. For the proof we identify each edge $e = [v_1^e, v_2^e] \simeq [0, l^e]$ as in Remark 3.1.

Proof of Theorem 3.1. The proof is based on an induction of the distance of the vertex v_1^e to the boundary (see Definition 2.1).

The base case is when $v_1^e \in \mathcal{V}_\partial^{in}$. The equality $y(v_1^e) = 0$ is satisfied by the boundary condition (recall that $u = 0$). Moreover, inside each edge the function y^e behaves like the solution to a classical transport equation. Consequently, as in [26, Chapter 2.1.2., p. 30], $y^e(t, x) = 0$ for all $x \in e = [v_1^e, v_2^e] \simeq [0, l^e]$ and $t \geq \frac{a^e x}{b^e}$, and in particular, $y^e(t, v_2^e) = 0$ for all $t \geq \hat{T}(v_2^e)$, as $\hat{T}(v_2^e) \geq \frac{a^e l^e}{b^e}$.

Let us now continue with the inductive case; that is, when $v_1^e \in \mathcal{V}_0$. The equality $y^{\bar{e}}(t, v_1^e) = 0$ is satisfied for all $t \geq \hat{T}(v_1^e)$ by the inductive hypothesis. Thus, from (1.3)₃ and (1.3)₄, we find that $y^e(t, v_1^e) = 0$ for all $t \geq \hat{T}(v_1^e)$. Moreover, inside the edge the function y^e behaves like a transport equation in a segment. Consequently, we get that $y^e(T, x) = 0$ for all $x \in [0, l^e]$ and $T \geq \hat{T}(v_1^e) + \frac{a^e x}{b^e}$, and in particular, $y^e(T, v_2^e) = 0$ for all $T \geq \hat{T}(v_2^e)$, as $\hat{T}(v_2^e) \geq \hat{T}(v_1^e) + \frac{a^e l^e}{b^e}$. \square

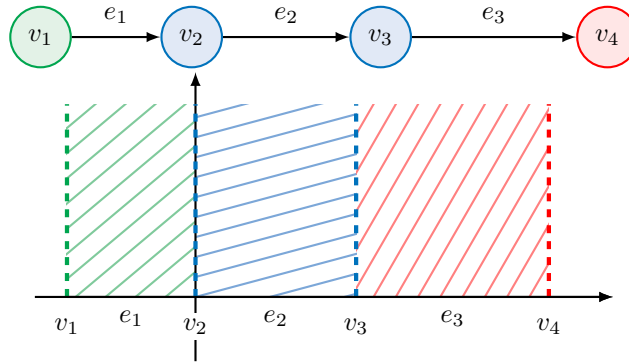


FIGURE 6. For the sake of simplicity, we illustrate the propagation (**bottom**) in a tree made up of three 1-1 junctions (**top**).

Remark 4.1 (Simplified proof for star-graphs). *The same proof as Theorem 3.1 and Proposition 3.1 is valid for star-shaped graphs, with the simplification that instead of an abstract induction argument, a two-step proof suffices.*

Remark 4.2 (Positivity assumption on a and b when they depend on the space variable). *The method of characteristics is valid when a^e and b^e depend on the space variable and $\inf_{\mathcal{E}} a, \inf_{\mathcal{E}} b > 0$, so Theorem 3.1 and Proposition 3.1 remain true under those assumptions. The positivity is needed because if the transport term gets null at some point, the characteristics do not leave the domain and we do not have the null controllability result for the hyperbolic problem. Correspondingly, in that situation the cost of the controllability explodes when $\varepsilon \rightarrow 0$.*

4.2. Optimal control times. We prove Proposition 3.1 with the method of characteristics.

Proof of Proposition 3.1. Let us consider the solution of (1.3) with initial value $y_0 = 1$. Then, we can prove by induction as in the proof of Theorem 3.1 that for any control u , any $v \in \mathcal{V}_0 \cup \mathcal{V}_\partial^{out}$ and any $e \in \mathcal{E}(v)$, $y^e(t, v) > 0$ on $[0, \tilde{T}(v)]$. \square

Finally, we discuss an improvement on the control time $\max_{v \in \mathcal{V}_\partial^{out}} \hat{T}(v)$ from Theorem 3.1 by means of non-zero boundary controls.

Remark 4.3 (Explicit computations of the optimal time in star graphs). *Let us consider a star-graph (see Figure 2), with n incoming edges and m outgoing edges. We identify the edges $e_i = [v_i, v_0] \simeq [-l_i, 0]$ for $i < 0$ and $e_i = [v_0, v_i] \simeq [0, l_i]$ for $i > 0$. Let us suppose, without loss of generality, that $v_{-1} \in \mathcal{V}_\partial^{in}$ satisfies $\frac{a^{e_{-1}} l_{-1}}{b_{e_{-1}}} = \min_{i \in \{-1, \dots, -n\}} \frac{a^{e_i} l_i}{b_{e_i}}$. We first remark that:*

$$(4.1) \quad \sup_{v \in \mathcal{V}_\partial^{out}} \tilde{T}(v) = \min_{i \in \{-1, \dots, -n\}} \frac{a^{e_i} l_i}{b_{e_i}} + \max_{j \in \{1, \dots, m\}} \frac{a^{e_j} l_j}{b_{e_j}},$$

$$(4.2) \quad \sup_{v \in \mathcal{V}_\partial^{out}} \hat{T}(v) = \max_{i \in \{-1, \dots, -n\}} \frac{a^{e_i} l_i}{b_{e_i}} + \max_{j \in \{1, \dots, m\}} \frac{a^{e_j} l_j}{b_{e_j}}.$$

Moreover, we might drive the control to 0 at a time

$$(4.3) \quad \bar{T} := \max \left\{ \min_{i \in \{-1, \dots, -n\}} \frac{a^{e_i} l_i}{b_{e_i}} + \max_{j \in \{1, \dots, m\}} \frac{a^{e_j} l_j}{b_{e_j}}, \max_{i \in \{-1, \dots, -n\}} \frac{a^{e_i} l_i}{b_{e_i}} \right\}.$$

This can be done by setting the controls

$$(4.4) \quad u^{v_{-1}}(s) = -\frac{1}{b_{e_{-1}}} \sum_{i=-n}^{-2} b^{e_i} y_0^{e_i} \left(-\frac{b^{e_i}}{a^{e_i}} \left(s + \frac{a^{e_{-1}}}{b_{e_{-1}}} l_{-1} \right) \right) \mathbb{1}_{(0, \frac{a^{e_i} b^{e_{-1}}}{b_{e_i} a^{e_{-1}}} l_{i-l_{-1}})}(s), \quad u^{v_{-2}} = \dots = u^{v_{-n}} = 0.$$

Indeed, in each incoming edge, the solution y satisfies

$$y^{e_i}(t, x) = y^0 \left(x - \frac{b^{e_i}}{a^{e_i}} t \right) \mathbb{1}_{\{x - \frac{b^{e_i}}{a^{e_i}} t > -l_i\}} + u^{v_i} \left(t - \frac{a^{e_i}}{b^{e_i}} (l_i + x) \right) \mathbb{1}_{\{x - \frac{b^{e_i}}{a^{e_i}} t < -l_i\}};$$

as a consequence, after $\min_{i \in \{-1, \dots, -n\}} \frac{a^{e_i} l_i}{b_{e_i}}$, the functions cancel at the junction and in the outgoing edges it behaves like the transport equation with null boundary data. Moreover, the time \bar{T} is optimal. If $T < \max_{i \in \{-1, \dots, -n\}} \frac{a^{e_i} l_i}{b_{e_i}}$, then clearly (1.3) is not null-controllable at an incoming edge; and, if $T < \min_{i \in \{-1, \dots, -n\}} \frac{a^{e_i} l_i}{b_{e_i}} + \max_{j \in \{1, \dots, m\}} \frac{a^{e_j} l_j}{b_{e_j}}$, then (1.3) is not null controllable by (4.1) and Proposition 3.1. Getting a similar general statement in the case of a tree \mathcal{G} is an interesting question that can be treated by an inductive argument.

Remark 4.4 (Non zero boundary controls and optimal time). *For all $0 < T_1 \leq T_2 < T_3 < T_1 + T_2$ or $T_1 = T_2 = T_3$ we can construct a star-graph such that $\max_{v \in \mathcal{V}_\partial^{out}} \tilde{T} = T_1$, $\min_{v \in \mathcal{V}_\partial^{out}} \tilde{T} = T_3$, and the optimal time in which (1.3) might be driven to 0 is T_2 .*

For the case $T_1 = T_2 = T_3$, it suffices to consider a star-graph with one incoming edge, an outgoing one, $a = b = 1$ and $l^{e_{-1}} = l^{e_1} = \frac{T_1}{2}$.

For the case $0 < T_1 \leq T_2 < T_3$, it suffices to consider a star-graph with two incoming edges, an outgoing one, $a = b = 1$ and $l_{-2} = T_2$, $l_{-1} = T_1 + T_2 - T_3$ and $l_1 = T_1 + T_2 - T_3$. This can be checked using (4.1) and (4.3).

Moreover, the hypothesis $T_3 < T_1 + T_2$ can be removed by considering more complex examples. In particular, if we consider $a = b = 1$, $n = \lfloor \frac{T_3 - T_2}{T_1} \rfloor$ and the graph with vertices $\mathcal{V} = \{v_i\}_{i=-n}^{n+1}$ and $\mathcal{E} = \{e_i\}_{i=1}^n \cup \{\tilde{e}_i\}_{i=0}^n$, for $e_i = [v_{-i}, v_i]$ and $\tilde{e}_i = [v_i, v_{i+1}]$. It suffices to consider the length $l_0 = T_2$, $l_{-1}, \dots, l_{n-1} = T_1$, $l_1 = \dots = l_{n-1} = T_1$, $l_n = T_3 - T_2 - nT_1$, $l_{-n} = T_1 - l_n$.

5. CONTROLLABILITY OF THE PARABOLIC PROBLEM

In order to prove the controllability of the parabolic problem (1.4), we are going to use the H.U.M. In particular, we are going to study the observability of the adjoint system with a Carleman inequality.

