CONTROL OF REACTION-DIFFUSION MODELS FROM BIOLOGY AND SOCIAL SCIENCES

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ABSTRACT. These lecture notes address the controllability under state constraints of reaction-diffusion equations arising in socio-biological contexts. We restrict our study to scalar equations with monostable and bistable nonlinearities.

The uncontrolled models, describing, for instance, population dynamics, concentrations of chemicals, temperatures, etc., intrinsically preserve pointwise bounds of the states that represent a proportion, volume-fraction, or density. This is guaranteed, in the absence of control, by the maximum or comparison principle.

Nevertheless, the presence of constraints introduces significant added complexity and produces barrier phenomena that the controls might not be able to overcome.

We focus on the classical controllability problem, in which one aims to drive the system to a final target, for instance, a steady-state, so that the state preserves the pointwise bounds of the uncontrolled dynamics. These constraints may force the needed control-time to be large enough or even make some natural targets to be unreachable.

We develop and present a general strategy to analyze these problems. We show how the combination of the various intrinsic qualitative properties of the systems' dynamics and, in particular, the use traveling waves and steady-states' paths can be employed to build controls driving the system to the desired target.

We also show how, depending on the Allee parameter and onthe size of the domain in which the process evolves, some natural targets might become unreachable. This is consistent with empirical observations in the context of minorized endangered languages and species in extinction.

Further recent extensions are presented, and open problems are settled. All the discussions are complemented with numerical simulations to illustrate the main methods and results.

Keywords Reaction-diffusion, control, steady-states, constraints, phase plane, traveling waves, comparison principle, Mathematical Biology.

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1. Introduction

1.1. **Motivation.** These lecture notes concern the control of reaction-diffusion equations with state constraints. Many quantities whose evolution is modeled by reaction-diffusion equations (such as those arising chemical reactions or biological populations) are positive, simply because they describe some density, concentration, distribution functions or volume fractions, etc. Then the maximum or comparison principle for parabolic equations plays an important role since it guarantees that the modeled dynamics fulfill these unilateral bounds.

In the context of control theory, the general goal is to find ways to interact with a given system to achieve a specific purpose. One of the prototypical problems is the controllability one: by the choice of an appropriate control, one aims to drive the dynamics from an initial condition to a target configuration in a given time horizon. Other objectives can also be pursued, such as feedback stabilization (see [18, Part 3]). The problem of controllability becomes challenging in the presence of constraints, something that might be unavoidable when dealing with the applications above where the state is intrinsically and naturally constrained to fulfill some given bounds.

In this work, we present the main challenges arising when dealing with controllability for reaction-diffusion equations with state constraints, some of the main results, and the techniques needed to handle them.

There is by now a fertile literature on the controllability of parabolic equations: [27, 30, 31, 35, 36, 41, 64] (and references therein). Roughly, using Fourier series techniques and Carlemnan inequalities, parabolic equations and systems can be controlled in an arbitrarily small time. However, most of the existing results do not guarantee that state constraints requirements such as the positivity of solutions are met. The study of controllability with state constraints for parabolic partial differential equations is a much recent research topic [69–71, 82, 91, 96, 104].

The presence of constraints requires the development of new methods for controllability. The staircase method or quasi-static control [20] which consists of controlling the system keeping the trajectory in a neighborhood of a path of steady-states connecting the initial and the final datum, is, by now, the most useful tool to achieve controllability under constraints. But this method requires the time-horizon to be long enough, something that, as we shall see, is in fact necessary to meet the constraints. In fact, in the presence of state constraints, the controllability requires a minimal controllability or waiting time, a fact that is not observed in the unconstrained setting.

Another relevant difficulty arising in the context of constraint controllability is the appearance barrier functions limiting the constrained dynamics, independently of the controls chosen. A barrier is a nontrivial steady-state that prevents any controlled trajectory from reaching specific targets due to the comparison principle.

The staircase method cannot overcome these barriers. We will explain how to build paths of steady-states, preserving the constraints, and therefore limited by the barrier functions, allowing to reach the full ensemble of reachable steady-states. Our study relies on the phase-plane analysis of the dynamical system associated with the elliptic equation that steady-states satisfy, initiated in [96].

In these lecture notes, we mainly focus on the one-dimensional case. However, the methods can also be applied to spatial dimensions and different nonlinearities. We will also briefly present the main multi-dimensional results in [104], those in [82] involving spatially heterogeneous drifts, and properties of the Allee interaction of [112] combined with a boundary control.

1.2. **Organization of the lecture notes.** The lecture notes are organized as follows.

(1) First of all, we complement this introduction by discussing different applications in which the control problems addressed in the present manuscript arise and by making a short description of various types of control problems.
(2) In Section 2, we present a model in which the control problem requires the fulfillment of state constraints.
(3) In Section 3, we review some of the most important properties of parabolic and elliptic equations that will be employed along with the paper.
(4) In Section 4, firstly, we discuss the well-posedness of the control problem and the controllability of parabolic equations. We also present the staircase method following [91].
(5) Section 5 is devoted to analyzing the existence of nontrivial solutions of the elliptic problem (1.2). Furthermore, depending on the measure of the domain, we discuss the possible existence of barrier functions.
(6) Section 6 gathers graphical illustrations of the energy functional associated with the elliptic problem (1.2).
(7) Section 7 is devoted to the construction of admissible paths of steady-states fulfilling the constraints. We shall mainly focus on the problem of driving the system towards the intermediate constant steady-state θ for the bistable nonlinearity. First, we introduce the strategy used in [96], and later, we extend the reasoning to build paths of symmetric (even with respect to the center point of the space-interval) steady-states. We discuss the length of the maximal path depending on the size of the domain.
(8) In Section 8 we summarize the results of the previous sections.
(9) In Section 9 some numerical simulations of the control process under consideration are presented.
(10) Most of the results presented in the paper can be extended to several space dimensions, [104], and to models involving spatially heterogeneous drift terms, [82]. These extensions are presented in Section 10, where we also describe how the boundary control and the Allee-Control introduced in [112] can be combined.
(11) In the last Section 11, we present some open problems and perspectives for future research.

1.3. **A model problem.** Let us first introduce the problem of the boundary control of some of the most common $1-d$ reaction-diffusion equations. This problem entails the core phenomenology arising in the context of constrained controllability.

Let $L > 0$ and consider the following problem:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v(0,t) = a_1(t) \quad v(L,t) = a_2(t) & t \in (0,T), \\ 0 \le v(x,0) \le 1, & x \in (0,L), \end{cases} \tag{1.1}$$

where $f \in C^2$ satisfying $f(0) = f(1) = 0$. We will mainly consider two types of nonlinearities f, (see Figure 1.3), namely:

- **Monostable**: if $f'(0) > 0$ and $f'(1) < 0$ with $f(s) > 0$ for $s \in (0,1)$ (see Figure 1.1 right). A prototypical example of monostable nonlinearity is $f(v) = v(1-v)$.
- **Bistable**: if there exists $\theta \in (0,1)$ such that $f(\theta) = 0$, $f'(\theta) > 0$ and $f'(0) < 0$ $f'(1) < 0$ with $f(s) < 0$ in $(0,\theta)$ and $f(s) > 0$ in $(\theta, 0)$ (see Figure 1.1 left). A typical example of bistable nonlinearity is $f(v) = v(1-v)(v-\theta)$.

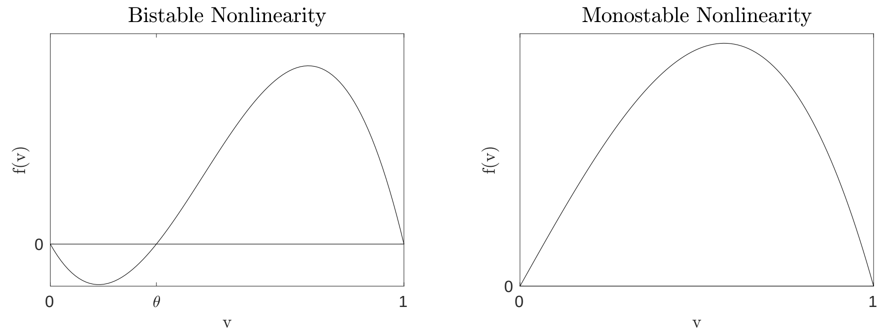

FIGURE 1.1. A bistable nonlinearity (left) and a monostable one (right).

Note that $v \equiv 1$ is a constant steady-state with $a_1 = a_2 = 1$, $v \equiv 0$ with $a_1 = a_2 = 0$ and $v \equiv \theta$ in the case of the bistable nonlinearities with $a_1 = a_2 = \theta$.

The state v typically represents a proportion or a population density. For this reason, we impose natural constraints on state of the system $v = v(x,t)$, namely $0 \leq v(x,t) \leq 1$. The boundary controls $a_j(t)$, $j = 1, 2$, are therefore limited by the same constraints, and, as we shall see, this limits significantly the possibility of controlling the system.

Most of these notes will be devoted to discussing whether the system can be driven, by the action of suitable controls, to reach these targets in a way that the constraints $0 \leq v \leq 1$ are preserved.

The classical methodology for controlling semilinear equations (e.g., [36]), based on using the linear controllability properties with careful estimates on the cost of control and fixed point arguments, do not apply since, typically, the controls obtained in this way will violate the constraints. Therefore, ad-hoc techniques, taking into account and exploiting the nonlinear dynamics of the system, will be developed to analyze the controllability properties under these constraints. One of the difficulties that we shall encounter is that barrier functions may arise as nontrivial solutions to the steady-state equation:

$$\begin{cases} -\partial_{xx} v = f(v) & x \in (0, L), \\ v(0) = 0, \quad v(L) = 0 \end{cases} \tag{1.2}$$

with null boundary conditions that correspond to zero boundary controls. Obviously, the trivial constant solution mentioned above, $v \equiv 0$, is a steady-state solution of the system. The existence of nontrivial solutions to (1.2), denoted by v_b, fulfilling the constraints $0 \leq v_b \leq 1$ depends, basically, on the length of L. Due to the comparison principle, when such a nontrivial barrier v_b exists, the final target $v = 0$ will not be reachable when the dynamics departs from an initial configuration above v_b, since, whatever the controls are, being nonnegative, the solution will always remain above this barrier. This is is an important warning since it indicates that the control results we might expect will vary depending on L.

This fact has a clear interpretation in applications. For instance, it is well-known that the survival of species depends on having sufficient area requirements [102] (see also [8]). In other words, if the area in which the population lives and evolves is too small, the population will very likely tend to extinction, while in larger domains, survival will be possible. This is due to the fact that, even if individuals reaching the boundary will die, the reproduction of the population inside the domain will suffice to compensate for their lost individuals, assuring the overall survival.

This can be understood in mathematical terms by the stability and attraction properties of the trivial steady-state $v \equiv 0$ that will attract, or not, the whole ensemble of initial data within the bounds $0 \leq v \leq 1$, depending on the length of L: when L is small enough, all initial data will be attracted towards $v \equiv 0$, while for L large, because of the barrier effects mentioned above, some initial data will evolve always remaining above v_b.

Of course, the stability properties of the system for long times and the nature of the set of steady-state solutions are intimately related. For instance, from Matano's Theorem [76] (see also [120]), we know that bounded solutions of one-dimensional reaction-diffusion equations converge to steady-states. This classical result, combined with the instability of 0 for large domains, is a way to understand the existence of a nontrivial solution to (1.2), a fact that can be directly explained in the elliptic context using the theory of critical points from Calculus of Variations.

We will mostly study target steady-states that are constant, 0, 1 for both models, and, in addition, θ for the bistable model. However, the main properties of such targets, for the bistable and monostable nonlinearities considered, will be

- targets steady-states that are in the border of the admissible set such as 0 or 1,
- targets steady-states in the interior which are unstable such as θ for the bistable case in certain situations,
- stable targets steady-states that are inaccessible for certain initial conditions of (1.1). We will see this situation, in particular, with the target 0 for the bistable model.

1.4. **Bibliographical notes on applications.** Reaction-diffusion systems frequently appear in natural sciences and applications. In this subsection, we describe some of the contexts where one can find them. Control problems can adopt different forms and formulations, depending on the application context.

- **Population dynamics and spatial ecology**. Kolmogorov [60] proposed a model in ecology in which a population diffuses in space and grows in a nonlinear manner, finding traveling wave solutions. A traveling wave is a solution of the form $s(x-ct)$ with a given profile $s(x)$ and a travel velocity c that may arise scalar reaction-diffusion equations as those considered here in the whole space $\mathbb{R}$. This pioneering work gave rise to a vast literature on reaction-diffusion, and spatial ecology [38]. For more general diffusion models in ecology, we refer to [5,6,17,40,83,86,90]. We also refer to [11] for a an empirical finding of traveling waves in ecology.
- **Chemical reactions** [90, 107]. In most cases, models in this context are constituted by systems and not scalar equations as those analyzed here, enjoying much richer dynamics. Turing patterns [115] is one of the paradigmatic phenomena, which emerges when there is a big contrast in the diffusivity constants of the various equations constituting the system. Alan Turing's model was originally proposed for morphogenesis, and it has been proven experimentally to be successful [87].
- **Magnetic systems** in material science [23]: Consider a material composed of magnetic dipole moments of atomic spins with two possible states. Assume that there is an infinite number of them placed in a lattice. A natural model in the context is the Ising model. The Ising model serves the study of a system in equilibrium. The physicist Roy J. Glauber [45] proposed a dynamical model to explain the shifting of spins from up to down and vice-versa, depending on the positioning of neighboring spins. Glauber's dynamics models a reaction between the spins and, considering a spin interchange between neighboring spins. In [23], it is formally derived a reaction-diffusion system for the evolution of the magnetization of the material from the stochastic processes of the spins.
- **Evolutionary game theory**: In this field, one seeks to understand how players change their strategies depending on the strategy of the other players [22, 54, 56, 89, 116]. When considering the possible spatial diffusion of the players, reaction-diffusion equations arise naturally, [53, 55].
- **Neuroscience:** traveling wave phenomena also arise when modeling nerve impulses [28].
- **Linguistics**: parabolic models may also be employed to analyze language shift, [100].

1.5. **Introduction to control problems.** Control problems arising in these fields can be presented in a diversity of forms and allow for various mathematical formulations. Here we briefly present some of them.

- **Interior Control**: This type of control action finds applications, for example, in parabolic equations arising in heat processes. Consider a bar of length L and a subinterval $\omega \subset (0, L)$ where a heater/cooler is placed, the evolution of the temperature of the bar follows:

$$\begin{cases} \partial_t v - \partial_{xx} v = \chi_\omega a, \\ v(0,t) = v(L,t) = 0, \\ v(\cdot, t=0) = v_0. \end{cases}$$

  here v_0 stands for the initial temperature distribution, while the temperature at the boundary is fixed. The heating control is modeled by $a = a(x,t)$ which acts locally ω but aiming with a global effect.

With the model above being fixed, several types of control problems can be considered. The controllability towards a steady-state (in particular, when $v \equiv 0$, the so-called null-control problem) has been extensively studied in the literature: [31, 32]. But the case where state constraints are imposed has been much less studied and only more recently: [71, 74].

- *Multiplicative and Bilinear Control*: In this case the control does not enter on the system as a linear right hand side term but rather as a potential $a(x,t)$ multiplying the state itself:

$$\begin{cases} \partial_t v - \partial_{xx} v = av, \\ v(0,t) = v(L,t) = 0, \\ v(\cdot, t=0) = v_0. \end{cases}$$

Of course, in this case, the impact of the control $a(x,t)$ on the system is much weaker. In particular, if the initial datum $v_0 \equiv 0$ is the trivial one, then the solution will also be $v \equiv 0$ for all time. Therefore at the final time, $t = T$ only the final target $v(\cdot, T) \equiv 0$ will be reached regardless of the control. We refer to [15, 25, 58, 59] and and to [24, 50, 73, 77–81, 84] for the analysis of bilinear control on population dynamics systems.

- *Allee Control*: In [112], adopting a micro-macro modeling perspective (as in [23, 26]), the role of the Allee threshold parameter θ of the bistable nonlinearity as an effective control of the system is justified. In practical applications, for instance, on the regulation of the propagation of invasive mosquito species, this Allee threshold can be regulated by releasing sterile mosquitoes. This leads to a model of the form

$$\begin{cases} \partial_t v - \partial_{xx} v = v(v - \theta(t))(1 - v) \quad x \in \mathbb{R}, \\ v(\cdot, t=0) = v_0 \in L^\infty(\Omega; [0,1]), \end{cases}$$

where $\theta = \theta(t)$ plays the role of control.

- *Boundary Control*: This action is another prototypical way of possible interaction in control systems. Boundary control mechanisms are closely related to interior controls acting on a neighborhood of the boundary. This can be rigorously justified through the classical extension-restriction argument, [108]. The corresponding control system then takes the form

$$\begin{cases} \partial_t v - \partial_{xx} v = 0, \\ v(0,t) = a_1(t), \quad v(L,t) = a_2(t), \\ v(\cdot, t=0) = v_0. \end{cases}$$

where $a_j = a_j(T)$, $j = 1, 2$ play the role of the controls.

Most of these lecture notes will be devoted to considering boundary control problems..

In all these contexts, control problems can be formulated differently. one may distinguish, in particular, the following goals and issues.

(a) Controllability problems, [18, 114, 124]: given an initial datum and a specific target, to find a control function that drives the system to the target in a given time-horizon.
(b) Stabilization, [18, Part III]: Given an unstable equilibrium configuration, can we find a control function in a feedback form (depending on the state) that stabilizes the systems towards this equilibrium?
(c) Optimal control. This problem can be formulated as a minimization one, for instance as a least-squares type of problem. One seeks to minimize a cost functional depending on the state and the control, typically, to lead the system towards a neighborhood of a target or a reference trajectory. Optimal control problems for parabolic problems have been widely considered in particular in [14, 65, 105, 113].

2. Parabolic models

In this section, we derive the model that will be considered and explain the relevance of constraints on the associated control problems.

2.1. **ODE versus PDE modelling.** The first Ordinary Differential Equation (ODE) considered in population dynamics exhibits an exponential growth of the population $P = P(t)$:

$$\frac{P'}{P} = \beta,$$

Verhulst [118] noticed that the competition for limited resources among individuals of the same population provides a more accurate model and leads to an upper threshold of the population growth:

$$P' = \beta P \left(1 - \frac{P}{\kappa}\right),$$

where κ is the capacity of the environment.

ODE can be adapted to the Partial Differential Equation (PDE) setting to model the movement and invasion of species. Assuming that the diffusion is homogeneous and that the resources in space are space invariant, the equation can be formulated as follows:

$$\begin{cases} \partial_t v - \partial_{xx} v = \beta v(1-v) & (x,t) \in (0,L) \times (0,T), \\ \partial_x v = 0 & (x,t) \in \{0,L\} \times (0,T), \\ v(\cdot, t=0) = v_0 \in L^\infty(\Omega; \mathbb{R}^+), \end{cases}$$

for $\beta > 0$. Here, null Neumann type boundary conditions are adopted to represent the fact that no population flow through the boundary is allowed.

The constraint on the state $v \leq 1$ is relevant when the quantity under consideration is a proportion. These leads to systems of the form

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ \partial_x v = 0 & (x,t) \in \{0,L\} \times (0,T), \\ v(\cdot, t=0) = v_0 \in L^\infty(\Omega; [0,1]), \end{cases}$$

where typically $f(1) = 0$.

2.2. **The bistable model.** Let $\Omega = (0,L) \subset \mathbb{R}$ and assume that two populations represented by non-negative functions V and W, interact in a nonlinear fashion while preserving the quantity $V + W$, and diffusing in space. We will discuss later possible contexts in which such a situation might occur. The corresponding model reads:

$$\begin{cases} \partial_t V - \partial_{xx} V = F(V,W) & (x,t) \in (0,L) \times (0,T), \\ \partial_t W - \partial_{xx} W = -F(V,W) & (x,t) \in (0,L) \times (0,T), \\ \partial_x V = \partial_x W = 0 & (x,t) \in \{0,L\} \times (0,T), \\ V(\cdot, t=0) = V_0 \in L^\infty((0,L); \mathbb{R}^+), \\ W(\cdot, t=0) = W_0 \in L^\infty((0,L); \mathbb{R}^+), \end{cases} \tag{2.1}$$

where F is a Lipschitz continuous function satisfying $F(V,0) = F(0,W) = 0$ for every $V, W \in \mathbb{R}$.

We first look for an approximation of (2.1) utilizing a single equation. Note that

$$P := V + W,$$

satisfies:

$$\begin{cases} \partial_t P - \partial_{xx} P = 0 & (x,t) \in (0,L) \times (0,T), \\ \partial_x P = 0 & (x,t) \in \{0,L\} \times (0,T), \\ P(\cdot, t=0) = V_0 + W_0, \end{cases} \tag{2.2}$$

whose long time asymptotic is given by the constant

$$\lim_{t \to \infty} P(t,x) = \frac{1}{L} \int_0^L [V(x,0) + W(x,0)] dx.$$

It is therefore natural to assume that P is actually constant.

Define $v := V/P$ as the proportion of the V type population in the whole population P. Then (2.1) reduces to:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ \partial_x v = 0 & (x,t) \in \{0,L\} \times (0,T), \\ v(\cdot, t=0) = v_0 \in L^\infty((0,L); [0,1]). \end{cases}$$

for a suitable f that can be easily derived out of F.

In this setting, the bilateral constraints $0 \leq v(x,t) \leq 1$ arise naturally for all (x,t). The zeros of the original nonlinearity F guarantee that $f(0) = f(1) = 0$. Then, by the comparison principle, when the initial datum v_0 satisfies these bilateral bounds, they are guaranteed to hold for all (x,t) (see next section, Section 3).

It is said that $f \in C^1$ is bistable if:

$$f(0) = f(\theta) = f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0, \quad f'(\theta) > 0,$$
$$f(s) < 0 \text{ for } s \in (0,\theta), \quad f(s) > 0 \text{ for } s \in (\theta, 1).$$

As an example, observe that if

$$F(V,W) = \frac{VW}{(V+W)^2}\left((1-\theta)V + \theta W\right)$$

for $\theta \in (0,1)$, then it corresponds to:

$$f(v) = v(1-v)(v-\theta)$$

which has such a bistable structure. This model has special interest, in particular, in game theory as we indicate below:

- *Game theoretical interpretation.* This nonlinearity arises naturally in the context of evolutionary game theory in coordinated games.

  The choice of languages in a social context is a prototypical example. Indeed, assume two bilingual individuals meet each other, and they establish a conversation in one of the two languages they manage. However, they may have different preferences for languages, and a payoff matrix models this.

  The replicator dynamics is a nonlinear ODE that models the change of strategy in time of players. Players continuously play the game, and they evaluate their success while comparing it with the other players' average success. The replicator dynamics for a two strategy game is represented by:

$$\frac{d}{dt}v = v\left((1,0)A\begin{pmatrix} v \\ 1-v\end{pmatrix} - (v, 1-v)A\begin{pmatrix} v \\ 1-v\end{pmatrix}\right)$$

  where $A \in M_2(\mathbb{R})$ is a payoff matrix.

  In this model, v is the proportion of players playing the first strategy, and $1-v$ that of those playing the remaining strategy. The first term on the right-hand side evaluates the success of the first strategy, while the second is the players' average success. If the first strategy has less success than the average, the proportion of players playing the first strategy will diminish in favor of the second strategy. The payoff matrix of a coordination two strategy game is

$$A = \begin{pmatrix} (1-\theta) & 0 \\ 0 & \theta \end{pmatrix},$$

  that leads to the nonlinearity $f(v) = v(1-v)(v-\theta)$. In this game, we see that if $\theta < 1/2$, players prefer to play the first strategy rather than the second. However, the preference does not uniquely determine the behavior of the system. One can see for instance that both consensus configuration $v = 0$ and $v = 1$ are stable. Indeed, $f'(0) < 0$ and $f'(1) < 0$, and this assures that the less wanted strategy is stable if enough players are currently playing it.

  For further details and other possible models we cite [22, 53, 54, 54, 55, 89].
- *Biological interpretation.*

  In a biological context, according to the so-called Allee effect (see [110]), when the population in a given habitat is lower than a given threshold θ, they cannot survive.

  A game-theoretical approach can also be adopted in this context. Assume the population presents two distinct genes. More successful genes will be transferred to the next generations, while the less successful ones will disappear. In this situation, the replicator dynamics apply.

  In ecology, spatial heterogeneities are omnipresent. Note that if A depends on x like

$$A(x) = \begin{pmatrix} (1-\theta(x)) & 0 \\ 0 & \theta(x) \end{pmatrix},$$

  $\theta(x)$ is determining which strategy is having an advantage depending on the spatial location. So that if $0 < s < \theta(x)$ then $f < 0$, but $f > 0$ if $\theta(x) < s < 1$. This leads to a model in which the nonlinearity depends on x. In ecological systems, one may also consider situations in which one gene is favorable in dry environments, and the other is favorable in humid habitats, for instance. This case will be briefly discussed later on.

One may also consider different diffusivities for each of the genes. Where the reduction to a scalar case might not be possible.

- *Chemical reactions.* A chemical reaction in which the number of reactants, constituted by particles of different classes, can also be represented by similar systems, [90, Chapter 1].

2.3. **Boundary control.** In these lecture notes we will mainly consider the situation in which the control acts on the boundary. The control can enter through the Neumann boundary condition, for instance, regulating the flux:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ \dfrac{\partial v}{\partial \nu} = b(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ 0 \le v(x,0) \le 1 & x \in (0,L). \end{cases}$$

But, obviously, a trajectory with Neumann control can also be understood as being submitted to a Dirichlet one. Therefore, equivalently, we may consider the same problems with Dirichlet controls that are somehow easier to handle.

