Research topics: control questions in Lotka-Volterra and cross-diffusion models

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3. (with S.Dipierro and E.Valdinoci) a class of evolution equations with both classic and fractional time derivatives; we study the asymptotic decay of solutions.

Today's talk

1. An aggressive competition Lotka-Volterra system
2. Cross-diffusion models
3. Reinforcement learning

Section 1

An aggressive competition Lotka-Volterra system

Two populations Lotka-Volterra competitive system

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u size of the first population, $u \geq 0$
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$\dot{u} = u(1 - u)$

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Interaction: if the populations live in the same environment, there is **competition for resources**:

$$\begin{cases} \dot{u} &= u\,(1 - u - v)\,, \quad t > 0, \\ \dot{v} &= v\,(1 - u - v)\,, \quad t > 0. \end{cases}$$

More **parameters**:

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r_u, r_v reproduction rates

k_u, k_v carrying capacities

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Complicated **dynamics**:

- If $k_u\alpha_u < 1$ and $k_v\alpha_v < 1$, coexistence equilibrium
- If $k_u\alpha_u < 1 < k_v\alpha_v$: u prevails
- If $k_u\alpha_u, k_v\alpha_v > 1$: phase plane splits into two basins

A model for civil war

We derive the model from basic principles.

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Two rational, strategist populations.

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Space is not taken into account (yet).

Modelling civil war

Parameters and how to use them:

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3. **Missing births**[1]: c_u and c_v

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   missing birth for the first population: $c_u au$
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We get:

$$\begin{cases} \dot{u} &= r_u u \left(1 - \dfrac{u + \alpha_u v}{k_u}\right) - a(c_u + \zeta)u, \quad t > 0, \\ \dot{v} &= r_v v \left(1 - \dfrac{v + \alpha_v u}{k_v}\right) - a(c_v + \theta)u, \quad t > 0, \end{cases}$$

that is very general but not easily treatable.

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Rescaling, we get:

$$\begin{cases} \dot{u} &= u(1 - u - v) - acu, \quad t > 0, \\ \dot{v} &= \rho v(1 - v - u) - au, \quad t > 0. \end{cases} \tag{1}$$

a **aggressiveness**, ρ **fitness** of the second population wrt the first, c **losses** of war for the first population wrt the second

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Observation: v may become **negative** in finite time, in contrast to what happen with Lotka-Volterra equations.

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We can define the **stopping time**

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$$\mathcal{B} := \Big\{ (u(0), v(0)) \ \Big| \ T_s(u(0), v(0)) = +\infty, (u(t), v(t)) \xrightarrow{t \to \infty} (0, 1) \Big\}.$$

Dynamics 1: $ac < 1$

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Equilibria:

$(0,0)$ unstable,
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$\left(\frac{1-ac}{1+\rho c}\rho c, \frac{1-ac}{1+\rho c}\right)$ saddle,
Γ its stable manifold

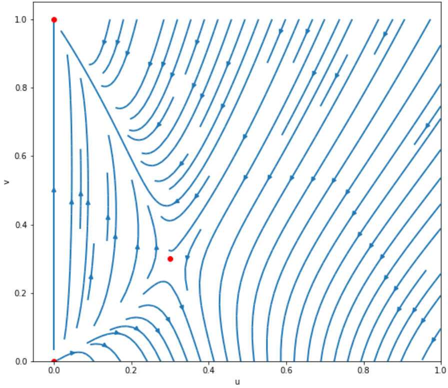

Figure: $a = 0.8,\ c = 0.5,\ \rho = 2$

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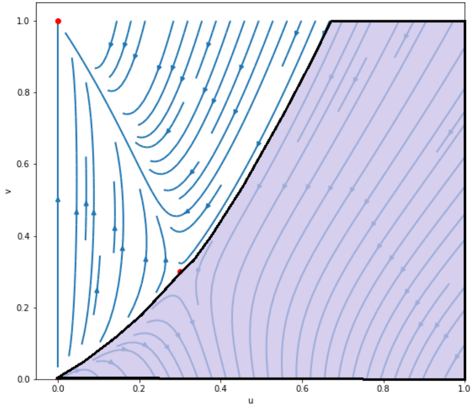

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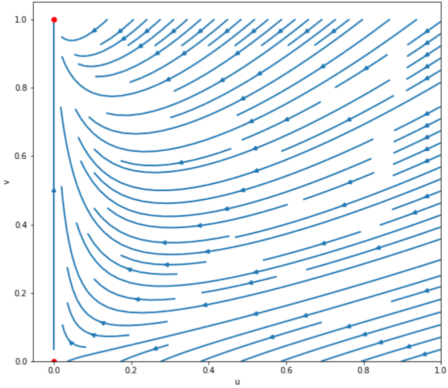

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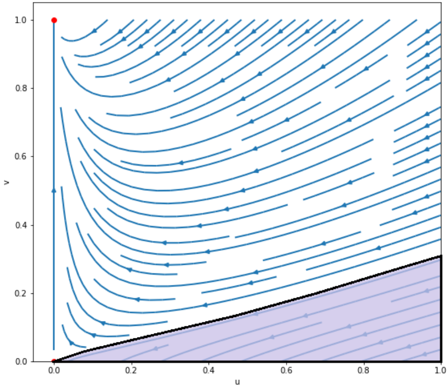

Figure: $a = 0.8$, $c = 3$, $\rho = 2$

Strategy

We sympathize with one of the two populations.