5.1. A Carleman inequality on networks. In order to simplify some computations, we first are going to prove a Carleman inequality for the solution of the system

$$(5.1) \quad \begin{cases} -a^e \partial_t \phi^e - \partial_{xx}^2 \phi^e = f, & \text{in } (0, T) \times e, e \in \mathcal{E}, \\ \phi^e(v) = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ \phi^{e_1}(t, v) = \phi^{e_2}(t, v), & t \in (0, T), v \in \mathcal{V}, \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} \partial_n^e \phi^e(t, v) = \tilde{\gamma}(t, v) \phi(t, v) & \text{on } (0, T) \times \mathcal{V}_0, \\ \phi^e(T, \cdot) = \phi_T^e, & \text{on } \mathcal{E}. \end{cases}$$

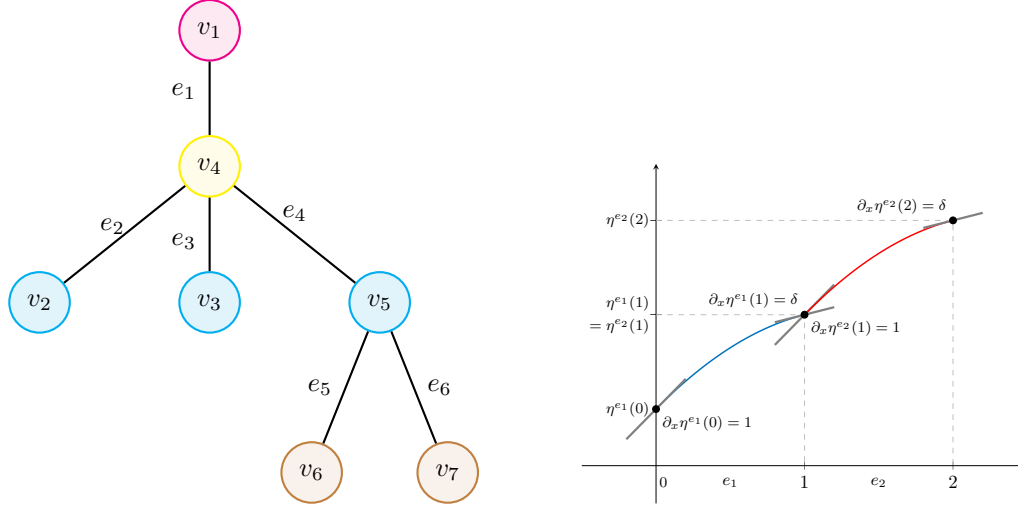


FIGURE 7. Construction of the auxiliary function $\eta \in C_{pw}^2(\bar{\mathcal{E}}) \cap C^0(\bar{\mathcal{E}})$ using the tree from Figure 1. In this tree, we distinguish four layers (**left** picture): the first (**magenta**) is $\{v_1\}$; the second (**yellow**) is layer $\{v_4\}$, the third (**cyan**) is $\{v_2, v_3, v_5\}$; and the fourth (**brown**) is $\{v_6, v_7\}$. Let us identify $e_1 := [0, 1] \simeq [v_1, v_4]$, $e_2 := [1, 2] \simeq [v_4, v_2]$, $e_3 := [1, 2] \simeq [v_4, v_3]$, $e_4 := [1, 2] \simeq [v_4, v_5]$, $e_5 := [2, 3] \simeq [v_5, v_6]$, $e_6 := [2, 5/2] \simeq [v_5, v_7]$. The auxiliary function η may be defined as follows: $\eta^{e_1}(x) = x - \frac{1-\delta}{2}x^2 + \frac{1-\delta}{2}$, $\eta^{e_2}(x) = \eta^{e_3}(x) = \eta^{e_4}(x) = (x-1) - \frac{1-\delta}{2}(x-1)^2 + 1$, $\eta^{e_5}(x) = \eta^{e_4}(2) + (x-2) - \frac{1-\delta}{2}(x-2)^2$, $\eta^{e_6}(x) = \eta^{e_4}(2) + (x-2) - (1-\delta)(x-2)^2$. As an example, in the **right** picture, we plot η^{e_1} and η^{e_2} with $\delta = \frac{1}{4}$. In the proof of Proposition 5.1, if \mathcal{G} is the graph illustrated above, we observe the vertex v_1 with v_4 ; v_4 with v_2, v_3, v_5 ; and v_5 with v_6, v_7 . So, by transitivity, we actually only need to observe using v_2, v_3, v_6, v_7 (i.e. we do not need the vertex in the 1-st layer).

Proposition 5.1 (Carleman inequality). *Let ϕ be the solution of (5.1) for $a > 0$, $f \in L^2((0, T) \times \mathcal{E})$, and $\tilde{\gamma} \in W^{1, \infty}((0, T) \times \mathcal{V}_0)$. Then, there exists a positive constant $C = C(\mathcal{G})$ such that ϕ satisfies the following inequality:*

$$\begin{aligned}
 (5.2) \quad & s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} (|\partial_t \phi|^2 + |\partial_{xx}^2 \phi|^2) dx dt \\
 & + s\tau^2 \iint_Q e^{-2s\alpha} \xi |\partial_x \phi|^2 dx dt + s^3 \tau^4 \iint_Q e^{-2s\alpha} \xi^3 |\phi|^2 dx dt \\
 & \leq C \sum_{v \in \mathcal{V}_\partial} s\tau \int_0^T e^{-2\tau\alpha} \xi(t, v) |\partial_{n^e(v)} \phi^e(t, v)|^2 dt + \iint_Q e^{-2\tau\alpha} |f|^2 dx dt,
 \end{aligned}$$

where $Q := (0, T) \times \mathcal{E}$, α, ξ are the Fursikov-Imanuvilov weights defined in (5.3), $\tau \geq C$ and $s \geq C(T + T^2)$.

The difficulty in the proof arises from the boundary terms at junctions. To suitably deal with them, we define the *Fursikov-Imanuvilov weights* ([35]) with a piecewise C^2 auxiliary function. Piecewise C^2 weights were first used for proving Carleman inequalities in [8], where the authors dealt with piecewise regular diffusivity; more recently, similar functions have been used to study coupled systems with Kirchhoff-type conditions in [10].

Proof. Step 0: Strategy of the proof and choice of the auxiliary functions. By Lemma 2.1, we identify each edge of \mathcal{E} with subintervals of \mathbb{R} . The main idea is to observe the nodes of the i -th layer with the nodes of the $(i+1)$ -th layer (see Definition 2.1). For that, we define an auxiliary function $\eta \in C_{pw}^2(\bar{\mathcal{E}}) \cap C^0(\bar{\mathcal{E}})$ recursively by the edges which joins the i -th and $(i+1)$ -th layer of the tree. Let us start with the base case: the edge $e = (v_1, v_2)$, for v_1 the root of the tree. We define η^e as a function satisfying $\partial_x \eta^e(v_1) = 1$, $\partial_x \eta^e > 0$ in $[v_1, v_2]$ and $\partial_x \eta^e(v_2) = \delta$, for $\delta > 0$ a small parameter depending on a and $\tilde{\gamma}$ to be defined later on. As for the inductive case, if v_1 is on the i -th layer and v_2 on the $(i+1)$ -th layer for $i \geq 2$, we define η^e such

that $\eta^e(v_1)$ is determined so that $\eta \in C^0(\bar{\mathcal{E}})$, $\partial_x \eta^e(v_1) = 1$, $\partial_x \eta^e > 0$ in $[v_1, v_2]$ and $\partial_x \eta^e(v_2) = \delta$, for $\delta > 0$ a small parameter depending on a and $\tilde{\gamma}$ to be defined later on (see Figure 7 for a sample illustration).

This auxiliary function allows us to define usual Fursikov-Imanuvilov weights:

$$(5.3) \quad \alpha(t, x) := \frac{e^{8\tau\|\eta\|_\infty} - e^{\tau(6\|\eta\|_\infty + \eta(x))}}{t(T-t)}, \quad \xi(t, x) := \frac{e^{\tau(6\|\eta\|_\infty + \eta(x))}}{t(T-t)},$$

for $\tau \in \mathbb{R}$ a fixed parameter (and in particular independent of the edge) that will be chosen later.

As in [35], we consider the change of variables $\psi = e^{-s\alpha}\phi$. From (5.1), we obtain that ψ satisfies the equation

$$(5.4) \quad L_1\psi + L_2\psi = L_3\psi,$$

where

$$(5.5) \quad \begin{cases} L_1\psi := -2s\tau^2|\partial_x\eta|^2\xi\psi - 2s\tau\xi\partial_x\eta\partial_x\psi + a\partial_t\psi, \\ L_2\psi := s^2\tau^2|\partial_x\eta|^2\xi^2\psi + \partial_{xx}^2\psi + sa\partial_t\alpha\psi, \\ L_3\psi := s\tau\partial_{xx}^2\eta\xi\psi - s\tau^2|\partial_x\eta|^2\xi\psi - e^{-s\alpha}f. \end{cases}$$

Indeed, by using the equality

$$(a\partial_t + \partial_{xx}^2)(e^{s\alpha}\psi) = f,$$

we obtain

$$(5.6) \quad a\partial_t\psi + as\partial_t\alpha\psi + \partial_{xx}^2\psi + 2s\partial_x\alpha\partial_x\psi + s\partial_{xx}\alpha\psi + s^2(\partial_x\alpha)^2\psi = -fe^{-s\alpha}$$

Now, combining (5.6) with $\partial_x\alpha = -\tau\partial_x\eta\xi$, $\partial_{xx}^2\alpha = -\tau\partial_{xx}^2\eta\xi - (\tau\partial_x\eta)^2\xi$, we obtain that ψ satisfies (5.4).

We now argue as in [5, 41], but paying extra attention to keep track of the boundary terms at junctions. As usual, we denote $(L_i\psi)_j$ the j -th term in the expression of $L_i\psi$ given above. From (5.4), we get

$$\|L_1\psi + L_2\psi\|_{L^2(Q)}^2 = \|L_1\psi\|_{L^2(Q)}^2 + \|L_2\psi\|_{L^2(Q)}^2 + 2 \sum_{i,j=1,2,3} ((L_1\psi)_i, (L_2\psi)_j)_{L^2(Q)} = \|L_3\psi\|_{L^2(Q)}^2.$$

In the next two steps, we need to estimate the product:

$$(L_1\psi, L_2\psi)_{L^2(Q)} := \sum_{e \in \mathcal{E}} ((a^e)^{-1/2}(L_1\psi)^e, (a^e)^{-1/2}(L_2\psi)^e)_{L^2((0,T) \times e)}.$$

In particular, we will show that the choice of the weights in (5.3) makes it positive up to several terms that can be controlled by the left-hand side (for a suitable choice of the parameters τ and s) and by the observation term.