Indeed, given the Neumann control $b(x,t)$, and the corresponding solution $v = v(x,t)$, its Dirichlet trace $a(x,t)$ can be understood as a Dirichlet control:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ 0 \le v(x,0) \le 1 & x \in (0,L). \end{cases} \tag{2.3}$$

Of course the constraints on v are automatically transferred to the Dirichlet controls

$$0 \le a(x,t) \le 1 \quad \text{for any } (x,t) \in \{0,L\} \times [0,T].$$

On the other hand, as we shall see, by the comparison principle, the constraints on the controls suffice for them to hold for the solution as well.

Thus, in the context of Dirichlet controls, imposing bilateral bounds is equivalent to imposing them on the controls.

3. Review of some results on parabolic equations

In this section, we gather some classical results on semilinear parabolic equations that are useful for control purposes. We will mainly expose results concerning

(1) convergence to steady-states,
(2) comparison results,
(3) traveling waves.

Nowadays, the primary tool to control reaction-diffusion equations with constraints to unstable targets is the staircase method, which uses paths of steady-states. For this reason, understanding whether or not there is a natural convergence to steady-states will be of practical use. We also mentioned that the existence of nontrivial steady-states would be a fundamental obstruction due to the comparison principle. These principles are also gathered in this section, and we will also use them to guarantee convergence to certain steady-states. In particular, we will employ the comparison principle between the solution of our reaction-diffusion equation with a section of a traveling wave. As mentioned earlier, traveling waves are special solutions to the Cauchy problem in the whole real line. The existence of traveling waves and their stability has been a classic topic in reaction-diffusion. Due to the impact of traveling waves on our systems, we will expose the main theorems below.

3.1. **Convergence to steady-states.** Consider the following one-dimensional semilinear heat equation.

$$\begin{cases} \partial_t v = \partial_x \left(a(x) \partial_x v \right) + f(x,v) & (x,t) \in (0,L) \times (0,T) \\ v(x,0) = v_0(x) & x \in (0,L), \\ v(0,t) = \beta_0, \quad v(L,t) = \beta_L & t \in (0,T), \end{cases} \tag{3.1}$$

β_0 and β_L being constants independent of t.

Let $t_e(v_0)$ be the maximum time of existence of the solution of (3.1). For general nonlinearities, solutions of (3.1) may blow-up in finite time.

Let $v_0 \in C([0,L];\mathbb{R})$ and suppose that the solution is global, so that the maximum time of definition of the solution is $+\infty$: $t_e(v_0) = \infty$. The ω-limit set of the solutions can be defined as the set of accumulation points of the trajectory $v(\cdot,t)$ in $C^1([0,L])$ as $t \to \infty$. It is by now well-known that any bounded trajectory converges to a steady-state. The result actually holds for more general boundary conditions [75] (see also [120] for an earlier reference).

Theorem 3.1 (Matano, Theorems A and B from [76])**.** *The ω-limit set of any function $v_0 \in C([0,L];\mathbb{R})$ contains at most one element.*

For any initial data $v_0 \in C([0,L];\mathbb{R})$, one and only one of the following three properties holds:

1. *The solution blows up in finite time $t_e(v_0) < \infty$ and $\lim_{t\to t_e(v_0)} \|v(t)\|_{L^\infty([0,L];\mathbb{R})} = \infty$.*
2. *The solution grows up as $t \to \infty$, $t_e(v_0) = \infty$ and $\lim_{t\to t_e(v_0)} \|v(t)\|_{L^\infty([0,L];\mathbb{R})} = \infty$.*
3. *The solution converges to a solution of the elliptic problem.*

$$\begin{cases} 0 = \partial_x \left(a(x)\partial_x v\right) + f(x,v) & x \in (0,L), \\ v(0) = \beta_0, \quad v(L) = \beta_L, \end{cases}$$

in the $C^1([0,L];\mathbb{R}) \cup C^2((0,L);\mathbb{R})$ topology.

In higher dimensions, the theorem above is no longer true. In [93, 94] counterexamples of the theorem above in several dimensions were constructed. However, if the nonlinearity is analytic, one can ensure the convergence to steady-states thanks to the Łojasiewicz gradient inequality [57, 109]. For further reading, see also [48, 68].

3.2. **Comparison results.** The comparison principle enables us to gain further understanding on the dynamics of the trajectories of the PDE under consideration [13, 39, 101] [38, Chapter 5]. We present simultaneously parabolic and elliptic comparison principles. We state them in one dimension; although they also hold in several dimensions.

Definition 3.2 (Parabolic sub- and supersolutions)**.** Consider the elliptic operator:

$$\mathcal{L} := \partial_{xx} + k(x)\partial_x$$

where $k : (0,L) \to \mathbb{R}$ is a smooth function. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth functions and $h : \{0,L\} \times (0,T) \to \mathbb{R}$. Consider the parabolic problem:

$$\begin{cases} \partial_t v - \mathcal{L}v = f(v) & (x,t) \in (0,L) \times (0,T) \\ v(t,x) = h(x,t) & (x,t) \in \{0,L\} \times (0,T) \\ v(0,x) = v_0(x) & x \in (0,L) \end{cases} \tag{3.2}$$

A subsolution $\underline{v}$ of (3.2) satisfies:

$$\begin{cases} \partial_t \underline{v} - \mathcal{L}\underline{v} \le f(\underline{v}) & (x,t) \in (0,L) \times (0,T) \\ \underline{v}(t,x) \le h(t,x) & (x,t) \in \{0,L\} \times (0,T) \\ \underline{v}(0,x) \le v_0(x) & x \in (0,L) \end{cases}$$

A supersolution $\overline{v}$ of (3.2) satisfies:

$$\begin{cases} \partial_t \overline{v} - \mathcal{L}\overline{v} \ge f(\overline{v}) & (x,t) \in (0,L) \times (0,T) \\ \overline{v}(t,x) \ge h(t,x) & (x,t) \in \{0,L\} \times (0,T) \\ \overline{v}(0,x) \ge v_0(x) & x \in (0,L) \end{cases}$$

Theorem 3.3 (Parabolic comparison principle [13])**.** *If $\underline{v}$ is a subsolution (respectively $\overline{v}$ a supersolution) to* (3.2) *and v is a solution such that $v \ge \underline{v}$ (respectively $v \le \overline{v}$) on $(x,t) \in \{0,L\} \times [0,T) \bigcup (0,L) \times \{0\}$ then $v \ge \underline{v}$ ($v \le \overline{v}$) in $(x,t) \in (0,L) \times (0,T)$.*

In the elliptic context the comparison principle reads as follows. Let $\mathcal{L} := a(x)\partial_{xx} + b(x)\partial_x + c(x)$ be an elliptic operator with coefficients in $C^{0,\alpha}([0,L])$. Furthermore, let $f : [0,L] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Consider the problem equation:

$$\begin{cases} -\mathcal{L}v(x) = f(x,v(x)) & x \in (0,L), \\ v(x) = 0 & x \in \{0,L\}. \end{cases} \tag{3.3}$$

Definition 3.4 (Elliptic sub- and supersolutions)**.** We say that $\underline{v}$ (respectively $\overline{v}$) is a subsolution (respectively a supersolution) if $\underline{v}$ (respectively $\overline{v}$) belongs to $C^0([0,L]) \cup C^2((0,L))$ and verifies that:

$$\begin{cases} -\mathcal{L}\underline{v}(x) \leq f(x, \underline{v}(x)) & x \in (0,L), \\ \underline{v} \leq 0 & x \in \{0, L\}. \end{cases}$$

Respectively,

$$\begin{cases} -\mathcal{L}\overline{v}(x) \geq f(x, \overline{v}(x)) & x \in (0,L), \\ \overline{v} \geq 0 & x \in \{0, L\}. \end{cases}$$

Theorem 3.5 (Elliptic comparison principle, Theorem 5.17 in [63])**.** *Assume that there exist a subsolution $\underline{v}$ and a supersolution $\overline{v}$ of* (3.3) *such that $\underline{v} \leq \overline{v}$.*

Then (3.3) *admits a minimal solution $\underline{v}_*$ and a maximal $\overline{v}^*$ (possibly equal) such that, $\underline{v} \leq \underline{v}_* \leq \overline{v}^* \leq \overline{v}$ and there exist no solution u between $\underline{v}$ and $\overline{v}$ such that at a certain point $x \in \Omega$ satisfies either $v(x) \leq \underline{v}_*(x)$ or $v(x) \geq \overline{v}^*(x)$*

As mentioned above, solutions of reaction-diffusion systems can blow up in finite time for certain initial data and specific nonlinearities, [29, Chapter 9, pp 547-550]. The following corollary guarantees the stability of the system if the initial data lies in $0 \leq v_0 \leq 1$.

Corollary 3.6 (Stability)**.** *Assume that $0 \leq v_0 \leq 1$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is locally Lipschitz continuous with $f(0) = f(1) = 0$. Then, the solution of the problem*

$$\begin{cases} \partial_t v = \partial_{xx} v + f(v) & (x,t) \in (0,L) \times (0,T), \\ v(x,t) = a(x) \in [0,1] & (x,t) \in \{0,L\} \times (0,T), \\ v(x,0) = v_0(x) & x \in (0,L), \end{cases}$$

is defined for all positive time and

$$0 \leq v(x,t) \leq 1.$$

Proof. The existence and uniqueness of solution follows from classical methods, existence and uniqueness for the linear problem and then a fixed point method for the nonlinear equation [29, Chapter 9].

Note that that the functions $\overline{v}, \underline{v} : (0,L) \times \mathbb{R}^+ \to \mathbb{R}$ defined as $\overline{v}(x,t) = 1$ and $\underline{v}(x,t) = 0$ are a supersolution and a subsolution respectively.

□

3.3. **Traveling waves.** We are considering Dirichlet boundary controls in both extremes of the domain of the definition of solutions: $x = 0, L$. This is, in fact, equivalent to not imposing boundary conditions at all, and simply reading-off the boundary traces of the solutions, provided they fulfill the bilateral bounds. Note that this argument is actually the one that allowed us to show the equivalence between the Neumann and the Dirichlet control problems. Observe, however, that this equivalence is no longer true when the control acts only on one extreme of the boundary, for instance.

Considering the Dirichlet boundary control problem from this perspective, i.e. ignoring the boundary conditions under the sole condition that the solution satisfies $0 \leq v(x,t) \leq 1$, allows for instance considering the Cauchy problem in the whole real line, and the restrictions of its solutions to the domain $(0,L)$ under consideration.

A particular relevant example of trajectories defined in the whole real line are the so-called traveling waves, [38, 60].

Consider the following Cauchy problem:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ v(\cdot, t=0) = v_0 \in L^\infty(\mathbb{R}). \end{cases} \tag{3.4}$$

Definition 3.7 (Traveling waves)**.** A traveling wave solution to (3.4) is a solution of the form $v(t,x) = U(x - ct)$ with $c \in \mathbb{R}$ being the wave speed and $U = U(s)$ its profile.

The existence of such functions was discovered by Kolmogorov [60] and since then they have been exhaustively studied.

Note that the C^2 profile U such that $U(+\infty) = 0$ and $U(-\infty) = 1$ defines a traveling wave solution iff

$$-cU'(s) + U''(s) = f(U(s)) \qquad s \in \mathbb{R} \tag{3.5}$$

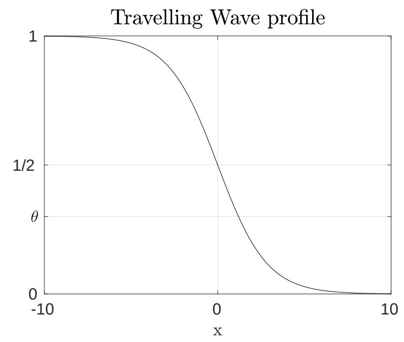

FIGURE 3.1. Profile for the traveling wave for the cubic nonlinearity $f(t) = t(1-t)(t-\theta)$ in the interval $[-10, 10]$. This profile is independent of the value of θ.

where $s = x - ct$. From (3.5) by multiplying by $U'(s)$ and integrating over $\mathbb{R}$ one observes,

$$-c\int_{\mathbb{R}}(U'(s))^2ds + \int_{\mathbb{R}} U''(s)U'(s)ds = \int_{\mathbb{R}} f(U(s))U'(s)ds.$$

This gives an implicit definition of the velocity of propagation of the profile. Indeed, using the fact that $U'(\pm\infty) = 0$, one can see that the above expression is reduced to:

$$c = \frac{F(1)}{\int_{\mathbb{R}}(U'(s))^2ds}, \tag{3.6}$$

where F is the primitive of f.

Equation (3.6) gives, in particular, the direction of the traveling wave. Indeed, if $F(1) > 0$, the traveling wave is moving to the right, so that eventually, as t increases, the value 1 invades the whole real line, while, if $F(1) = 0$, the profile defines a steady-state solution. The existence of traveling waves for bistable nonlinearities can be proved, for instance, using phase-plane techniques, understanding (3.5) as a dynamical system and looking for a trajectory that connects $(U(-\infty) = 1, U'(-\infty) = 0)$ with $(U(-\infty) = 0, U'(-\infty) = 0)$.

The next theorem guarantees the existence of traveling waves for bistable nonlinearities:

Theorem 3.8 (Traveling waves for the bistable nonlinearity, Theorem 4.9 in [90], Theorem 4.15 [38])**.** *Assume that:*

$$f(0) = 0, \quad f'(0) < 0, \quad f(\theta) = 0, \quad f(1) = 0, \quad f'(1) < 0$$
$$f(v) < 0 \quad \textit{for} \quad 0 < v < \theta, \quad f(v) > 0 \quad \textit{for} \quad \theta < v < 1$$

Then, there exists a unique traveling wave (c^*, v) of (3.4) with v decreasing and satisfying:

$$c^* > 0 \textit{ for } F(1) > 0, \quad c^* < 0 \textit{ for } F(1) < 0, \quad c^* = 0 \textit{ for } F(1) = 0$$

where $F(s) = \int_0^s f(v)dv$.

For the cubic nonlinearity $f(t) = t(1-t)(t-\theta)$ the profile $U(x)$ has an explicit expression

$$U(x) = \frac{\exp\{-x/\sqrt{2}\}}{1 + \exp\{-x/\sqrt{2}\}}$$

shown in Figure 3.1 and its traveling speed is

$$c^* = \sqrt{2}\left(\frac{1}{2} - \theta\right).$$

Traveling wave solutions connect the steady-state 0 with the steady-state 1. Traveling waves can act as attractors for the dynamical system. Indeed, there is a wide class of initial data that exponentially converges to a traveling wave.

Theorem 3.9 (Theorem 4.16 in [38])**.** *Assume that f bistable is in the theorem above. Then if the initial data of (3.4), v_0 satisfies:*

$$\limsup_{x\to-\infty} v_0(x) < \theta \qquad \liminf_{x\to+\infty} v_0(x) > \theta,$$

there exist constants $C > 0$, $\mu > 0$ and $x_0 \in \mathbb{R}$ such that:

$$|u(x,t) - U(x - ct - x_0)| < Ce^{-\mu t}.$$

Assuming that f is monostable:

$$f(0) = f(1) = 0, \quad f(s) > 0 \quad s \in (0,1); \quad f'(0) > 0, \quad f'(1) < 0,$$

there is no uniqueness of the traveling waves

Theorem 3.10 (Theorem 4.15 in [38])**.** *Let f be monostable, there exist $c^* > 0$ such that:*

- *there exist a traveling wave solution with $U(-\infty) = 1$, $U(+\infty) = 0$ if $c \geq c^*$,*
- *if $c < c^*$ there does not exist any traveling wave.*

Note that, there exist infinitely many traveling waves. This issue has been extensively analysed in some specific models like Fisher-KPP equation below, [60]:

$$\begin{cases} \partial_t v - \partial_{xx} v = rv(1-v) & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ 0 \leq v(x,0) \leq 1 & x \in \mathbb{R} \end{cases} \tag{3.7}$$

Concerning the stability of traveling wave solutions in the monostable case there is a specific traveling wave that enjoys stability properties:

Theorem 3.11 ([60])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be monostable fulfilling $f'(0) \geq f'(s)$ for all $0 \leq s \leq 1$ and let $v = v(x,t)$ be the solution of*

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ v(\cdot, t = 0) = v_0 \end{cases}$$

where

$$v_0(x) = \begin{cases} 0 & \textit{for} \quad x > 0 \\ 1 & \textit{for} \quad x < 0. \end{cases}$$

Then there exist a function $\psi \in C^1$ such that

$$|u(x,t) - U(x - c^* t - \psi(t))| \to 0 \quad \textit{as} \quad t \to \infty \tag{3.8}$$

uniformly in x and:

$$\lim_{t \to \infty} \psi'(t) = 0$$

We refer to [106], [103], [37], among others, for further results on the asymptotic stability of solutions.

Traveling waves play a very interesting role in the context of control since its restriction to the bounded domain $[0, L]$ yields a trajectory linking an arc near the steady-state 0 with another one near 1. Traveling waves can also naturally be used to construct sub-solutions for controlled trajectories.

4. Well-posedness of the control problem and controllability

4.1. **Comments on the well-posedness.** Let us first discuss the well-posedness of the main control problem considered in these lecture notes, namely the boundary control of the scalar reaction-diffusion equation:

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v(0,t) = a_1(t), \quad v(L,t) = a_2(t) & t \in (0,T), \\ v(x,0) = v_0(x), \quad v_0 \in L^\infty((0,L);[0,1]). \end{cases} \tag{4.1}$$

where $a_i \in L^\infty((0,T);\mathbb{R})$ for $i = 1,2$ are control functions. Splitting the solution into two subproblems, $v = w + y$, as in [91], where

$$\begin{cases} \partial_t w - \partial_{xx} w = 0 & (x,t) \in (0,L) \times (0,T), \\ w(0,t) = a_1(t), \quad w(L,t) = a_2(t) & t \in (0,T), \\ w(x,0) = 0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \partial_t y - \partial_{xx} y = f(y + w) & (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, \quad y(L,t) = 0 & t \in (0,T), \\ y(x,0) = v_0(x), \quad v_0 \in L^\infty((0,L);[0,1]), \end{cases} \tag{4.3}$$

the existence and uniqueness of a weak solution can be determined. In the presence of boundary control weak solutions are defined as follows:

Definition 4.1 (Weak solution)**.** Consider the space:

$$\mathcal{T} := \{\varphi \in C^\infty([0,T]\times[0,L]) : \quad \varphi(\cdot,T)=0, \quad \varphi(x,t)=0 \quad (x,t)\in\{0,L\}\times[0,T]\}\,.$$

For $v_0 \in L^\infty([0,L];[0,1])$ and for $a_i \in L^\infty((0,T))$ for $i=1,2$,

$$v \in C^0((0,T),H^{-1}((0,L))) \cap L^\infty((0,L)\times(0,T))$$

is a weak solution of system (4.1) if for every $\varphi \in \mathcal{T}$ one has that:

$$\int_0^T\int_0^L v\big(-\partial_t\varphi - \partial_{xx}\varphi\big) - f(v)\varphi dxdt = \int_0^L v_0\varphi(x,0)dx + \int_0^T a_1(t)\partial_x\varphi(0,t) - a_2(t)\partial_x\varphi(L,t)dt.$$

Note that, in the construction above, solving the non-homogeneous boundary-value problem (4.2), due to the low regularity of the boundary controls, requires to first introduce the notion of transposition solution [66, Ch.13].

Let us introduce the following notation:

$$\Omega=(0,L), \quad Q=\Omega\times(0,T), \quad \Sigma=\partial\Omega\times(0,T)$$

and the adjoint problem

$$\begin{cases} -\partial_t p - \partial_{xx}p = \phi & (x,t)\in(0,L)\times(0,T),\\ p(0,t)=0, \quad p(L,t)=0 & t\in(0,T),\\ p(x,T)=0 \end{cases} \tag{4.4}$$

for $\phi \in L^2((0,T);L^2((0,L)))$.

The well-posedness of the adjoint equation (4.4) can be addressed by classical methods, [29, Chapter 7 pp.378]. In fact, under the inversion of the time variable ($t\to -t$) this system becomes an homogeneous Dirichlet problem for the forward heat equations. The solution then belongs to:

$$p\in L^2((0,T);H_0^1((0,L))), \quad \frac{d}{dt}p \in L^2((0,T);H^{-1}((0,L))).$$

On the other hand, multiplying (4.2) by ϕ and integrate over Q:

$$\begin{aligned}\int_0^T\int_0^L w\phi dxdt &= \int_0^T\int_0^L w(-\partial_t p - \partial_{xx}p)dxdt = \int_0^T w\frac{\partial p}{\partial\nu}\Big|_{x=0}^{x=L}dt\\ &= \int_0^T a_2(t)\partial_x p(L,t) - a_1(t)\partial_x p(0,t)dt\\ &= \langle a, \Lambda\phi\rangle_{L^2(\Sigma)},\end{aligned}$$

where $a=(a_1,a_2)$. The map $\Lambda : L^2(Q)\to L^2((0,T);L^2(\partial\Omega))$, $\Lambda(\phi)=(-\partial_x p(0,t), \partial_x p(L,t))$, such that p solves (4.4), is linear and continuous. A transposition solution w of (4.2) is a distribution w satisfying the following relationship for every $\phi\in L^2(Q)$

$$\int_0^T\int_0^L w\phi = \langle a,\Lambda\phi\rangle_{L^2(\Sigma)}. \tag{4.5}$$

By duality the existence and uniqueness of the transposition solution holds. For details see [66, Chapter 4, Sections 8, 12.3, 13 and Section 15 Example 1] and [65, page 202].

The well-posedness of problem (4.3) can then be achieved as an application of Banach Fixed-point [91] to

$$\begin{cases} \partial_t y - \partial_{xx} y = f(\xi + w) & (x,t)\in(0,L)\times(0,T)\\ y(0,t)=y(L,t)=0 & t\in(0,T)\\ y(x,0)=v_0(x), \quad v_0\in L^\infty((0,L);[0,1]) \end{cases} \tag{4.6}$$

The solution $\psi(\xi)$ of (4.6), defines a map $\psi : B_R \longrightarrow B_R$ where $B_R\subset L^\infty(Q)$ is a ball of radius R.

Using the variations of constants formula with the semigroup generated by the heat equation, a contraction map is defined for T small enough, leading to local existence and uniqueness. When the nonlinearity is assumed to be globally Lipschitz, the solution is globally defined in time. Blow up may occur when the nonlinearity is superlinear at infinity.

4.2. **Null Controllability of linear problems.** We present some of the main features of the null controllability of scalar parabolic equations presenting the main arguments and references.

The first method to deal with the controllability of the heat equation in one dimension is based on the use of biorthogonal functions, [31].

We present the extension-restriction principle that allows to translate the boundary control problem into an interior control one. The argument consists of extending the system to a larger domain in which the control acts in the interior. Once the solution in the extended domain is controlled, the restriction to the original domain leads to a controlled trajectory with Dirichlet boundary controls.

Here we shall only consider scalar equations. For extensions to parabolic systems we refer to [1, 33, 46, 51, 61].

The domain $\Omega = (0, L)$ is extended to $\Omega_E = (-1, L+1)$. Let $\omega \subset (-1, 0)$ (see Figure 4.1) and consider the following linear parabolic control problem:

$$\begin{cases} \partial_t v - \partial_{xx} v - b(x,t)v = \chi_\omega h & (x,t) \in \Omega_E \times (0,T), \\ v(-1,t) = 0, \quad v(L+1,t) = 0 & t \in (0,T), \\ v(x,0) = v_0(x), \quad v_0 \in L^\infty(\Omega_E; [0,1]) & x \in \Omega_E, \end{cases} \tag{4.7}$$

where the initial datum is a smooth extension by zero of compact support of v_0 and $b \in L^2((0,T), L^2(\Omega_E))$.

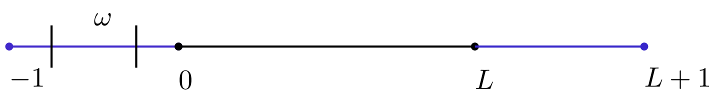

FIGURE 4.1. Original domain $\Omega = [0, L]$, extended domain $\Omega_E = (-1, L+1)$ and the control region ω.

We now consider the extended adjoint equation:

$$\begin{cases} \partial_t p + \partial_{xx} p + b(x,t)p = 0 & (x,t) \in \Omega_E \times (0,T), \\ p(-1,t) = 0, \quad p(L+1,t) = 0 & t \in (0,T), \\ p(x,T) = p^T(x) & x \in \Omega_E. \end{cases} \tag{4.8}$$

Integrating by parts we get

$$0 = \int_{\Omega_E} v(T)p(T)dx - \int_{\Omega_E} v(0)p(0)dx - \int_0^T \int_{\Omega_E} v\left(\partial_t p + \partial_{xx} p + bp)\right) dxdt - \int_0^T \int_\omega hpdxdt.$$

Hence,

$$0 = \int_{\Omega_E} v(T)p(T)dx - \int_{\Omega_E} v(0)p(0)dx - \int_0^T \int_\omega hpdxdt.$$

A control $\overline{h}$ driving the solution v to the null state, i.e. $v(T) \equiv 0$, is characterized by the duality identity:

$$-\int_{\Omega_E} v(0)p(0)dx - \int_0^T \int_\omega \overline{h}pdxdt = 0$$

for all p^T such that the solution of (4.8) satisfies $p \in L^2(\omega \times (0,T))$. Observe that the condition $p \in L^2(\omega \times (0,T))$ does not imply that $p^T \in L^2(\Omega_E)$. In fact, p^T belongs to a much larger space, [36].

