BLOW-UP RESULTS FOR A LOGARITHMIC PSEUDO-PARABOLIC $p(.)$-LAPLACIAN TYPE EQUATION

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Abstract. In this paper, we consider an initial-boundary value problem for the following mixed pseudo-parabolic $p(.)$-Laplacian type equation with logarithmic nonlinearity:

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right) = |u|^{q(.)-2}u\ln(|u|), \quad (x,t)\in\Omega\times(0,+\infty),$$

where $\Omega\subset\mathbb{R}^n$ is a bounded and regular domain, and the variable exponents $p(.)$ and $q(.)$ satisfy suitable regularity assumptions. By adapting the first-order differential inequality method, we establish a blow-up criterion for the solutions and obtain an upper bound for the blow-up time. In a second moment, we show that blow-up may be prevented under appropriate smallness conditions on the initial datum, in which case we also establish decay estimates in the $H_0^1(\Omega)$-norm as $t\to+\infty$.

1. Introduction

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we look at the following pseudo-parabolic $p(.)$-Laplacian equation with logarithmic nonlinearity:

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right) = |u|^{q(.)-2}u\ln(|u|), & (x,t)\in\Omega\times(0,+\infty) \\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,+\infty) \\ u(x,0)=u_0(x)\neq 0, & x\in\Omega \end{cases} \tag{1.1}$$

In (1.1), $p(.)$ and $q(.)$ are given measurable functions on $\overline{\Omega}$ satisfying:

$$2\le p_1\le p(x)\le p_2<q_1\le q(x)\le q_2<p^*(x),$$

where we have denoted

$$p_1 := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_2 = \operatorname*{ess\,sup}_{x\in\Omega} p(x)$$

$$q_1 := \operatorname*{ess\,inf}_{x\in\Omega} q(x), \quad q_2 = \operatorname*{ess\,sup}_{x\in\Omega} q(x).$$

and

$$p^*(x) := \begin{cases} \dfrac{np(x)}{n-p(x)}, & \text{if } n>p_2, \\ +\infty, & \text{if } n\le p_2 \end{cases}. \tag{1.2}$$

Moreover, we assume that $p(.)$ and $q(.)$ satisfy the following Zhikov-Fan uniform local continuity condition: there exist a constant $M>0$ such that

$$|q(x)-q(y)|\le\frac{M}{|\log|x-y||}, \quad \text{for all } (x,y)\in\Omega\times\Omega \text{ with } |x-y|<\frac{1}{2}. \tag{1.3}$$

Key words and phrases. Pseudo-parabolic equation, Logarithmic nonlinearity, Variable coefficients, Blow-up time, Bounds of the blow-up time.

The work of A.R. is supported by the General Direction of Scientific Research and Technological Development (DGRSDT), Algeria. The work of U.B. is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement NO: 694126-DyCon), by the Air Force Office of Scientific Research (AFOSR) under Award NO: FA9550-18-1-0242, by Grant MTM2017-92996-C2-1-R COSNET of MINECO (Spain) and by the Elkartek grant KK-2020/00091 CONVADP of the Basque government.

Finally, the initial datum in (1.1) is assumed to be any given function $u_0 \in W_0^{1,p(.)}(\Omega)$, where $W_0^{1,p(.)}(\Omega)$ denotes the extension to the variable exponent case of the Sobolev space $W_0^{1,p}(\Omega)$. We refer to Section 2 for more detail.

The nonlinear term $\operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right)$ in (1.1) is the so-called $p(.)$-Laplacian operator, which is sometimes symbolized by

$$\Delta_{p(.)}u = \operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right)$$

and is a generalization of the well-known p-Laplacian, which corresponds to taking $p(.) \equiv p \in \mathbb{R}$ constant.

This operator can be prolonged to a monotone operator between the space $W_0^{1,p(.)}(\Omega)$ and its dual space $W^{-1,p'(.)}(\Omega)$ as follows

$$\begin{cases} -\Delta_{p(.)}u : W_0^{1,p(.)}(\Omega) \to W^{-1,p'(.)}(\Omega) \\ \langle -\Delta_{p(.)}u, \phi\rangle_{p(.)} = \displaystyle\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\phi \,\mathrm{d}x, \quad 2 \le p_1 \le p(x) \le p_2 < +\infty, \end{cases}$$

where $\langle .,.\rangle_{p(.)}$ indicates the duality pairing between $W_0^{1,p(.)}(\Omega)$ and $W^{-1,p'(.)}(\Omega)$, and $p'(.)$ denotes the conjugate exponent such that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Nonlinear pseudo-parabolic equations like (1.1) arise in the description of several problems in hydrodynamics, thermodynamics, nonlinear elasticity, image processing and filtration theory (see [4, 22, 30, 32, 33]). Furthermore, we can mention interesting applications in the study of electrorheological fluids, whose viscosity depends on the electric field in the fluid itself (see [2, 18--20, 26, 45]).

In the case in which the exponents $p(.)$ and $q(.)$ are constants, that is when $p(.) \equiv p \in \mathbb{R}$ and $q(.) \equiv q \in \mathbb{R}$, the corresponding PDE

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{q-2}u\ln(|u|), \quad (x,t) \in \Omega \times (0,+\infty) \tag{1.4}$$

has been largely studied by the mathematical community.

In [46, 49], the linear version of (1.4) (i.e. with the right-hand side equal to zero) has been considered in the case $p = 2$ in which the p-Laplacian $-\Delta_p$ becomes the standard Laplace operator $-\Delta$. In particular, results of existence, uniqueness and regularity of the solutions were obtained.

Later on, these results have been extended to the general case of (1.4). In several papers, the asymptotic behavior of weak solutions to (1.4) with initial datum $u_0 \in W_0^{1,p}(\Omega)$ has been studied. In more detail,

- The case $p = q > 2$ has been considered in [37].
- The case $1 < p = q <$ has been considered in [50].
- The case $2 < p < q < p(1 + \frac{2}{n})$ has been considered in [27].
- The case $1 < p \le q < p^*$, with p^* given by (1.2), has been considered in [21].

In all the aforementioned references, the global-in-time existence or the finite time blow-up of such solutions has been characterized in terms of the values of the exponents p and q. We shall stress, however, that the methodology employed in [27, 37, 50] to prove the blow-up of weak solutions to (1.4) does not allow the authors to address the relevant issue of estimating the blow-up time and rate (we refer to [36, 38--41, 47] for the analysis of these issues in the case of other interesting non-linear evolution equations).

A partial answer to these relevant questions has been recently given in [16] where, under the same conditions as in [27] on the exponents $p, q \in \mathbb{R}$ (that is, $2 < p < q < p(1 + \frac{2}{n})$), explicit estimates for the blow-up time are provided.

Finally, we refer to [11--13, 15--17, 25, 29, 31, 34, 42, 51] for the study of the asymptotic behavior of weak solutions to (1.4) under high initial energy level conditions.

As for the case of variable exponents $p(.)$ and $q(.)$, to the best of our knowledge, there are no results in the existing literature in the spirit of those that we have just recalled. As a matter of fact, many of the techniques employed in the previous references to deal with (1.4) become unsuccessful when $p(.)$ and $q(.)$ are measurable functions on $\overline{\Omega}$ and, therefore, are not immediately extendable to treat (1.1).

The goal of this paper is precisely to address this variable measurable coefficients framework. In particular, we shall identify sufficient conditions on $p(.)$, $q(.)$ and the initial datum u_0 for which the blow-up and the non-blow-up phenomenon of solutions to (1.1) occur, also providing estimates for the blow-up time.