Step 1: Estimates in the interior. In this step we perform integrations by parts in the spirit of [5, 41], but keeping track of the boundary terms appearing at the vertices.

To begin with,

$$(5.7) \quad ((L_1\psi)_1, (L_2\psi)_1)_{L^2(Q)} = -2s^3\tau^4 \iint_Q a^{-1}|\partial_x\eta|^4\xi^3|\psi|^2 dx dt.$$

Secondly, we compute

$$(5.8) \quad \begin{aligned} ((L_1\psi)_2, (L_2\psi)_1)_{L^2(Q)} &= -2s^3\tau^3 \iint_Q a^{-1}|\partial_x\eta|^2\partial_x\eta\xi^3\psi\partial_x\psi dx dt \\ &= 3s^3\tau^4 \iint_Q a^{-1}|\partial_x\eta|^4\xi^3|\psi|^2 dx dt \\ &\quad - \underbrace{2s^3\tau^3 \sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} \int_0^T (a^e)^{-1}|\partial_x\eta^e|^2\partial_{n^e(v)}\eta^e(\xi^e)^3|\psi^e|^2(t, v) dt}_{=: J_1} \\ &\quad + o\left(s^3\tau^4 \iint_Q \xi^3|\psi|^2 dx dt\right). \end{aligned}$$

Thirdly, integration by parts (with respect to the time variable) yields, for $\tau \geq C$ and $s \geq C(T+T^2)$,

$$(5.9) \quad ((L_1\psi)_3, (L_2\psi)_1)_{L^2(Q)} = s^2\tau^2 \iint_Q |\partial_x\eta|^2\xi^2\psi\partial_t\psi dx dt = o\left(s^3\tau^4 \iint_Q \xi^3|\psi|^2 dx dt\right).$$

To continue with, we find that

$$\begin{aligned}
((L_1\psi)_1, (L_2\psi)_2)_{L^2(Q)} &= -2s\tau^2 \iint_Q a^{-1} |\partial_x \eta|^2 \xi \psi \partial_{xx}^2 \psi \, dx \, dt \\
&= 2s\tau^2 \iint_Q a^{-1} |\partial_x \eta|^2 \xi |\partial_x \psi|^2 \, dx \, dt \\
(5.10) \quad &+ \underbrace{\sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} 2s\tau^2 \int_0^T (a^e)^{-1} |\partial_x \eta^e|^2 \xi^e \psi^e \partial_{n^e(v)} \psi^e(t, v) \, dt}_{=: J_2} \\
&+ o\left(s^3 \tau^4 \iint_Q \xi^3 |\psi|^2 \, dx \, dt\right) + o\left(s\tau^2 \iint_Q \xi |\nabla \psi|^2 \, dx \, dt\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
((L_1\psi)_2, (L_2\psi)_2)_{L^2(Q)} &= -2s\tau \iint_Q a^{-1} \partial_x \eta \xi \partial_{xx}^2 \psi \partial_x \psi \, dx \, dt \\
(5.11) \quad &= s\tau^2 \iint_Q a^{-1} \partial_x \eta \xi |\partial_x \psi|^2 \, dx \, dt - \underbrace{\sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} s\tau \int_0^T (a^e)^{-1} \partial_{n^e(v)} \eta^e \xi^e |\partial_x \psi^e|^2(t, v) \, dt}_{=: J_3} \\
&+ o\left(s\tau^2 \iint_Q \xi |\partial_x \psi|^2 \, dx \, dt\right).
\end{aligned}$$

Moreover, we can prove that:

$$(5.12) \quad ((L_1\psi)_3, (L_2\psi)_2)_{L^2(Q)} = \iint_Q \partial_t \psi \partial_{xx}^2 \psi \, dx \, dt = \underbrace{\sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_{n^e(v)} \psi^e \partial_t \psi^e(t, v) \, dt}_{=: J_4}.$$

Finally,

$$\begin{aligned}
(L_1\psi, (L_2\psi)_3)_{L^2(Q)} &= - \underbrace{\sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} s^2 \tau \int_0^T \xi^e \partial_{n^e(v)} \eta^e \partial_t \alpha^e |\psi^e|^2(t, v) \, dt}_{=: J_5} \\
(5.13) \quad &+ o\left(s^3 \tau^4 \iint_Q \xi^3 |\psi|^2 \, dx \, dt\right).
\end{aligned}$$

Step 2: Estimation of the boundary terms. In this part of the proof, we estimate the boundary terms J_1, \dots, J_5 . In particular, we need to make a distinction between *exterior vertices*, which can be treated as in [5, 41] (since they correspond to the boundary terms appearing in a classical IBVP), and *junctions*, which require new more precise computations. As we are going to see, the terms corresponding to the exterior vertices $v \in \mathcal{V}_\partial$ either vanish (due to the zero Dirichlet boundary condition in (5.1)) or can be moved to the right hand-side of the Carleman estimate (corresponding to the ‘‘classical’’ boundary terms that appear in [41]). The interior junction terms at $v \in \mathcal{V}_0$ are more critical: they are in the left-hand side of the Carleman estimate and we need to show that they are non-negative. To this end, we will rely on the properties of the auxiliary function η and on the Kirchhoff junction condition (5.1)₄. At the end of the computations, all these boundary terms at the internal junctions can be absorbed into the expression in the right-hand side of (5.20), which is non-negative.

To begin with, let us deal with the boundary term J_1 in (5.8). If $v \in \mathcal{V}_\partial$, we obtain that $\psi = 0$ from the Dirichlet boundary conditions. Otherwise, for each interior node $v \in \mathcal{V}_0$, we get that

$$(5.14) \quad -2\varepsilon^2 s^3 \tau^3 \int_0^T \xi^3 \left(\sum_{e \in \mathcal{E}(v)} (a^e)^{-1} |\partial_x \eta^e|^2 \partial_{n^e(v)} \eta^e \right) |\psi^e|^2(t, v) \, dt \geq c\varepsilon^2 s^3 \tau^3 \int_0^T \xi^3 |\psi^e|^2(t, v) \, dt.$$

Indeed, the function ξ is continuous at junctions, $\partial_{n^{\tilde{e}}(v)}\eta^{\tilde{e}}(v) = \delta$ for \tilde{e} the edge going joining the previous layer to v and $\partial_{n^e(v)}\eta^e(v) = -1$ if $e \in \mathcal{E}(v) \setminus \{\tilde{e}\}$, so we obtain (5.14) by choosing δ small enough. With the right-hand side of (5.14) we can absorb the boundary term J_5 in (5.13) for $s \geq C(T + T^2)$ and $\tau \geq C$.

To continue with, let us study the boundary term J_3 given in (5.11) for each $v \in \mathcal{V}$, i.e.

$$-s\tau \int_0^T \xi \sum_{e \in \mathcal{E}(v)} (a^e)^{-1} \partial_{n^e(v)} \eta^e |\partial_{n^e(v)} \psi^e|^2(t, v) dt.$$

If $v \in \mathcal{V}_\partial$, that term can be moved to the right-hand side of the Carleman estimate. On the other hand, if $v \in \mathcal{V}_0$, there is one edge \tilde{e} (the edge going joining the previous layer to v) for which $\partial_{n^{\tilde{e}}}\eta^{\tilde{e}} = \delta$ and such that $\partial_{n^e}\eta^e = -1$ for all $e \in \mathcal{E}(v) \setminus \{\tilde{e}\}$. So, we have to absorb the boundary term of the edge \tilde{e} . For that, we use $\phi = \psi e^{-s\alpha}$ to get

$$(5.15) \quad \begin{aligned} & s\tau \int_0^T (a^{\tilde{e}})^{-1} \xi^{\tilde{e}} |\partial_{n^{\tilde{e}}}\eta^{\tilde{e}}| |\partial_{n^{\tilde{e}}}\psi^{\tilde{e}}|^2(t, v) dt \\ & \leq 2\delta s\tau \int_0^T (a^{\tilde{e}})^{-1} \xi^{\tilde{e}} |\partial_{n^{\tilde{e}}}\alpha^{\tilde{e}}|^2 |\psi|^2(t, v) dt + 2\delta s\tau \int_0^T (a^{\tilde{e}})^{-1} \xi^{\tilde{e}} e^{-2s\alpha^{\tilde{e}}} |\partial_{n^{\tilde{e}}}\phi|^2(t, v) dt. \end{aligned}$$

We can absorb the first term in the right-hand side of (5.15) by (5.14). As for the second one, it equals

$$(5.16) \quad \begin{aligned} & \delta s\tau \int_0^T (a^{\tilde{e}})^{-1} \xi^{\tilde{e}} e^{-2s\alpha^{\tilde{e}}} \left| \sum_{e \in \mathcal{V}(e) \setminus \tilde{e}} \partial_{n^e} \phi^e \right|^2(t, v) dt \\ & \leq \delta s\tau \int_0^T (a^{\tilde{e}})^{-1} \sum_{e \in \mathcal{V}(e) \setminus \tilde{e}} \xi^e |\partial_{n^e} \psi^e|^2(t, v) dt + C\delta s^3 \tau^3 \int_0^T \xi^3 |\psi^e|^2(t, v) dt. \end{aligned}$$

Indeed, we have used the continuity of functions ξ and α on the junction, the condition on the junction (5.1)₄, the fact $\partial_x \alpha = -\tau \partial_x \eta \xi$ and that $\gamma^e \in L^\infty((0, T) \times \mathcal{V}_0)$. Taking δ small enough, we can absorb the term (5.16) by (5.14) and

$$(5.17) \quad -s\tau \int_0^T \xi \sum_{e \in \mathcal{E}(v) \setminus \tilde{e}} (a^e)^{-1} \partial_{n^e(v)} \eta^e |\partial_{n^e(v)} \psi^e|^2(t, v) dt = s\tau \int_0^T \xi \sum_{e \in \mathcal{E}(v) \setminus \tilde{e}} (a^e)^{-1} |\partial_{n^e(v)} \psi^e|^2(t, v) dt.$$