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Difficulty: the problem is not controllable; we use geometrical methods.

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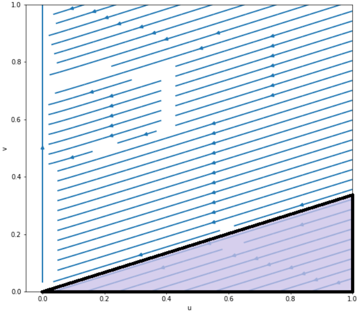
Case "$a = +\infty$"

$$\mathcal{E} \xrightarrow{a\to\infty} \{(u, v) \mid v < \tfrac{u}{c}\},$$



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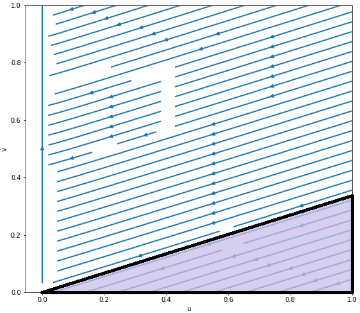
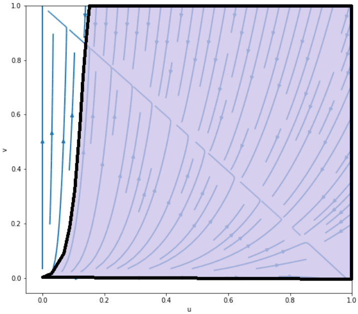


Case "$a = 0$"

$$\mathcal{E} \xrightarrow{a \to 0} \{(u, v) \mid v < ku^{\rho}\},$$

where $k = \frac{(1+\rho c)^{\rho-1}}{(\rho c)^{\rho}}$



Theorem: Characterisation of $\mathcal{V}$

[A., Dipierro, Rossi, Valdinoci, 2020]

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$$\mathcal{E}(a) = \left\{ (u, v) \in [0,1] \times [0,1] \;\middle|\; v < \frac{u}{c} \right\},$$

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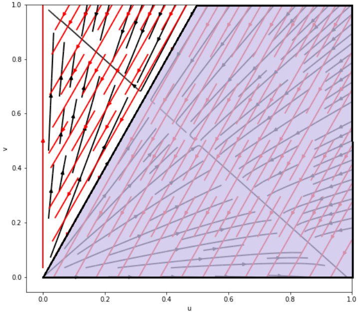

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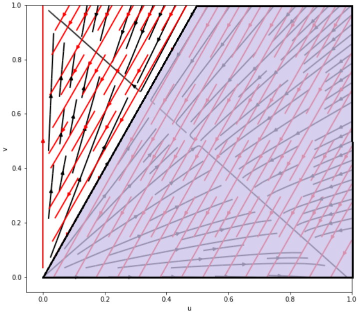
If the population are **indistinguishable** up to the aggressiveness, the initial condition determines the outcome.



For $\rho < 1$, we have that

$$\mathcal{V} = \left\{ (u,v) \mid v < \gamma_0(u) \text{ if } u \leq u_s^0, v < \frac{1}{c}u + \frac{1-\rho}{1+\rho c} \text{ if } u \geq u_s^0 \right\},$$

where $\gamma_0(u) := \frac{v_s^0}{(u_s^0)^\rho} u^\rho$, $u_s^0 = \frac{\rho c}{1+\rho c}$.

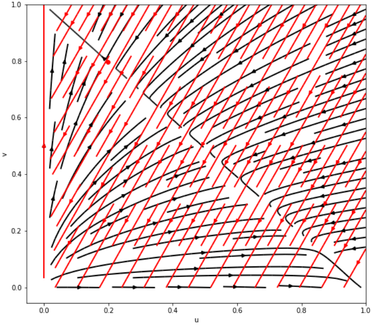


Figure: $\rho = 0.5$, $c = 0.5$, $a_1 = 0.01$, $a_2 = 100$

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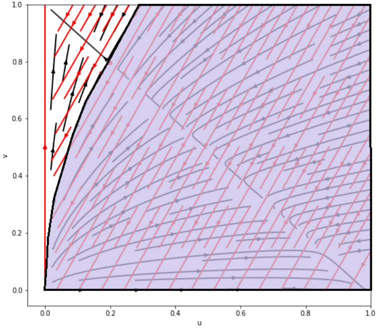