At this regard, we shall mention that this kind of results are already available for the hyperbolic version of (1.1). The interested reader may refer for instance to [1, 5, 7--10, 14, 35, 43, 44, 48]. In particular, in [5, 8], the authors have discussed the Dirichlet problem for following equation

$$u_{tt} = \mathrm{div}\left(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\right) + \alpha\Delta u_t + b(x,t)|u|^{\sigma(x,t)-2}u + f(x,t), \quad (x,t) \in \Omega \times (0,+\infty)$$

with negative initial energy and, under suitable conditions on the functions a, b, f, p, σ, they have used Galerkin and energy methods to establish local existence, global existence and blow-up of the solutions.

The present paper is organized as follows: in Section 2, we will introduce some preliminary concepts and notations that will be of use in our further analysis. Secondly, we will study the finite-time blow-up of solutions in Section 3. Finally, we will conclude this paper by addressing the global existence of weak solutions to (1.1) in Section 4.

2. Preliminaries

In this section, we present some preliminary concepts and notations that we shall employ in our further analysis. Let us start by introducing the variable-order Lebesque space $L^{p(.)}(\Omega)$, which is defined for all $p : \Omega \to [1,+\infty]$ measurable as

$$L^{p(.)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)} \, \mathrm{d}x < +\infty \right\}.$$

We then know that $L^{p(.)}(\Omega)$ is a Banach space, equipped with the Luxemburg-type norm

$$\|u\|_{p(.)} := \inf\left\{ \lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Next, we define the variable-order Sobolev space $W^{1,p(.)}(\Omega)$ as

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) \ : \ \nabla u \in L^{p(.)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(.)}(\Omega)} = \left(\|u\|^2_{p(.)} + \|\nabla u\|^2_{p(.)}\right)^{\frac{1}{2}}. \tag{2.1}$$

Moreover, in what follows we will need the following embedding result.

Lemma 2.1 ([19, 23])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. It holds the following.*

1. *If $p \in C(\overline{\Omega})$ and $q : \Omega \to [1,+\infty)$ is a measurable function such that*
$$\underset{x\in\Omega}{ess\ inf}(p^*(x) - q(x)) > 0,$$
with p^* defined as in (1.2), then $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ with continuous and compact embedding.
2. *If p satisfy (1.3), then $\|u\|_{p(.)} \le C\|\nabla u\|_{p(.)}$ for all $u \in W_0^{1,p(.)}(\Omega)$. In particular, $\|u\|_{1,p(.)} := \|\nabla u\|_{p(.)}$ defines a norm on $W_0^{1,p(.)}(\Omega)$ which is equivalent to (2.1).*

Let us now introduce our notion of weak solution to (1.1), and give a first result of local-in-time existence and uniqueness.

Definition 2.1. *Let $u_0 \in W_0^{1,p(.)}(\Omega)$. A function*

$$u \in L^\infty\left(0, T_0; W_0^{1,p(.)}(\Omega)\right) \ \text{with } u_t \in L^2\left(0, T_0; H_0^1(\Omega)\right),$$

is called a weak solution to (1.1) if the identity

$$\langle u_t, v\rangle + \langle \nabla u_t, \nabla v\rangle + \left\langle |\nabla u|^{p(.)-2}\nabla u, \nabla v\right\rangle = \int_\Omega |u|^{q(.)-2} u \ln(|u|) v \, \mathrm{d}x$$

holds for any $v \in W_0^{1,p(.)}(\Omega)$, and for a.e. $t \in [0,T]$.

Theorem 2.1. *For all* $u_0 \in W_0^{1,p(.)}(\Omega)$*, the problem* (1.1) *admits a unique weak solution* u *on* $[0,T]$.

Proof. The result can be easily obtained by using the Faedo-Galerkin approach and combining the ideas from [3, 20, 24, 28] with the ones from [6] (see also [50]). We leave the details to the reader. ☐

Let now $\sigma > 0$ be a positive constant satisfying

$$0 < \sigma < \begin{cases} p^* - q_2, & \text{if } p(.) < n, \\ +\infty, & \text{if } p(.) \geq n. \end{cases} \tag{2.2}$$

Then, from Lemma 2.1 we have that $W_0^{1,q(.)+\sigma}(\Omega) \hookrightarrow L^{q(.)+\sigma}(\Omega)$, and there exists a positive constant $B_\sigma > 0$ such that

$$\|u\|_{q(.)+\sigma} \leq B_\sigma \|\nabla u\|_{q(.)+\sigma}. \tag{2.3}$$

Finally, let us indicate with α_1, B_1 and E_1 the following positive constants:

$$\alpha_1 = \left(\frac{e\sigma q_1}{q_1 + \sigma} B_1^{-(q_1+\sigma)} \right)^{\frac{p_2}{q_1 - p_2 + \sigma}} \tag{2.4a}$$

$$B_1 = \max(1, B_\sigma) \tag{2.4b}$$

$$E_1 = \left(\frac{1}{p_2} - \frac{1}{q_1 + \sigma} \right) \alpha_1. \tag{2.4c}$$

3. Finite-time blow-up of solutions

In this section, we obtain a blow-up criterion for the solutions to (1.1), also providing an upper bound for the blow-up time. To this end, let us introduce the energy associated with the solution of our problem (1.1), which is defined as follows:

$$E(t) := \int_\Omega \frac{1}{p(.)} |\nabla u|^{p(.)} \, \mathrm{d}x - \int_\Omega \frac{1}{q(.)} |u|^{q(.)} \ln(|u|) \, \mathrm{d}x + \int_\Omega \frac{1}{q^2(.)} |u|^{q(.)} \, \mathrm{d}x \tag{3.1}$$

We have the following result providing upper and lower bounds for $E(t)$.

Lemma 3.1. *Let* $u_0 \in W^{1,p(.)}(\Omega)$ *and* $E(t)$ *be the energy associated with the corresponding solution of* (1.1)*, as defined in* (3.1)*. Let* $g : [0, +\infty) \to \mathbb{R}$ *be given by*

$$g(\xi) := \frac{1}{p_2} \min\left(\xi^{\frac{p_1}{p_2}}, \xi\right) - \frac{1}{e\sigma q_1} \max\left(B_1^{q_2+\sigma} \xi^{\frac{q_2+\sigma}{p_2}}, B_1^{q_1+\sigma} \xi^{\frac{q_1+\sigma}{p_2}}\right), \tag{3.2}$$

where σ *and* B_1 *are the constants defined in* (2.2) *and* (2.4b)*, respectively. Let* $\alpha : [0, +\infty) \to [0, +\infty)$ *be defined as*

$$\alpha(t) = \|\nabla u(t)\|_{p(.)}^{p_2}. \tag{3.3}$$

Then, we have

$$g(\alpha) \leq E(t) \leq E(0). \tag{3.4}$$

Proof. First of all, a direct computation yields that

$$\frac{dE(t)}{dt} = -\int_\Omega u_t^2 \, \mathrm{d}x - \int_\Omega |\nabla u_t|^2 \, \mathrm{d}x \leq 0. \tag{3.5}$$

Hence, the function E is decreasing with respect to t, and the inequality $E(t) \leq E(0)$ immediately holds for all $t \geq 0$. As for the lower estimate in (3.4), we have that

$$\int_\Omega \frac{1}{q(.)} |u|^{q(.)} \ln(|u|) \, \mathrm{d}x \leq \frac{1}{q_1} \int_\Omega |u|^{q(.)} \ln(|u|) \, \mathrm{d}x \leq \frac{1}{e\sigma q_1} \int_\Omega |u|^{q(.)+\sigma} \, \mathrm{d}x, \tag{3.6}$$

where we used that

$$\ln(\xi) \leq \frac{1}{e\sigma} \xi^\sigma, \quad \text{for all } \xi, \sigma > 0. \tag{3.7}$$

Therefore, we get from (3.1), (3.2), (3.3) and (3.6) that

$$E(t) \geq \frac{1}{p_2}\min\left(\|\nabla u\|_{p(.)}^{p_1}, \|\nabla u\|_{p(.)}^{p_2}\right) - \frac{1}{e\sigma q_1}\max\left(\|u\|_{q(.)+\sigma}^{q_2+\sigma}, \|u\|_{q(.)+\sigma}^{q_1+\sigma}\right) \tag{3.8}$$

$$\geq \frac{1}{p_2}\min\left(\|\nabla u\|_{p(.)}^{p_1}, \|\nabla u\|_{p(.)}^{p_2}\right) - \frac{1}{e\sigma q_1}\max\left(\left(B_1\|\nabla u\|_{p(.)}\right)^{q_2+\sigma}, \left(B_1\|\nabla u\|_{p(.)}\right)^{q_1+\sigma}\right) = g(\alpha).$$

This concludes the proof of (3.4). □

Next, we state some technical lemmas which will be needed in the proof of our main result.