To conclude, let us study the boundary term J_4 in (5.12). If $v \in \mathcal{V}_\partial$, then $\partial_t \psi = 0$ because of the Dirichlet boundary conditions. Otherwise, if $v \in \mathcal{V}_0$, from (3.9)₄ we obtain

$$(5.18) \quad \begin{aligned} & \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_{n^e} \psi^e \partial_t \psi^e(t, v) dt \\ & = \int_0^T \tilde{\gamma} \partial_{n^e} \xi^e \psi^e \partial_t \psi^e(t, v) dt + \sum_{e \in \mathcal{E}(v)} \int_0^T \gamma \psi^e \partial_t \psi^e(t, v) dt \\ & = I_1 + I_2 \end{aligned}$$

We can absorb I_1 by (5.14) considering that $\tilde{\gamma} \in W^{1,\infty}((0, T) \times \mathcal{V}_0)$. Moreover,

$$(5.19) \quad \begin{aligned} I_2 & = s^2 \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_{n^e} \xi^e \partial_t \alpha^e e^{-2s\alpha^e} |\phi^e|^2(t, v) dt + s \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_{n^e} \xi^e e^{-2s\alpha^e} \frac{\partial_t (|\phi^e|^2)}{2}(t, v) dt \\ & = s^2 \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_{n^e} \xi^e \partial_t \alpha^e |\psi^e|^2(t, v) dt - s \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_t (\partial_{n^e} \xi^e) \frac{|\psi^e|^2}{2}(t, v) dt, \end{aligned}$$

which can be absorbed by (5.14). To sum up the result of this step, we have proved that, for $s \geq C(T + T^2)$ and $\tau \geq C$, the following estimate holds:

$$(5.20) \quad \begin{aligned} \sum_{\ell=1}^5 J_\ell &\geq c \sum_{v \in \mathcal{V}_0} s^3 \tau^3 \int_0^T \xi^3 |\psi|^2(t, v) dt + c \sum_{v \in \mathcal{V}_0} s \tau \int_0^T \xi \sum_{e \in \mathcal{E}(v) \setminus \bar{e}} (a^e)^{-1} |\partial_{n^e(v)} \psi^e|^2(t, v) dt \\ &\quad - C \sum_{v \in \mathcal{V}_\partial} s \tau \int_0^T \xi (a^e)^{-1} \partial_{n^e(v)} \eta^e |\partial_{n^e(v)} \psi^e|^2(t, v) dt. \end{aligned}$$

Step 3: Conclusion of the proof. Combining (5.7)-(5.20) and the fact that $|\partial_x \eta| > 0$, we obtain the following estimate:

$$(5.21) \quad \begin{aligned} &s \tau^2 \iint_Q \xi |\partial_x \psi|^2 dx dt + s^3 \tau^4 \iint_Q \xi^3 |\psi|^2 dx dt \\ &\leq (L_1 \psi, L_2 \psi)_{L^2(Q)} + \sum_{v \in \mathcal{V}_\partial} s \tau \int_0^T \xi^e |\partial_{n^e(v)} \psi^e|^2(t, v) dt. \end{aligned}$$

From (5.21), it is classical to obtain (5.2) as in [35, 41]. We add $\frac{1}{2}(\|L_1 \psi\|_{L^2(Q)}^2 + \|L_2 \psi\|_{L^2(Q)}^2)$ at both sides of (5.21); we consider that $\|L_1 \psi + L_2 \psi\|_{L^2(Q)}^2 = \|L_3 \psi\|_{L^2(Q)}^2$; we absorb $(L_3 \psi)_1$ and $(L_3 \psi)_2$; we estimate the terms on $\partial_t \psi$ and $\partial_{xx}^2 \psi$ by considering (5.5)₁ and (5.5)₂ respectively; and, finally, reverting the transformation $\phi = \psi e^{-s\alpha}$. \square

Remark 5.1 (Networks with loops). *In networks with loops, the difficulty is to construct an auxiliary function η^e that ensures that J_2 is estimated as in (5.20).*

Remark 5.2 (Reducing the number of vertices in the right-hand side of the Carleman inequality). *The pathological case given in Proposition 3.2 shows that the Carleman inequality may be false if we observe fewer vertices. This shows that the computation (5.20) – and, in particular, the estimate of J_2 – is quite sharp and difficult to improve, at least for general trees.*

Remark 5.3 (Special proof for star-shaped graphs). *The Carleman can be simplified when the graph \mathcal{G} is star-shaped. In that situation it suffices to consider an auxiliary function η that increases from the central vertex to the outwards ones. In that case we can replicate the proof in [27]. The only difficult term is when we treat $\iint_Q \partial_t \psi \partial_{xx}^2 \psi$, but the boundary term that appears in the central vertex can be estimated as in (5.18).*

5.2. Conclusion of the proof of Theorem 5.1. From (5.2), we can easily obtain a Carleman inequality for the solution of (3.1). In fact, it suffices to divide (3.1)₁ by μ^e and consider $\tilde{\gamma} = (\gamma - \sum n^e b^e)(\mu^e)^{-1}$ and $f = (-\partial_x b^e \partial_x \varphi^e - b^e \partial_x \varphi^e + c^e(t, x) \varphi^e)(\mu^e)^{-1}$. In that case the source term can be absorbed by the right hand side of the Carleman inequality (as $s\xi < 1$ for $s \geq C(T^2 + T)$, it suffices to consider τ large enough). In the end, we obtain that

$$(5.22) \quad \begin{aligned} &s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} (|\partial_t \varphi|^2 + |\partial_{xx}^2 \varphi|^2) dx dt \\ &\quad + s \tau^2 \iint_Q e^{-2s\alpha} \xi |\partial_x \varphi|^2 dx dt + s^3 \tau^4 \iint_Q e^{-2s\alpha} \xi^3 |\varphi|^2 dx dt \\ &\leq C \sum_{v \in \mathcal{V}_\partial} s \tau \int_0^T e^{-2\tau\alpha} \xi(t, v) |\partial_{n^e(v)} \varphi^e(t, v)|^2 dt, \end{aligned}$$

which implies the following observability inequality:

$$\|\varphi(0, \cdot)\|_{L^2(\mathcal{E})} \leq C \iint_{(0, T) \times \mathcal{V}} |\partial_n \varphi^e|^2.$$

By duality, this completes the proof of Theorem 3.2.

Remark 5.4 (Results when a^e depends on the time and space variable). *The Carleman inequality is also true when a^e depends on the time and space variable as long as $\inf a^e > 0$ and $a^e \in L^\infty(0, T; W_{pw}^{1, \infty}(\mathcal{E})) \cap W^{1, \infty}(0, T; L^\infty(\mathcal{E}))$. In fact, the terms in which $(a^e)^{-1}$ is differentiated when integrating by parts can be*

absorbed by taking $\tau \geq C$ and $s \geq C(T + T^2)$. Thus, we conclude that Theorem 2.1 is also true assuming $\inf a^e > 0$, $a^e \in L^\infty(0, T; W_{pw}^{1,\infty}(\mathcal{E})) \cap W^{1,\infty}(0, T; L^\infty(\mathcal{E}))$, $\inf \mu^e > 0$ and $\mu^e \in L^\infty(0, T; W_{pw}^{2,\infty}(\mathcal{E})) \cap W^{1,\infty}(0, T; L^\infty(\mathcal{E}))$.

5.3. Controls acting in the interior. In a similar manner, we may also study the controllability properties of the system

$$(5.23) \quad \begin{cases} a^e \partial_t y^e - \mu^e \partial_{xx}^2 y^e + b^e(t, x) \partial_x y^e + c^e(t, x) y^e = f \mathbf{1}_\omega, & \text{in } (0, T) \times e, \quad e \in \mathcal{E}, \\ y^e(t, v) = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ y^{e_1}(t, v) = y^{e_2}(t, v), & t \in (0, T), \quad v \in \mathcal{V}, \quad \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} \mu^e \partial_x y^e n^e = \gamma(t, v) y(t, v) & \text{on } (0, T) \times \mathcal{V}_0, \\ y(0, \cdot) = y_0, & \text{on } \mathcal{E}, \end{cases}$$

where the control f acts on an interior domain ω . Similarly to controls acting on the boundary, it does not suffice to act on any arbitrary subdomain. Indeed, we can prove the following result arguing as in Proposition 3.3.

Proposition 5.2 (Lack of null-controllability for general graphs and controls acting on the interior). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph in Figure 4, formed by the vertices v_1, v_2, v_3, v_4 , the edges $e_1 = [v_1, v_2] \simeq [0, 1]$, $e_2 = [v_2, v_3] \simeq [0, 1]$, and $e_3 = [v_2, v_4] \simeq [0, 1]$, and coefficients $b^{e_1} = a^{e_1} = a^{e_2} = a^{e_3} = 1$, $\mu = 1$, $b^{e_2} = b^{e_3} = \frac{1}{2}$, and $c = 0$. Then, system (5.23) is not approximately controllable by acting only on ω for any $\omega \subset e_1$.*

To obtain a positive result on controllability, we may consider the following situations (see Figure 8 for a sample illustration).

Hypothesis 5.1. *The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a tree and all edges e_i with an end in \mathcal{V}_∂ (except at most one) satisfy that $\omega^e := \omega \cap e_i \neq \emptyset$.*

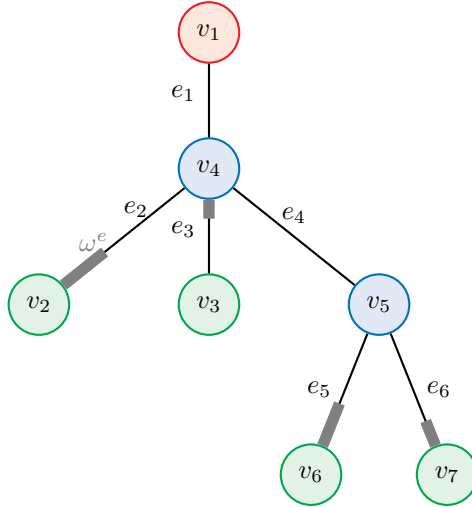


FIGURE 8. Example illustration of the situation from Hypothesis 5.1 in the case of a star-graph: the domains $\omega^{e_i} := \omega \cap e_i$ are illustrated in gray.