The control $\overline{h}$ fulfilling the above identity can be obtained minimizing the functional

$$\begin{aligned} J &: H \longrightarrow \mathbb{R}, \\ p^T &\longrightarrow J(p^T) = \int_0^T \int_\omega p^2 dxdt + \int_{\Omega_E} p(0)v(0)dx. \end{aligned}$$

where,

$$H := \left\{ p^T \text{ such that } p \text{ solves (4.8) and } \int_0^T \int_\omega p^2 dxdt < +\infty \right\},$$

The Euler-Lagrange equation satisfied by the minimizer $p^{*,T}$ of J is given by

$$DJ(p^{*,T})[\xi^T] = \int_0^T \int_\omega \xi p^* dxdt + \int_{\Omega_E} \xi(0)v(0)dx = 0,$$

This leads to the desired control $\overline{h} = p^*$, which is of minimal L^2- norm among all possible controls.

In order to show that J has a minimizer we observe that J is continuous and convex. Its coercivity is equivalent to the so-called observability inequality for the adjoint system:

$$\|p(0)\|^2_{L^2(\Omega_E)} \leq C \int_0^T \int_\omega p^2 dxdt, \quad \forall p^T \in H. \tag{4.9}$$

This kind of inequalities for parabolic equations in one and several space dimensions was proved using Carleman inequalities in [41] (see also [34]).

This leads to the control of the system (4.7) such that $v(x, T; h) = 0$. Taking its restriction to the original domain $(0, L)$ we find the boundary controls

$$a(0,t) = v(0,t;h) \quad a(L,t) = v(L,t;h)$$

leading to the null control of the original system

$$\begin{cases} \partial_t v - \partial_{xx} v - b(x,t)v = 0 & (x,t) \in \Omega \times (0,T) \\ v(0,t) = a(0,t), \quad v(L,t) = a(L,t) & t \in (0,T) \\ v(x,0) = v_0, & x \in \Omega \end{cases}$$

so that

$$v(x,T) = 0, \, \forall x \in (0,L).$$

Note however that this arguments do not guarantee that the controls and/or the controlled states fulfill the bilateral constraints.

4.3. **Null controllability for the semilinear problem and further comments.** Once the linear control problem has been solved the semilinear one can be addressed using either Schauder's or Kakutani's fixed point [27]. This fixed point argument was originally employed for the semilinear wave equation, see for instance [121, 122] and references therein (see also [62]).

The fixed point argument can be implemented as follows. Given $\xi \in L^2(Q)$, we introduce the bounded potential b_ξ

$$b_\xi(x,t) = \begin{cases} \dfrac{f(\xi(x,t))}{\xi(x,t)} & \text{if } \xi(x,t) \neq 0, \\ f'(0) & \text{otherwise.} \end{cases}.$$

Then, one considers the linear controlled problem

$$\begin{cases} \partial_t v - \partial_{xx} v - b_\xi(x,t)v = \chi_\omega h_\xi & (x,t) \in \Omega_E \times (0,T), \\ v(-1,t) = v(L+1,t) = 0 & t \in (0,T), \\ v(x,0) = v_0, \quad v(x,T) = 0 & x \in \Omega_E. \end{cases} \tag{4.10}$$

Applying the linear methods above, this system can be controlled, and in this way we can define the linear map $\psi : L^2(Q) \to L^2(Q)$ defined as $\xi \to \psi(\xi) = v$ solution of (4.10). This map turns out to be continuous and compact. Under suitable growth conditions on the nonlinearity, in particular, when f is globally Lipschitz, it can be shown that this map is invariant in a sufficiently larger ball. This allows to apply Schauder fixed point. The fixed point corresponds to a controlled trajectory for the nonlinear system.

There is by now an extensive literature in linear and semilinear parabolic control problems. We refer to [64] for an alternative approach to Carleman inequalities, based on spectral decompositions, and to [35, 36] for the control of weakly blowing up semilinear heat equations.

As mentioned above, these arguments do not yield estimates in the controls and controlled states allowing to assure that the bilateral constraints are fulfilled. A careful analysis of the Carleman inequalities allows to obtain estimates on the cost of control of the form

$$\|a\|_{L^2((0,T),\mathbb{R}^2)} \leq \exp\left\{C\left(\frac{1}{T} + \|b\|_{L^\infty} T + \|b\|^{2/3}_{L^\infty}\right)\right\}$$

As expected, the size of controls increases exponentially when the time horizon T tends to zero leading to oscillations on controls and states that are incompatible with the bilateral constraints considered in these lecture notes (see Figure 4.2).

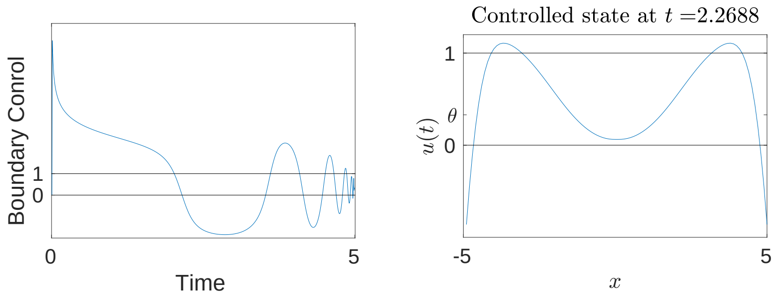

FIGURE 4.2. (Left) Control function $a(0,t)$ steering the cubic bistable equation to the steady-state $w \equiv \theta$ in time $T = 5$. (Right) Snapshot at time $t = 2.288$ of the controlled trajectory $u_a(\cdot,t)$ violating the constraints $0 \leq u \leq 1$.

Important further developments are needed in order to understand the controllability of these systems under bilateral constraints of the form $0 \leq u \leq 1$.

4.4. **Constrained controllability results.** In this subsection, we present the main Theorem given in [91], which ensures that under certain assumptions, and, in particular, for long time-horizons, the problem of high amplitude oscillations observed in Figure 4.2 can be overpassed and bilateral constraints can be assured. However, this will require a much more detailed analysis of the dynamics of the system, in particular, to build paths of steady-states linking one steady-state to another.

Consider

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ v(0,x) = v_0(x) \in [0,1] & x \in (0,L), \end{cases} \tag{4.11}$$

we say that v_1 and v_0 are path connected steady-states if there exists a continuous function from $[0,1]$ to the set of admissible steady-states S endowed with the L^∞ topology, $\gamma : [0,1] \to S$, such that $\gamma(0) = v_0$ and $\gamma(1) = v_1$. Denote by $\overline{v}^s := \gamma(s)$.

Theorem 4.2 (Theorem 1.2 in [91])**.** *Let be v_0 and v_1 be path-connected admissible bounded steady-states. Assume there exists $\nu > 0$ such that:*

$$\nu \leq \overline{v}^s(x) \leq 1 - \nu \quad \textit{for } x \in \{0, L\}, \tag{4.12}$$

for any $s \in [0,1]$. Then, if T is large enough, there exist a control function $a \in L^\infty\left((0,T);[0,1]^2\right)$ such that the problem (4.11) with initial datum v_0 and control function a admits unique solution verifying $v(\cdot,T) = v_1$.

Figure 4.3 shows the strategy qualitatively. If one has a connected path of steady-states, one can use local controllability to control sequentially along elements in the path in short time intervals. The strategy is based on using an L^∞ bound on the control in terms of the L^∞ norm of the difference between the initial datum and the target. This allows the extraction of a finite number of steady-states along the path and applying local controllability from one to another without breaking the constraints. This strategy, by construction, requires a large time, while the controllability of the linear and semilinear heat equation can be achieved in arbitrary small time.

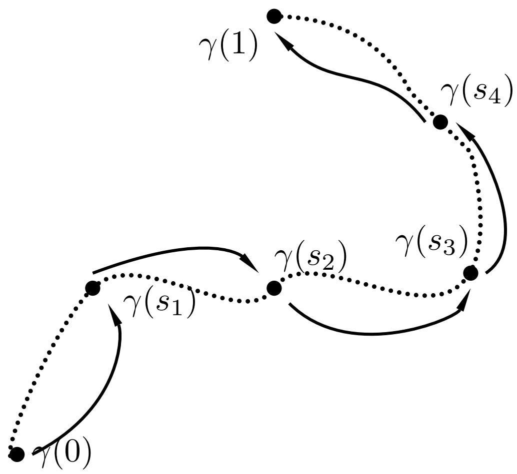

FIGURE 4.3. Qualitative representation of the Staircase strategy. The dashed blue line represents the connected path of steady-states and the red line the state under the control.

4.5. **Minimal controllability time.**

Let us consider the problem (4.11) where f is Lipschitz. As mentioned before, when state constraints are not present, the controllability problem to some steady-state can be achieved in arbitrary small time [36]. Hence if one seeks to control in $T << 1$, the cost is becoming exponentially large. However, the results of [36] concern the dynamics without constraints in the state. When the system has positivity constraints, there is a minimal controllability time for the linear heat equation (see [71] and also see [69] for systems) and for the semilinear (see [91]). Here, we will provide a schematic proof of the existence of a minimal control time when there are bilateral bounds.

Theorem 4.3 (Positivity of the minimal controllability time for the semilinear equation [91])**.** *Let us consider a steady-state target* $\overline{v}$*:*

$$\begin{cases} -\overline{v}_{xx} = f(\overline{v}) & x \in (0, L), \\ 0 < \overline{v} < 1 & x \in (0, L), \\ \overline{v}(0) = a_1, \quad \overline{v}(L) = a_2. \end{cases}$$

Furthermore, consider the boundary control problem (4.11) *with target function* $\overline{v}$ *and* $v_0 \neq \overline{v}$*. Then the controllability cannot be achieved in arbitrary small time if the control function satisfies* $0 \leq a \leq 1$*.*

Proof. The result is an application of the comparison principle.

Let $\overline{v}$ be an admissible target steady-state. Assume that the admissible initial condition v_0 is different from the target. Then, one has that there exists an open interval $I \subset (0, L)$ such that either

$$\begin{cases} (A) & \overline{v} < v_0 \quad x \in I, \\ (B) & \overline{v} > v_0 \quad x \in I. \end{cases}$$

(A) For any control strategy a, one has that by the comparison principle

$$v(t; v_0, a) \geq v(t; v_0, a = 0)$$

where $v(t; v_0, a)$ is the solution of (4.11) with control a. Moreover, one has that there exists a nonnegative test function $\phi \in H_0^1$ such that:

$$\int_I (v_0 - \overline{v})\phi dx > 0$$

Since the solution of (4.11) is continuous with values in $H^{-1}((0, L))$ one has that there exists $T^* > 0$ for which

$$\int_I (v(t; v_0, a = 0) - \overline{v})\phi > 0 \quad t \in (0, T^*)$$

Hence,

$$\int_I (v(t;v_0,a)-\overline{v})\phi > \int_I (v(t;v_0,a=0)-\overline{v}) > 0 \quad t\in(0,T^*)$$

Therefore, for any admissible control a, controllability cannot hold before T^*

(B) The same paradigm arises, but by employing the control $a=1$ for the comparison.

□

Remark 4.4. If we have unilateral constraints, the result is still valid. However, it requires employing duality techniques and using the continuity of the normal derivative of the adjoint equation (see [91]).

5. Barriers and multiplicity of steady-states

5.1. **Barriers.**

The first notion that we should emphasize is that a fundamental lack of controllability can occur when state-constraints are present. We will see that the existence of nontrivial elliptic solutions depends on the length of the domain. These nontrivial solutions, by the comparison principle, will impede specific trajectories to reach the prescribed target.

The comparison principle, presented in Subsection 3.2, ensures that a solution of the elliptic equation:

$$\begin{cases} -\partial_{xx}w = f(w) & x\in(0,L),\\ 0<w(x)<1 & x\in(0,L),\\ w(0)=w(L)=0, \end{cases} \tag{5.1}$$

will always be below the solution of the following parabolic problem

$$\begin{cases} \partial_t v - \partial_{xx}v = f(v) & (x,t)\in(0,L)\times(0,T),\\ v(x,t)=a(x,t)\geq 0 & (x,t)\in\{0,L\}\times(0,T),\\ v(x,0)\geq w(x). \end{cases}$$

Then we see that $w(x)$ is an intrinsic obstruction to the controllability for the initial data above w to targets that have at least an interval below w. We say that $w(x)$ acts as a barrier since

$$w(x)\leq v(x,t).$$

As a consequence, we cannot reach any state below $w(x)$ with positive control functions $a(x,t)$.

Recall that our control a physically means to modify the proportion in the boundary, and for that reason, we have the constraint $0\leq a(x,t)\leq 1$.

Hence, the existence of solutions for the Dirichlet condition equals to zero (5.1) is problematic in terms of the controllability to 0 (see Figure 5.1). We will see that the existence of multiple solutions depends basically on the length of the domain L.

The dependence on the domain L can be understood intuitively in the application context. There is a population that reproduces in the interior of a domain while being killed at the boundary. These terms are competing. If the domain is big enough, the reproduction inside can compensate for individuals' loss through the boundary.

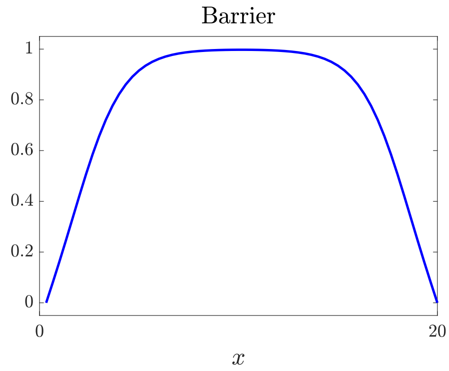

FIGURE 5.1. Simulation of the semilinear heat equation with cubic nonlinearity $f(y) = y(1-y)(y-1/3)$ in the interval $(0,20)$ finding the barrier.

Note that nontrivial solutions with a boundary value 1 will have the same effect for reaching the state 1. This section is devoted to studying the existence and nonexistence of nontrivial solutions. We will restrict the study in the following section in the one-dimensional case, even though the results also hold in several dimensions by the same techniques.

5.2. **Rescaling.**

Consider an interval $(0, L)$ and the evolution equation

$$\begin{cases} \partial_t v - \partial_{xx} v = f(v) & (x,t) \in (0,L) \times (0,T), \\ v(x,t) = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ v(\cdot, t=0) = v_0 \in L^\infty((0,L),[0,1]). \end{cases} \tag{5.2}$$

First, let us reparameterize equation (5.2) for considering it into the interval $(0,1)$. We apply the spatial transformation $s(x) = x/L$ and the time transformation $\tau(t) = t/L^2$. By setting $v(L^2\tau, Ls) = u(\tau, s)$ the problem reads:

$$\begin{cases} u_\tau - \partial_{ss} u = L^2 f(u) & (x,t) \in (0,1) \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \{0,1\} \times (0,T), \\ u(\cdot, t=0) = u_0 \in L^\infty((0,1),[0,1]). \end{cases} \tag{5.3}$$

We will make an abuse of notation, and we will use t instead of τ and x instead of s and we denote $\lambda := L^2$. In [67] the existence of positive solutions for the semilinear elliptic problem and its multiplicity is studied, we collect here the results of the work [67] that can be applied in our steady-state problem (5.4).

Consider the interval $(0,1)$ and the boundary value problem:

$$\begin{cases} -\partial_{xx} u = \lambda f(u) & x \in (0,1), \\ 0 < u < 1 & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{5.4}$$

5.3. **Variational formulation.**

The weak formulation of the boundary value problem:

$$\begin{cases} -\partial_{xx} v = \lambda f(x,v) & x \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{5.5}$$

is:

$$\int_0^1 v_x h_x - \lambda f(x,v) h dx = 0 \qquad \forall h \in H_0^1((0,1)).$$

which corresponds to look for critical points of the energy functional:

$$\begin{aligned} I : H_0^1((0,1)) &\longrightarrow \mathbb{R} \\ v &\longrightarrow I[v] := \int_0^1 \frac{1}{2}v_x^2 - \lambda F(x,v)dx, \end{aligned}$$

where $F(x,v) = \int_0^{v(x)} f(x,s)ds$.

Theorem 5.1 (Coercivity of I)**.** *Assume that:*

$$\limsup_{|s|\to\infty} \lambda \frac{f(x,s)}{s} < \lambda_1((0,1)), \tag{5.6}$$

where $\lambda_1((0,1)) = \pi^2$, the first eigenvalue of the Dirichlet Laplacian. Then I is coercive.

Proof. For simplicity we will take care only on the case in which $s \to +\infty$, the case $s \to -\infty$ follows similarly. Since

$$\limsup_{s\to\infty} \lambda \frac{f(x,s)}{s} < \lambda_1((0,1)),$$

we know that there exist $R > 0$ such that

$$\lambda \frac{f(x,s)}{s} < \lambda_1 \quad \forall s \geq R$$

using also the fact that $f(x,0) = 0$ we can write:

$$\begin{aligned} \int_0^v \lambda f(s)ds &= \int_0^R \lambda f(s)ds + \int_R^v \lambda \frac{f(s)}{s} sds \\ &\leq \int_0^R \lambda f(s)ds + \frac{p}{2}\left(u^2 - R^2\right) \\ &\leq \frac{p}{2}u^2 + C(R,f,\lambda) \end{aligned}$$

for a certain $p < \lambda_1((0,L))$. So

$$\begin{aligned} I[u] &= \int_0^1 \frac{1}{2}u_x^2 - F(u)dx \\ &\geq \int_0^1 \frac{1}{2}u_x^2 - \frac{p}{2}u^2 - C(R,f,\lambda)dx \\ &\geq \frac{1}{2}\int_0^1 \left(1 - \frac{p}{\lambda_1((0,1))}\right) u_x^2 - C(R,f,\lambda)dx \\ &\geq \frac{1}{2}\left(1 - \frac{p}{\lambda_1((0,1))}\right)\int_0^1 u_x^2dx - C(R,f,\lambda) \\ &\geq c\|u\|^2_{H_0^1([0,1])} - C(R,f,\lambda) \end{aligned}$$

where $c > 0$ because $p < \lambda_1((0,1))$. So I is coercive.

□

Theorem 5.2 (Existence of a minimizer)**.** *Under the assumptions of Theorem 5.1, I has a minimizer.*

Proof. Note that the Lagrangian $L(p,z,x) = \frac{1}{2}|p|^2 - F(z)$ is convex with respect to the variable p. Therefore I is weakly lower-semicontinuous (Theorem 1 Ch.8 pp.468 in [29]) and the functional has a minimizer $u \in H_0^1((0,L))$. □

Remark 5.3. The hypothesis (5.6) is only needed for proving that the functional is coercive. For the monostable and bistable nonlinearities, since f has a positive zero, one can extend f by zero afterwards.

Note that for both the monostable case and the bistable case we can redefine f to be constant before $s = 0$ ($f(x,s) = 0$ for $s \leq 0$) and after $s = 1$ ($f(x,s) = 0$ for $s \geq 1$). In general we are interested on minimizers such that $0 \leq v(x) \leq 1$. After redefining f (if needed) by

$$\tilde{f}(s) := \begin{cases} f(s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \\ 0 & \text{if } s > 1, \end{cases}$$

we define the functional:

$$\begin{aligned} I : H_0^1((0,1)) &\longrightarrow \mathbb{R} \\ v \longrightarrow I[v] &:= \int_0^1 \frac{1}{2} v_x^2 - \lambda \tilde{F}(v) dx \end{aligned}$$

where $\tilde{F}(v) = \lambda \int_0^v \tilde{f}(s) ds$

Indeed $I[0] = 0$. Thanks to the redefinition of f, we ensure that whenever 0 is not a minimizer, the minimizer satisfies $0 < v < 1$ and the Euler-Lagrange equations:

$$\begin{cases} -\partial_{xx} u = \lambda \tilde{f}(u) & x \in (0,1) \\ u > 0 & x \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Indeed, if the minimizer would not satisfy $0 < v < 1$, then one can reach a contradiction. Suppose that v takes values outside $[0,1]$, then the function

$$\underline{v} = \begin{cases} v(x) & \text{if } v(x) \in [0,1] \\ 1 & \text{if } v(x) > 1 \\ 0 & \text{if } v(x) < 0 \end{cases}$$

has a lower value for the gradient part $\int_0^1 \underline{v}_x^2 < \int_0^1 v_x^2$ while the nonlinear part remains equal thanks to the redefinition of f. However, by the strong maximum principle $\underline{v}$ cannot be a solution of the elliptic problem. Therefore the minimizer must satisfy $0 < v < 1$. Since $0 < u < 1$, $\tilde{f} = f$ in this range and therefore is a solution of our original problem as well.

Definition 5.4. We define λ^* as the infimum value $\lambda \in \mathbb{R}^+$ for which there is a solution of:

$$\begin{cases} -\partial_{xx} v = \lambda f(v) & x \in (0,1), \\ 0 < v < 1 & x \in (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{5.7}$$

The following classical result [67] makes use of subsolutions to prove that, if for a certain λ_1 there exists a nontrivial solution to (5.7), then, for any $\lambda > \lambda_1$ there will exist a nontrivial solution. Recall that $\lambda = L^2$.

Proposition 5.5. *For every $\lambda > \lambda^*$ there exist a nontrivial solution to* (5.7)

Proof. Recall that $\lambda = L^2$. We prove it making use of different domains. We will first assume that there exists a nontrivial solution of the elliptic problem for a certain $L_1 > 0$. Then we will see that there exists a nontrivial solution for any $L > L_1$, which will correspond to say that if there exists a nontrivial solution for λ_1, for any $\lambda > \lambda_1$, there will also exist a nontrivial solution. Assume that there exists a solution to:

$$\begin{cases} -\partial_{xx} v_1 = f(v_1) & x \in (0, L_1), \\ 0 < v_1 < 1 & x \in (0, L_1), \\ v_1(0) = v_1(L) = 0. \end{cases}$$

To prove that for any $L > L_1$ there exists a nontrivial solution, one can construct a subsolution of:

$$\begin{cases} -\partial_{xx} v = f(v) & x \in (0, L), \\ 0 < v < 1 & x \in (0, L), \\ v(0) = v(L) = 0, \end{cases}$$

using v_1. One can extend v_1 by zero in (L_1, L),

$$\tilde{v}(x) = \begin{cases} v_1(x) & \text{if } x \in (0, L_1), \\ 0 & \text{if } x \in (L_1, L). \end{cases}$$

$\tilde{v}$ is a weak subsolution for the problem in $(0, L)$, note that 1 is always a supersolution, therefore we have that if a nontrivial solution exists for $(0, L_1)$ for some $L_1 > 0$, then for every $L > L_1$ there will exist a nontrivial solution. □

5.4. **Monostable nonlinearity.**

5.4.1. *Dirichlet condition* 0. Using the variational structure of the problem, we will be able to give estimates from above for λ^*. Furthermore, the existence of positive solutions and some estimates can also be given for the case in which nonlinearities are depending on x, i.e., $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$.

The following Theorem was proven in [10, Theorem II.1], however, the proof given below is using a variationa.

Theorem 5.6 (One Upper bound of λ^*). [1] *Assume that*

$$\lim_{s \to 0^+} \lambda \frac{f(x, s)}{s} > \lambda_1((0, 1)) \text{ uniformly in } x \in [0, 1]$$

and

$$\overline{\lim_{s \to \infty}} \lambda \frac{f(x, s)}{s} < \lambda_1((0, 1)) \quad \forall x \in [0, 1]$$

Then, for all $\lambda > \frac{\lambda_1((0,1))}{\min_{x \in [0,1]} f'(0,x)}$ *there exist a solution of the problem* (5.4). *Therefore:*

$$\lambda^* \leq \frac{\lambda_1((0, 1))}{\min_{x \in [0,1]} f'(0, x)}.$$

Proof. The main idea of the proof is simple: To prove the existence of a local minimizer for I that takes values in $0 \leq v \leq 1$ and to prove that 0 is not a minimizer. Since we have 0 boundary condition, the function $v = 1$ is not a possible candidate. Let e_1 be the first eigenfunction of the operator $A = (-\partial_{xx})$. We know that this function is positive. Set $v = \epsilon e_1$ for $\epsilon > 0$ to be chosen later on.

By hypothesis, we assumed that:

$$\lim_{s \to 0^+} \lambda \frac{f(x, s)}{s} > \lambda_1((0, 1)) \text{ uniformly in } x \in [0, 1]$$

this means that there exists $r \in \mathbb{R}^+$ such that

$$\frac{\lambda f(x, s)}{s} > \lambda_1((0, 1)) \quad \forall s \in [0, r].$$

Choose ϵ small enough such that $\epsilon e_1 < r$ and evaluate the functional:

$$\begin{aligned} I[\epsilon e_1] &= \int_0^1 \frac{\epsilon^2}{2} \partial_x e_1^2 - \int_0^{\epsilon e_1} \lambda \frac{f(x, s)}{s} s ds dx \\ &\leq \int_0^1 \frac{\epsilon^2}{2} \partial_x e_1^2 - p \frac{\epsilon^2 e_1^2}{2} dx \end{aligned}$$

for some $p > \lambda_1((0, 1))$. Then integrating by parts $\partial_x e_1^2$, using the fact that $-\partial_{xx} e_1 = \lambda_1 e_1$ and imposing that $I[\epsilon e_1] < 0$ i.e.

$$I[\epsilon e_1] \leq \frac{\epsilon^2}{2} \int_0^1 (\lambda_1((0, 1)) - p) \, e_1^2 dx < 0$$

we obtain the existence of a nontrivial solution. Therefore we know that for all $\lambda > \frac{\lambda_1((0,1))}{\min_{x \in [0,L]} f'(0,x)}$ there exists a positive solution, this means that

$$\lambda^* \leq \frac{\lambda_1((0, 1))}{\min_{x \in [0,L]} f'(0, x)}.$$

[1] Here we provide a weaker version of the original theorem and moreover the proof given is a variational argument. In [10] one can find also a proof that does not rely on a variational argument

□

The following Theorem is a lower bound for this splitting in the case in which the nonlinearities are concave and twice differentiable.