Lemma 3.2. *Let α_1, B_1 and E_1 be given as in* (2.4a), (2.4b) *and* (2.4c), *respectively. Define the function $h : [0, +\infty) \to \mathbb{R}$ as*

$$h(\xi) = \frac{1}{p_2}\xi - \frac{1}{e\sigma q_1}B_1^{q_1+\sigma}\xi^{\frac{q_1+\sigma}{p_2}}. \tag{3.9}$$

Then, the following assertions hold:

1. *h is increasing for $0 < \xi < \alpha_1$ and decreasing for $\xi > \alpha_1$.*
2. *$h(\alpha_1) = E_1$.*
3. *$\lim_{\xi\to+\infty} h(\xi) = -\infty$.*

Proof. First of all, from the definition (3.9) of the function h we can easily compute

$$h'(\xi) = \frac{1}{p_2} - \frac{q_1+\sigma}{p_2}\frac{1}{e\sigma q_1}B_1^{q_1+\sigma}\xi^{\frac{q_1+\sigma}{p_2}-1}.$$

Then, it is straightforward to check that $h'(\xi) > 0$ for $\xi < \alpha_1$ and $h'(\xi) < 0$ for $\xi > \alpha_1$. Besides, the fact that $h(\alpha_1) = E_1$ follows directly from (2.4c) and (3.9). Finally, since by definition of p_2, q_1 and σ we have $q_1 - p_2 + \sigma > 0$, we can readily check that

$$\lim_{\xi\to+\infty} h(\xi) = \lim_{\xi\to+\infty} \xi\left(\frac{1}{p_2} - \frac{1}{e\sigma q_1}B_1^{q_1+\sigma}\xi^{\frac{q_1-p_2+\sigma}{p_2}}\right) = -\infty.$$

This concludes our proof. □

Lemma 3.3. *Let σ, α_1, B_1 and E_1 be given as in* (2.2), (2.4a), (2.4c) *and* (2.4c), *respectively. Assume that the initial value u_0 is chosen so that*

$$0 \leq E(0) < \frac{p_2}{q_1+p_2}\left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)}\right) \quad \text{and} \quad \alpha_1 < \|\nabla u_0\|_{p(.)}^{p_2} \leq B_1^{-p_2}. \tag{3.10}$$

Then ,there exists a positive constant $\alpha_2 > \alpha_1$ such that

$$\|\nabla u\|_{p(.)}^{p_2} \geq \alpha_2, \quad \text{for all } t \geq 0 \tag{3.11a}$$

$$\int_\Omega \frac{1}{q(.)}|u|^{q(.)}\ln(|u|)\mathrm{d}x \geq \frac{1}{e\sigma q_1}B_1^{q_1+\sigma}\alpha_2^{\frac{q_1+\sigma}{p_2}} \tag{3.11b}$$

$$\frac{\alpha_2}{\alpha_1} \geq \left((q_1+\sigma)\left(\frac{1}{p_2} - \frac{E(0)}{\alpha_1}\right)\right)^{\frac{p_2}{q_1+\sigma-p_2}} > 1. \tag{3.11c}$$

Proof. Since from (3.10) we have

$$0 \leq E(0) < \frac{p_2E_1}{q_1+p_2} < E_1,$$

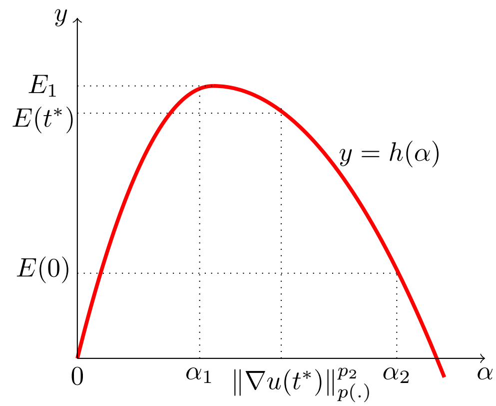
it follows from Lemma 3.2 that there exists a positive constant $\alpha_2 > \alpha_1$ such that the initial energy $E(0)$ satisfies $E(0) = h(\alpha_2)$ (see Figure 1). Moreover, since $B_1 > 1$ (see (2.4b)), it follows again from (3.10) that

$$\alpha_0 = \|\nabla u_0\|_{p(.)}^{p_2} \leq B_1^{-p_2} < 1.$$

Therefore, recalling the definitions (3.2) and (3.9) of the functions g and h, since $p_1 < p_2$, and using (3.8), we can conclude that

$$h(\alpha_0) = g(\alpha_0) \leq E(0) = h(\alpha_2).$$

FIGURE 1. Diagram for the proof of Lemma 3.3. We plot the function h highlighting the quantities α_1, α_2 and $\|\nabla u(t^*)\|_{p(.)}^{p_2}$ with the corresponding values of the energy (E_1, $E(0)$ and $E(t^*)$, respectively).



Hence, it follows from Point 1 in Lemma 3.2 that $\alpha_0 \geq \alpha_2$, and (3.11a) holds for $t = 0$.

To prove (3.11a) for all $t \geq 0$, we argue by contradiction. Suppose that there exists some strictly positive time $t^* > 0$ such that $\|\nabla u(t^*)\|_{p(.)}^{p_2} < \alpha_2$. Then, by the continuity of the $L^{p(.)}$-norm, and since $\alpha_2 > \alpha_1$, we may take t^* such that

$$\alpha_2 > \|\nabla u(t^*)\|_{p(.)}^{p_2} > \alpha_1.$$

Hence, it follows from Lemma 3.2 and (3.8) that

$$E(0) = h(\alpha_2) < h\left(\|\nabla u(t^*)\|_{p(.)}^{p_2}\right) \leq E(t^*).$$

This contradicts the fact that the energy $E(t)$ is decreasing, as we proved in Lemma 3.1. Therefore, (3.11a) holds.