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where $u_\infty = \frac{c}{c+1}$, $\zeta(u) := \frac{(1+c)^{\rho-1}}{c^\rho} u^\rho$

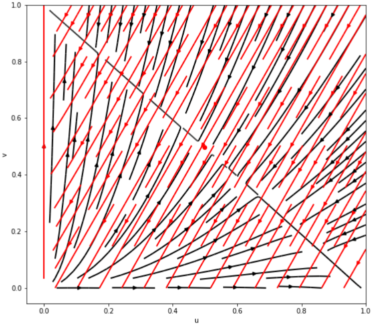

Figure: $\rho = 2$, $c = 0.5$, $a_1 = 0.01$, $a_2 = 100$

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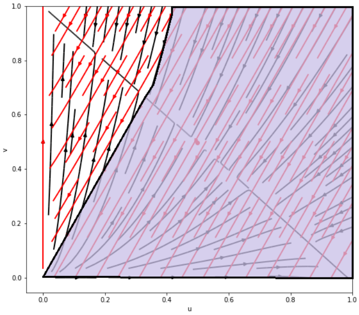

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Section 2

Cross-diffusion models

Cross-diffusion systems

y_1 size of the first population

y_2 size of the second population

Lotka-Volterra with diffusion (due to natural displacement):

$$\begin{cases} \partial_t y_1 - d_1 \Delta y_1 &= y_1 \left(1 - y_1 - y_2\right), \\ \partial_t y_2 - d_2 \Delta y_2 &= y_2 \left(1 - y_1 - y_2\right). \end{cases}$$

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Cross-diffusion: impact of the presence of the first species on the movement of the individuals of the second one through a **repulsive effect**; new diffusion rate:

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Presents strong **nonlinear coupling**, resulting in Turing instability and causing the pattern formation typical of **segregation of populations**

My problem in cross-diffusion models

Problem: **adding some control terms on the boundary**

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Question: Can we find a control (u_1, u_2) that drives an initial condition (y_1^0, y_2^0) in a target state $(\tilde{y}_1, \tilde{y}_2)$

1. in finite time?
2. approximately?

Solution strategy:

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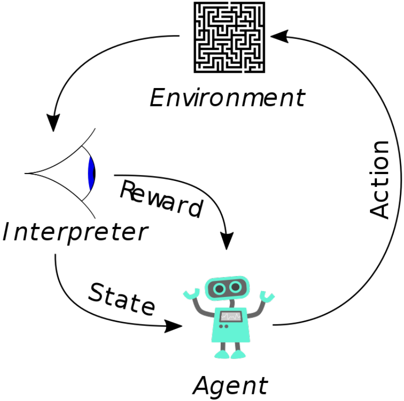
Microscopic model: The cross-diffusion effect is approximated by the existence of **two states** for the repulsed population; the stressed state y_{2B} has higher diffusivity. Then, one writes again the model for y_1 and y_{2A}, y_{2B} such that $y_2 = y_{2A} + y_{2B}$

Section 3

Reinforcement learning

What is reinforcement learning?

Reinforcement learning is an area of machine learning concerned with how intelligent agents ought to take actions in an environment in order to **maximize a cumulative reward**, but **without knowing the exact dynamics** without knowing the exact dynamics of the environment.



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Dynamics of the environment: $y_{t+1} = Ay_t + Bu_t$,
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The dynamics is linear in both the state and the control

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Cost to minimize: $\frac{1}{T}\sum_{t=1}^{T} y_t^* Q y_t + u_{t-1}^* R u_{t-1}$
with Q and R definite positive matrices.
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Dynamics: $\dot{y} = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}x + \chi_\omega u$ for $x \in \Omega,\ t > 0$
with $A = [a_{ij}]_{ij}$ definite positive (so that the operator is elliptic)
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Thank you for your attention!

Section 4

Other results on civil wars

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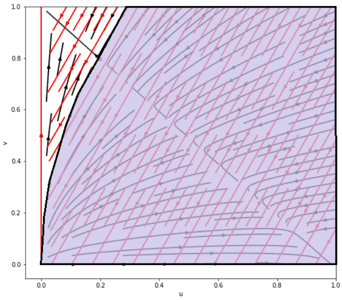
For $\rho = 1$, for all $(u_0, v_0) \in \mathcal{V}$, (all) **constants work**.
For $\rho \neq 1$, there exists a point in $\mathcal{V}$ for which **no constant strategy works**.



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Theorem [A., Dipierro, Rossi, Valdinoci, 2020]

For all points in $\mathcal{V}$, u can win with an **Heaviside** strategy.

If $(u_0, v_0) \in \mathcal{V}$, then either there exists a constant winning strategy or u wins for a strategy

$$a(t) = \begin{cases} a_1 & t < T \\ a_2 & t \geq T \end{cases}$$

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We can win the war with bang-bang strategies.