Theorem 5.7 (A Lower bound of λ^*)**.** *Let f be twice differentiable such that $f'(0) > 0$ and concave, i.e. $f''(t) \leq 0$. Then if:*

$$\lambda < \frac{\lambda_1((0,1))}{f'(0)}$$

there cannot be any positive solution to Problem (5.4)*, Therefore,*

$$\lambda^* \geq \frac{\lambda_1((0,1))}{f'(0)} > 0$$

Proof. Multiply the equation by v and integrate over the domain and integrate by parts:

$$\int_0^1 -v\partial_{xx}v dx = \lambda \int_0^1 f(v)vdx,$$

$$\int_0^1 v_x^2 - \lambda \int_0^1 f(v)v = 0.$$

By the Poincaré inequality,

$$\int_0^1 \lambda_1((0,1))v^2 - \lambda f(v)v \leq 0$$

Now consider the Taylor formula on f

$$f(v) = f(0) + f'(0)v + \int_0^v f''(t)(v-t)dt$$

Due to the fact that $f(0) = 0$ we end up with

$$\int_0^1 (\lambda_1((0,1)) - \lambda f'(0))\, v^2 - \lambda v \int_0^v f''(t)(v-t)dtdx \leq 0$$

since $v > 0$ we have that $v - t \geq 0$. Moreover, by assumption $f''(t) \leq 0$ and we obtain that the second term is bigger or equal than zero, hence, we can conclude that a necessary condition to have a positive solution is:

$$\int_0^1 (\lambda_1((0,1)) - \lambda f'(0))\, v^2 \leq 0$$

which concludes the proof.

□

Remark 5.8 (Space dependent nonlinearity)*.* After a subtle change in the proof of 5.7, one can see that indeed the result also holds for the following problem:

$$\begin{cases} -\partial_{xx}v = \lambda f(v,x) & x \in (0,1) \\ v > 0 & x \in (0,1) \\ v(0) = v(L) = 0, \end{cases} \tag{5.8}$$

where $f(v,x)$ is twice differentiable with respect to v, concave with respect to v. Following the argument one arrives at the following condition

$$\int_0^1 (\lambda_1((0,1)) - \lambda f'(0,x))\, v^2 \leq 0.$$

If we assume that

$$0 < \min_{x\in[0,L]} f'(0,x) \leq f'(0,x) \leq \max_{x\in[0,L]} f'(0,x) < +\infty$$

We see that in this case we obtain also a lower bound for λ^*:

$$\lambda^* \geq \frac{\lambda_1((0,1))}{\max_{x\in[0,1]} f'(0,x)} > 0$$

Theorem 5.9 (λ^* for monostable concave nonlinearities)**.** *When f is monostable, concave and does not depend on x we have:*

$$\lambda^* = \frac{\lambda_1((0,1))}{f'(0)} > 0$$

When f monostable and concave but depending on x,

$$\frac{\lambda_1((0,1))}{\max_{x\in[0,1]} f'(x,0)} \leq \lambda^* \leq \frac{\lambda_1((0,1))}{\min_{x\in[0,1]} f'(x,0)}$$

Remark 5.10 (Uniqueness of positive solutions for concave nonlinearities)*.* When f is concave and a positive solution exists it is unique [9, 67].

Remark 5.11 (Non uniqueness of positive solutions for non concave nonlinearities)*.* When f is not concave uniqueness of the nontrivial positive solution might not hold. Assuming that $f'(0) > 0$, if $\lambda^* < \frac{\lambda_1((0,1))}{f'(0)}$ we have that for all λ such that $\lambda^* < \lambda < \lambda_1((0,1))$ there exists a second positive solution. This is proven using topological degree arguments [67].

5.4.2. *Phase Portrait.* In one dimension the elliptic equation

$$-\partial_{xx} u = f(u),$$

can be interpreted as a dynamical system by considering the ordinary differential equation:

$$\frac{d}{dx}\begin{pmatrix} u \\ v_x \end{pmatrix} = \begin{pmatrix} v_x \\ -f(u) \end{pmatrix}. \tag{5.9}$$

For the monostable nonlinearity, we notice that $(1,0)$ is a topological saddle for the nonlinear system, and $(0,0)$ is a center for the linearized system. The differential matrix is:

$$DF(u,v) = \begin{pmatrix} 0 & 1 \\ -\partial_u f(u) & 0 \end{pmatrix}.$$

Since by definition of monostable, $\partial_u f(u)\big|_{u=1} < 0$ and $\partial_u f(u)\big|_{u=1} > 0$, we have that

$$DF(1,0) = \begin{pmatrix} 0 & 1 \\ -\partial_u f(u)\big|_{u=1} > 0 & 0 \end{pmatrix}, \qquad DF(0,0) = \begin{pmatrix} 0 & 1 \\ -\partial_u f(u)\big|_{u=0} < 0 & 0 \end{pmatrix},$$

by symmetry with respect to the horizontal axis, we can also conclude that $(0,0)$ is a center for the nonlinear system. Moreover, by the first integral of the system, we know that the separatrix of the saddle is given by:

$$u_x = \pm\sqrt{2(F(1) - F(u))}.$$

The following Figure 5.2 is a representation of the phase portrait in the monostable case

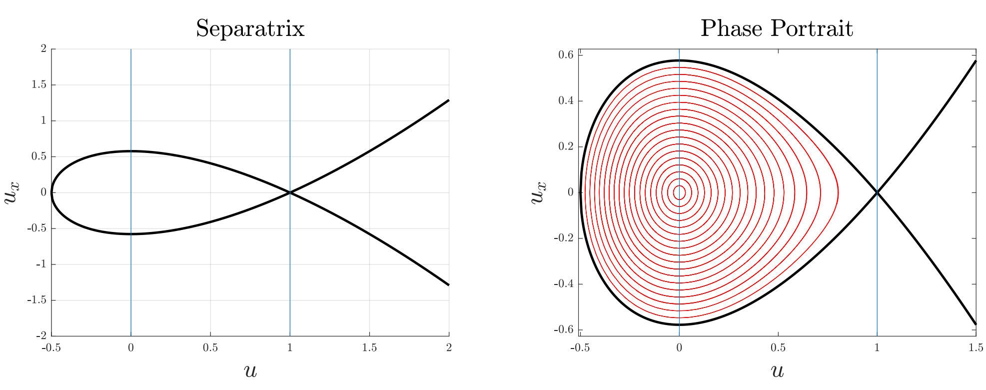

FIGURE 5.2. Left, separatrix of the system where the blue lines limit the admissible region $f(s) = s(1-s)$. Right, the phase portrait inside limit curves of the separatrix.

5.4.3. *Bifurcation diagram.* In Figure 5.3 (left), one can qualitatively visualize the bifurcation diagram for a concave nonlinearity, while in Figure 5.3 (right) the case in which it is not concave. In the horizontal axis, $\lambda = L^2$ is represented, and in the vertical one, the infinity norm of the nontrivial solutions when they exist (for further examples, see [67]).

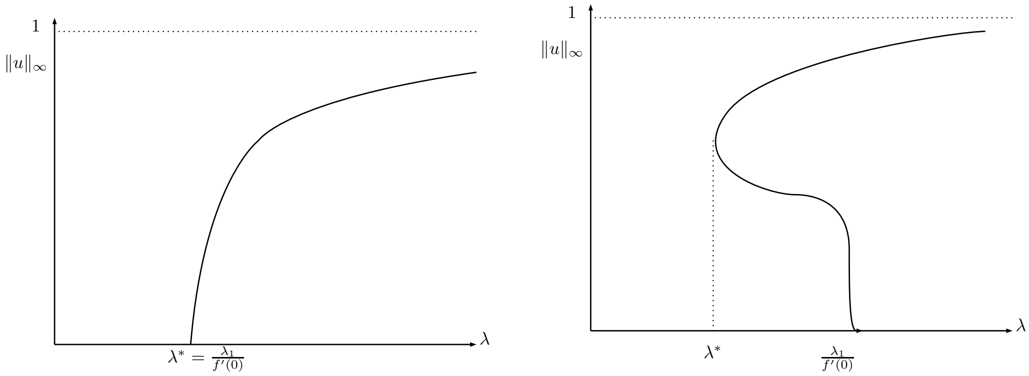

FIGURE 5.3. Qualitative Bifurcation diagram for the stationary solutions

5.4.4. *Dirichlet condition* 1. As said before, a nontrivial solution around the boundary value 1 would have the same blocking effect. However, we shall see that in this context, such a solution does not exist.

To prove the nonexistence of a solution to the problem:

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 0 < u < 1 & x \in (0,1), \\ u(0) = u(L) = 1. \end{cases} \tag{5.10}$$

we will show that for any initial datum between 0 and 1 the solution of the parabolic problem:

$$\begin{cases} \partial_t u - \partial_{xx}u = \lambda f(u) & (x,t) \in (0,1) \times \mathbb{R}^+, \\ 0 < u < 1 & (x,t) \in (0,1) \times \mathbb{R}^+, \\ u(0,t) = u(1,t) = 1 & t \in \mathbb{R}^+, \\ 0 \leq u(x,0) \leq 1 & x \in (0,1), \end{cases}$$

goes asymptotically to the constant solution $u = 1$. This will imply that there is no solution to (5.10). Indeed we can write the semilinear parabolic problem as a gradient system:

$$u_t = -\nabla_u I[u],$$

where I is:

$$I[u] = \int_0^1 \frac{1}{2}u_x^2 - \lambda F(u)dx.$$

Convergence to 1 for any admissible initial data implies that $I[u]$ has no critical point for any $u \in H^1((0,1))$ satisfying $0 < u < 1$ with Dirichlet trace equals to 1. Therefore, it does not exist any weak solution of (5.10).

For monostable nonlinearities, a Lyapunov functional exists [95]:

$$V(t) := \int_0^1 u - 1 - \log(u)dx$$

indeed,

$$\frac{d}{dt}V(t) = -\int_0^1 \frac{u_x^2}{u^2}dx - \int_0^1 \lambda f(u)\frac{1-u}{u}dx \leq 0$$

Remark 5.12 (Comparison with traveling waves). Another way to check it is by using the comparison principle with the traveling wave solution for the Cauchy problem. Indeed we know that a decreasing traveling wave function exists for monostable nonlinearities [90, Ch.4 Th. 4.5 pp.67]. Since it is decreasing and connecting 0 and 1 for any initial data $u(x,0) > 0$ of the parabolic problem, we can choose a section of the traveling wave that is strictly under $u(x,0)$. Then, the boundary conditions of this section of the traveling wave will be below 1, so this restriction of the traveling wave is a subsolution of the parabolic problem with Dirichlet data equals to 1. This argument enables us to conclude that the solution of the parabolic problem will converge to 1.

5.5. **Bistable nonlinearity.**

5.5.1. *Dirichlet condition* 0. Now we turn our attention to bistable nonlinearities. The structure of the proofs estimates will be similar.

Theorem 5.13 (A Lower bound for λ^*). *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, assume that f is bounded uniformly with respect to x. Assume furthermore that $f(x,0)=0$. Consider:*

$$\begin{cases}-\partial_{xx}v=\lambda f(x,v) & x\in(0,1),\\ 1>v>0 & x\in(0,1),\\ v(0)=v(1)=0.\end{cases}$$

Then,

$$\frac{\lambda_1((0,1))}{\max_{(x,s)\in[0,1]^2}\frac{f(x,s)}{s}}\le\lambda^*$$

Proof.

$$\lambda_1((0,1))\int_0^1 v^2dx\le\int_0^1(\partial_x v)^2dx=\int_0^1\lambda f(x,v)vdx\le\int_0^1\lambda Pv^2dx \tag{5.11}$$

with $P=\max_{(x,s)\in[0,1]^2}f(x,s)/s$. Note that if

$$\int_0^1\lambda Pv^2dx<\lambda_1((0,1))\int_0^1 v^2dx \tag{5.12}$$

we would violate (5.11). Therefore, for any $\lambda<\lambda((0,1))/P$ there cannot be a nontrivial solution, hence a lower bound on λ^* is

$$\frac{\lambda_1((0,1))}{\max_{(x,s)\in[0,1]^2}\frac{f(x,s)}{s}}\le\lambda^*$$

□

Theorem 5.14 (An upper bound for λ^*). *Assume that $f(0)=f(\theta)=f(1)=0$, and that $f'(0)<0$, $f'(1)<1$, $f'(\theta)>0$. Moreover consider $F(v)=\int_0^v f(s)ds$ and assume that $F(1)>0$. Consider the interval $[0,1]$. The following problem:*

$$\begin{cases}-\partial_{xx}u=\lambda f(u) & x\in(0,1),\\ u>0 & x\in(0,1),\\ u(0)=u(1)=0.\end{cases} \tag{5.13}$$

has a solution for every $\lambda\ge 8\frac{F(1)-F(\theta)}{F(1)^2}$.
This implies an upper bound of λ^*:

$$\frac{\pi^2}{\max_{s\in[0,1]}\frac{f(s)}{s}}\le\lambda^*\le 8\frac{F(1)-F(\theta)}{F(1)^2},$$

where the lower bound comes from Theorem 5.13.

Proof. We know that 0 is a solution of the Euler-Lagrange equations of the corresponding functional.

$$I[u]=\frac{1}{2}\int_0^1 u_x^2dx-\lambda\int_0^1F(u)dx.$$

The strategy is similar than the one of Theorem 5.6. We construct a family of functions $v_\delta\in H_0^1((0,1))$. We define v_δ in the following way:

$$v_\delta(x)=\begin{cases}\frac{2}{\delta}x & \text{if } x\in\left[0,\frac{\delta}{2}\right]\\ 1 & \text{if } x\in\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)\\ 1-\frac{2}{\delta}\left(x-\left(1-\frac{\delta}{2}\right)\right) & \text{if } x\in\left[1-\frac{\delta}{2},1\right].\end{cases}$$

One can see the function v_δ in Figure 5.4. The functions have an inner interval in which there are equal to one. Furthermore, the functions are zero in the boundary and increase linearly to 1 (where $F(1) > 0$). δ is related to the length of the inner interval in which the function is not constant.

The idea is: first ensure under which conditions on δ we have that:

$$\int_0^1 F(v(x)) > c > 0,$$

once we have this, we choose λ in order to dominate the term:

$$\frac{1}{2}\int_0^1 v_x^2 dx.$$

that will depend only on the δ chosen before and thus we constructed v such that:

$$I[v] < 0.$$

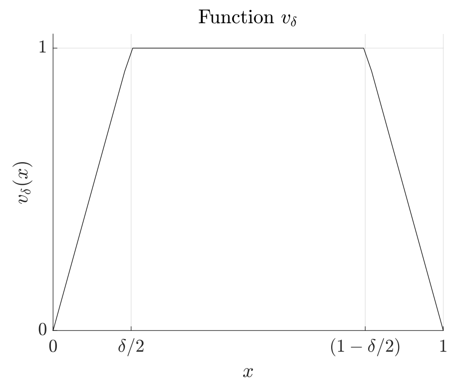

FIGURE 5.4. Function v_δ

Note that $v_\delta \in H_0^1((0,1))$. Then we have that:

$$(\partial_x v_\delta)^2 = \begin{cases} 0 & \text{if } x \in \left(\frac{\delta}{2}, 1-\frac{\delta}{2}\right) \\ \frac{4}{\delta^2} & \text{if } x \in \left(0, \frac{\delta}{2}\right) \cup \left(1-\frac{\delta}{2}, 1\right) \end{cases}$$

we want to find a pair (λ, δ) for which:

$$I[v] = \int_0^1 \frac{1}{2}|\partial_x v|^2 - \lambda \int_0^{v(x)} f(s) ds dx < 0.$$

For doing so, first we choose $\delta > 0$ to be small enough such that:

$$\int_0^1 \int_0^{v(x)} f(s) ds dx > c > 0$$

we split the space integral in two parts:

$$\begin{aligned} \int_0^1 \int_0^{v(x)} f(s) ds dx &= \int_{\frac{\delta}{2}}^{1-\frac{\delta}{2}} \int_0^{v(x)} f(s) ds dx + \int_{[0,1]\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} \int_0^1 f(s) ds dx \\ &\geq \int_{[0,1]\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} F(\theta) dx + F(1)(1-\delta) \\ &= F(1)(1-\delta) + F(\theta)\delta. \end{aligned}$$

So, it will suffice if we require that:

$$0 < \delta < \frac{F(1)}{F(1) - F(\theta)}. \tag{5.14}$$

We fix δ fulfilling (5.14) and now or goal is to choose λ big enough so that the space integral on $F(v(x))$ dominates the gradient part.

$$\begin{aligned} I[v_\delta] &= \int_0^1 \frac{1}{2}|\partial_x v_\delta|^2 - \lambda F(v_\delta(x))dx \\ &\leq \int_0^1 \frac{1}{2}|\partial_x v_\delta|^2 dx - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right) \\ &= \int_{(0,1)\setminus\left(\frac{\delta}{2},1-\frac{\delta}{2}\right)} \frac{2}{\delta^2} dx - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right) \\ &= \frac{2}{\delta} - \lambda\left(F(1)(1-\delta) + \delta F(\theta)\right) \end{aligned}$$

so, it will be sufficient if:

$$\lambda > \frac{2}{\delta\left(F(1)(1-\delta) + F(\theta)\delta\right)} \tag{5.15}$$

Hence, any pair (λ, δ) that satisfies both (5.14) and (5.15) will guarantee the existence of a nontrivial solution. Now the question is: For which choice of δ can we obtain the minimum upper bound in terms of λ?

We choose the δ in the interval $0 < \delta < \frac{F(1)}{F(1)-F(\theta)}$ such that maximizes the denominator:

$$\delta\left(F(1)(1-\delta) + F(\theta)\delta\right).$$

We note that $-F(1) + F(\theta)$ is negative, hence we have a polynomial in δ that attains its maximum in:

$$\delta^* = \frac{F(1)}{2(F(1) - F(\theta))}$$

This optimal δ^* satisfies the requirement on δ:

$$\delta^* = \frac{F(1)}{2(F(1) - F(\theta))} < \frac{F(1)}{F(1) - F(\theta)}$$

So we have that:

$$\lambda > \frac{8(F(1) - F(\theta))}{F(1)^2},$$

will be enough.

□

Remark 5.15. Notice that the structure of the proof of Theorem 5.14 also works for the monostable case. When bounding by above the integral of the primitive, we will have $F(1)$ instead of $F(1) - F(\theta)$ because the primitive in the monostable case is monotone.

Corollary 5.16 (Unexpected Corollary)**.** *Let f by any $C^2(\mathbb{R};\mathbb{R})$ function satisfying:*

- $f(0) = f(\theta) = f(1) = 0$ *with* $0 < \theta < 1$
- *consider* $F(v) = \int_0^v f(s)ds$ *and suppose that* $F(1) > 0$
- $f'(0) < 0$, $f'(\theta) > 0$ *and* $f'(1) < 0$
- $f < 0$ *in* $(0, \theta)$ *and* $f > 0$ *in* $(\theta, 1)$

Then,

$$\pi^2 \leq 8\frac{F(1) - F(\theta)}{F(1)^2} \max_{s\in[0,1]} f'(s) := Q$$

Proof. Indeed this is a corollary from the theorem proven before,

$$\pi^2 \leq 8\frac{F(1) - F(\theta)}{F(1)^2} \max_{s\in[0,1]} \frac{f(s)}{s} \leq 8\frac{F(1) - F(\theta)}{F(1)^2} \max_{s\in[0,1]} f'(s)$$

since, by the mean value theorem $\max_{s\in[0,1]} \frac{f(s)}{s} \leq \max_{s\in[0,1]} f'(s)$. □

Remark 5.17 (Open question)*.* The remarkable issue here is that Corollary 5.16 has a very mild reminiscence of the PDE theory, $\lambda_1 = \pi^2$ in $[0, 1]$. How can this general estimate be proven without the PDE theory?

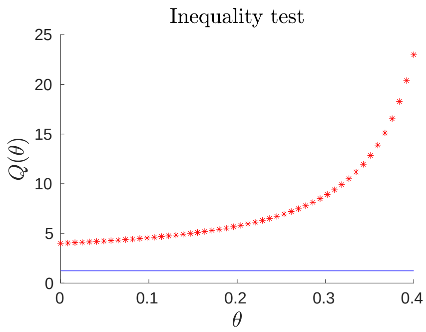

FIGURE 5.5. The blue line is π, the red dots is the quantity Q depending on θ for the prototypical example $f(s) = s(1-s)(s-\theta)$. For this nonlinearity $F(1) > 0$ is equivalent to $\theta < 1/2$. The values between 0.4 and 0.5 are not shown because the function blows up since for $\theta = 1/2$, one has that $F(1) = 0$.

Remark 5.18 (Existence of positive solutions for nonlinearities depending on space)*.* Similar arguments can be applied to show the existence of positive solutions with space-dependent nonlinearities. Consider:

$$\begin{cases} -\partial_{xx}u = f(x,u) & x \in (0,1), \\ 1 > u > 0 & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Consider that exist smooth functions $\theta, K : [0,1] \longrightarrow \mathbb{R}^+$ such that:

- $\theta(x) < K(x)$ for all x
- $f(\theta(x), x) = 0$ for all x
- $f(K(x), x) = 0$ for all x
- $f(s, x) < 0$ for all $0 < s < \theta(x)$
- $f(s, x) > 0$ for all $\theta(x) < s < K(x)$
- There exist $x^* \in [0,1]$ such that $F(K(x^*), x^*) > 0$.

Then there is a finite λ for which the positive solution exists.

Remark 5.19 (Survival of the gene)*.* The physical interpretation of the previous result is the following: Consider a population with two characteristics. Each characteristic is advantageous only in some parts of the environment. If the set in which one trait is advantageous is big enough, we cannot control to zero this characteristic employing a boundary control.

Remark 5.20 (Double blocking phenomenon)*.* Note that by constructing spatial heterogeneities like this one, one can generate nontrivial solutions with boundary 1 and 0. Setting different values of $\theta(x)$ above and below $\frac{1}{2}$ for the nonlinearity $f(y) = y(1-y)(y-\theta(x))$ one could apply these methods to prove the existence of both nontrivial solutions. By the comparison principle, this example leads to a double blocking phenomenon, already observed in [82].

Proposition 5.21 (Maximum of positive solutions)**.** *Let u be a solution to:*

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 1 > u > 0 & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

with f being bistable then the maximum of u in $[0,1]$ is above θ:

$$\max_{x\in[0,1]} u(x) > \theta.$$

Proof. The proof follows by contradiction. Assume that the maximum of u is lower or equal than θ, then the energy estimate gives us the contradiction:

$$0 < \int_0^1 u_x^2 dx = \lambda \int_0^1 uf(u) < 0$$

where the strict inequality in the left-hand side comes from the assumption that the solution is not trivial, and the right-hand side inequality comes from the fact that f is negative in $(0, \theta)$.

□

5.5.2. *Dirichlet condition* θ. Note that these results imply the thresholds for λ_θ^* for the bistable nonlinearities i.e. when the nontrivial solution to:

$$\begin{cases} -\partial_{xx} v = \lambda f(v) & x \in (0,1), \\ v > \theta & x \in (0,1), \\ v(0) = v(1) = \theta. \end{cases}$$

By hypothesis, we know that $f > 0$ in $(\theta, 1)$, $f'(\theta) > 0$, $f'(1) < 0$ and $f(\theta) = f(1) = 0$. f is monostable in $[\theta, 1]$, indeed $w = v - \theta$ does satisfy the criteria of the previous theorems.

Theorem 5.22 (The estimates of λ_θ^* for bistable nonlinearities)**.**

$$\frac{\pi}{\max_{s\in[\theta,1]} \frac{f(s+\theta)}{s-\theta}} \leq \lambda_\theta^* \leq \min\left\{\frac{2}{F(1) - F(\theta)}, \frac{\pi}{f'(\theta)}\right\}$$

If f is convex in $(0, \theta)$ and concave in $(\theta, 1)$ then:

$$\lambda_\theta^* = \frac{\pi}{f'(\theta)}$$

Leaving aside the estimates given above, one, in general, has the following result.

Theorem 5.23 (Order in the thresholds)**.** *When $F(1) > 0$ we have that:*

$$\lambda_\theta^* \leq \lambda^*$$

and fixing $\lambda > \lambda^*$ we denote by v_θ and v_0 the maximum nontrivial solution bounded by 1 of the elliptic problem with Dirichlet boundary conditions equal to θ and 0 respectively, then:

$$v_\theta \geq v_0$$

Proof. The result follows from the elliptic comparison principle, together with the fact that any nontrivial solution of the boundary value problem has its maximum above θ. □

5.5.3. *Dirichlet condition* 1. Here we will consider the problem:

$$\begin{cases} -\partial_{xx} u = \lambda f(u) & x \in (0,1), \\ 1 > u > 0 & x \in (0,1), \\ u(0) = u(1) = 1 \end{cases} \tag{5.16}$$

As mentioned before, by using comparison principles to sections of traveling waves, we can prove that we converge to 1 for any domain size.