Let us now prove (3.11b). At this regard, from the definition of the energy (3.1), and since $E(t) \leq E(0)$, we obtain

$$\begin{aligned}\int_\Omega \frac{1}{q(.)}\, |u|^{q(.)} \ln(|u|)\mathrm{d}x &= \int_\Omega \frac{1}{p(.)}\, |\nabla u|^{p(.)}\, \mathrm{d}x + \int_\Omega \frac{1}{q^2(.)} |u|^{q(.)}\, \mathrm{d}x - E(t) \\ &\geq \int_\Omega \frac{1}{p(.)}\, |\nabla u|^{p(.)}\, \mathrm{d}x - E(0) \geq \frac{1}{p_2} \min\left(\|\nabla u\|_{p(.)}^{p_2}, \|\nabla u\|_{p(.)}^{p_1}\right) - E(0) \\ &\geq \frac{1}{p_2} \min\left(\alpha_2^{\frac{p_1}{p_2}}, \alpha_2\right) - E(0). \end{aligned} \tag{3.12}$$

Moreover, since $\alpha_2 \leq \alpha_0 < 1$ and $p_1 \leq p_2$, we have

$$\min\left(\alpha_2^{\frac{p_1}{p_2}}, \alpha_2\right) = \alpha_2.$$

Hence, it follows from (3.9) and (3.12) that

$$\int_\Omega \frac{1}{q(.)}\, |u|^{q(.)} \ln(|u|)\mathrm{d}x \geq \frac{1}{p_2}\alpha_2 - E(0) = \frac{1}{p_2}\alpha_2 - h(\alpha_2) = \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \alpha_2^{\frac{q_1+\sigma}{p_2}}.$$

Finally, let us prove (3.11c). To this end, since $E(0) < E_1$, recalling the definition of E_1 given in (2.4c) we can readily check that

$$\left((q_1+\sigma)\left(\frac{1}{p_2} - \frac{E(0)}{\alpha_1}\right)\right)^{\frac{p_2}{q_1-p_2+\sigma}} > \left((q_1+\sigma)\left(\frac{1}{p_2} - \frac{E_1}{\alpha_1}\right)\right)^{\frac{p_2}{q_1-p_2+\sigma}} = 1.$$

Hence, the second inequality in (3.11c) holds. As for the first inequality, we can easily compute

$$E(0) = h(\alpha_2) = \alpha_2\left(\frac{1}{p_2} - \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \alpha_2^{\frac{q_1+\sigma}{p_2}-1}\right) = \alpha_1 \frac{\alpha_2}{\alpha_1}\left(\frac{1}{p_2} - \frac{1}{e\sigma q_1} B_1^{q_1+\sigma} \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{q_1+\sigma}{p_2}-1} \alpha_1^{\frac{q_1+\sigma}{p_2}-1}\right).$$

Recalling the definition of α_1 given in (2.4a), and since $\alpha_2 > \alpha_1$ we then have that

$$E(0) = \alpha_1 \frac{\alpha_2}{\alpha_1}\left(\frac{1}{p_2} - \frac{1}{q_1+\sigma}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{q_1+\sigma-p_2}{p_2}}\right) \geq \alpha_1\left(\frac{1}{p_2} - \frac{1}{q_1+\sigma}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{q_1+\sigma-p_2}{p_2}}\right).$$

Therefore,

$$\frac{E(0)}{\alpha_1} \geq \frac{1}{p_2} - \frac{1}{q_1+\sigma}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{q_1+\sigma-p_2}{p_2}},$$

from which it follows immediately that

$$\frac{\alpha_2}{\alpha_1} \geq \left((q_1+\sigma)\left(\frac{1}{p_2} - \frac{E(0)}{\alpha_1}\right)\right)^{\frac{p_2}{q_1+\sigma-p_2}}$$

Our proof is then concluded. □

Let us define

$$H(t) := E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)} - E(t), \qquad \text{for } t \geq 0. \tag{3.13}$$

We have the following result.

Lemma 3.4. *Let σ, α_1, B_1 and E_1 be given as in* (2.2), (2.4a), (2.4c) *and* (2.4c), *respectively. Assume that the initial value u_0 is chosen so that*

$$0 \leq E(0) < \frac{p_2}{q_1+p_2}\left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)}\right) \qquad \text{and} \qquad \alpha_1 < \|\nabla u_0\|_{p(.)}^{p_2} \leq B_1^{-p_2}.$$

Then, the functional $H(t)$ defined in (3.13) *satisfies the following estimates:*

$$0 < H(0) \leq H(t) \leq \int_\Omega \frac{1}{q(.)}|u|^{q(.)}\ln(|u|)\mathrm{d}x, \quad t \geq 0. \tag{3.14}$$

Proof. By Lemma 3.1, we know that $H(t)$ is non-decreasing for all $t \geq 0$. Hence

$$H(t) \geq H(0) = E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)} - E(0). \tag{3.15}$$

By the definition (3.1) of $E(t)$, we have

$$\begin{aligned} H(t) - \int_\Omega \frac{1}{q(.)}|u|^{q(.)}\ln(|u|)\,\mathrm{d}x &= E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)} - \int_\Omega \frac{1}{p(.)}|\nabla u|^{p(.)}\,dx - \int_\Omega \frac{1}{q^2(.)}|u|^{q(.)}\,\mathrm{d}x \\ &< E_1 - \int_\Omega \frac{1}{p(.)}|\nabla u|^{p(.)}\,\mathrm{d}x. \end{aligned}$$

Hence, using (2.4a), (2.4c), (3.11a) and the fact that $\alpha_2 > \alpha_1$, for all $t \geq 0$ we have

$$H(t) - \int_\Omega \frac{1}{q(.)}|u|^{q(.)}\ln(|u|)\mathrm{d}x < E_1 - \frac{1}{p_2}\min\left(\|\nabla u\|_{p(.)}^{p_2}, \|\nabla u\|_{p(.)}^{p_1}\right) < \left(\frac{1}{p_2} - \frac{1}{q_1+1}\right)\alpha_1 - \frac{1}{p_2}\alpha_1 < 0. \tag{3.16}$$

The inequalities (3.14) follow immediately from (3.15) and (3.16). □

We are now ready to state and prove our main result of this Section, concerning the blow-up of solutions for (1.1).

Theorem 3.1. *Let σ, α_1, B_1 and E_1 be given as in* (2.2), (2.4a), (2.4c) *and* (2.4c), *respectively. Assume that the initial value u_0 is chosen so that* (3.10) *holds. Then, the corresponding solution of problem* (1.1) *will blow-up in finite time T^*. Furthermore, we have the following upper estimate for T^*:*

$$0 < T^* < \frac{\|u_0\|_{H_0^1(\Omega)}^{p_2}}{(\mathcal{B}+p_2-2)\left(\frac{q_1-p_2}{q_1+\sigma}\alpha_1 - (\mathcal{B}+p_2)E(0)\right)},$$

where we have denoted

$$\mathcal{B} = \mathcal{B}(\alpha_1, \alpha_2, p_2, q_1, \sigma) := (q_1 - p_2)\left(1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\sigma+q_1}{p_2}}\right) > 0, \tag{3.17}$$

with α_2 *as in Lemma 3.3.*

Proof. Let us define the function

$$\varphi(t) = \frac{1}{2}\int_\Omega u^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x = \frac{1}{2}\|u\|^2_{H_0^1(\Omega)}.$$

Then, using (1.1) and integration by parts, the derivative of $\varphi'(t)$ satisfies

$$\begin{aligned}\varphi'(t) &= \int_\Omega u u_t \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla u_t \,\mathrm{d}x \\ &= \int_\Omega u\left(\Delta u_t + \operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right) + |u|^{q(.)-2} u \ln(|u|)\right)\,\mathrm{d}x - \int_\Omega u\Delta u_t \,\mathrm{d}x \\ &= -\int_\Omega |\nabla u|^{p(.)}\,\mathrm{d}x + \int_\Omega |u|^{q(.)}\ln(|u|)\,\mathrm{d}x.\end{aligned}$$

Recalling the definitions (3.1) and (3.13) of the energy $E(t)$ and the function $H(t)$, from the above identity we can estimate

$$\begin{aligned}\varphi'(t) &\geq -p_2 E(t) - p_2\int_\Omega \frac{1}{q(.)}|u|^{q(.)}\ln(|u|)\mathrm{d}x + \int_\Omega |u|^{q(.)}\ln(|u|)\,\mathrm{d}x \\ &\geq -\left(p_2 E_1 - \frac{\sigma\alpha_1}{q_1+\sigma}\right) + p_2 H(t) + \frac{q_1 - p_2}{q_1}\int_\Omega |u|^{q(.)}\ln(|u|)\,\mathrm{d}x.\end{aligned} \tag{3.18}$$