Under such assumption, we can then prove the following theorem.

Theorem 5.1 (Controllability of parabolic systems of networks). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a network satisfying Hypothesis 5.1 and $y_0 \in L^2(\mathcal{E})$. Then, there exists $f \in L^2((0, T) \times \omega)$ such that the solution of (5.23) satisfies $y(T, \cdot) = 0$.*

Theorem 5.1 can be proved as Theorem 3.2 with the usual differences when the control acts on the interior (see [35] for further information). The construction of the auxiliary function η the only difficult task: as

before, the function η can be constructed by induction on the layers of the tree. For the edge $e = [v_1, v_2]$ on the first layer, η satisfies $\partial_{n^e}\eta^e < 0$ on $\{v_1, v_2\}$ and $\inf_{e \setminus \omega} |\partial_x \eta^e| > 0$. Then, on an edge $e \simeq [v_1, v_2]$ from v_1 on the i -th to v_2 on the $(i+1)$ -th layer, if $\omega \cap e = \emptyset$, we define η^e so that it is continuous in v_1 , satisfies $\inf \partial_x \eta > 0$ and $\partial_x \eta(v_2) = \delta$, where δ is a sufficiently small parameter. Moreover, if $\omega \cap e \neq \emptyset$, we define η^e so that it is continuous in v_1 , $\partial_{n^e}\eta^e < 0$ on $\{v_1, v_2\}$, and $\inf_{e \setminus \omega} |\partial_x \eta^e| > 0$.

Remark 5.5 (Particular geometries). *Sharper results can be obtained by considering more specific geometries or metrics for the trees. Of course, the most difficult task is the construction of an appropriate auxiliary function η . However, doing an in-detail classification and the construction of such functions is out of the scope of this paper.*

6. BLOW-UP OF THE COST OF CONTROLLABILITY

In this Section we prove Claims (1) and (2) of Theorem 3.3. Since both proofs are very similar, we prove Claim (1) and only outline the key difference for Claim (2).

6.1. Agmon-type inequality. We start by proving an Agmon-type inequality (see [1, Theorem 5.9]), which gives an exponentially weighted energy estimate.

Lemma 6.1 (Agmon-type inequality). *Let $\zeta \in H_{pw}^2(\mathcal{E}) \cap H^1(\mathcal{E})$ satisfy*

$$(6.1) \quad -a_e \partial_t \zeta^e + |\partial_x \zeta^e|^2 - b_e \partial_x \zeta^e \geq 0.$$

Then, any solution φ of the adjoint system (3.6) satisfies the following Agmon-type inequality:

$$\begin{aligned} & \sum_{e \in \mathcal{E}} \frac{a_e}{2} \int_e |e^{\zeta^e/\varepsilon} \varphi^e(t, x)|^2 dx + \varepsilon \sum_{e \in \mathcal{E}} \int_t^T \int_e |\partial_x (e^{\zeta^e/\varepsilon} \varphi^e(t, x))|^2 dx ds \\ & \leq \sum_{e \in \mathcal{E}} \frac{a_e}{2} \int_e |e^{\zeta^e(T, x)/\varepsilon} \varphi^e(T, x)|^2 dx. \end{aligned}$$

Proof. We compute $\frac{d}{dt} \frac{a_e}{2} \int_e |e^{\zeta^e/\varepsilon} \varphi^e(t, x)|^2 dx$ as follows:

$$\begin{aligned} & \frac{d}{dt} \frac{a_e}{2} \int_e |e^{\zeta^e/\varepsilon} \varphi^e(t, x)|^2 dx \\ & = a_e \int_e e^{\zeta^e/\varepsilon} \partial_t e^{\zeta^e/\varepsilon} |\varphi^e(t, x)|^2 dx + \frac{a_e}{2} \int_e e^{\zeta^e/\varepsilon} \partial_t |\varphi^e(t, x)|^2 dx \\ & = a_e \int_e e^{\zeta^e/\varepsilon} \partial_t e^{\zeta^e/\varepsilon} |\varphi^e(t, x)|^2 dx + \int_e e^{2\zeta^e/\varepsilon} \varphi^e (-b^e \partial_x \varphi^e - \varepsilon \partial_{xx}^2 \varphi^e) dx \\ & = a_e \int_e e^{\zeta^e/\varepsilon} \partial_t e^{\zeta^e/\varepsilon} |\varphi^e(t, x)|^2 dx + \frac{b_e}{2} \int_e (\varphi^e)^2 \partial_x e^{2\zeta^e/\varepsilon} dx - \frac{b_e}{2} [(\varphi^e)^2 e^{2\zeta^e/\varepsilon}]_e \\ & \quad + \varepsilon \int_e (\partial_x \varphi^e)^2 e^{2\zeta^e/\varepsilon} dx + \varepsilon \int_e \partial_x \varphi^e \varphi^e \partial_x e^{2\zeta^e/\varepsilon} dx - \varepsilon [\partial_x \varphi^e \varphi^e e^{\zeta^e/\varepsilon}]_e \\ & = \frac{1}{\varepsilon} \int_e e^{2\zeta^e/\varepsilon} (\varphi^e)^2 \underbrace{\left(a_e \partial_t \zeta^e + b_e \partial_x \zeta^e - |\partial_x \zeta^e|^2 \right)}_{\leq 0} dx + \varepsilon \int_e (\partial_x (\varphi^e e^{\zeta^e/\varepsilon}))^2 dx \\ & \quad - \frac{b_e}{2} [(\varphi^e)^2 e^{2\zeta^e/\varepsilon}]_e - \varepsilon [\varphi^e \partial_x \varphi^e e^{2\zeta^e/\varepsilon}]_e. \end{aligned}$$

Summing up over $e \in \mathcal{E}$, using Dirichlet boundary conditions, the junction conditions, and the flux balance condition (1.2), we obtain

$$\begin{aligned} \frac{d}{dt} \sum_{e \in \mathcal{E}} \frac{a_e}{2} \int_e |e^{\zeta^e/\varepsilon} \varphi^e(t, x)|^2 dx & = \frac{1}{\varepsilon} \sum_{e \in \mathcal{E}} \int_e |e^{\zeta^e/\varepsilon} \varphi^e(t, x)|^2 \underbrace{\left(a_e \partial_t \zeta^e + b_e \partial_x \zeta^e - |\partial_x \zeta^e|^2 \right)}_{\leq 0} dx \\ & \quad + \varepsilon \sum_{e \in \mathcal{E}} \int_e (\partial_x (\varphi^e e^{\zeta^e/\varepsilon}))^2 dx. \end{aligned}$$

By integrating this expression in (t, T) , we conclude the proof. \square

Remark 6.1 (Choice of the auxiliary function ζ). *As in [6, Lemma 3.5], we will choose*

$$(6.2) \quad \zeta^e(t, x) = -\delta b^e x - \frac{(b^e)^2}{a^e}(\delta + \delta^2)(T - t),$$

which is precisely a solution of the equality in (6.1).

6.2. Non-degeneracy of the solution. As a second preliminary tool, we establish that the mass of $\varphi(0, \cdot)$ is bounded away from zero.

Lemma 6.2 (Non-degeneracy of the solution). *Let $v \in \mathcal{V}_\partial^{out}$ be the vertex which maximizes \tilde{T} (the minimal propagation time introduced in Definition 3.1). Then, for all $T < \tilde{T}$ and all $\delta \in (0, T - \tilde{T})$ there is $\varphi_T \in C^\infty(\Omega)$ and $c > 0$ such that $\text{supp}(\varphi_T) \subset [v - \delta, v]$ and such that for all $\varepsilon \in (0, 1)$:*

$$(6.3) \quad \|\varphi_\varepsilon(0, \cdot)\|_{L^2(\mathcal{E})} \geq c,$$

for φ_ε the solution of (3.6) with initial value φ_T .

Proof. We prove (6.3) by contradiction. First of all, we remark that, thanks to the backwards uniqueness of (3.6) (which can be derived from the backwards uniqueness of (3.9)), we obtain that $\|\varphi_\varepsilon(0, \cdot)\|_{L^2(\mathcal{E})} > 0$ for all $\varepsilon \in (0, 1)$. In addition, since the solution is continuous with respect to ε (which can be proved with the Fourier series solutions of (3.9) as in [6]), the only possibility is that there is $\varepsilon_k \rightarrow 0$ such that

$$\|\varphi_{\varepsilon_k}(0, \cdot)\|_{L^2(\mathcal{E})} \rightarrow 0.$$

Let us suppose that there exists such a sequence and derive a contradiction. Since φ_{ε_k} is bounded on $L^2((0, T) \times \mathcal{E})$ we may suppose that it converges weakly to some function $\gamma \in L^2((0, T) \times \mathcal{E})$. We can prove by considering the weak solution of parabolic equation that γ is a solution of the adjoint problem of (1.3) with $\varphi(0) = 0$ and initial value φ_T , which is impossible by the method of characteristics. \square

6.3. Proof of Claim (1) of Theorem 3.3. Using the tools developed in the previous sections, we complete the proof of Claim (1) of Theorem 3.3.

Proof of Theorem 3.3, Claim (1). Let φ be the solution of the adjoint problem (3.6) and let us assume that $\text{supp} \varphi_T \subset [v - \tilde{\delta}, v]$ for $e \in \mathcal{E}(v)$, with $v \in \mathcal{V}_\partial^{out}$ which maximizes \tilde{T} (see Figure 9 (left) for an illustration).