Proposition 5.24 (Convergence to 1)**.** *For any interval $(0, L)$, the solution of the reaction diffusion system* (5.2) *with Dirichlet data equals to* 1 *converges to $u = 1$.*

Proof. We know that the problem:

$$\begin{cases} u_t - \partial_{xx} u = f(u) & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ 0 \leq u(0,x) \leq 1 & x \in \mathbb{R}. \end{cases}$$

Has a traveling wave solution, and by Theorem 3.8, we know that the traveling wave profile is a monotone function decreasing in the direction of the velocity vector. The idea is to use a segment of the traveling wave as a parabolic subsolution to our problem. Now we come back to our (parabolic) problem;

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(x,t) = 1 & (x,t) \in \{0,L\} \times (0,T), \\ 0 < u(x,0) < 1 & x \in (0,L). \end{cases} \tag{5.17}$$

Since the traveling wave profile is monotone decreasing, we can consider a segment of the traveling wave below to $u(x,0)$ in $[0,L]$. Let us denote by $TW(x)$ a traveling wave profile that satisfies:

$$TW(x) \leq u(x,0) \quad \forall x \in [0,L].$$

Now we note that the following problem:

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times \mathbb{R}^+, \\ u(x,t) = TW(x-ct) & (x,t) \in \{0,L\} \times \mathbb{R}^+ \\ u(x,0) = TW(x) & x \in (0,L), \end{cases} \tag{5.18}$$

is a subsolution of (5.17), then by the parabolic comparison principle we have that the solution of (5.17) will be above (5.18) and therefore the solution of (5.17) will converge to 1.

□

5.5.4. *Phase Portrait.* Here we study the ODE dynamics of system (5.9) for bistable nonlinearities. First of all we notice that the points $(0,0)$ (corresponding to the stationary solution $u(x) = 0$) and $(1,0)$ (corresponding to the stationary solution $u(x) = 1$) are saddles for all values of $\theta \in (0,1)$. Indeed our nonlinearity fulfills:

$$\frac{\partial}{\partial u} f(u)\big|_{u=0} < 0, \qquad \frac{\partial}{\partial u} f(u)\big|_{u=1} < 0, \qquad \frac{\partial}{\partial u} f(u)\big|_{u=\theta} > 0$$

The corresponding linearized system around $(0,0)$ and the one around $(1,0)$ have the following matrices:

$$\begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial u} f(u)\big|_{u=0} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial u} f(u)\big|_{u=1} & 0 \end{pmatrix}$$

The eigenvalues of those matrices are real and with a different sign. Then we know that for the nonlinear system, we have topological saddles. On the other hand, for $(\theta,0)$, we have that the corresponding linearized system is a center since the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ -\frac{\partial}{\partial u} f(u)\big|_{u=\theta} & 0 \end{pmatrix}$$

lie in the imaginary axis. This is not enough to conclude, but observing that the system is symmetric with respect to the horizontal axis, we have that $(\theta,0)$ is a center for the nonlinear system.

One can easily find a first integral of the system:

$$E(u,v) = \frac{1}{2}v^2 + F(u)$$

where $F(u) = \int_0^u f(s)ds$.

For $F(1) > 0$, notice that the separatrix of the saddle in 0 is the same trajectory and encloses $(\theta,0)$. Indeed, $E(0,0) = 0$, hence we have the curves $v = \pm\sqrt{-2F(u)}$ below and above the horizontal axis. At the point $0 < \theta_1 < 1$ that fulfills $F(\theta_1) = 0$, these curves meet. One can see this since $F(1) > 0$ and since $F(u) < 0$ for all $0 < u < \theta$, we know that there exist a $1 > \theta_1 > \theta$ such that $F(\theta_1) = 0$.

Notice that when $F(1) = 0$ one has that $F(u) < 0$ for all $0 < u < 1$. This means that the separatrix is split into two trajectories that connect 0 and 1 (These are the traveling wave profiles for $F(1) = 0$). We will call Γ the region in the phase-plane that the separatrix $v_{E=0}(u) = \pm\sqrt{-2F(u)}$ encloses.

Notice that Γ is included in $[0,1] \times \mathbb{R}$ which means that all arcs of any length inside Γ are admissible for our constraints.

Doing the same procedure for finding the separatrix that exit from $(1,0)$ we end up with the curves $v_{E=1}(u) = \pm\sqrt{2(F(1) - F(u))}$. At the vertical axis, $u = 0$, they take values $v_{E=1}(0) = \pm\sqrt{2F(1)}$.

Notice that, in the case of the cubic nonlinearity $f(u) = u(1-u)(u-\theta)$, as we increase θ towards $\frac{1}{2}$, $v_{E=1}(0)$ is decreasing until arriving to 0 for $\theta = \frac{1}{2}$, while, at the same time θ_1 goes to 1.

Moreover, we can also find the separatrix outside our admissible domain. We can see that

$$v_{E=1} = \pm\sqrt{2(F(1) - F(u))}$$

is well defined for $u > 1$ since $F(u)$ is a strictly decreasing function so $-F(u)$ is increasing. This means that both separatrix do not cross the horizontal axis anymore after $u = 1$. This is valid also for $F(1) = 0$.

Furthermore, the separatrix going out from 0 towards $-\infty$,

$$v_{E=0} = \pm\sqrt{-2F(u)}$$

is well defined for $u < 0$. Notice that this argument also holds for $F(1) = 0$.

See Figure 5.6 and 5.7 for the phase-portrait representation and the graphical representation of the separatrix.

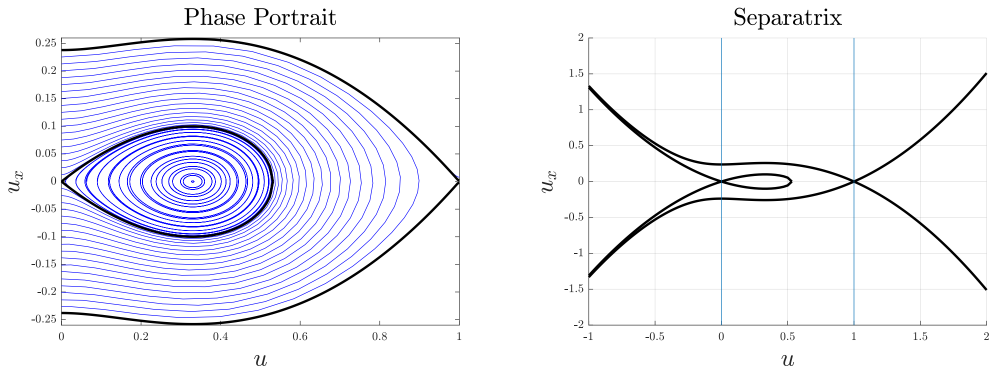

FIGURE 5.6. Phase portraits for $f(s) = s(1-s)(s-\theta)$, when $\theta < 1/2$ ($F(1) > 0$), left in the admissible region, right separatrix. The blue lines in the right hand side plot denote the admissible region.

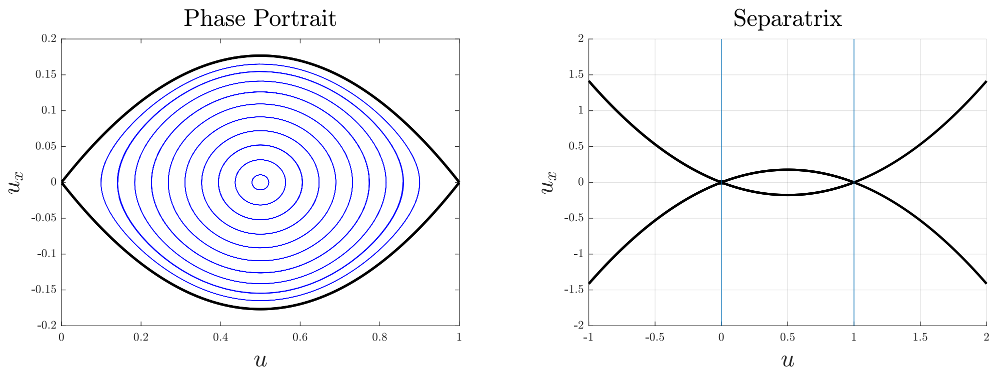

FIGURE 5.7. Phase portraits for $f(s) = s(1-s)(s-\theta)$ when $\theta = 1/2$ ($F(1) = 0$), left in the admissible region, right separatrix. The blue lines in the right hand side plot denote the admissible region.

5.5.5. *An expression for L in the phase plane.* In the particular case of one dimension, one can obtain an expression for the length L (time L in the ODE setting) in the phase portrait using the first integral of the system. We restrict our study in the curves that cross the vertical axis. The parameter $\alpha \in (0, F(1))$ is introduced.

Notice that any trajectory starting from the vertical axis until it reaches is maximum is strictly increasing. This means that it is a C^1 diffeomorphism from the time interval $[0, 1/2L(\alpha)]$ to its altitude $[0, u_{max}(\alpha)]$. Let us denote this diffeomorphism (that depends on α) by $U : [0, 1/2L(\alpha)] \to [0, u_{max}(\alpha)]$. Then noticing that $u_{max}(\alpha) = Y(L(\alpha)/2)$ and that $0 = Y(0)$:

$$\begin{aligned} L(\alpha) &= 2\int_0^{L(\alpha)/2} dz = 2\int_{U^{-1}(0)}^{U^{-1}(L(\alpha)/2)} dz \\ &= 2\int_0^{u_{max}(\alpha)} \left(U^{-1}\right)'(u)du = 2\int_0^{u_{max}(\alpha)} \frac{1}{U'(U^{-1}(u))}du. \end{aligned}$$

Now, we make use of the first integral, notice that here x plays the role of time, so the term $v = u_x = U'$. Moreover, $u_{max}(\alpha) = F^{-1}(\alpha)$ and $Y' = \sqrt{2}\sqrt{\alpha - F(u)}$. For the bistable nonlinearity F is not invertible but in the case of looking for trajectories that cross the vertical axis and that are below the separatrix going out from 1 we notice that in that set is invertible,

$$F : [\theta_1, \sqrt{2F(1)}] \to [0, F(1)]$$

is monotone increasing.

$$L(\alpha) = \sqrt{2}\int_0^{F^{-1}(\alpha)} \frac{1}{\sqrt{\alpha - F(U(U^{-1}(u))}} du = \sqrt{2}\int_0^{F^{-1}(\alpha)} \frac{1}{\sqrt{\alpha - F(u)}} du$$

Changing the way of parameterizing an expression for L^* can be written in the following way:

$$L^* = \inf_{\beta\in(\theta_1,1)} \sqrt{2}\int_0^{\beta} \frac{du}{\sqrt{F(\beta) - F(u)}}$$

Using the aforementioned expression in [96] the authors prove the following thresholds

Proposition 5.25 (Proposition 4 in [96])**.** *If f is C^2, monostable and the following holds:*

$$f^2 \geq 2Ff' \quad \text{on } [0,1]$$

then,

$$L^* = \frac{\pi}{\sqrt{\frac{\pi^2}{f'(0)}}}$$

5.5.6. *Bifurcation diagrams.* This subsection shows bifurcation diagrams for some bistable nonlinearities and graphics of the bounds obtained before.

The blue and the red lines in Figure 5.8 represent the nontrivial solutions for Dirichlet boundary 0 and θ respectively. For the blue curve, the vertical axis is the infinity norm. For the red curve, the vertical axis is the infinity norm for when the curve is above θ and $\theta - \|u-\theta\|_\infty$ when the curve is under θ. In this way, it is showing the minimum value taken since it corresponds to solutions for the following problem:

$$\begin{cases} -\partial_{xx}v = \lambda f(v) & x \in (0,1) \\ 0 < v < \theta & x \in (0,1) \\ v(0) = v(1) = \theta \end{cases}$$

Figure 5.9 show the bounds for the nonlinearity $f(s) = s(1-s)(s-\theta)$ for different values of θ.

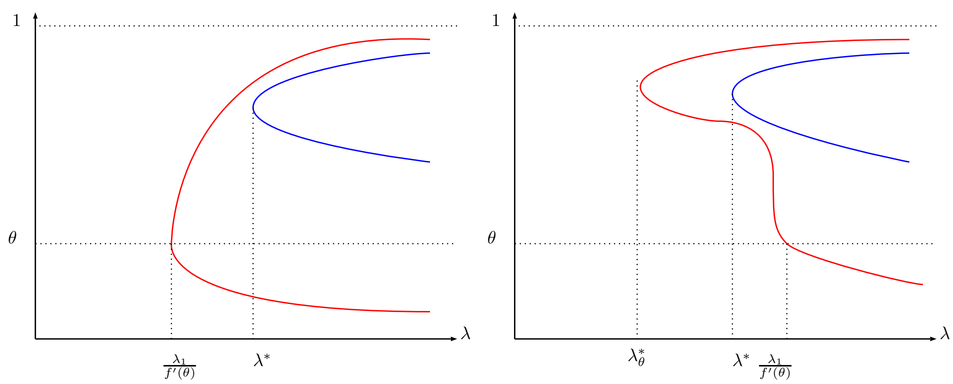

FIGURE 5.8. Qualitative Bifurcation diagram for the stationary solutions for a bistable nonlinearity that is convex in $(0,\theta)$ and concave in $(\theta,1)$ (Left), and a bistable nonlinearity that is convex in $(0,p)$ with $p > \theta$ and concave in (p,θ) as $f(s) = s(1-s)(s-\theta)$ for $\theta < 1/2$.

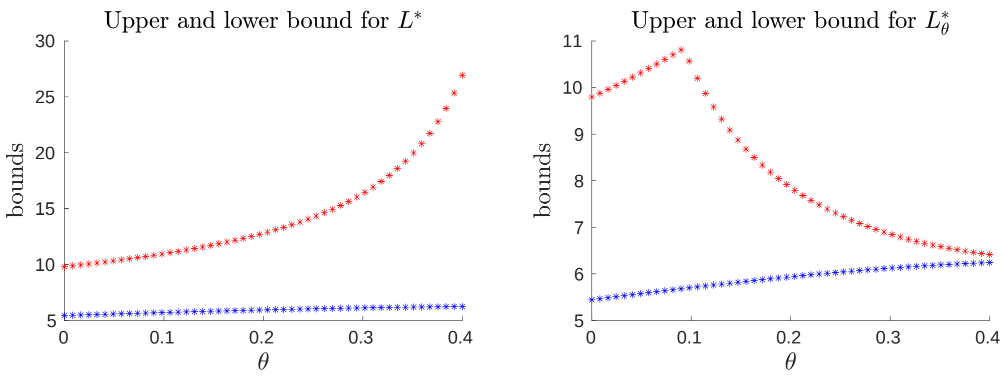

FIGURE 5.9. Bounds on L^* (Left) and L^*_θ (Right) for different values of θ for the nonlinearity $f(s) = s(1-s)(1-\theta)$. The red dots represent upper bounds and the blue dots lower bounds.

Remark 5.26. Note that the lower bound proven for λ^* and λ^*_θ is the same, in general, to our knowledge, we do not know if $\frac{\lambda_1((0,1))}{f'(\theta)}$ is smaller or bigger than λ^*.

Remark 5.27 (Further bifurcations). More bifurcations for the boundary value θ can occur increasing λ, the solutions that bifurcate and have oscillations around θ are not represented in the diagrams. It has to be said that those solutions will appear after the nontrivial solution above θ and the nontrivial solution below θ have appeared. The reason is that an oscillatory solution of:

$$\begin{cases} -\partial_{xx}u = \lambda f(u) & x \in (0,1), \\ 0 < u < 1 & x \in (0,1), \\ u(0) = u(1) = \theta, \end{cases}$$

is above/below θ in a set smaller measure than the original set. Therefore, the domain should be big enough for this smaller subset to be above the thresholds. This reasoning holds for any dimension using the monotonicity of the eigenvalues. Indeed, we have that if $D \subset \Omega$ then $\lambda_1(\Omega) \leq \lambda_1(D)$ (see [49, Section 1.3.2])

Remark 5.28 (Harmonic Oscillator). Note that if we linearize the ODE dynamics associated with the elliptic problem around $(\theta, 0)$, one obtains the harmonic oscillator. Indeed the linearized system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -f(\theta)u \end{pmatrix},$$

corresponds to

$$\partial_{xx}u = -f'(\theta)u \tag{5.19}$$

We observe that $f'(\theta)$ corresponds to the frequency of the oscillations. In this way, one can intuitively understand how nontrivial solutions that are close to $v \equiv \theta$ can exist or not. The general solution of (5.19) is:

$$u(x) = A\sin\left(\sqrt{f'(\theta)}x\right) + B\cos\left(\sqrt{f'(\theta)}x\right).$$

Imposing Dirichlet conditions lead us to the choice of $B = 0$. Then, one can see that a critical length is $L = \frac{\pi}{\sqrt{f'(\theta)}}$.

This length is the critical length for which the stationary solution $v \equiv \theta$ becomes unstable in the first eigenfunction. Consider

$$\begin{cases} \partial_t u - \partial_{xx}u = f(u) & (x,t) \in (0,L) \times (0,T) \\ u(0) = u(L) = \theta \end{cases}$$

and linearize around $u \equiv \theta$:

$$\begin{cases} \partial_t \tilde{u} - \partial_{xx}\tilde{u} = f'(\theta)\tilde{u} & (x,t) \in (0,L) \times (0,T), \\ \tilde{u}(0) = \tilde{u}(L) = 0. \end{cases} \tag{5.20}$$

The first eigenvalue of (5.20) is $\frac{\pi^2}{L^2} - f'(\theta)$, which becomes unstable when $L^2 > \frac{\pi^2}{f'(\theta)}$. The same reasoning can be applied for further bifurcations while linking it with the instability.

6. Numerical visualization of the barriers for the 1-D problem

In this section, we aim to illustrate with figures the results obtained in the previous section. We will consider this specific 1-D problem

$$\begin{cases} -\partial_{xx} u = \lambda u(1-u)(u-\theta) & x \in (0,1) \\ u(0) = u(1) = a \\ 0 \leq u \leq 1 & x \in (0,1). \end{cases}$$

By doing the change of variables $v = u - a$, we want to see the change of the energy functional

$$J_\lambda(v, a) = \int_0^1 \frac{1}{2} v_x^2 - \lambda \left\{ \int_0^{v(x)} f(s+a) ds \right\} dx,$$

depending on the parameter λ and the Dirichlet condition a.

For representing these energy functionals, we consider the energy functional along the first and third eigenvector of the Dirichlet Laplacian. We will plot J in the finite-dimensional subspace V generated by:

$$e_1(x) := \sin(\pi x), \qquad e_3(x) := \sin(3\pi x).$$

The functional $J : V \to \mathbb{R}$ can be expressed in terms of the coordinates with respect to the eigenfunctions showed before:

$$J_\lambda^{e_1,e_3}(\alpha, \beta, a) := J_\lambda(\alpha e_1 + \beta e_3, a) = \int_0^1 \frac{1}{2} |\alpha \partial_x e_1(x) + \beta \partial_x e_3(x)|^2 - \lambda \left\{ \int_0^{\alpha e_1 + \beta e_3} f(s+a) ds \right\} dx,$$

α and β are the coordinates with respect to the e_1 and e_3 direction respectively.

Moreover, we observe how other critical points may appear for the Dirichlet condition 0 as increasing the measure of the domain $\lambda = L^2$ (see Figure 6.1). We have to emphasize that these plots show the functional along with two directions, and they are only illustrative.

A Mountain pass is a saddle point of a functional. Observe that, despite the finite dimensional representation of the functional, critical points cannot be local maxima. A steady-state can not be a local maximum since there will always exist a high-frequency perturbation for which the steady-state will be stable, which is a consequence of the fact that the eigenvalues of the Laplacian tend to infinity, $\lambda_n \to +\infty$. In Figure 6.3, one can see two local minima three Mountain Passes.

The intention of Figure 6.2 is to represent how the functional evolves when one changes λ for understanding why the solution of:

$$\begin{cases} -\partial_{xx} u = \lambda f(u) & x \in (0,1), \\ 1 > u > \theta & x \in (0,1), \\ u = \theta, \end{cases}$$

appears for a smaller value of λ than the solution of:

$$\begin{cases} -\partial_{xx} u = \lambda f(u) & x \in (0,1), \\ 0 < u < \theta & x \in (0,1) \\ u = \theta. \end{cases} \tag{6.1}$$

In Figure 6.4, it has been performed a minimization with IpOpt [119] of the discretized functional:

$$\begin{cases} \min\limits_{v \in H_0^1(0,1) \cap C} \displaystyle\int_0^1 \frac{1}{2} v_x^2 - \lambda(F(v+a) - F(a)) dx \\ C := \{v \in L^\infty(0,1) \quad \text{s.t.} \quad -b_1 < v(x) < b_2, \quad b_1, b_2 \geq 0\} \end{cases}$$

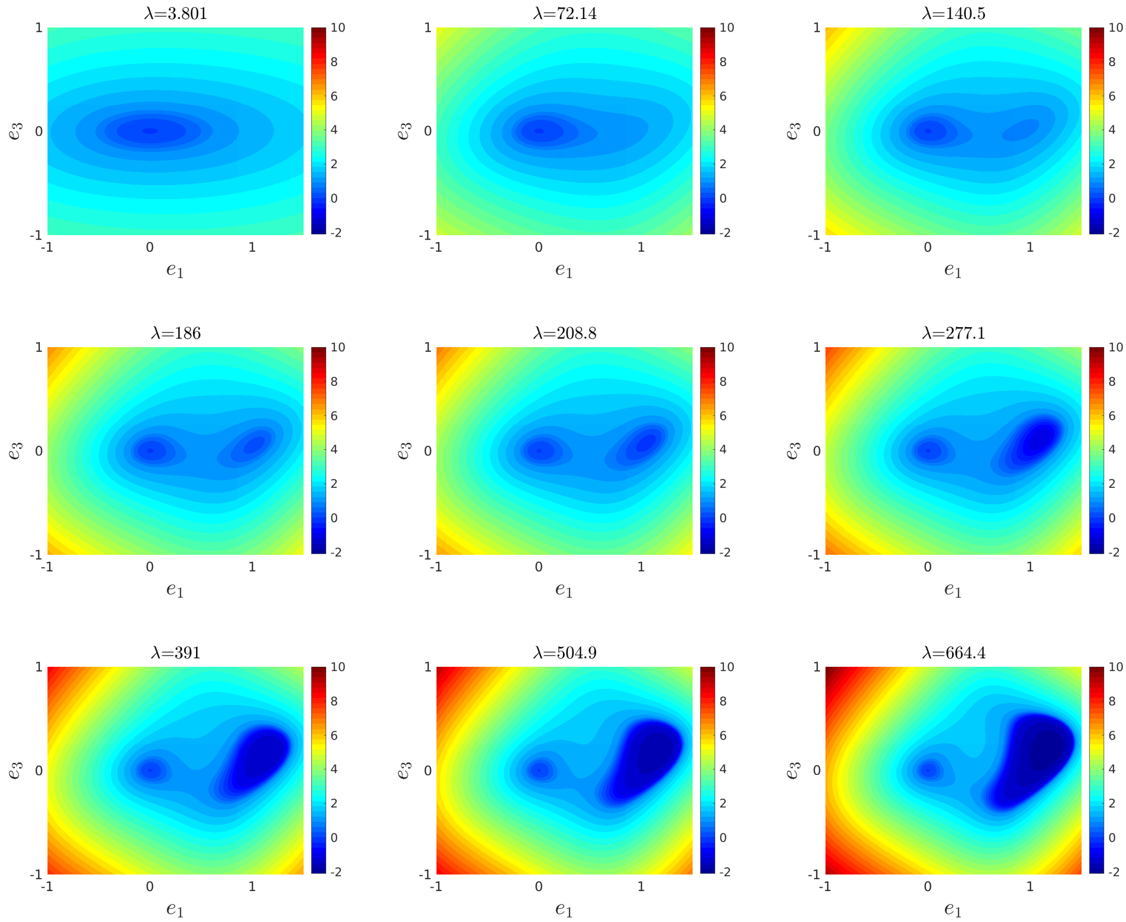

FIGURE 6.1. Functional $J_\lambda^{e_1,e_3}(\alpha,\beta,0)$ for different values of λ

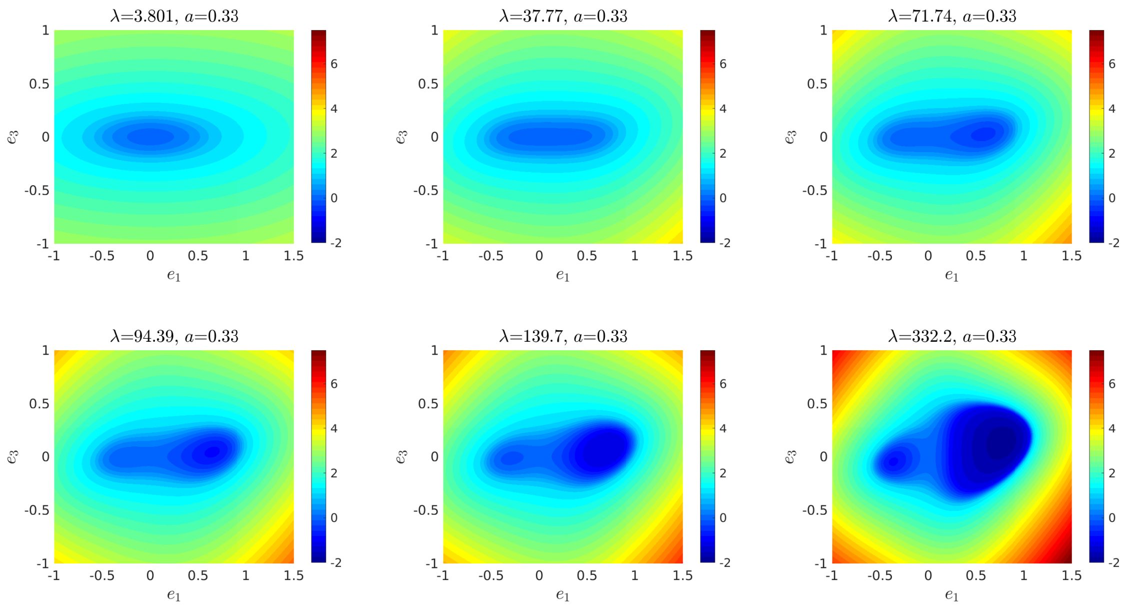

FIGURE 6.2. In these figures the value of $J_\lambda^{e_1,e_3}(\alpha,\beta,\theta = 0.33)$ is contrasted against different pairs of α,β for different λ

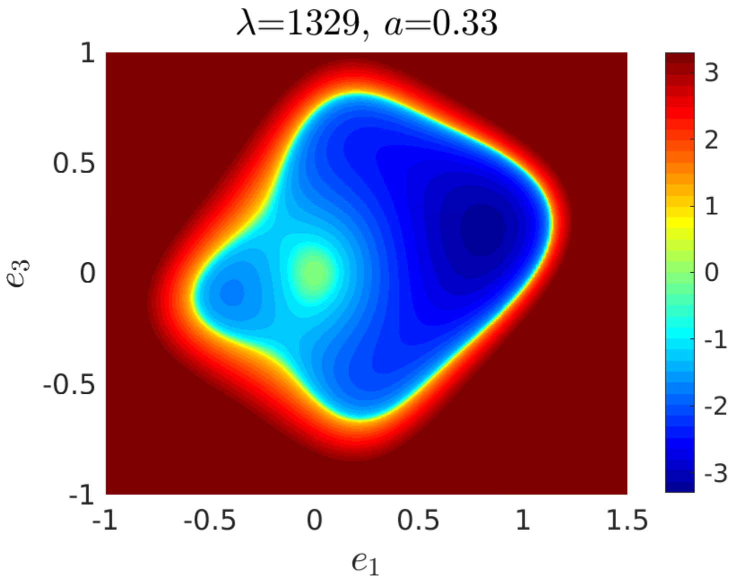

FIGURE 6.3. Energy functional $J_\lambda^{e_1,e_3}(\alpha,\beta,\theta)$. The values above 3 are not represented in the picture.