Moreover, from (2.4a), (2.4c) and (3.11b) we have

$$\begin{aligned}p_2E_1 - \frac{\sigma\alpha_1}{q_1+\sigma} &= \frac{q_1-p_2}{q_1+\sigma}\alpha_1 = \frac{q_1-p_2}{q_1+\sigma}\alpha_1^{\frac{q_1+\sigma}{p_2}}\alpha_1^{-\frac{q_1-p_2+\sigma}{p_2}} = \frac{q_1-p_2}{q_1+\sigma}\alpha_1^{\frac{q_1+\sigma}{p_2}}\left(\frac{e\sigma q_1}{q_1+\sigma}B_1^{-(q_1+\sigma)}\right)^{-1} \\ &= \left(\frac{e\sigma q_1}{q_1+\sigma}\right)^{-1}\frac{q_1-p_2}{q_1+\sigma}B_1^{q_1+\sigma}\alpha_1^{\frac{q_1+\sigma}{p_2}} = \frac{q_1-p_2}{q_1}\left(\frac{1}{e\sigma}B_1^{q_1+\sigma}\alpha_1^{\frac{q_1+\sigma}{p_2}}\right) \\ &= \frac{q_1-p_2}{q_1}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{q_1+\sigma}{p_2}}\left(\frac{1}{e\sigma q_1}B_1^{q_1+\sigma}\alpha_2^{\frac{q_1+\sigma}{p_2}}\right) \leq \frac{q_1-p_2}{q_1}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{q_1+\sigma}{p_2}}\int_\Omega |u|^{q(.)}\ln(|u|)\,\mathrm{d}x.\end{aligned} \tag{3.19}$$

Then, it follows (3.18) and (3.19) that

$$\varphi'(t) \geq \frac{q_1-p_2}{q_1}\left(1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{q_1+\sigma}{p_2}}\right)\int_\Omega |u|^{q(.)}\ln(|u|)\,\mathrm{d}x + p_2 H(t). \tag{3.20}$$

Besides, using (2.2), we can readily check that

$$\zeta := \frac{q_1-p_2}{q_1}\left(1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{q_1+\sigma}{p_2}}\right) > 0. \tag{3.21}$$

Now, let us define the function

$$\psi(t) := -\zeta q_1 E(t) + p_2 H(t). \tag{3.22}$$

Then, using (3.5) and (3.13), we have

$$\psi'(t) = -\zeta q_1 E'(t) + p_2 H'(t) = -(\zeta q_1 + p_2)E'(t) = (\zeta q_1 + p_2)\int_\Omega \left(|u_t|^2 + |\nabla u_t|^2\right)\,\mathrm{d}x > 0. \tag{3.23}$$

Hence, using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}\varphi(t)\psi'(t) &= \frac{\zeta q_1 + p_2}{2}\left(\int_\Omega \left(|u|^2 + |\nabla u|^2\right) \mathrm{d}x\right)\left(\int_\Omega \left(|u_t|^2 + |\nabla u_t|^2\right) \mathrm{d}x\right)\\ &\geq \frac{\zeta q_1 + p_2}{2}\left(\int_\Omega u u_t \mathrm{d}x + \int_\Omega \nabla u \nabla u_t \mathrm{d}x\right)^2\\ &= \frac{\zeta q_1 + p_2}{2}\varphi'(t)^2. \end{aligned} \tag{3.24}$$

Since, by definition, $\zeta < 1$, we have from (3.10) that

$$E(0) < \frac{p_2}{q_1 + p_2}\left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)}\right) < \frac{p_2}{\zeta q_1 + p_2}\left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)}\right).$$

Hence, we obtain that

$$\psi(0) = -\zeta q_1 E(0) + p_2 H(0) = -(\zeta q_1 + p_2)E(0) + p_2\left(E_1 - \frac{\sigma\alpha_1}{p_2(q_1+\sigma)}\right) > 0 \tag{3.25}$$

which, thanks to (3.23), yields $\psi(t) > 0$ for all $t \geq 0$.

Moreover, by employing (3.20) and (3.22) it is easy to check that $\varphi'(t) \geq \psi(t)$. Hence from the inequality (3.24), we get

$$\varphi(t)\psi'(t) \geq \frac{\zeta q_1 + p_2}{2}\varphi'(t)\psi(t)$$

which can be written as

$$\frac{\psi'(t)}{\psi(t)} \geq \frac{\zeta q_1 + p_2}{2}\frac{\varphi'(t)}{\varphi(t)}. \tag{3.26}$$

Integrating (3.26) from 0 to t and using (3.22), we have

$$\frac{\varphi'(t)}{\varphi(t)^{\frac{\zeta q_1 + p_2}{2}}} \geq \frac{\psi(0)}{\varphi(0)^{\frac{\zeta q_1 + p_2}{2}}}. \tag{3.27}$$

Integrating also (3.27) from 0 to t, we obtain

$$-\frac{2}{\zeta q_1 + p_2 - 2}\,\varphi(s)^{-\frac{\zeta q_1 + p_2}{2}-1}\Big|_{s=0}^{s=t} \geq \frac{\psi(0)}{\varphi(0)^{\frac{\zeta q_1 + p_2}{2}}}t,$$

that is,

$$\frac{1}{\varphi(t)^{\frac{\zeta q_1 + p_2 - 2}{2}}} \leq \frac{1}{\varphi(0)^{\frac{\zeta q_1 + p_2 - 2}{2}}} - \frac{\zeta q_1 + p_2 - 2}{2}\frac{\psi(0)}{\varphi(0)^{\frac{\zeta q_1 + p_2}{2}}}t,$$

From the above inequality, by means of some algebraic manipulations, we finally get

$$\varphi(t)^{\frac{\zeta q_1 + p_2 - 2}{2}} \geq \frac{\varphi(0)^{\frac{\zeta q_1 + p_2}{2}}}{\varphi(0) - \frac{\zeta q_1 + p_2 - 2}{2}\psi(0)t}.$$

Let

$$0 < T^* = \frac{2\varphi(0)}{(\zeta q_1 + p_2 - 2)\psi(0)}. \tag{3.28}$$

Then

$$\varphi(0) - \frac{\zeta q_1 + p_2 - 2}{2}\psi(0)T^* = 0,$$

yielding the blow-up of $\varphi(t)$ at time T^*. By definition of φ, we can then conclude that $u(x,t)$ blows-up in $H_0^1(\Omega)$-norm at $t = T^*$.

Finally, notice that by means of (3.17), (3.21) and (3.25), and recalling the definition of φ, we can easily obtain from (3.28)

$$T^* = \frac{\|u_0\|^2_{H_0^1(\Omega)}}{(\mathcal{B} + p_2 - 2)\left(\frac{q_1 - p_2}{q_1+\sigma}\alpha_1 - (\mathcal{B} + p_2)E(0)\right)}.$$

This conclues our proof. $\square$

4. Non-blow-up case

In this section, we present some non blow-up conditions for the solution of (1.1). In particular, we are going to show that, when the initial datum u_0 is small enough in $H_0^1(\Omega)$, blow-up cannot manifest at any time $t \geq 0$. In addition to that, we will also provide explicit time-decay estimates for the $H_0^1(\Omega)$-norm of the solution.

In order to state our main result in this section, let us denote with λ_1 the first eigenvalue of the Dirichlet Laplacian on Ω and introduce the following constants

$$\mathrm{A} = |\Omega|^{\frac{2-p_1}{2}} \left(\frac{\lambda_1}{2(\lambda_1+1)} \right)^{\frac{p_1}{2}}, \quad \mathrm{B} = \frac{4}{e\sigma} B_\sigma \mathcal{C}_1^{q_1+\sigma}, \quad \mathrm{C} = \frac{4}{e\sigma} B_\sigma \mathcal{C}_2^{q_2+\sigma}. \tag{4.1}$$

In (4.1), $\mathcal{C}_1, \mathcal{C}_2 > 0$, are the best embedding constants of $H_0^1(\Omega)$ into $W_0^{1,q_1+\sigma}(\Omega)$ and $W_0^{1,q_2+\sigma}(\Omega)$, respectively. Moreover, σ is defined as in (2.2) and B_σ is the constant introduced in (2.3) for the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)+\sigma}(\Omega)$. We then have the following non blow-up result for the weak solutions of (1.1).