Using the assumption on $\text{supp} \varphi_T^e$, we deduce

$$\begin{aligned} T \int_e |e^{-2\delta x/\varepsilon} \varphi_T^e|^2 dx &\leq e^{-2\tilde{\delta}\delta/\varepsilon} \int_e |\varphi_T^e|^2 dx, \\ e^{2\delta(T-\delta T)/\varepsilon} \int_{(0, T) \times \mathcal{V}_\partial^{in}} |\partial_\nu \varphi|^2 dt &\leq \int_0^T \int_e |e^{(-\delta x - (\delta + \delta^2)(T-t))/\varepsilon} \varphi^e|^2 dx dt. \end{aligned}$$

Combining these with the Agmon identity of Lemma 6.1, choosing ζ as in (6.2), we deduce

$$(6.4) \quad \int_{(0, T) \times \mathcal{V}_\partial^{in}} |\partial_\nu \varphi|^2 dt \leq e^{2\delta(T+\delta T+\tilde{\delta}+\delta)/\varepsilon} \|\varphi_T^e\|_{L^2(\mathcal{E})}^2.$$

Plugging (6.3) and (6.4) into (3.7), we deduce that (3.3) holds. \square

Proof of Theorem 3.3, Claim (2). When proving Claim (2) of Theorem 3.3 the only difference from the argument above is that the function φ^T must be supported near the interior nodes instead of near the exterior ones (see Figure 9 (right)). \square

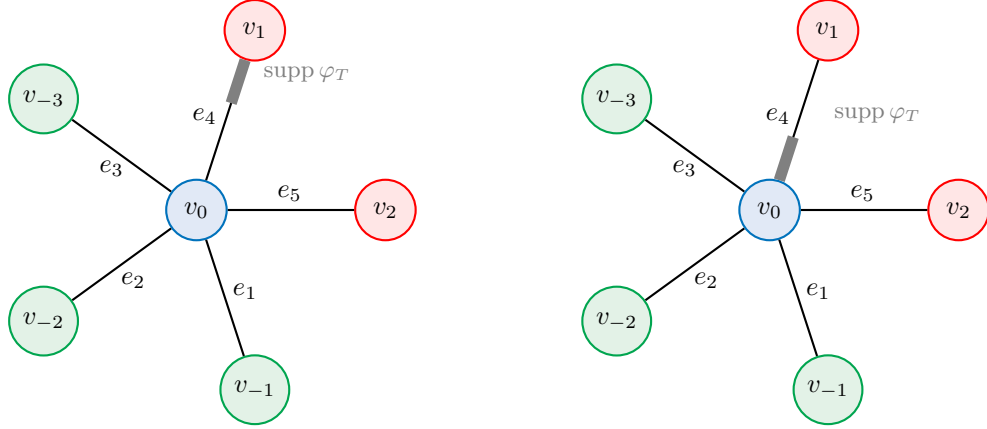


FIGURE 9. Example illustration of the support of φ_T in the case of a star-graph. **Left:** choice of $\text{supp } \varphi_T$ for the proof of Claim (1) of Theorem 3.3. **Right:** choice of $\text{supp } \varphi_T$ for the proof of Claim (2). In both cases, the (hyperbolic) observation domain is $\{v_{-1}, v_{-2}, v_{-3}\}$, the green vertices.

7. DECAY OF THE COST OF CONTROLLABILITY

7.1. The decay property for the parabolic problem. We start by proving a decay property for the solution of (1.1).

Proposition 7.1 (Decay property). *There are $c, C > 0$ such that the solution of the parabolic problem (3.6) satisfies the following decay property for all $\varphi_T \in L^2(\mathcal{E})$ and all $t < T$:*

$$(7.1) \quad \int_{\mathcal{E}} |\varphi_{\varepsilon}(t, x)|^2 dx \leq \exp\left(\frac{C - c(T - t)}{\varepsilon}\right) \|\varphi_T\|_{L^2(\mathcal{E})}^2.$$

The proof of the decay property is based on classical energy estimates for the symmetrized system (3.9) with Gronwall's inequality.

Proof. In order to prove (7.1), we first obtain a decay property for the symmetrized system (3.9). Multiplying the PDEs in (3.9) by z_{ε}^e , integrating by parts and summing up all the edges we obtain that:

$$-\frac{d}{dt} \frac{1}{2} \left(\int_{\mathcal{E}} a^e |z_{\varepsilon}(t, x)|^2 dx \right) + \varepsilon \int_{\mathcal{E}} |\partial_x z_{\varepsilon}|^2 dx + \int_{\mathcal{E}} \frac{|b^e|^2}{4\varepsilon} |z_{\varepsilon}|^2 dx = 0.$$

Consequently, by using $b^e > 0$,

$$-\frac{d}{dt} \left(\int_{\mathcal{E}} a^e |z_{\varepsilon}|^2 dx \right) \leq -\frac{c}{\varepsilon} \int_{\mathcal{E}} |z_{\varepsilon}|^2 dx.$$

Using backwards Gronwall's inequality in (t, T) we obtain that:

$$\int_{\mathcal{E}} |z_{\varepsilon}(t, x)|^2 dx \leq \exp\left(-\frac{c(T - t)}{\varepsilon}\right) \int_{\mathcal{E}} |z_T|^2 dx.$$

Finally, reverting the change of variables (3.8), we obtain the decay property (7.1). \square

7.2. A Carleman inequality. In order to prove the Carleman estimate, we focus on the viscosity dynamics and do not take into account the information provided by the velocity on the flow of the transport system.

Proposition 7.2 (Carleman inequality). *Let φ_ε be the solution of (3.6). Then there exists a positive constant $C = C(\mathcal{G})$ such that φ_ε satisfies the following inequality:*

$$(7.2) \quad \begin{aligned} & s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} (\varepsilon^{-2} |\partial_t \varphi_\varepsilon|^2 + |\partial_{xx}^2 \varphi_\varepsilon|^2) dx dt \\ & + s\tau^2 \iint_Q e^{-2s\alpha} \xi |\partial_x \varphi_\varepsilon|^2 dx dt + s^3 \tau^4 \iint_Q e^{-2s\alpha} \xi^3 |\varphi_\varepsilon|^2 dx dt \\ & \leq C \sum_{v \in \mathcal{V}_\partial} s\tau \int_0^T e^{-2\tau\alpha} \xi(t, v) |\partial_{n^e(v)} \varphi_\varepsilon^e(t, v)|^2 dt, \end{aligned}$$

where $Q := (0, T) \times \mathcal{E}$, α, ξ are the Fursikov-Imanuvilov weights defined in (5.3), $\tau \geq C$ and $s \geq C(T + T^2)\varepsilon^{-1}$.

We cannot use Proposition 5.1 directly to deduce Proposition 7.2 because we need to keep track of the dependence of s and τ with respect to ε , and in Proposition 5.1 there is no track of the parameters of the equation. However, the proof remains similar to the one in Section 5, and so, we just outline the main estimates.

Proof. Step 0: Strategy of the proof and choice of the auxiliary functions. Using the notations and auxiliary functions established in the proof of Proposition 5.1 but with δ small enough depending on a and b , we consider the change of variables $\psi = e^{-s\alpha} z_\varepsilon$. From (3.9), we obtain that ψ satisfies the equation

$$(7.3) \quad L_1\psi + L_2\psi = L_3\psi,$$

where

$$(7.4) \quad \begin{cases} L_1\psi := -2\varepsilon s\tau^2 |\partial_x \eta|^2 \xi \psi - 2\varepsilon s\tau \xi \partial_x \eta \partial_x \psi + a \partial_t \psi, \\ L_2\psi := \varepsilon s^2 \tau^2 |\partial_x \eta|^2 \xi^2 \psi + \varepsilon \partial_{xx}^2 \psi + sa \partial_t \alpha \psi - \frac{|b|^2}{4\varepsilon} \psi, \\ L_3\psi := \varepsilon s\tau \partial_{xx}^2 \eta \xi \psi - \varepsilon s\tau^2 |\partial_x \eta|^2 \xi \psi. \end{cases}$$

As in the proof of Proposition 5.1, we need to estimate the product

$$(L_1\psi, L_2\psi)_{L^2(Q)} := \sum_{e \in \mathcal{E}} ((a^e)^{-1/2} (L_1\psi)^e, (a^e)^{-1/2} (L_2\psi)^e)_{L^2((0, T) \times e)}.$$

In particular, we will show that the choice of the weights in (5.3) makes it positive up to several terms that can be controlled by the left-hand side for a suitable choice of the parameters τ and s .

Step 1: Estimates in the interior. To begin with,

$$(7.5) \quad ((L_1\psi)_1, (L_2\psi)_1)_{L^2(Q)} = -2\varepsilon^2 s^3 \tau^4 \iint_Q a^{-1} |\partial_x \eta|^4 \xi^3 |\psi|^2 dx dt.$$

Secondly, we compute

$$(7.6) \quad \begin{aligned} ((L_1\psi)_2, (L_2\psi)_1)_{L^2(Q)} &= -2\varepsilon^2 s^3 \tau^3 \iint_Q a^{-1} |\partial_x \eta|^2 \partial_x \eta \xi^3 \psi \partial_x \psi dx dt \\ &= 3\varepsilon^2 s^3 \tau^4 \iint_Q a^{-1} |\partial_x \eta|^4 \xi^3 |\psi|^2 dx dt \\ &\quad - \underbrace{2\varepsilon^2 s^3 \tau^3 \sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} \int_0^T (a^e)^{-1} |\partial_x \eta^e|^2 \partial_{n^e(v)} \eta^e (\xi^e)^3 |\psi^e|^2(t, v) dt}_{=: J_1} \\ &\quad + o\left(\varepsilon^2 s^3 \tau^4 \iint_Q \xi^3 |\psi|^2 dx dt\right). \end{aligned}$$

Thirdly, integration by parts (with respect to the time variable) yields, for $\tau \geq C$ and $s \geq C(T + T^2)$,

$$(7.7) \quad ((L_1\psi)_3, (L_2\psi)_1)_{L^2(Q)} = \varepsilon s^2 \tau^2 \iint_Q |\partial_x \eta|^2 \xi^2 \psi \partial_t \psi dx dt = o\left(\varepsilon^2 s^3 \tau^4 \iint_Q \xi^3 |\psi|^2 dx dt\right).$$