In Figure 6.4, one can observe different solutions with different boundary conditions, and, in green, we show a section of the nontrivial solution in the whole domain $\mathbb{R}$

$$\begin{cases} \partial_{xx}u = f(u) & x \in \mathbb{R}, \\ 0 < u < 1 & x \in \mathbb{R}. \end{cases}$$

The solution in the whole $\mathbb{R}$ corresponds to the homoclinic orbit around $(0,0)$ in the phase plane already presented in the previous section.

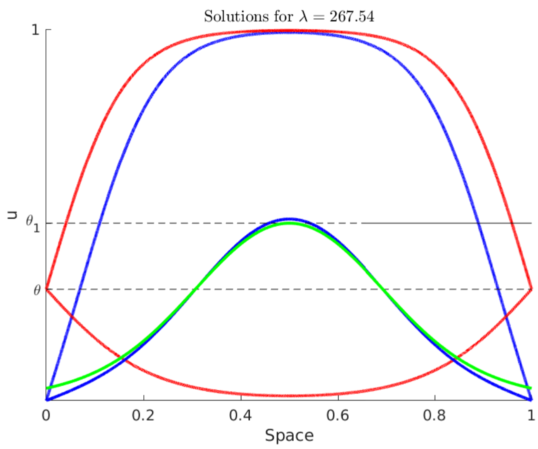

FIGURE 6.4. The blue (red) lines are for nontrivial solutions with Dirichlet condition 0 (respectively θ). In green, a section of the solution whole space $\mathbb{R}$.

7. The construction of paths of steady-states and the control strategy

In this section, we will focus on the bistable nonlinearity and its control to the constant steady-state θ as a paradigm of study for controlling to unstable equilibria using the staircase method.

In the previous sections, we have understood when barriers can appear. Whenever we are in such a setting, we cannot expect that for all initial data, we can reach steady-states such as 0 or θ.

The steady-states 0 and 1 are linearly stable; this can be seen from the linearization and the fact that $f'(1) < 0$ and $f'(0) < 0$. However, since the steady-state 0 or 1 are respectively a subsolution and a supersolution, we cannot attain these steady-states in finite time.

On the other hand, the constant steady-state θ might be unstable or might not be the unique elliptic solution with boundary value θ. For these reasons, understanding how we can attain θ without violating the physical constraints is not obvious. When there is multiplicity of elliptic solutions with boundary value θ, the trivial strategy of setting the boundary condition equal to θ does not work. In this section, we will understand how we can reach the steady-state θ, avoiding both the instability and the multiplicity issues.

The staircase method ensures that if we have a connected path of admissible steady-states in the interior of the admissibility region, we can control from one steady-state to the other. Here we will devote some attention on the understanding the construction of admissible paths of steady-states for the one-dimensional problem. Then, once the paths have been constructed, the control strategy consists of two phases:

1. *Dynamic phase.* For any initial data, find a control function a such that is able to steer system to an element of the path.
2. *Quasistatic phase* Application of the staircase method in the constructed path to reach the target in large time.

We will split the discussion in two situations, when $F(1) > 0$ (remind that $F(t) = \int_0^t f(s)ds$), in such case barriers can exist and one cannot have controllability to θ for all admissible initial data and the situation in which $F(1) = 0$. In such case, barriers do not exist and the controllability to θ holds for any admissible initial data regardless of the stability of θ. The case $F(1) < 0$ is analogous to the $F(1) > 0$ by reversing the roles of 0 and 1. We will also analyze a bit further which states are path connected to the steady-state 0 and the steady-state θ.

7.1. Case $F(1) > 0$.

Recall that, we understand by bistable the following: $f < 0$ on $(0, \theta)$ and $f > 0$ on $(\theta, 1)$ assuming that $f'(0) < 0$, $f'(1) < 0$, and that $f'(\theta) > 0$. In this subsection, we assume that the primitive, $F(u) = \int_0^u f(s)ds$, evaluated at 1 is positive, $F(1) > 0$. We denote by θ_1 the value different from 0 such that $F(\theta_1) = 0$. In the prototypical example of bistable nonlinearity, $f(u) = u(1-u)(u-\theta)$, one has that $F(1) = 0$ when $\theta = 1/2$ and $F(1) > 0$ corresponds to $\theta < 1/2$.

Hereafter we present the strategy in [96] for finding the connected path of steady-states. The authors make use of the phase portrait to find a path of steady-states for ensuring under certain conditions the constrained controllability of the one-dimensional problem:

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = a_1(t) & t \in (0,T), \\ u(L,t) = a_2(t) & t \in (0,T), \\ 0 \le u(x,0) \le 1 & x \in (0,L). \end{cases}$$

We are looking for a path of steady-states with fixed length that connects to the stationary solution θ

$$\begin{cases} -\partial_{xx} u = f(u) & x \in (0,L), \\ u(0) = a_1, \quad u(L) = a_2, \end{cases} \tag{7.1}$$

We emphasize that it might not always be possible to do this for any initial data. Indeed, if it appears a stationary solution, then, by the parabolic comparison principle, we will not be able to steer any initial data to a steady-state with a small maximum. In particular, when this solution appears, its maximum is above θ therefore it also blocks the controllability to the steady-state θ.

7.1.1. *Control strategy in [96].* The authors choose L so that 0 is the only steady-state of the problem. In this way, setting the control to 0, one approaches asymptotically this steady-state. Afterwards, whenever we are close to 0 the phase of attachment to the path starts.

Definition 7.1. We call Γ the region in the phase-plane that the homoclinic orbit $v_{E=0}(u) = \pm\sqrt{-2F(u)}$ encloses i.e.

$$\Gamma := \left\{(u,v) \in [0,\theta_1] \times \mathbb{R} : \quad -\sqrt{-2F(u)} \le v \le \sqrt{-2F(u)}\right\}.$$

with $\theta_1 \in (0,1)$ being $F(\theta_1) = 0$.

More specifically one proceeds in the following way:

(1) **Stabilize to** 0. Set both controls to 0. For $L < L^*$ the solution will approach the stationary solution 0 in the L^∞ norm.
(2) **Attach to a path of steady-states**. Whenever the maximum is below θ, say ϵ, set ϵ as a boundary value. $v \equiv \epsilon$ is a parabolic supersolution, for this reason, the solution is going to converge to a steady-state with boundary value ϵ which is below ϵ and above 0.

$$\begin{cases} -\partial_{xx} v = f(v) & x \in (0,L) \\ v(0) = \epsilon = v(L) \\ 0 < v < \epsilon \end{cases}$$

Let us refer to these nontrivial steady-states as v_ϵ. Wait a long time and apply local controllability to the steady-state v_ϵ at which we were converging. In this process, we can guarantee that we do not violate the constraints provided that we wait enough time (see [91, Lemma 8.3]).
(3) **Construct the path to** θ. Now we find the connected path of steady-states that bring us to the stationary solution θ for the Dirichlet condition being equal to θ. Take the Neumann trace of v at 0 and consider the following family of boundary conditions for the second-order ODE of the steady-states:

$$\begin{cases} v^s(0) = (1-s)\epsilon + s\theta \\ \partial_x v^s(0) = \partial_x (1-s) v_\epsilon \end{cases}$$

solve from $x = 0$ to $x = L$:

$$\begin{cases} -\partial_{xx} v^s = f(v) \\ v^s(0) = (1-s)\epsilon + s\theta \\ \partial_x v^s(0) = \partial_x (1-s) v_\epsilon \end{cases}$$

The set of boundary conditions that we are looking for is

$$\begin{cases} u_1^s = v^s(0) \\ u_2^s = v^s(L) \end{cases}$$

This path ends at θ by uniqueness of the solution of the ODE system:

$$\begin{cases} -\partial_{xx} v = f(v) \\ v(0) = \theta \\ \dfrac{\partial}{\partial x} v(0) = 0 \end{cases}$$

(4) **Application of the staircase method** with the family generated before.

The set of solutions is connected due to continuous dependence of the initial data.

The last property to verify is that the boundary condition $u_2^s = v^s(L)$ and the solution v^s is between 0 and 1. This is a consequence of these two facts:

- There is an invariant region Γ such that $(0,0) \in \partial\Gamma$ and $(\theta, 0) \in \Gamma$. Moreover, Γ is included in the admissible set of states and it is star-shaped with respect to $(\theta, 0)$. This is a consequence of the fact that $f(s) < 0$ in $(0,\theta)$ and $f(s) > 0$ in $(0,\theta_1)$.
- In step (2), we have reached a stationary curve that lies inside Γ. (Proof in Proposition 7.2)

- The control strategy in [96] is based on tracing a line between the Neumann and Dirichlet traces in one extreme of the curve obtained in step 2 and the Dirichlet and Neumann trace of the target (the point $(\theta, 0)$). This line lies inside the invariant region Γ; hence, by solving the ODE problem until $x = L$, we obtain a set of stationary solutions such that all of them lie in Γ and hence are admissible.

Proposition 7.2 (Invariant region Γ)**.** Γ *is an invariant region.*

Proof. Γ is enclosed by a homoclinic curve; by the uniqueness of the ODE, the result follows. □

Remark 7.3. Note that the convexity assumption on Γ is not needed. Indeed we need to require that there can exist a continuous curve $l : [0,1] \to \mathbb{R}^2$ such that $l(s) \in \Gamma$ for all $s \in [0,1]$ connecting the point $l(0) = (v(0), \partial_x v(0))$ to $l(1) = (\theta, 0)$.

Figure 7.1 is an illustration of the procedure described before.

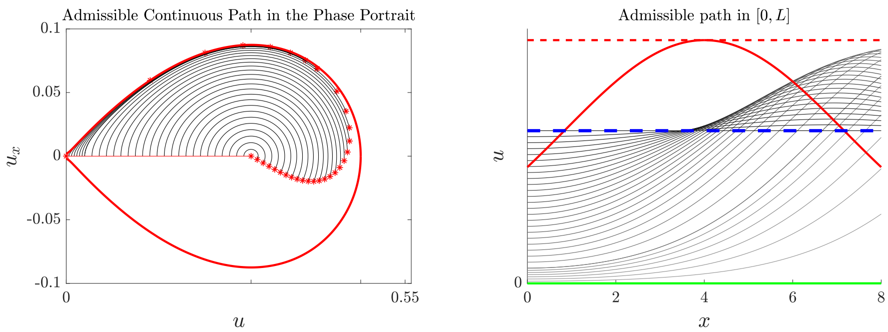

FIGURE 7.1. Strategy of [96] in the phase portrait of system (5.9). In black the representation of the steady-states of (7.1) that are part of a connected path that connects to the stationary solution θ. In red the values of the control that can be taken. $L = 8 > L_\theta$, $\theta = 0.33$. In the right, the stationary path plotted in the space domain, the red curve is the curve of maximum value in the invariant region Γ, in green the initial condition.

7.1.2. *Connected symmetric path.* A connected path of steady-states can be constructed following another strategy than the one used in [96] that provides controls that take the same value on both sides. The improvement is that we only require a function instead of two and enables us to control the problem with the control on one side and Neumann boundary conditions on the other boundary and the construction of the path is independent on the geometry of Γ. Moreover, we will see that looking at symmetric paths leads to an easier extension to several dimensions using radial steady-states.

The idea follows from the phase plane's symmetry (and the symmetry of radial solutions of the Poisson equation).

- Instead of considering the boundary conditions at one extreme $x = 0$ as done in [96]:

$$\begin{cases} v^s(0) = (1-s)\epsilon + s\theta \\ \partial_x v^s(0) = \partial_x (1-s) v_\epsilon, \end{cases}$$

  we consider the condition at the middle point $x = L/2$. By the radial symmetry, we know is a critical point of the solution of ODE, and hence it lies in the horizontal axis of the phase portrait. We now solve the ODE from $x = L/2$ to $x = L$ (or backward to $x = 0$) for obtaining the necessary conditions in the boundary. The set of steady-states is then

$$v^s(L) = v^s(0) = a^s \quad s \in [0,1]$$

  where a^s is the projection in the first component of the solution at $x = L$ of the Cauchy problem. In other words, let $\Phi\left(x; \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, L/2\right)$ be the solution at x of $\frac{d}{dx}\begin{pmatrix} v \\ v_x \end{pmatrix} = \begin{pmatrix} v_x \\ -f(v) \end{pmatrix}$ with initial

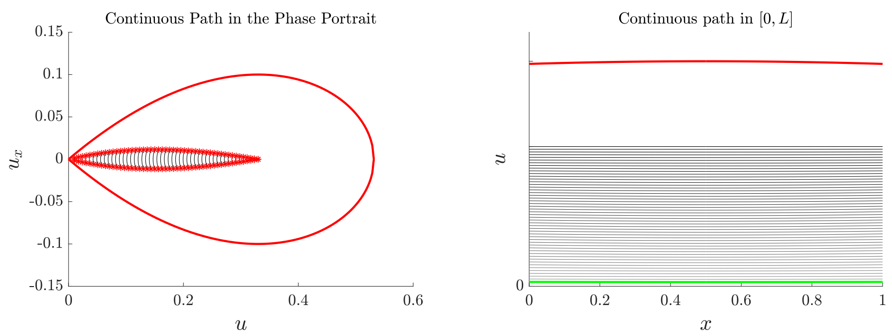

FIGURE 7.2. The even strategy is represented in the phase portrait of system (5.9). In black, the representation of the steady-states of (7.1) that are part of a connected path that connects to the stationary solution θ. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 1$, $\theta = 0.33$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

data at $x = L/2$ being $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and consider:

$$a^s := P_1\Phi\left(L; \begin{pmatrix}(1-s)v_\epsilon(L/2) + s\theta \\ 0\end{pmatrix}, L/2\right),$$

where P_1 is the projection on the first component, and v_ϵ is the even stationary solution in which we arrived after step 2 in the procedure above.

- a^s is continuous with respect to s due to continuous dependence of the initial data. There is a continuously connected set of solutions to the boundary value problem associated with a^s by construction.
- Notice that the trajectories lie in Γ since the path $(v_\epsilon(L/2)(1-s) + s\theta)$ is in the horizontal axis inside the invariant region Γ.

Remark 7.4. Notice that this procedure does not depend on L; the only requirement is to be able to reach an even stationary state inside the invariant region Γ. This means that for every $L > 0$ there exists a path connecting 0 with θ.

Figures 7.2 7.4 7.6 are the representations of the proof for different values of L, for $L < L^*_\theta$, for $L^*_\theta < L < L^*$ and $L^* < L$. Figures 7.3 7.5 and 7.7 represent the value at the boundary of the continuous path of steady-states for different values of L. Finally Figures 7.5 and 7.7 are bifurcation diagrams depending on a, i.e. as we rise the Dirichlet condition bifurcations or unifications of solutions might appear. Indeed the connected path uses this feature to reach θ from 0.

The key question is if we are able to reach Γ for certain initial conditions of the reaction-diffusion system even though the barrier has already appeared.

Do they exist connected paths of steady-states with even controls that connect certain steady-states to Γ ? For which initial conditions of the reaction-diffusion system, can we reach one of these connected components?

7.2. **A non returning path.**

The following result basically is giving an insight on which are the admissible steady-states from which a connected path of steady-states can be constructed towards θ and 0.

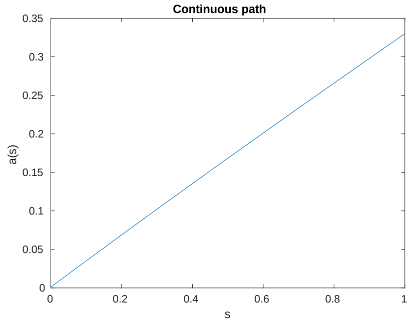

FIGURE 7.3. Connected path of steady-states. The value in the boundary in the vertical axis, in the horizontal axis the parameter $s \in [0,1]$ in the vertical axis the value of the boundary

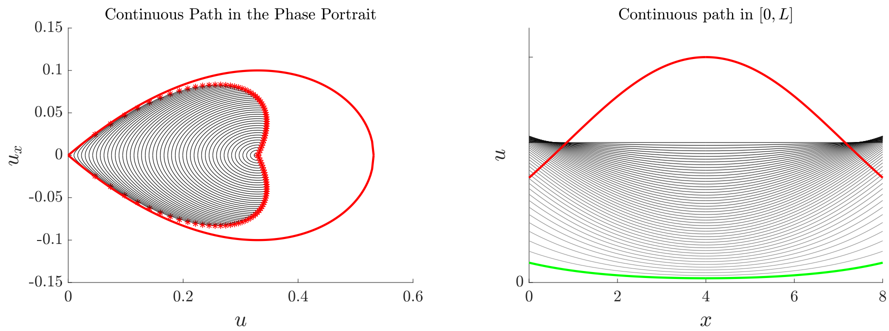

FIGURE 7.4. The even strategy is represented in the phase portrait of system (5.9). In black, the representation of the steady-states of (7.1) that are part of a connected path that connects to the stationary solution θ. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 8$, $\theta = 0.33$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

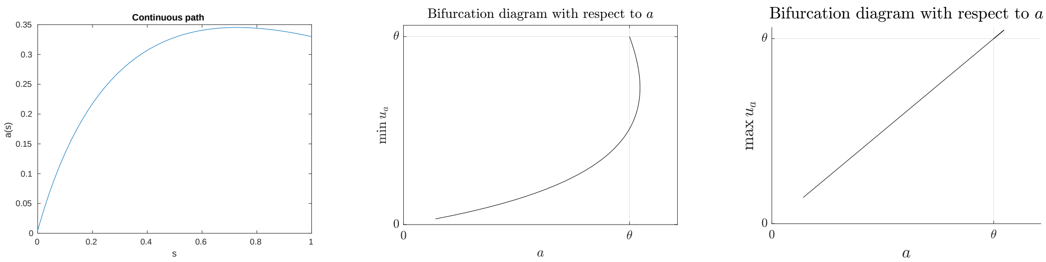

FIGURE 7.5. (Left) Connected path of steady-states. The value in the boundary in the vertical axis, in the horizontal axis the parameter $s \in [0,1]$ in the vertical axis the value of the boundary. (Center) The minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. (Right) The maximum value $\max_{x\in[0,L]} u(x)$ against a

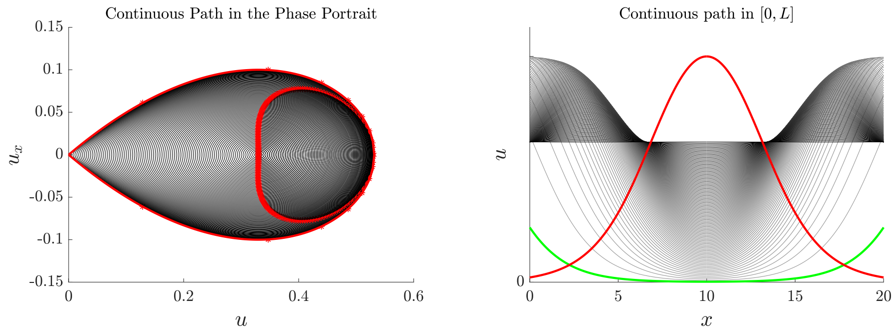

FIGURE 7.6. The even strategy is represented in the phase portrait of system (5.9). In black, the representation of the steady-states of (7.1) that are part of a connected path that connects to the stationary solution θ. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 20$, $\theta = 0.33$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

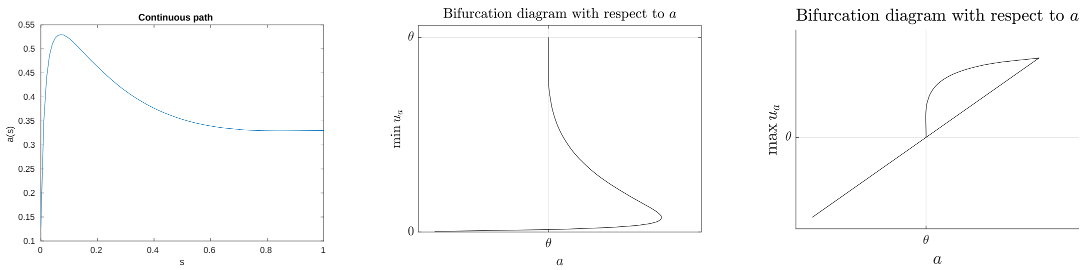

FIGURE 7.7. (Left) Connected path of steady-states. The value in the boundary in the vertical axis, in the horizontal axis the parameter $s \in [0,1]$ in the vertical axis the value of the boundary (Center) The minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. (Right) The maximum value $\max_{x\in[0,L]} u(x)$ against a.

Proposition 7.5. *Let w be an admissible steady-state fulfilling any admissible boundary conditions. Let $\underline{u}_{L^*}$ be the minimum solution with respect to the L^∞ norm of the problem:*

$$\begin{cases} -\partial_{xx}\underline{u}_{L^*} = f(\underline{u}_{L^*}) & x \in (0,L), \\ 0 < \underline{u}_{L^*} < 1 & x \in (0,L), \\ \underline{u}_{L^*}(0) = \underline{u}_{L^*}(L^*) = 0 \end{cases}$$

- *If* $\max w(x) \le \max_{x\in[0,L]} \underline{u}_{L^*}$ *and*
- $$\frac{1}{2}w_x(0)^2 + F(w(0)) \le F\left(\max_{x\in[0,L]} \underline{u}_{L^*}\right)$$

Then there is a connected path of steady-states from w to θ and from w to 0. Moreover, if w is symmetric, the second condition is not needed, and there exists a path that maintain even boundary conditions.

Proof. (Sketch) Let Λ denote the closed region between $\underline{u}_{L^*}$ and the vertical axis in the phase plane. Given any point x in Λ fulfills conditions of Proposition 7.5.

The idea of the proof is to generate first a path towards a symmetric steady-state $w^*(x)$. This can be done using the trajectory from where the initial steady-state belongs to in the phase portrait (note that the first integral $\frac{1}{2}w_x^2+F(w)$ is preserved along the trajectory). Then since $\frac{1}{2}w_x(0)^2+F(w(0)) \le F(\max_{x\in[0,L]} \underline{u}_{L^*})$ we know that $F(\max_{x\in[0,L]} w^*(x)) \le F(\max_{x\in[0,L]} \underline{u}_{L^*})$. Then we use the same argument as in the previous section (taking the Dirichlet and Neumann conditions in $L/2$) to build a connected path of steady-states. The admissibility of them follows by the fact that $\underline{u}_{L^*}$ is the minimum solution of the elliptic problem and that the boundary of Γ is a homoclinic orbit (hence it is associated with a solution to the problem in the whole real line). Indeed, any symmetric admissible trajectory in the phase plane in $\mathrm{Int}(\Lambda - \Gamma)$ will not attain the axis $u = 0$; otherwise, $\underline{u}_{L^*}$ will not be the minimum solution. □

The Staircase method requires a path that such that its boundary is not saturating the constraints. Here we show that this assumption is necessary by finding a path violating such condition and for which there is no controllability. Indeed we prove the following propositions.