Theorem 4.1. *Let $u_0 \in W_0^{1,p(.)}(\Omega)$ satisfy*

$$\|u_0\|_{H_0^1(\Omega)} < \min \left(\left(\frac{\mathrm{A}}{2\mathrm{B}} \right)^{\frac{1}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}} \right)^{\frac{1}{q_2-p_1+\sigma}} \right), \tag{4.2}$$

where the constants A, B *and* C *have been defined in* (4.1)*. Then, the corresponding weak solution u of* (1.1) *cannot blow-up in $H_0^1(\Omega)$-norm, and the following decay estimates hold*

$$\|u(t)\|_{H_0^1(\Omega)} \leq \left(\frac{4}{\mathrm{A}(p_1-2)t + 4\|u_0\|_{H_0^1(\Omega)}^{2-p_1}} \right)^{\frac{1}{p_1-2}}, \quad \text{if } p_1 > 2 \tag{4.3a}$$

$$\|u(t)\|_{H_0^1(\Omega)} \leq \left(\frac{A\|u_0\|_{H_0^1(\Omega)}^{q_1-2+\sigma}}{\mathrm{A} - B\|u_0\|_{H_0^1(\Omega)}^{q_1-2+\sigma}} \right)^{\frac{1}{q_1-2+\sigma}} e^{-\frac{A}{2}t}, \quad \text{if } p_1 = 2. \tag{4.3b}$$

Proof. Define the auxiliary function

$$\theta(t) := 2\varphi(t) = \int_\Omega u^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x = \|u\|_{H_0^1(\Omega)}^2. \tag{4.4}$$

We then have from (1.1) and integration by parts that

$$\theta'(t) = 2\int_\Omega u u_t \,\mathrm{d}x + 2\int_\Omega \nabla u \nabla u_t \,\mathrm{d}x = -2\int_\Omega |\nabla u|^{p(.)} \,\mathrm{d}x + 2\int_\Omega |u|^{q(.)} \ln(|u|) \,\mathrm{d}x. \tag{4.5}$$

For any $t > 0$, we split the domain Ω into $\Omega = \Omega_1 \cup \Omega_2$ with

$$\Omega_1 := \{x \in \Omega : |\nabla u(x,t)| < 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |\nabla u(x,t)| \geq 1\}.$$

We then have

$$\int_\Omega |\nabla u|^{p(.)} \,\mathrm{d}x = \int_{\Omega_1} |\nabla u|^{p(.)} \,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{p(.)} \,\mathrm{d}x \geq \int_{\Omega_1} |\nabla u|^{p_2} \,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{p_1} \,\mathrm{d}x \geq \int_{\Omega_2} |\nabla u|^{p_1} \,\mathrm{d}x. \tag{4.6a}$$

$$\int_\Omega |\nabla u|^{\rho+\sigma} \,\mathrm{d}x = \int_{\Omega_1} |\nabla u|^{\rho+\sigma} \,\mathrm{d}x + \int_{\Omega_2} |\nabla u|^{\rho+\sigma} \,\mathrm{d}x \leq 2\int_{\Omega_2} |\nabla u|^{\rho+\sigma} \,\mathrm{d}x. \tag{4.6b}$$

Using (2.3), (3.7), (4.6a) and (4.6b), from (4.5) we can then estimate

$$\begin{aligned}\theta'(t) &\leq -2\int_{\Omega_2} |\nabla u|^{p_1}\ \mathrm{d}x + \frac{2}{e\sigma}\int_{\Omega} |u|^{q(.)+\sigma}\ \mathrm{d}x \qquad (4.7)\\ &\leq -2\int_{\Omega_2} |\nabla u|^{p_1}\ \mathrm{d}x + \frac{2}{e\sigma}B_\sigma\int_{\Omega} |\nabla u|^{q(.)+\sigma}\ \mathrm{d}x\\ &\leq -2\int_{\Omega_2} |\nabla u|^{p_1}\ \mathrm{d}x + \frac{2}{e\sigma}B_\sigma\left(\int_{\Omega_1} |\nabla u|^{q_1+\sigma}\ \mathrm{d}x + \int_{\Omega_2} |\nabla u|^{q_2+\sigma}\ \mathrm{d}x\right)\\ &\leq -2\int_{\Omega_2} |\nabla u|^{p_1}\ \mathrm{d}x + \frac{4}{e\sigma}B_\sigma\left(\int_{\Omega_2} |\nabla u|^{q_1+\sigma}\ \mathrm{d}x + \int_{\Omega_2} |\nabla u|^{q_2+\sigma}\,,\mathrm{d}x\right)\end{aligned}$$

By Hölder's inequality, we have

$$|\Omega|^{\frac{2-p_1}{2}}\left(\int_{\Omega_2} |\nabla u|^2\ \mathrm{d}x\right)^{\frac{p_1}{2}} \leq \int_{\Omega_2} |\nabla u|^{p_1}\ \mathrm{d}x,$$

while Poincaré's inequality gives

$$\int_{\Omega} |\nabla u|^2\ \mathrm{d}x \geq \lambda_1\int_{\Omega} |u|^2\ \mathrm{d}x,$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian on Ω. Thus θ satisfies

$$\begin{aligned}\theta(t) &\leq \left(\frac{1}{\lambda_1}+1\right)\int_{\Omega} |\nabla u|^2\ \mathrm{d}x\\ &= \left(\frac{1}{\lambda_1}+1\right)\int_{\Omega_1} |\nabla u|^2\ \mathrm{d}x + \left(\frac{1}{\lambda_1}+1\right)\int_{\Omega_2} |\nabla u|^2\ \mathrm{d}x \leq 2\frac{\lambda_1+1}{\lambda_1}\int_{\Omega_2} |\nabla u|^2\ \mathrm{d}x.\end{aligned}$$

Moreover, from the Sobolev embedding we have

$$\begin{aligned}\int_{\Omega_2} |\nabla u|^{q_1+\sigma}\ \mathrm{d}x + \int_{\Omega_2} |\nabla u|^{q_2+\sigma}\ \mathrm{d}x &\leq \int_{\Omega} |\nabla u|^{q_1+\sigma}\ \mathrm{d}x + \int_{\Omega} |\nabla u|^{q_2+\sigma}\ \mathrm{d}x\\ &\leq C_1^{q_1+\sigma}\theta(t)^{\frac{q_1+\sigma}{2}} + C_2^{q_2+\sigma}\theta(t)^{\frac{q_2+\sigma}{2}}.\end{aligned}$$

Thus, (4.7) gives

$$\begin{aligned}\theta'(t) &\leq -2\,|\Omega|^{\frac{2-p_1}{2}}\left(\frac{\lambda_1}{2(\lambda_1+1)}\right)^{\frac{p_1}{2}}\theta(t)^{\frac{p_1}{2}} + \frac{4}{e\sigma}B_\sigma\left(C_1^{q_1+\sigma}\theta(t)^{\frac{q_1+\sigma}{2}} + C_2^{q_2+\sigma}\theta(t)^{\frac{q_2+\sigma}{2}}\right)\\ &= -\theta(t)^{\frac{p_1}{2}}\left(\mathrm{A}-\mathrm{B}\theta(t)^{\frac{q_1-p_1+\sigma}{2}}\right) - \theta(t)^{\frac{p_1}{2}}\left(\mathrm{A}-\mathrm{C}\theta(t)^{\frac{q_2-p_1+\sigma}{2}}\right). \qquad (4.8)\end{aligned}$$