To continue with,

$$\begin{aligned}
((L_1\psi)_1, (L_2\psi)_2)_{L^2(Q)} &= -2\varepsilon^2 s\tau^2 \iint_Q a^{-1} |\partial_x \eta|^2 \xi \psi \partial_{xx}^2 \psi \, dx \, dt \\
&= 2\varepsilon^2 s\tau^2 \iint_Q a^{-1} |\partial_x \eta|^2 \xi |\partial_x \psi|^2 \, dx \, dt \\
(7.8) \quad &+ \underbrace{\sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} 2\varepsilon^2 s\tau^2 \int_0^T (a^e)^{-1} |\partial_x \eta^e|^2 \xi^e \psi^e \partial_{n^e(v)} \psi^e(t, v) \, dt}_{=: J_2} \\
&+ o\left(\varepsilon^2 s^3 \tau^4 \iint_Q \xi^3 |\psi|^2 \, dx \, dt\right) + o\left(\varepsilon^2 s\tau^2 \iint_Q \xi |\nabla \psi|^2 \, dx \, dt\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
((L_1\psi)_2, (L_2\psi)_2)_{L^2(Q)} &= -2\varepsilon^2 s\tau \iint_Q a^{-1} \partial_x \eta \xi \partial_{xx}^2 \psi \partial_x \psi \, dx \, dt \\
(7.9) \quad &= \varepsilon^2 s\tau^2 \iint_Q a^{-1} \partial_x \eta \xi |\partial_x \psi|^2 - \underbrace{\sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} \varepsilon^2 s\tau \int_0^T (a^e)^{-1} \partial_{n^e(v)} \eta^e \xi^e |\partial_x \psi^e|^2(t, v) \, dt}_{=: J_3} \\
&+ o\left(\varepsilon^2 s\tau^2 \iint_Q \xi |\partial_x \psi|^2 \, dx \, dt\right).
\end{aligned}$$

Moreover, we can prove that

$$(7.10) \quad ((L_1\psi)_3, (L_2\psi)_2)_{L^2(Q)} = \varepsilon \iint_Q \partial_t \psi \partial_{xx}^2 \psi \, dx \, dt = \underbrace{\sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} \varepsilon \int_0^T \partial_{n^e(v)} \psi^e \partial_t \psi^e(t, v) \, dt}_{=: J_4}.$$

Finally,

$$\begin{aligned}
(L_1\psi, (L_2\psi)_3 + (L_2\psi)_4)_{L^2(Q)} &= \underbrace{\sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{E}(v)} s\tau \int_0^T \xi^e \partial_{n^e(v)} \eta^e \left(s\varepsilon \partial_t \alpha^e + \frac{|b^e|^2}{4} \right) |\psi^e|^2(t, v) \, dt}_{=: J_5} \\
(7.11) \quad &+ o\left(\varepsilon^2 s^3 \tau^4 \iint_Q \xi^3 |\psi|^2 \, dx \, dt\right).
\end{aligned}$$

Step 2: Estimation of the boundary terms. To begin with, let us deal with the boundary term J_1 in (7.6). If $v \in \mathcal{V}_\partial$, we get that $\psi = 0$ from the Dirichlet boundary conditions. Otherwise, for each interior node $v \in \mathcal{V}_0$, we get that

$$(7.12) \quad -2\varepsilon^2 s^3 \tau^3 \int_0^T \xi^3 \left(\sum_{e \in \mathcal{E}(v)} (a^e)^{-1} |\partial_x \eta^e|^2 \partial_{n^e(v)} \eta^e \right) |\psi^e|^2(t, v) \, dt \geq c\varepsilon^2 s^3 \tau^3 \int_0^T \xi^3 |\psi^e|^2(t, v) \, dt.$$

Indeed, the function ξ is continuous at junctions, $\partial_{n^{\tilde{e}(v)}} \eta^{\tilde{e}}(v) = \delta$ for \tilde{e} the edge going joining the previous layer to v and $\partial_{n^e(v)} \eta^e(v) = -1$ if $e \in \mathcal{E}(v) \setminus \{\tilde{e}\}$, so we obtain (7.12) by choosing δ small enough. With the right-hand side of (7.12) we can absorb the boundary term J_5 in (7.11) for $s \geq C(T + T^2)$ and $\tau \geq C$.

To continue with, let us study the boundary term J_3 given in (7.9) for each $v \in \mathcal{V}$, i.e.

$$-\varepsilon^2 s\tau \int_0^T \xi \sum_{e \in \mathcal{E}(v)} (a^e)^{-1} \partial_{n^e(v)} \eta^e |\partial_{n^e(v)} \psi^e|^2(t, v) \, dt.$$

If $v \in \mathcal{V}_\partial$, that term can be moved to the right-hand side of the Carleman estimate. On the other hand, if $v \in \mathcal{V}_0$ there is one edge \tilde{e} (the edge going joining the previous layer to v) for which $\partial_{n^{\tilde{e}}} \eta^{\tilde{e}} = \delta$ and such that

$\partial_{n^e}\eta = -1$ for all $e \in \mathcal{E}(v) \setminus \{\tilde{e}\}$. So, we have to absorb the boundary term of the edge \tilde{e} . For that we use $z = \psi e^{-s\alpha}$ to get:

$$(7.13) \quad \begin{aligned} & \varepsilon^2 s\tau (a^{\tilde{e}})^{-1} \int_0^T \xi^{\tilde{e}} |\partial_{n^{\tilde{e}}}\eta^{\tilde{e}}| |\partial_{n^{\tilde{e}}}\psi^{\tilde{e}}|^2(t, v) dt \\ & \leq 2\delta\varepsilon^2 s\tau (a^{\tilde{e}})^{-1} \int_0^T \xi^{\tilde{e}} |\partial_{n^{\tilde{e}}}\alpha^{\tilde{e}}|^2 |\psi|^2(t, v) dt + 2\delta s\tau (a^{\tilde{e}})^{-1} \int_0^T \xi^{\tilde{e}} e^{-2s\alpha^{\tilde{e}}} |\varepsilon \partial_{n^{\tilde{e}}} z_\varepsilon|^2(t, v) dt. \end{aligned}$$

We can absorb the first term in the right-hand side of (7.13) by (7.12). As for the second one, it equals

$$(7.14) \quad \begin{aligned} & \delta\varepsilon^2 s\tau (a^{\tilde{e}})^{-1} \int_0^T \xi^{\tilde{e}} e^{-2s\alpha^{\tilde{e}}} \left| \sum_{e \in \mathcal{V}(e) \setminus \tilde{e}} \partial_{n^e} z_\varepsilon^e \right|^2(t, v) dt \\ & \leq \delta\varepsilon^2 s\tau (a^{\tilde{e}})^{-1} \int_0^T \sum_{e \in \mathcal{V}(e) \setminus \tilde{e}} \xi^e |\partial_{n^e}\psi^e|^2(t, v) dt + C\delta\varepsilon^2 s^3 \tau^3 \int_0^T \xi^3 |\psi^e|^2(t, v) dt. \end{aligned}$$

Indeed, we use the continuity of functions ξ and α on the junction, and the condition on the vertex (3.9)₄ and the fact $\partial_x \alpha = -\tau \partial_x \eta \xi$. Taking δ small enough, we can absorb the term (7.14) by (7.12) and

$$(7.15) \quad \begin{aligned} & -\varepsilon^2 s\tau \int_0^T \xi \sum_{e \in \mathcal{E}(v) \setminus \tilde{e}} (a^e)^{-1} \partial_{n^e(v)} \eta^e |\partial_{n^e(v)} \psi^e|^2(t, v) dt \\ & = \varepsilon^2 s\tau \int_0^T \xi \sum_{e \in \mathcal{E}(v) \setminus \tilde{e}} (a^e)^{-1} |\partial_{n^e(v)} \psi^e|^2(t, v) dt. \end{aligned}$$

To conclude, let us study the boundary term J_4 in (7.10). If $v \in \mathcal{V}_\partial$, then $\partial_t \psi = 0$ because of the Dirichlet boundary conditions. Otherwise, if $v \in \mathcal{V}_0$, from (3.9)₄, we obtain that

$$(7.16) \quad \begin{aligned} & \sum_{e \in \mathcal{E}(v)} \int_0^T \varepsilon \partial_{n^e} \psi^e \partial_t \psi^e(t, v) dt = \sum_{e \in \mathcal{E}(v)} \int_0^T \varepsilon \partial_{n^e} \xi^e e^{-s\alpha^e} z_\varepsilon^e \partial_t (e^{-s\alpha^e} z_\varepsilon^e)(t, v) dt \\ & = s^2 \varepsilon \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_{n^e} \xi^e \partial_t \alpha^e e^{-2s\alpha^e} |z_\varepsilon^e|^2(t, v) dt + s\varepsilon \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_{n^e} \xi^e e^{-2s\alpha^e} \frac{\partial_t (|z_\varepsilon^e|^2)}{2}(t, v) dt \\ & = s^2 \varepsilon \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_{n^e} \xi^e \partial_t \alpha^e |\psi^e|^2(t, v) dt - s\varepsilon \sum_{e \in \mathcal{E}(v)} \int_0^T \partial_t (\partial_{n^e} \xi^e) \frac{|\psi^e|^2}{2}(t, v) dt, \end{aligned}$$

which can be absorbed by (7.12). To sum up the result of this step, we have proved that, for $s \geq C(T+T^2)\varepsilon^2$ and $\tau \geq C$, the following estimate holds:

$$(7.17) \quad \begin{aligned} \sum_{\ell=1}^5 J_\ell & \geq c \sum_{v \in \mathcal{V}_0} \varepsilon^2 s^3 \tau^3 \int_0^T \xi^3 |\psi|^2(t, v) dt + c \sum_{v \in \mathcal{V}_0} \varepsilon^2 s\tau \int_0^T \xi \sum_{e \in \mathcal{E}(v) \setminus \tilde{e}} (a^e)^{-1} |\partial_{n^e(v)} \psi^e|^2(t, v) dt \\ & \quad - C \sum_{v \in \mathcal{V}_\partial} \varepsilon^2 s\tau \int_0^T \xi (a^e)^{-1} \partial_{n^e(v)} \eta^e |\partial_{n^e(v)} \psi^e|^2(t, v) dt. \end{aligned}$$

Step 3: Conclusion of the proof. Combining (7.5)-(7.17) and the fact that $|\partial_x \eta| > 0$, we obtain the estimate:

$$(7.18) \quad \begin{aligned} & \varepsilon^2 s\tau^2 \iint_Q \xi |\partial_x \psi|^2 dx dt + \varepsilon^2 s^3 \tau^4 \iint_Q e^{-2s\alpha} \xi^3 |\psi|^2 dx dt \\ & \leq (L_1 \psi, L_2 \psi)_{L^2(Q)} + \sum_{v \in \mathcal{V}_\partial} \varepsilon^2 s\tau \int_0^T \xi^e |\partial_{n^e(v)} \psi^e|^2(t, v) dt. \end{aligned}$$

From (7.18), we conclude the proof as in Proposition 5.1. \square

Remark 7.1 (On the remarks for parabolic equations). *The analogous of Remarks 5.1, 5.2 and 5.3 are valid for this kind of Carleman inequality too.*

As an immediate consequence of Proposition 7.2, we deduce the following result.