Proposition 7.6. *When $L > L^*$ there is an admissible connected path of steady-states continuous with respect to the L^∞ topology that connects the stationary solution $u = 0$ with the minimum solution of the following problem:*

$$\begin{cases} -\partial_{xx} u = f(u) & x \in (0, L) \\ u(0) = u(L) = 0 \\ 1 > u > 0 & x \in (0, L) \end{cases}$$

Proof. Consider the following ODE system depending on the parameter $s \in [0, 1]$

$$\begin{cases} y_x = v \\ y_{xx} = v_x = -f(y) \\ y(L/2) = s \\ y_x(L/2) = 0 \end{cases} \tag{7.2}$$

As before, we will use this system to construct a path of stationary steady-states. We start for $s = 0$. By the uniqueness of the ODE, we have that $y(x) = 0$ is the only solution. Then, consider the family of solutions of (7.2) depending upon the parameter s that are defined on $x \in [0, L]$ (solving the ODE forward in time and backward). Let $s^* \in [0, 1]$ be such:

$$s^* = \min_{s\in[0,1]} \{\|y^s\|_{L^\infty([0,L])} \text{ such that } y^s \text{ is a solution of (7.2) and } y^s(L) = y^s(0) = 0 \text{ and } \|y^s\|_{L^\infty([0,L])} \leq 1\}$$

This $s^* \in [0, 1]$ exists because $L > L^*$. The path of steady-states is:

$$\begin{aligned} \gamma : [0, s^*] &\longrightarrow L^\infty([0, L]) \\ s &\longrightarrow y^s \end{aligned}$$

By construction, this path is admissible. Moreover, by continuous dependence on the initial data, we have that the path is continuous: Let

$$\begin{aligned} y' &= F(y) \\ y(0) &= y_0 \end{aligned}$$

and let

$$\begin{aligned} z' &= F(z) \\ z(0) &= z_0 \end{aligned}$$

we have that:

$$\begin{aligned} \|y(x) - z(x)\|_\infty &= \left\| y_0 - z_0 + \int_0^x F(y(r)) - F(z(r))dr \right\|_\infty \\ &\leq \|y_0 - z_0\| + \int_0^x \|F(y(r)) - F(z(r))\|_\infty dr \\ &\leq \|y_0 - z_0\| + \int_0^x \sup_{x\in\mathbb{R}^2:\|x\|\leq\max\{\|y\|_{L^\infty},\|z\|_{L^\infty}\}} \|\nabla F\|_\infty \|y(r) - z(r)\| dr \\ &\leq C(L, \|y\|_\infty, \|z\|_\infty)\|y_0 - z_0\| \end{aligned}$$

□

Proposition 7.7. *It does not exist any admissible control function such that brings u^* to 0.*

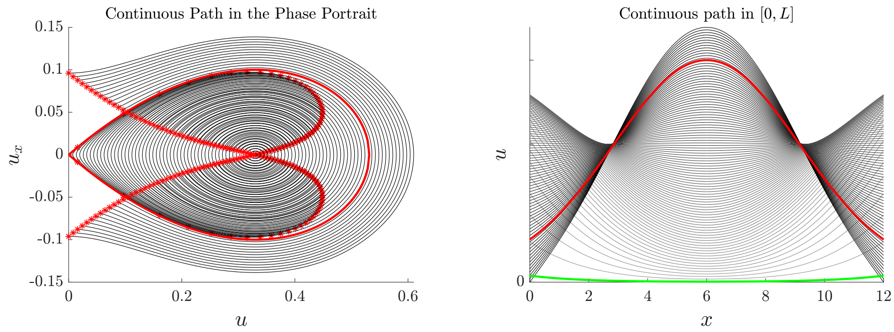

FIGURE 7.8. Connected path of steady-states connecting 0 and $\underline{u}_L$

Proof. Consider the stationary solution y^{s^*}:

$$\begin{cases} -\partial_{xx} y^{s^*} = f(y^{s^*}) \\ y^{s^*}(0) = y^{s^*}(L) = 0 \end{cases}$$

and consider the parabolic problem under discussion:

$$\begin{cases} \partial_t y - \partial_{xx} y = f(y) \\ y(0) = y(L) = a(t) \\ y(x,0) = y^{s^*}(x) \end{cases}$$

with $a \in L^\infty((0,T);[0,1])$. By the comparison principle 3.3 we have that

$$y(t) \geq y^{s^*} \qquad \forall a(t) \in L^\infty((0,T);[0,1])$$

□

Remark 7.8. As it was anticipated, we note that the assumption (4.12) of Theorem 4.2 is needed.

Remark 7.9. There is another procedure to follow a path of steady-states that does not rely on Theorem 4.2. Indeed, in [20], the authors construct a feedback law that stabilizes the quasi-static deformation while moving from a path of steady-states. This feedback law, in our case, can violate the constraints. However, it can be a useful alternative to follow a path of steady-states that needs to be mentioned.

Remark 7.10. In this section, we showed that we can build a path of admissible steady-states outside Γ that connect to θ and 0.

7.3. Case $F(1) = 0$, even and not even controls.

In the case in which $F(1) = 0$ (for the prototypical example corresponds to $\theta = 1/2$), we have a positive result for the admissible paths of steady-states for any length L. For any even initial data of the reaction-diffusion system, we can bring it to a connected path of steady-states that connect 0, θ, and 1. The last result follows from the fact that for $F(1) = 0$, the traveling wave that connects 0 to 1 is a stationary solution.

Hereafter we point out the main relevant factors for the $F(1) = 0$ case.

(1) The region Γ is defined by the two standing traveling waves that connect the points $(0,0)$ and $(1,0)$.
(2) By uniqueness of the ODE and the fact that the traveling waves connect the points $(0,0)$ and $(1,0)$, there cannot exist an admissible nontrivial solution with boundary values 0 or 1.
(3) Any symmetric admissible steady-state is inside Γ.

Remark 7.11 (A simpler dynamic strategy). The path of admissible steady-states that connect any even solution to 1 or 0 is not needed. Indeed we can directly use a dynamic strategy for the parabolic problem. Note that in this case, by Matano, we already have the convergence to 0 and to 1 in infinite time since they are the only stationary solutions that take boundary values 0 and 1, respectively. From the phase portrait in the $F(1) = 0$ case, it is clear that the only solutions that take the same values in both

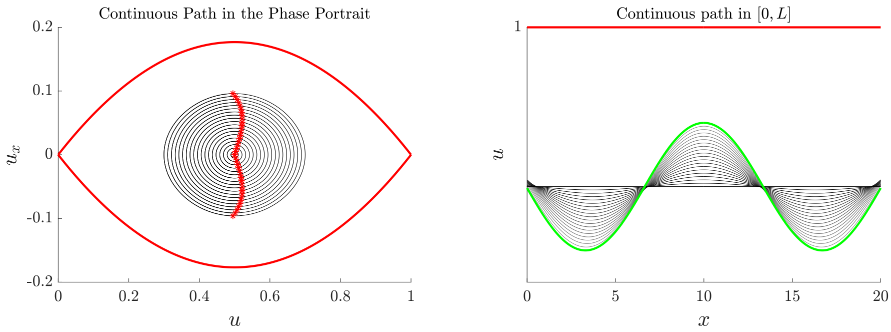

FIGURE 7.9. The even strategy is represented in the phase portrait of system (5.9). In black, the representation of the steady-states of (7.1) that are part of a connected path that connects to the stationary solution θ. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 20$, $\theta = 0.5$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

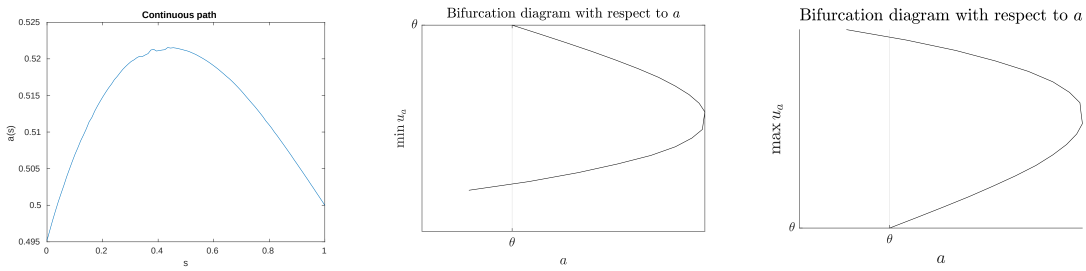

FIGURE 7.10. Connected path of steady-states, value in the boundary, in the horizontal axis the parameter $s \in [0, 1]$ in the vertical axis the value of the boundary In the left, the minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. At the right, the maximum value $\max_{x\in[0,L]} u(x)$ against a

boundaries should be inside the invariant region defined by the stationary traveling waves or separatrix that connect 0 to 1 and vice versa.

Proposition 7.12. *Let $(0, L) \subset \mathbb{R}$, and consider that $F(1) = 0$. Then there exists a connected path of steady-states $u^s(x)$ such that connects 0 and 1.*

Proof. For the case in which $F(1) = 0$ we know that the traveling wave v is a stationary solution for the problem in $\mathbb{R}$:

$$\begin{cases} -\partial_{xx} v = f(v) & x \in \mathbb{R} \\ v(+\infty) = 0, \quad v(-\infty) = 1 \\ v(0) = 1/2 \end{cases}$$

(the heteroclinic orbits in the phase portrait connecting 0 and 1). The restriction of v to $[0, L]$ is a stationary solution. Note that $v(x + c)$ is also a stationary solution for any $c \in \mathbb{R}$. Choose a direction in $c \in \mathbb{R}$ and by considering the restriction of $v(x + cs)$ on $[0, L]$ we obtain a connected path of steady-states. □

The following Figures 7.9 7.11 and 7.13 are illustrations of the paths connecting a steady-state to the stationary solution θ to 1 and to 0. Figures 7.10 7.12 and 7.14 show its respective values at the boundary of the continuous paths of steady-states while Figures 7.10 7.12 and 7.14 represent their bifurcation diagrams.

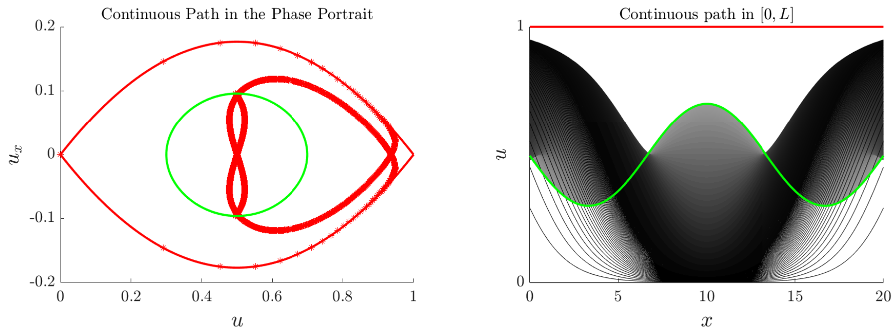

FIGURE 7.11. The even strategy is represented in the phase portrait of system (5.9). In black, the representation of the steady-states of (7.1) that are part of a connected path that connects to the stationary solution 0. In red, the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 20$, $\theta = 0.5$. In the right, the stationary path plotted in the space domain, the red curve is the curve of the maximum value in the invariant region Γ, in green the initial condition.

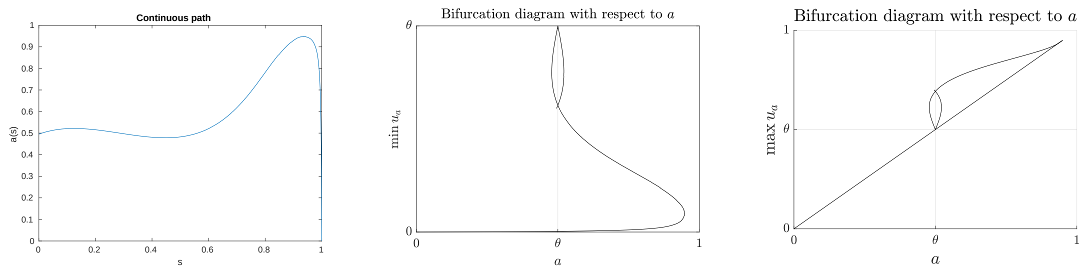

FIGURE 7.12. (Left) Connected path of steady-states. The value in the boundary in the vertical axis, in the horizontal axis the parameter $s \in [0, 1]$ in the vertical axis the value of the boundary (Center) The minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. (Right) The maximum value $\max_{x\in[0,L]} u(x)$ against a.

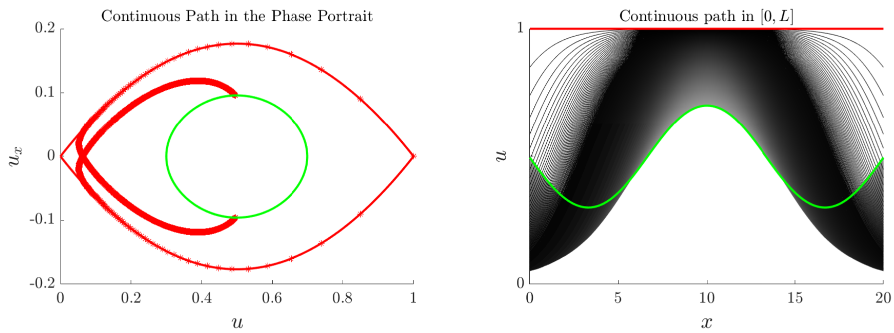

FIGURE 7.13. Even strategy represented in the phase portrait of system (5.9). In black the representation of the steady-states of (7.1) that are part of a connected path that connects to the stationary solution θ. In red the values of the control that can be taken. Red points represent the value that has to be taken in the other extreme for the Dirichlet control. $L = 20$, $\theta = 0.5$. In the right, the stationary path plotted in the space domain, the red curve is the curve of maximum value in the invariant region Γ, in green the initial condition.

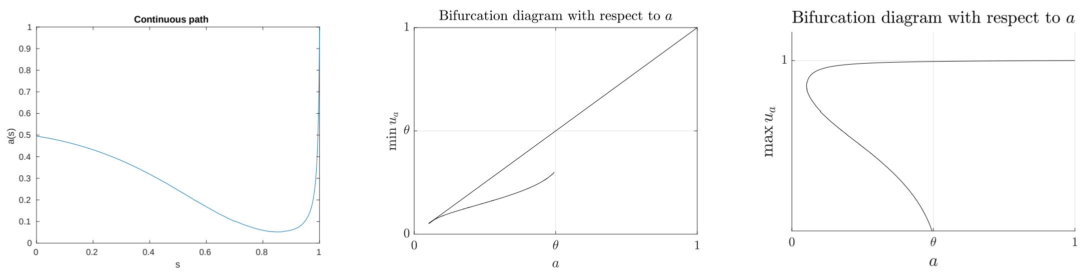

Figure 7.14. Connected path of steady-states, value in the boundary, in the horizontal axis the parameter $s \in [0,1]$ in the vertical axis the value of the boundary. In the left, the minimum value of $u_a(x)$, $\min_{x\in[0,L]} u_a(x)$ is represented against a, for the connected path of steady-states. At the right, the maximum value $\max_{x\in[0,L]} u(x)$ against a

8. Summary of the control results

In these lecture notes, we have seen how nontrivial elliptic solutions can arise and their control consequences when state-constraints are present. Moreover, we have seen how to construct paths of steady-states for ensuring controllability to unstable steady-states.

8.1. Bistable Nonlinearities.

The Figures 8.1 and 8.1 are a visual summary of the study. The direction of the traveling waves between constant stationary solutions is shown together with connected paths of steady-states between those constant solutions in a domain $[0, L]$. In the case of $F(1) = 0$, the traveling wave is stationary. Therefore, as discussed previously, its restrictions on domains of length L give a connected path of steady-states between 0 and 1 naturally. However, note that this path does not pass through the stationary solution θ.

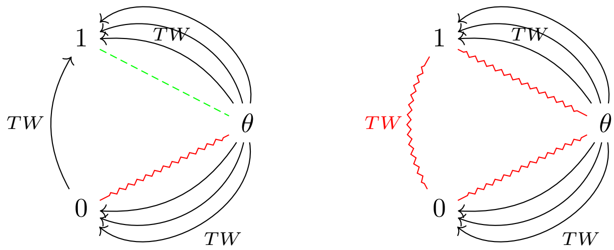

FIGURE 8.1. (Left) Connectivity map for $F(1) > 0$. In red, it is shown an admissible continuous path of steady-states (for any L) connecting stationary solutions. In green, it is an admissible and continuous path of steady-states connecting two stationary solutions, but in this case, its existence depends on L. In black, the existence of traveling waves for the Cauchy problem is shown. The traveling wave from 0 to 1 is unique, while the traveling waves from θ to 1 or to 0 are infinitely many. (Right) Connectivity map for $F(1) = 0$. In red, it is shown an admissible continuous path of steady-states (for any L) connecting stationary solutions. The traveling wave from 0 to 1 is unique and stationary, giving a continuous path of admissible steady-states connecting 0 and 1. In black, the existence of non-stationary traveling waves for the Cauchy problem is shown. The traveling waves from θ to 1 or to 0 are infinitely many.

Consider the following reaction-diffusion equation:

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(0) = a_1(t), \quad u(L) = a_2(t) & t \in (0,T), \\ u(x,0) = u_0(x) \in [0,1], \end{cases} \tag{8.1}$$

where f is a C^2 bistable function satisfying $f(0) = f(1) = f(\theta) = 0$ for a certain $\theta \in (0,1)$. Let F be defined as $F(t) = \int_0^t f(s)ds$ and $a_1, a_2 \in L^\infty((0,T);[0,1])$.

Condition 8.1. Let w be any stationary solution of (8.1) satisfying:

$$\begin{cases} \dfrac{1}{2} w_x(0)^2 + F(w(0)) \leq F\left(\max\limits_{x\in[0,L]} \underline{u}_{L^*}\right), \\ \max w(x) < \max\limits_{x\in[0,L]} \underline{u}_{L^*}, \end{cases}$$

where $\underline{u}_{L^*}$ is the minimum solution with respect to the infinity norm to the problem:

$$\begin{cases} -\partial_{xx} \underline{u}_{L^*} = f\left(\underline{u}_{L^*}\right) & x \in (0,L), \\ \underline{u}_{L^*}(0) = \underline{u}_{L^*}(L^*) = 0, \\ 1 > \underline{u}_{L^*} > 0 & x \in (0,L), \end{cases} \tag{8.2}$$

where $L^* \in \mathbb{R}^+$ is the minimum value for which the problem (8.2) has a solution.

In the case that the solution of (8.2) does not exist, it is assumed that, given a steady-state, the condition is always fulfilled.

Note that Condition 8.1, in particular, is asking to have the maximum below the maximum of the minimum nontrivial solution. With the energy requirement, one can construct a parabolic supersolution with a steady-state that is below the nontrivial solution and then either approach to 0 or use the staircase method with the path constructed in Subsection 7.2.

Theorem 8.2. *For f bistable.*
If $F(1) > 0$.

- *The solution of the system* (8.1) *can be driven asymptotically to* 0 *using the controls* $a_1, a_2 = 0$
  - *for any initial data u_0 iff $L < L^*$.*
  - *for any $L > 0$ if u_0 is below some steady-state that fulfills Condition 8.1.*
- *The solution of the system* (8.1) *can be driven asymptotically to* 1 *using the controls* $a_1, a_2 = 1$ *for any initial data u_0 and for any L.*
- *There exists $T^*_{u_0,L} > 0$ such that for every $T \geq T^*_{u_0,L}$ the solution of the system* (8.1) *can be controlled to θ iff u_0 asymptotically goes to* 0 *when $a \equiv 0$.*

Furthermore one has the following estimate for L^*:

$$\frac{\pi}{\sqrt{\max_{s\in[0,1]} f'(s)}} \leq L^* \leq 2\sqrt{2}\sqrt{\frac{F(1) - F(\theta)}{F(1)^2}}.$$

If $F(1) = 0$ then

- *the solution of the system* (8.1) *can be driven asymptotically to* 0 *(or* 1*) using the controls $a_1, a_2 =$ 0 ($a_1, a_2 = 1$) for any $L > 0$ and any admissible u_0.*
- *there exists $T^*_{u_0,L} > 0$ such that for every $T \geq T^*_{u_0,L}$ the solution of the system* (8.1) *can be controlled to θ for every u_0 and every $L > 0$ and any admissible u_0.*

Remark 8.3. As noted in [104], 0 and θ have the same ω-limit when the control is set to 0 ($a = 0$). Moreover, one can only build admissible steady-state paths between steady-states whose ω limit for the control $a = 0$ are not in comparison. The proof follows by contradiction. Assume that there is an admissible path of steady-states between two admissible steady-states whose ω-limits for the $a = 0$ control are in comparison. Since, by hypothesis, a path exists, there exists a positive control function that, with finite time, is able to bring the first steady-state to the second. However, there also exists another control function that can bring the second to the first in finite time by means of a positive control. This enters in contradiction with the comparison principle.

Remark 8.4. When one takes a steady-state that fulfills Condition 8.1 as an initial condition with control $a = 0$, the solution asymptotically goes to 0 (due to Remark 8.3 and the fact that the steady states fulfilling condition 8.1 are path-connected with the steady-state θ). Therefore, by comparison, all initial conditions below a steady-state fulfilling Condition 8.1 will also asymptotically go to 0 with a $a = 0$ control.

Remark 8.5. Moreover, if the initial data is symmetric with respect $L/2$ taking the same value in $u_0(0) = u_0(L)$ the path can be constructed in a symmetric way $a_1(t) = a_2(t) = a(t)$.

Remark 8.6. The construction of a symmetric path allows to control the bistable equation with one boundary control in one extreme and Neumann condition to the other extreme. To generate a path for several dimensions is done in [104]. However, how to proceed if we act only in a part of the boundary is an open question (see Section 11).

Remark 8.7. Note that the continuous path of steady-states between 0 and 1 can exist if we allow ourselves to break the constraints (see Figure 8.2).

In general, an important concern is if the ODE dynamics is going to blow up for a finite L. This would stop any continuous path to exist regardless of the constraints.

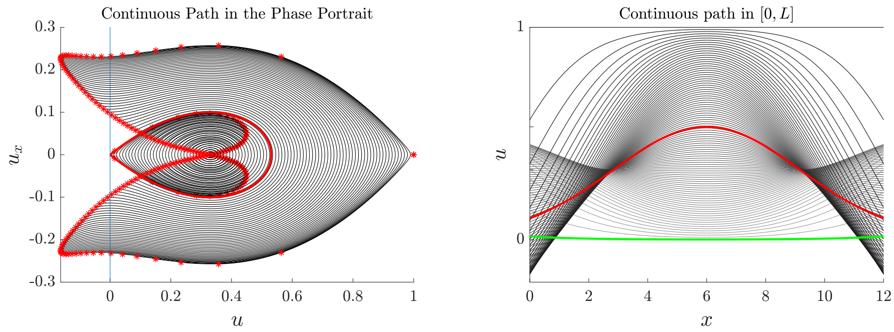

FIGURE 8.2. Non admissible continuous path from 0 to 1

8.2. **Monostable Nonlinearities.**

Previously we gave precise thresholds and conditions for when the nontrivial positive solutions of the semilinear elliptic problem appear, then for the monostable nonlinearity one can write:

Theorem 8.8. *For f monostable.*
Let $[0, L]$ be a domain, the system (8.1) can be asymptotically driven to:

- 1 *for any initial condition and any measure of the domain.*
- 0 *for any initial condition if $L^2 < \lambda^*$.*

where:

$$\frac{\lambda_1((0,1))}{\max_{s\in[0,1]} f'(s)} \leq \lambda^* \leq \overline{\lambda} < +\infty$$

where $\lambda_1((0,1))$ is the first eigenvalue of the Dirichlet Laplacian in $[0,1]$.

Remark 8.9. Note that for $\lambda > \lambda^*$ one can still have that 0 is stable.

Following the computations in 1-D for the bistable case, it can be shown that for the monostable case in dimension 1 one has $\overline{\lambda} = \frac{8}{F(1)}$.

9. NUMERICAL SIMULATIONS

This section is devoted to seeing numerical implementations of optimal control strategies. We want to observe the results showed above, together with some qualitative experiments. The numerical implementations performed are optimization problems of the form:

$$\min_{a\in\mathcal{A}} J(u_a; u_0) \tag{9.1}$$

for different $\mathcal{A}$ and where u_a solves:

$$\begin{cases} \partial_t u - \partial_{xx} u = u(1-u)(u-1/3) & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = a_1(t), \quad u(L,t) = a_2(t), & t \in (0,T), \\ u(x,0) = u_0(x) & \end{cases}$$

We have used IpOpt ([119]) to do the optimization procedures. The order will be the following:

(1) The control strategy for $L_\theta < L < L^*$. Here we take:

$$J = \|u(T;a) - \theta\|^2,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\},$$

and we observe in Figure 9.1 how the control takes values for an open time interval below $\theta = 1/3$. This is natural since for $L > L_\theta$ there is a nontrivial solution above θ that has to be overpassed. Furthermore, it can be observed that the final state is close to θ as the theory supports, Figure 9.1.

(2) The lack of controllability towards 0 for $L > L^*$ for $u_0 = 1$ due to the emergence of a barrier, Figure 9.2. The functional employed in the figure is:

$$J = \|u(T;a)\|^2,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\}.$$

However, one can reach θ when starting from $u_0 = 0$. In Figure 9.3, the functional:

$$J = \|u(T;a) - \theta\|^2,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\},$$

is being minimized.