We now claim that

$$\theta(t) < \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{2}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{2}{q_2-p_1+\sigma}}\right), \quad \text{for all } t \geq 0. \qquad (4.9)$$

Indeed, if we suppose (4.9) is not satisfied, by the continuity of $\theta(t)$ and (4.2), there exists $t_0 > 0$ such that

$$\theta(t_0) = \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{2}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{2}{q_2-p_1+\sigma}}\right), \qquad (4.10)$$

and

$$\theta(t) < \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{2}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{2}{q_2-p_1+\sigma}}\right) \quad \text{for } 0 \leq t < t_0. \qquad (4.11)$$

Since we are assuming in (4.2) that

$$\theta(0) < \min\left(\left(\frac{\mathrm{A}}{2\mathrm{B}}\right)^{\frac{2}{q_1-p_1+\sigma}}, \left(\frac{\mathrm{A}}{2\mathrm{C}}\right)^{\frac{2}{q_2-p_1+\sigma}}\right),$$

we get from (4.10) that $\theta'(t) > 0$ for $0 < t < t_0$. On the other hand, (4.8) and (4.11) implies that $\theta'(t) < 0$ for $0 < t < t_0$. We then found a contradiction, meaning that (4.9) holds.

In particular, we have that $\theta'(t) < 0$ for all $t \geq 0$. Hence, the $H_0^1(\Omega)$-norm of the solution to (1.1) decays in time and cannot blow-up.

To obtain the decay estimates (4.3a) and (4.3b), without losing generality let us assume that $\theta(t) \neq 0$ for all $t \geq 0$. We are allowed to do that because, if there exists $t_0 \geq 0$ such that $\theta(t_0) = 0$ then, since $\theta(t)$ is non-negative and decreasing for all $t \geq 0$, we would also have

$$\theta(t) = 0 \quad \text{for all } t \geq t_0$$

and (4.3a) and (4.3b) would be trivially satisfied.

If, on the other hand, $\theta(t) \neq 0$ for all $t \geq 0$, then from the differential inequality (4.8) and (4.9) we can write

$$1 \leq -\frac{\theta'(t)}{\theta(t)^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\theta(t)^{\frac{q_1-p_1+\sigma}{2}}\right)}. \tag{4.12}$$

We shall now distinguish two cases.

Case 1: $p_1 > 2$**.** Integrating (4.12) over the interval $(0, t)$, and taking into account that $\theta(t) \leq \theta(0)$ for all $t \geq 0$, we obtain

$$t \leq -\int_{\theta(0)}^{\theta(t)} \frac{d\gamma}{\gamma^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}\right)} = \int_{\theta(t)}^{\theta(0)} \frac{d\gamma}{\gamma^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}\right)}, \tag{4.13}$$

which is equivalent to

$$t \leq \frac{1}{\mathrm{A}}\int_{\theta(t)}^{\theta(0)} \left(\frac{1}{\gamma^{\frac{p_1}{2}}} + \frac{\mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}}{\gamma^{\frac{p_1}{2}}\left(\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}\right)}\right) d\gamma. \tag{4.14}$$

Moreover, since by (4.9) we have

$$\frac{\mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}}{\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-p_1+\sigma}{2}}} \leq 1 \quad \text{for all } \gamma \in [\theta(t), \theta(0)],$$

we obtain from (4.14) that

$$t \leq \frac{2}{\mathrm{A}}\int_{\theta(t)}^{\theta(0)} \frac{d\gamma}{\gamma^{\frac{p_1}{2}}} = \frac{4}{A(2-p_1)}\left(\theta(0)^{\frac{2-p_1}{2}} - \theta(t)^{\frac{2-p_1}{2}}\right) = \frac{4}{A(2-p_1)}\left(\|u_0\|^{2-p_1} - \|u(.,t)\|^{2-p_1}\right).$$

From the above inequality, (4.3a) easily follows by means of simple algebraic manipulations.

Case 2: $p_1 = 2$**.** When $p_1 = 2$, we obtain from (4.13) that

$$t \leq \int_{\theta(t)}^{\theta(0)} \frac{d\gamma}{\gamma\left(\mathrm{A} - \mathrm{B}\gamma^{\frac{q_1-2+\sigma}{2}}\right)}. \tag{4.15}$$

For simplicity of notation, we shall denote

$$\varpi := \frac{q_1 - 2 + \sigma}{2}. \tag{4.16}$$

Then if we introduce the change of variables $\xi = \gamma^{\varpi}$, we can compute

$$\int_{\theta(t)}^{\theta(0)} \frac{d\gamma}{\gamma\left(\mathrm{A} - \mathrm{B}\gamma^{\varpi}\right)} = \frac{1}{\varpi}\int_{\theta(t)^{\varpi}}^{\theta(0)^{\varpi}} \frac{d\xi}{\xi\left(\mathrm{A} - \mathrm{B}\xi\right)} = \frac{1}{\mathrm{A}\varpi}\ln\left(\frac{\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}}\frac{\mathrm{A} - \mathrm{B}\theta(t)^{\varpi}}{\theta(t)^{\varpi}}\right). \tag{4.17}$$

We then get from (4.15) and (4.17) that

$$\frac{\theta(t)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(t)^{\varpi}} \leq \frac{\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} e^{-\mathrm{A}\varpi t}.$$

Since by (4.2) we have

$$\mathrm{A} - \mathrm{B}\theta(t)^{\varpi} > 0,$$

we then obtain that

$$\theta(t)^{\varpi} \leq \left(\frac{\mathrm{A}\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} - \frac{\mathrm{B}\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} \theta(t)^{\varpi} \right) e^{-\mathrm{A}\varpi t} \leq \frac{\mathrm{A}\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} e^{-\mathrm{A}\varpi t}.$$

Hence

$$\theta(t) \leq \left(\frac{\mathrm{A}\theta(0)^{\varpi}}{\mathrm{A} - \mathrm{B}\theta(0)^{\varpi}} \right)^{\frac{1}{\varpi}} e^{-\mathrm{A}t}.$$

Since, by definition, $\theta(t) = \|u\|^2_{H_0^1(\Omega)}$, (4.3b) follows immediately from the above estimate and (4.16). Our proof is then concluded. □

References

[1] R. Aboulaich, D. Meskine and A. Souissi, *New diffusion models in image processing*, Comput. Math. Appl., **56.4** (2008), pp. 874-882.

[2] E. Acerbi and G. Mingione, *Regularity results for electrorheological fluids: the stationary case*, C. R. Math., **334.9** (2002), 817-822.

[3] G. Akagi and M. Ôtani, *Evolution inclusions governed by subdifferentials in reflexive Banach spaces*, J. Evol. Equ., **4.4** (2004), pp. 519-541.

[4] A.B. Al'shin, M.O. Korpusov and A.G. Sveshnikov, *Blow-up in nonlinear Sobolev type equations*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 15, Walter de Gruyter & Co., Berlin, 2011.

[5] P. Amorim and S. Antontsev, *Young measure solutions for the wave equation with $p(x,t)$-Laplacian: existence and blow-up*, Nonlin. Anal. Theo. Methods Appl., **92** (2013), pp. 153-167.

[6] S.N. Antonsev, *Blow up of solutions to parabolic equations with nonstandard growth conditions*, J. Comput. Appl. Math., **234.9** (2010), pp. 2633-2645.

[7] S. Antontsev, *On the existence, uniqueness and stability of solutions for semi-linear generalized elasticity equation with general damping term*, Differ. Equ. Appl., **3.4** (2011), pp. 503-525.