Corollary 7.1 (Observability inequality in one unit of time). *Let $T > 1$. Then, there is $C > 0$ independent of T such that for all $\varphi^T \in L^2(\mathcal{E})$ the solution φ of (3.6) satisfies:*

$$(7.19) \quad \|\varphi_\varepsilon^e(T-1, \cdot)\|_{L^2(\mathcal{E})}^2 \leq e^{C/\varepsilon} \sum_{v \in \mathcal{V}_\partial} \int_{T-1}^T |\partial_\nu \varphi_\varepsilon^e(t, v)|^2 dt.$$

Proof. This can be proved with (7.2), usual bounds on the exponential weights and Theorem 2.1 applied to $\chi(T-t)\varphi_\varepsilon^e$, for χ a cut-off function supported in $(-\infty, 1/2]$. \square

7.3. Proof of Claim (3) of Theorem 3.3. Using the Carleman inequality of the previous section, we now conclude the proof of the main result.

Proof of Theorem 3.3, Claim (3). This result is direct consequence of Corollary 7.1 and of Proposition 7.1. In fact, combining the decay estimate (7.1) and the observability inequality (7.19), we conclude that

$$\|\varphi_\varepsilon(0, \cdot)\|_{L^2(\mathcal{E})}^2 \leq e^{(C-c(T-t))/\varepsilon} \|\varphi_\varepsilon(T-1, \cdot)\|_{L^2(\mathcal{E})}^2 \leq \sum_{v \in \mathcal{V}_\partial} e^{(C-c(T-t))/\varepsilon} \int_{T-1}^T |\partial_\nu \varphi_\varepsilon^e(t, v)|^2 dt.$$

As a consequence, we obtain that (3.5) holds for a sufficiently large time $T > 0$. \square

Remark 7.2 (The case of non-constant coefficients). *The same techniques that we have developed in this work apply to prove analogous results if the coefficients b and a depend on the time and space variables as long as $a \in C^1([0, T] \times \mathcal{E})$ such that $\inf_{\mathcal{E}} a > 0$ and $b \in C_{pw}^1([0, T] \times \mathcal{E})$ satisfying $\sum_{e \in \mathcal{E}(v)} b^e(v) = 0$ in all $v \in \mathcal{V}_0$ and $\inf_{\mathcal{E}} b > 0$ (see Remark 4.2 for the necessity of the positivity of b). In fact, the decay property is proved with the transformation*

$$(7.20) \quad z_\varepsilon^e = \varphi_\varepsilon^e \exp \left(\frac{\int_0^x b^e(\xi) d\xi + c^e}{2\varepsilon} \right),$$

for c^e the right constants so that z^e is continuous at the junctions. The existence of such constants c^e can be proved similarly to Lemma 2.1. With this change of variables we obtain the system

$$(7.21) \quad \begin{cases} -a^e \partial_t z_\varepsilon^e + \left(\frac{|b^e|^2}{4\varepsilon} - \frac{\partial_x b^e}{2} \right) z_\varepsilon^e - \varepsilon \partial_{xx}^2 z_\varepsilon^e = 0, & \text{in } (0, T) \times e, \quad e \in \mathcal{E}, \\ z_\varepsilon^e(v) = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ z_\varepsilon^{e_1}(t, v) = z_\varepsilon^{e_2}(t, v), & t \in (0, T), \quad v \in \mathcal{V}, \quad \forall e_1, e_2 \in \mathcal{E}^{out}(v), \\ \sum_{e \in \mathcal{E}(v)} \varepsilon \partial_{n^e} z_\varepsilon^e = 0 & \text{on } (0, T) \times \mathcal{V}_0, \\ z_\varepsilon(0, \cdot) = z_0, & \text{on } \mathcal{E}. \end{cases}$$

Then, the computations for decay and Carleman estimates are still valid; indeed, we just get some lower order terms that can be easily absorbed by putting them in the source term and taking ε small enough.

Remark 7.3 (Singular limits and internal control). *By doing an analogous proof, we can show that the cost of null-controllability for the system*

$$\begin{cases} a^e \partial_t y_\varepsilon^e + b^e \partial_x y_\varepsilon^e - \varepsilon \partial_{xx}^2 y_\varepsilon^e = \mathbb{1}_\omega f_\varepsilon, & \text{in } (0, T) \times e, \quad e \in \mathcal{E}, \\ y_\varepsilon^e(t, v) = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ y_\varepsilon^{e_1}(t, v) = y_\varepsilon^{e_2}(t, v), & t \in (0, T), \quad v \in \mathcal{V}, \quad \forall e_1, e_2 \in \mathcal{E}(v), \\ \sum_{e \in \mathcal{E}(v)} (b^e y_\varepsilon^e - \varepsilon \partial_x y_\varepsilon^e) n^e = 0, & \text{on } (0, T) \times \mathcal{V}_0, \\ y_\varepsilon(0, \cdot) = y_0, & \text{on } \mathcal{E}, \end{cases}$$

decays exponentially for sufficiently large time provided that the intersection of a sufficient large number of edges with ω is not empty. The only difference in the argument is that we need to consider the auxiliary function defined in Section 5.3. Moreover, if $\omega \neq \mathcal{E}$, by an analogous proof, we also obtain that the cost of null-controllability explodes exponentially as $\varepsilon \rightarrow 0$ for sufficiently small times.

8. CONCLUSIONS AND FUTURE WORK

In this paper, we have obtained the first results on uniform controllability of linear advection-diffusion equations on a tree (and, in particular, on a star-graph) as the diffusivity parameter vanishes. This extends the classical results contained in [27, 41]. Many related problems remain open. In particular, the following ones are of special relevance.

- **Networks containing loops.** In our proofs, it is essential to assume that our network contains no loop (i.e. that we are on a tree). Namely, Lemma 2.1 does not hold when there are loops, which would not allow us to do the transformation (3.8). In addition, it can be proved that we do not have the decay property in Proposition 7.1 for cycles, where the mass is preserved, so we cannot expect that the cost of the control decay for general graphs (see Example 3.5). Moreover, in the Carleman estimate, cycles cause problems to define the auxiliary function, as explained in Remark 5.1. It would be interesting to characterize the controllability properties of (1.4) by the topology or metric properties of the network (see [29, Chapter 5] for the wave equation).
- **Alternative junction conditions.** In addition to the continuity condition that we have imposed at the junctions, there are alternative transmission conditions that are physically relevant, such as those in [40]. It would be interesting to see if analogous results on the controllability and on the cost could be obtained in the framework of [40]. The main challenge to overcome is that the continuity condition was pivotal in estimating the boundary terms in the integrations by parts arising in both the decay property and the Carleman inequality.
- **Balance relation of velocity coefficients in the junction.** The balance relation (1.2), introduced in [32], has been frequently used in many integration by parts throughout the paper and is pivotal to obtain energy dissipation in the junction (see Remark 3.6). As it concerns a lower order term, (1.2) is not needed for the qualitative controllability properties of the parabolic system (see Theorem 3.2) nor for the well-posedness of either system. It is, however, an interesting question to see if the same convergence results hold if we omit such hypothesis.
- **Nonlinear conservation laws.** It would be interesting to extend the analysis of the present paper to nonlinear conservation laws (as in [37, 57]), for which existence results on networks are available in [23, 22, 3, 34, 17].
- **Dispersive effects.** We have considered only a model that includes advection and diffusive phenomena, but the PDEs may also have dispersive effect (as in the Korteweg-de Vries equation analyzed in [39, 38]). The main challenge to deal with the KdV (or KdV-Burgers) equation on a network is to identify suitable transmission conditions at the junction.
- **Optimal time T for the decay.** Establishing sharp estimates on the time T that separates blow-up and decay is an interesting question, though this problem seems really challenging as it is still open even on networks which consist on a single edge. The first result, in dimension one and for a constant velocity, was obtained in [27]; afterwards, the same problem was studied in [41] in any dimension and with a transport flow belonging to $W^{1,\infty}(\mathbb{R}^+ \times \Omega)$. More recently, better approximations have been given for the optimal time in which the cost of the control decays: the lower bound was improved first in [64], through complex analysis and properties of the entire functions, and in [56] through semi-classical and spectral analysis; and the upper bound was improved in [36, 63] (in the first one through complex analysis and, in the second one, by transforming the original equation into the heat equation without convection terms).
- **Cost of approximate controllability on the heat equation for a sufficiently large time.** For a tree-shaped graph \mathcal{G} , the limit system (1.3) is exactly controllable for a sufficiently large time. Consequently, it looks natural that the cost of approximate controllability of (1.1) should decay for a sufficiently large time. We would like to remark that some relevant works for the cost of parabolic equations are [33] (for the heat equation) and [55] (for hypoelliptic equations).
- **Parabolic singular limit of dissipative wave equation.** An interesting extension of this work could be the study of parabolic singular limits – namely, the singular limit of $\varepsilon \partial_{tt}^2 u - \partial_{xx}^2 u + \partial_t u = 0$ as $\varepsilon \rightarrow 0^+$ – on networks. Regarding the convergence properties of those systems in domains belonging to \mathbb{R}^d , they were first studied in [72], and then in [73]. In addition, the controllability properties of those system, and in particular the asymptotic behaviour, was studied in [67] and [66].

- **Controllability under positivity constraints.** It also remains open to prove that, for positive initial data, we can take the solution to equilibrium without losing the positivity, as in [65, 71].

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