[8] S. Antontsev, *Wave equation with $p(x,t)$-Laplacian and damping term: existence and blow-up*, J. Difference Equ. Appl., **3.4** (2011), pp. 503-525.

[9] S.N. Antontsev and S.I. Shmarev, *Blow-up of solutions to parabolic equations with nonstandard growth conditions*, J. Comput. Appl. Math., **234.9** (2010), pp. 633-2645.

[10] S.N. Antontsev and V. Zhikov, *Higher integrability for parabolic equations of $p(x,t)$-laplacian type*, Adv. Differential Equations, **10.9** (2009), pp. 1053-1080.

[11] J.M. Ball, *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations*, Quart. J. Math., **28.4** (1977), pp. 473-486.

[12] Y. Cao and J. Yin, *Small perturbation of a semilinear pseudo-parabolic equation*, Discrete Contin. Dyn. Syst. Ser. A, **36.2** (2016), pp. 631-642.

[13] H. Chen, P. Luo and G. Liu, *Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. Appl., **422.1** (2015), pp. 84-98.

[14] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., **66.4** (2006), pp. 1383-1406.

[15] H. Chen and S. Tian, *Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity*, J. Differential Equations, **258.12** (2015), pp. 4424-4442.

[16] P. Dai, C. L. Mu and G. Y. Xu, *Blow-up phenomena for a pseudo-parabolic equation with p-Laplacian and Logarithmic nonlinearity terms*, J. Math. Anal. Appl., **481.1** (2020), p. 123439.

[17] H. Di, Y. Shang and X. Zheng, *Global well-posedness for a fourth order pseudo-parabolic equation with memory and source terms*, Discrete Contin. Dyn. Syst. Ser. B, **21.3** (2016), pp. 781-801.

[18] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Springer Lecture Notes, Vol. 2017, Springer-Verlag, Berlin, 2011.

[19] L. Diening, P. Hästo, P. Harjulehto and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Springer-Verlag: Berlin, 2017.

[20] L. Diening and M. Ruzicka, *Calderon-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and problems related to fluid dynamics*, J. Reine Angew. Math., **563** (2003), pp. 197-220.

[21] H, Ding and J, Zhou, *Global existence and blow-up for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity*, J. Math.Anal.Appl., **478.2** (2019), pp. 393-420.
[22] E.S. Dzektser, *A generalization of equations of motion of underground water with free surface*, Dokl. Akad. Nauk SSSR, **202.5** (1972), pp. 1031-1033.
[23] X.L. Fan, J.S. Shen and D. Zhao, *Sobolev embedding theorems for spaces* $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl., **262.2** (2001), pp. 749-760.
[24] Y. Fu, *The existence of solutions for elliptic systems with nonuniform growth*, Studia Math., **151** (2002), pp. 227-246.
[25] V.R. Gopala Rao and T.W. Ting, *Solutions of pseudo-heat equations in the whole space*, Arch. Ration. Mech. Anal., **49** (1972), pp. 57-78.
[26] T.C. Halsey, *Electrorheological fluids*, Science, **258** (1992), pp. 761-766.
[27] Y. J. He, H. H. Gao and H. Wang, *Blow-up and decay for a class of pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity*, Comput. Math. Appl., **75.2** (2018), pp. 459-469.
[28] W. Hua and H. Yijun, *On blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy*, Appl. Math. Lett., **26.5** (2013), pp. 1008-1012.
[29] S. Ji, J. Yin and Y. Cao, *Instability of positive periodic solutions for semilinear pseudo-parabolic equations with logarithmic nonlinearity*, J. Differential Equations, **261.10** (2016), pp. 5446-5464.
[30] M. Kbiri Alaoui, T. Nabil and M. Altanji, *On some new nonlinear diffusion model for the image filtering*, Appl. Anal., **93.2** (2014), pp. 269-280.
[31] S. Khomrutai, *Global well-posedness and grow-up rate of solutions for a sublinear pseudoparabolic equation*, J. Differential Equations, **260.4** (2016), pp. 3598-3657.
[32] M.O. Korpusov and A.G. Sveshnikov, *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics*, Zh. Vychisl. Mat. Mat. Fiz., **43.12** (2003), pp. 1835-1869.
[33] M.O. Korpusov and A.G. Sveshnikov, *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics II*, Zh. Vychisl. Mat. Mat. Fiz., **44.11** (2004), pp. 2041-2048.
[34] C. N. Le and X. T. Le, *Global solution and blow-up for a class of p-Laplacian evolution equations with logarithmic nonlinearity*, Acta Appl. Math., **151.1** (2017), pp. 149-169.
[35] S. Lian, W. Gao, C. Cao and H. Yuan, *Study of the solutions to a model porous medium equation with variable exponent of nonlinearity*, J. Math. Anal. Appl., **342.1**, (2008), pp. 27-38.
[36] Y. Liu, *Lower bounds for the blow-up time in a non-local reaction diffusion problem under nonlinear boundary conditions*, Math. Comput. Modelling, **57.3-4** (2013), pp. 926-931.
[37] L.C. Nhan and L.X. Truong, *Global solution and blow-up for a class of pseudo p-Laplacian evolution equations with logarithmic nonlinearity*, Comput. Math. Appl., **73.9** (2017), pp. 2076-2091.
[38] L.E. Payne, G.A. Philippin and S.V. Piro, *Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition I*, Z. Angew. Math. Phys., **761.6** (2010), pp. 999-1007.
[39] L.E. Payne, G.A. Philippin and S.V. Piro, *Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition II*, Nonlinear Anal., **73.4** (2010), pp. 971-978.
[40] L.E. Payne and P.W. Schaefer, *Lower bounds for blow-up time in parabolic problems under Neumann conditions*, Appl. Anal., **85.10** (2006), pp. 1301-1311.
[41] L.E. Payne and P.W. Schaefer, *Lower bounds for blow-up time in parabolic problems under Dirichlet conditions*, J. Math. Anal. Appl., **328.2** (2007), pp. 1196-1205.
[42] X. Peng, Y. Shang and X. Zheng, *Blow-up phenomena for some nonlinear pseudo-parabolic equations*, Appl. Math. Lett., **56** (2016), pp. 17-22.
[43] A. Rahmoune, *Semilinear hyperbolic boundary value problem associated to the nonlinear generalized viscoelastic equations*, Acta Math. Vietnam., **43.2** (2018), pp. 219-238.
[44] A. Rahmoune, *Existence and asymptotic stability for the semilinear wave equation with variable-exponent nonlinearities*, J. Math. Phys. **60.12** (2019), p. 122701.
[45] M. Ruzicka, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, Vol. 1748, Springer, 2000.
[46] R.E. Showalter and T.W. Ting, *Pseudoparabolic partial differential equations*, SIAM J. Math. Anal., **1.1** (1970), pp. 1-26.

[47] J.C. Song, *Lower bounds for the blow-up time in a non-local reaction-diffusion problem*, Appl. Math. Lett., **24.5** (2011), pp. 793-796.

[48] L. Sun, Y. Ren and W. Gao, *Lower and upper bounds for the blow-up time for nonlinear wave equation with variable sources*, Comput. Math. Appl., **71.1** (2016), pp. 267-277.

[49] T.W. Ting, *Parabolic and pseudo-parabolic partial differential equations*, J. Math. Soc. Japan, **21.3** (1969), pp. 440-453.

[50] C. Yang and C. Liu., *Initial boundary value problem for a mixed pseudo parabolic $p(.)$-Laplacian type equation with logarithmic nonlinearity*, Electron. J. Differ. Equ., **116.2018** (2018), pp. 1-19.

[51] X. Zhu, F. Li, Z. Liang and T. Rong, *A sufficient condition for blowup of solutions to a class of pseudo-parabolic equations with a nonlocal term*, Math. Methods Appl. Sci., **39.13** (2016), pp. 3591-3606.

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