(3) The existence of a minimal controllability time and the control in minimal time. First, we observe the lack of controllability in a short time horizon (Figure 9.4) for

$$J = \|u(T;a) - \theta\|^2,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1])\Big\},$$

In Figure 9.5 the functional minimized is the time horizon

$$J = T,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1]) : \text{ the trajectory fulfills } \theta - \epsilon \le u_a(T;u_0) \le \theta + \epsilon\Big\}.$$

Numerically this can be achieved by first generating an initial guess minimizing the L^2 norm of the difference of the final datum with the target. Then, minimizing δt, the discretization time step. The control showed in Figure 9.5 points out that the control strategy associated with the control in minimal time is a bang-bang function.

(4) A quasistatic control that approximately follows the path of steady-states constructed above. We set T large, and we minimize

$$J = \int_0^T \|a_t\|^2 dt,$$
$$\mathcal{A} = \Big\{a_1, a_2 \in L^\infty([0,T];[0,1]) : \text{ the trajectory fulfills} \theta - \epsilon \le u_a(T;u_0) \le \theta + \epsilon\Big\}.$$

Moreover, in Figure 9.7, one can see snapshots of the parabolic controlled state being close in the phase plane to the elliptic ones.

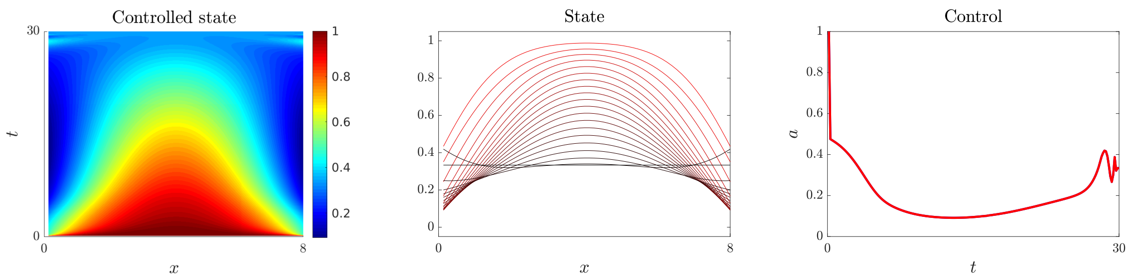

FIGURE 9.1. Controlled state for $L = 8$, $T = 30$ and initial datum $u_0(x) = 1$.

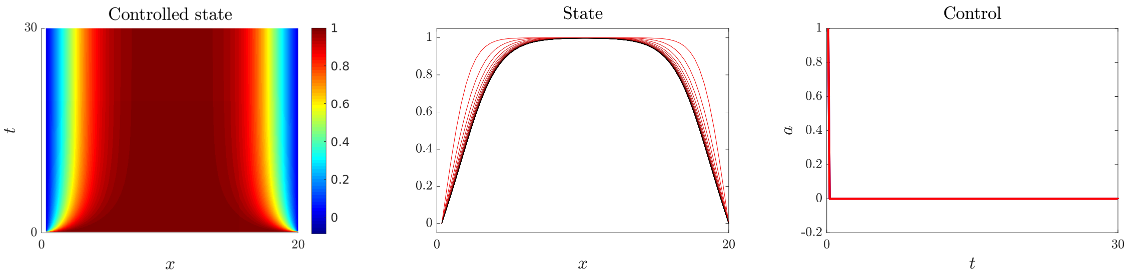

FIGURE 9.2. Controlled state for $L = 20$, $T = 30$ and initial datum $u_0(x) = 1$.

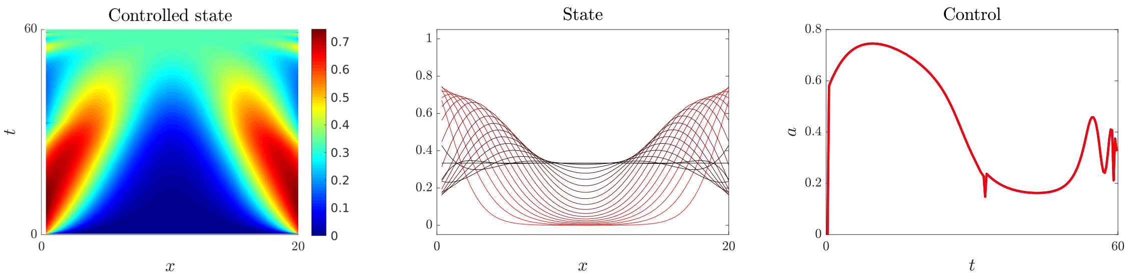

FIGURE 9.3. Controlled state for $L = 20$, $T = 60$ and initial datum $u_0(x) = 0$.

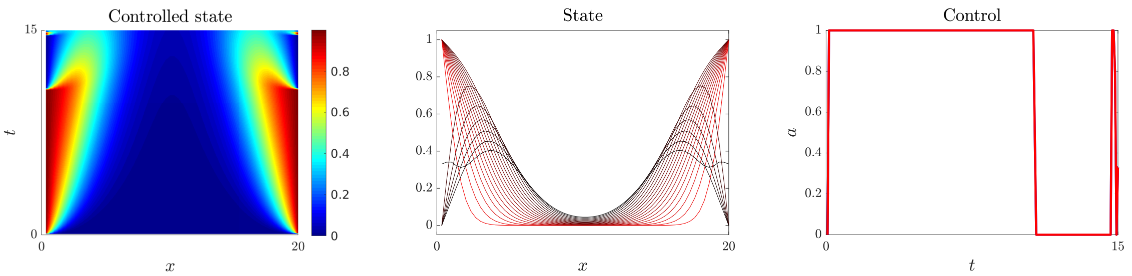

FIGURE 9.4. Controlled state for $L = 20$, $T = 15$ and initial datum $u_0(x) = 0$.

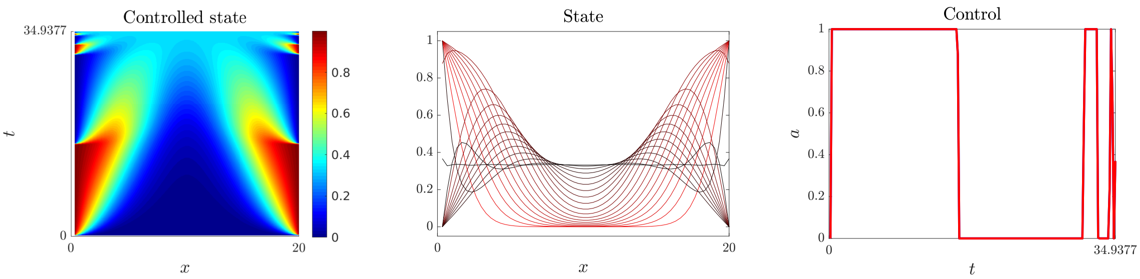

FIGURE 9.5. Controlled state in minimal time for $L = 20$ and initial datum $u_0(x) = 1$.

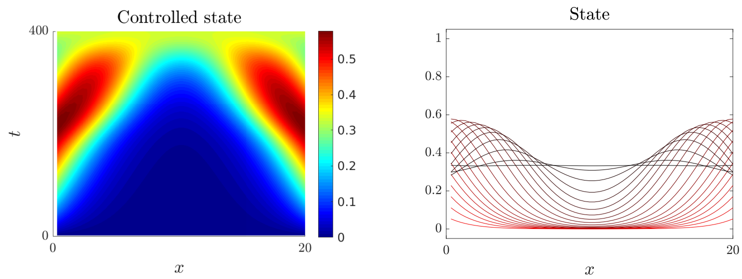

FIGURE 9.6. Controlled state for $L = 20$, $T = 400$ and initial datum $u_0(x) = 0$.

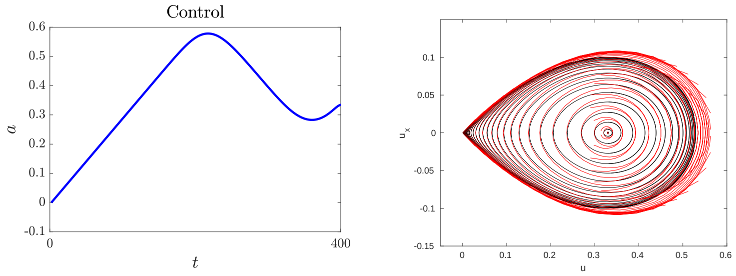

FIGURE 9.7. (Left) Optimal constrained control associated with Figure 9.6. (Right) in black steady-states in the phase plane, in red snapshots of the parabolic state in the phase plane

10. Extensions and other problems

10.1. Combining Allee Control and Boundary Control.

10.1.1. *Preliminaries.* The goal of this subsection is to present the control strategy in [112] schematically. In the reference above, the authors present the modeling of the evolution of a population of mosquitoes when sterile mosquitoes are released. This way of acting to the system leads to an interaction to the nonlinearity, more precisely to the Allee parameter θ:

$$\begin{cases} \partial_t v - \partial_{xx} v = v(1-v)(v-\theta(t)) & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ 0 \leq \theta(t) \leq 1, \\ 0 \leq v(x,0) \leq 1. \end{cases}$$

The goal of the authors is to control approximately to a traveling wave in large time. They assume that the limits $\lim_{x\to\pm\infty} v(x,0) = l_\pm$ exist and they are different. The strategy of the proof relies on the following arguments:

- Fix $\theta \in (\min(l_-, l_+), \max(l_-, l_+))$ and let the system converge exponentially to a traveling wave profile.
- Using that for any θ, the profile of the traveling wave for the cubic nonlinearity is the same, one can move the control θ to move the center of the traveling wave. Note that if $\theta < 1/2$, the traveling wave moves in the direction such that will make 1 invade and if $\theta > 1/2$ the opposite.

10.1.2. *Combined Strategy.* Combining the two interactions more targets can be achieved. In particular, barriers can be broken by changing the value of θ. Furthermore, one can easily obtain controllability in large time, to any steady-state inside Γ for any θ and to sections of traveling waves. We consider the following problem:

$$\begin{cases} \partial_t v - \partial_{xx} v = v(1-v)(v-\theta(t)) & (x,t) \in (0,L)\times(0,T), \\ v(0,t) = a_1(t) \quad v(L,t) = a_2(t) & t \in (0,T), \\ 0 \le v(x,0) \le 1. \end{cases}$$

The control strategy is:

- Take $\theta = 1/2$, $a_1 = a_2 = 0$ for long time. Matano's Theorem ensures the convergence to 0, remind that, for this value of θ, there is no barrier for reaching 0.
- Change θ to the desired value between $(0,1/2)$.
- set $a_1 = a_2 = 0$ and apply the strategy described in the previous sections with the staircase method.

Furthermore for $\theta = 1/2$, the traveling waves are stationary and they are seen in the phase portrait connecting $(0,0)$ and $(1,0)$. We can have exact controllability to the traveling waves. Using the staircase method, we can approach a traveling wave arc and reach it in finite time for $\theta = 1/2$. Then changing θ, we can move the traveling wave in the direction that we desire.

10.2. **Multi dimensional case.**

The results and techniques explained before can be extended in several dimensions [104]. Let us consider a bounded domain $\Omega \subset \mathbb{R}^d$ with C^2 boundary of $|\Omega| = 1$ where the following dynamics is taking place:

$$\begin{cases} \partial_t u - \Delta u = \lambda f(u) & (x,t) \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T), \\ 0 \le u(x,t) \le 1 & (x,t) \in \Omega \times [0,T], \end{cases} \tag{10.1}$$

where $\lambda > 0$.

Note that in this context, the bounds derived in Section 5 also work by adapting the proofs properly.

One should note that the first Dirichlet eigenvalue depends on the shape of the domain. This feature has more relevance than the measure of the domain since we can consider a set with very large measure but with a very big first eigenvalue. Proof of Theorem 5.14 is adapted by finding the biggest ball inside the domain and making the computations for finding a subsolution. While the proof of Theorem 5.13 holds the same, and the result will depend on the first eigenvalue that, in turn, depend on the domain geometry. Therefore, the existence of barriers is a general feature of these problems see Figure 10.1.

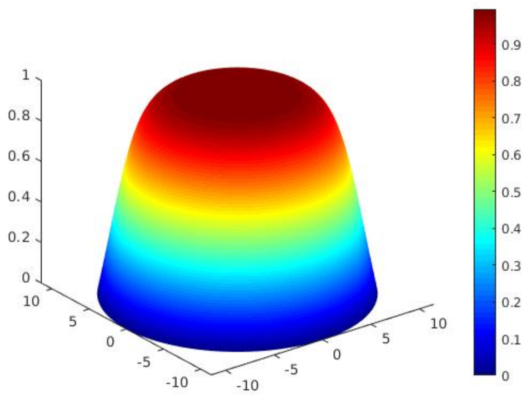

FIGURE 10.1. Simulation of the semilinear heat equation with nonlinearity $f(s) = s(1-s)(1-\theta)$ with boundary value $a(x,t) = 0$ finding a barrier.

The proof of the convergence to the constant steady-state 1 in case that $F(1) > 0$ follows by the traveling wave solutions as in the one-dimensional case Proposition 5.24. Note that by extending the one-dimensional traveling wave to be constant in all other $d-1$ remaining dimensions, one obtains a traveling wave profile for the multi-dimensional case.

The construction of the continuous path of steady-states relies now on extending the domain to a bigger ball, Figure 10.2,

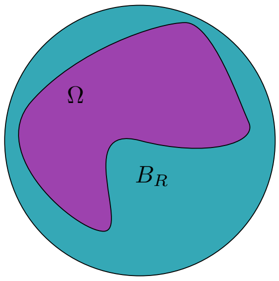

FIGURE 10.2. Ball containing the original domain.

and write the problem (10.2)

$$\begin{cases} -\Delta u = \lambda f(u) & x \in B_r \subset \mathbb{R}^d, \\ u(0) = a, \\ Du(0) = 0, \end{cases} \tag{10.2}$$

as an ODE problem:

$$\begin{cases} u_{rr}(r) + \dfrac{d-1}{r} u_r(r) = -f(u(r)) & r \in [0, R_{B,\lambda}), \\ u(0) = a, \\ u_r(0) = 0. \end{cases} \tag{10.3}$$

where $R_{B,\lambda}$ is the radius of the ball after rescaling for absorbing λ. The solutions of a semilinear elliptic equation in a ball are radially symmetric [29, Ch. 9, pp.555]. This is why the transformation is allowed.

Furthermore, the local well-posedness of the ODE (10.3) comes from Banach contraction and Grönwall inequality. The key points are the following:

(1) Radial solutions of the problem (10.3) dissipate:
Assume that $F(1) \geq 0$ and consider the energy

$$E(u, v) = \frac{1}{2} v^2 + F(u)$$

where $F(u) = \int_0^u f(s)ds$. Now we define the region:

$$M := \{(u, v) \in \mathbb{R}^2 \quad \text{such that} \quad E(u, v) \leq 0\}$$

Note that the region defined by

$$\Gamma := \{(u, v) \in [0, \theta_1] \times \mathbb{R} \quad \text{such that} \quad |v| \leq \sqrt{-2F(u)}\}$$

satisfies $\Gamma \subset M$. In fact, this region is the same than in the one dimensional case.
Now one considers an initial datum of the form $(u_0, 0) \in \Gamma$, then the solution of (10.3) with initial datum $(u_0, 0)$ satisfies:

$$\frac{d}{dr} E(u, v) = v v_r + f(u) v = -\frac{d-1}{r} v^2 < 0.$$

As a consequence $(u, v) \in \Gamma$ for all $r > 0$ and the path is admissible and globally defined.
In Figure 10.3 one can see the construction of the construction of the path.
Then, restricting to our original domain Ω, one obtains the desired path.

(2) Another important feature is how to control to the path. For this, one realizes that the minimum solution to

$$\begin{cases} -\Delta u = \lambda f(u) & x \in \Omega, \\ 0 < u < 1 & x \in \Omega, \\ u = a(x) & x \in \partial\Omega, \end{cases}$$

is radial with respect to some ball, and therefore it belongs to a path of steady-states. The argument follows by contradiction using a comparison argument [104].

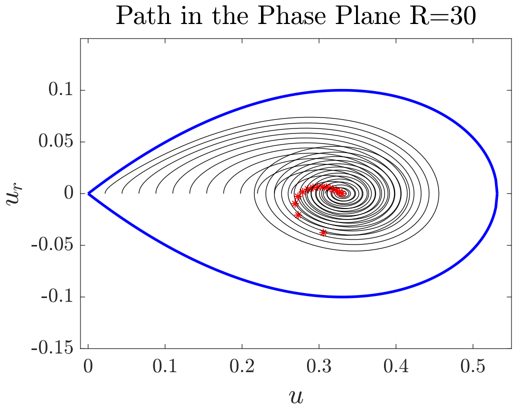

FIGURE 10.3. In blue the invariant region in the phase space, in black the radial trajectories forming the continuous path of steady-states where the red stars indicate the condition in the boundary. $\theta = 1/3$, $R = 30$ and $d = 2$.

10.3. **Spatially Heterogeneous case.**

In this section, we sketch a more general model studied in [82]:

$$\begin{cases} \partial_t u - \Delta u + \left\langle \frac{2\nabla N}{N}, \nabla u \right\rangle = f(u) & x \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & x \in \partial\Omega \times (0,T), \\ 0 \leq u(x,t) \leq 1, \end{cases}$$

where $N : \mathbb{R}^d \to (0, +\infty)$ be a C^∞ function.

10.3.1. *Modeling.* Note that in the modelling in Section 2, one noted that we have assumed that the whole population N was behaving as a homogeneous heat equation. If we consider, instead, that N follows a nonhomogeneous semilinear heat equation:

$$\begin{cases} \partial_t N - \Delta N = N(\kappa(x) - N) & (x,t) \in \Omega \times (0,T), \\ \frac{\partial N}{\partial \nu} = 0 & (x,t) \in \partial\Omega \times (0,T), \\ N(x,0) = N_0(x) \geq 0, \end{cases} \tag{10.4}$$

where κ is a C^∞ function, then, it is known that there is only one positive steady-state of (10.4) []. Since the set of set of positive steady states is a singleton and the steady state 0 is unstable, and due to the gradient structure of the semilinear heat equation the solution of (10.4) converges to (10.5) [16],

$$\begin{cases} -\Delta N = N(\kappa(x) - N) & x \in \Omega, \\ \frac{\partial N}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases} \tag{10.5}$$

Taking N from (10.5) and employing the same approximation done in Section 2, one arrives at the following heterogeneous bistable equation

$$\begin{cases} \partial_t u - \Delta u + \left\langle \frac{2\nabla N}{N}, \nabla u \right\rangle = f(u) & (x,t) \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in \Omega \times [0,T], \end{cases}$$

where u is a proportion.

The effect of the mother population N on the state depends on its shape. Figure 10.4 shows the effect of the drift produced by N. In the right hand side of Figure 10.4, one can see that the drift pushes from the boundary to the interior. For this reason, one intuitively expects that the controllability is easier in such setting. In the left hand side of Figure 10.4, on can see that the effect of the boundary control is diminished by the drift. This will lead to the existence of new nontrivial solutions as it is shown in the subsequent subsections.

The new barriers block controllability from 0 to θ in Figure 10.5 (left) one can see how the minimal controllability time blows up when varying a parameter.

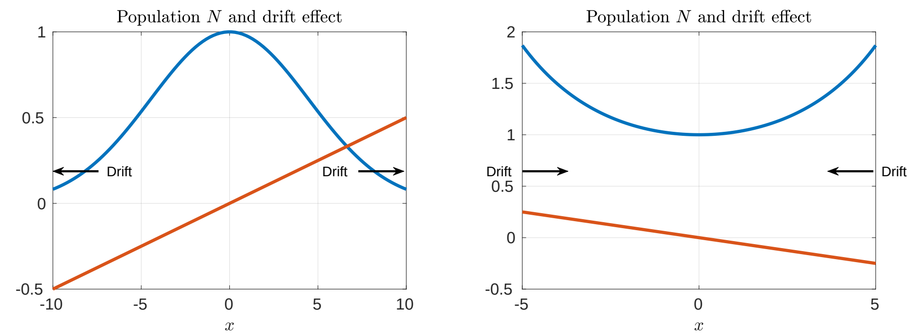

FIGURE 10.4. In blue the curve $N(x)$, in orange the quotient $-\frac{N_x(x)}{N(x)}$ responsible of the drift effect. Left $N(x) = e^{-\frac{x^2}{\sigma}}$, right $N(x) = e^{\frac{x^2}{\sigma}}$.

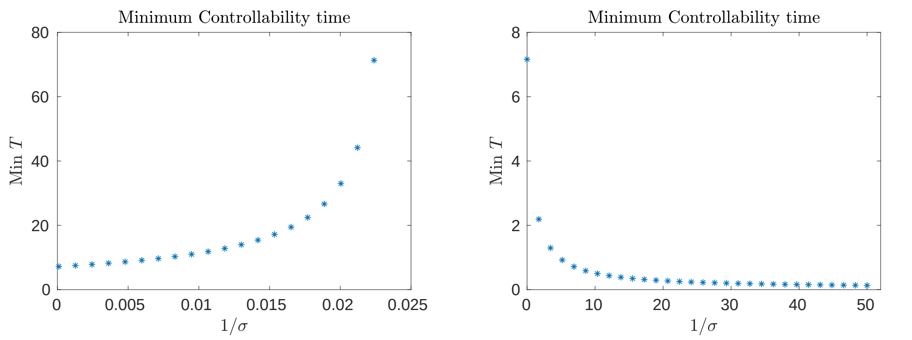

FIGURE 10.5. Minimal controllability time from 0 to $\theta = \frac{1}{3}$ versus $\epsilon = \frac{1}{\sigma}$. In the right the minimal controllability time for the Gaussian case $N(x) = e^{-\frac{x^2}{\sigma}}$, in the right $N(x) = e^{\frac{x^2}{\sigma}}$.

10.3.2. *Small drifts.* In [82], a method concerning perturbations of the homogeneous path is developed to guarantee the controllability for small drifts. Take finite sequence on the path of steady-states for the homogeneous equation and consider small drift equation:

$$\begin{cases} \partial_t u - \Delta u + \epsilon \langle \nabla p(x), \nabla u \rangle = f(u) & x \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & x \in \partial\Omega \times (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in \Omega \times [0,T]. \end{cases}$$

After making domain perturbations, the implicit function theorem can be applied to all the elements selected from the homogeneous path to obtain a sequence of steady-states for (10.3.2) in a way that remain close enough to apply the staircase method (see Figure 10.6), see [82, Theorem 2] for details. It has to be remarked that we do not know if a continuous path for the perturbed problem exists, the only requirement to apply the staircase method is to have a sequence whose elements are successively close, see Figure 10.6 for an intuitive representation.

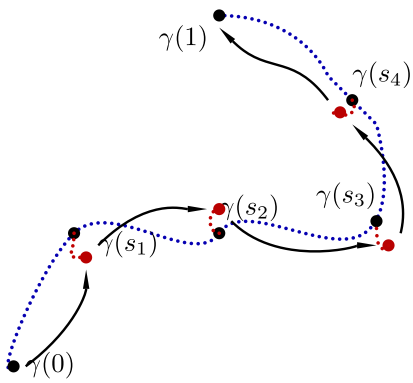

FIGURE 10.6. The blue dotted line represents the continuous path of steady-states for the homogeneous equation. In red, the perturbed steady-states, linked to the unperturbed steady-states (black) that belong to a continuous path for the homogeneous equation.

This perturbative argument is an improvement of the Staircase method [91].

The method explained depends on the smallness of ϵ, however, Figure 10.5 Figure 10.5 points that we cannot expect this method to work for big ϵ. We shall see in the subsection below the apearence of such obstructions.

10.3.3. *Big drifts and new barriers.* In the case of radial drifts in [82], one can conclude that if the following differential inequality is fulfilled, then one also has an invariant region in the phase portrait where the stationary states 0 and θ belong to. In case of multi-D one has that the energy $E(u,v) = \frac{1}{2}v^2 + F(u)$ follows:

$$\frac{d}{dr}E = -\frac{N'}{N}v^2 - \frac{d-1}{r}v^2$$

If

$$\frac{d}{dr}N(r) \geq -\frac{d-1}{r}N(r),$$

then the energy decreases and the set:

$$E(u,v) \leq 0$$

is positively invariant (see [82, Theorem 3]).

However, in the opposite case, new barriers can appear in the one-dimensional case. Moreover, a scheme of the proof of the upper barrier, Figure 10.7, is also shown.

Let us consider the a Gaussian drift $N(x) = e^{-x^2/\sigma}$. From the modeling perspective, this amounts to say that there is a big concentration of individuals around $x = 0$. Consider, in addition, the following boundary value problem:

$$\begin{cases} -\partial_{xx}u + 4\left(\frac{x}{\sigma}\right)u_x = f(u) & x \in (-L,L), \\ u(-L) = u(L) = a \in \{0,1\}. \end{cases}$$

Theorem 10.1 (Theorem 4 [82]). *Let $F(1) > 0$. For every $\sigma > 0$, there exists a solution of* (10.3.3) *satisfying the state constraints $0 \leq u \leq 1$ with $a = 1$ for L big enough. Moreover, there exist another nontrivial solution for certain $L > 0$ with $a = 0$.*

To prove the existence of the barrier function to reach 0, one can proceed with the same variational arguments discussed previously. However, for proving the existence of the upper barrier, the same methods do not apply, since the stationary solution 1 is a global minimum of the energy functional.

For this reason, in [82] Theorem 10.1 is proved using phase plane techniques (shooting method), namely find an initial condition for the system:

$$\begin{cases} \dfrac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -f(u) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{4}{\sigma}xv \end{pmatrix}, \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \end{cases}$$

for which at a certain L the trajectory of 10.3.3 reaches 1.

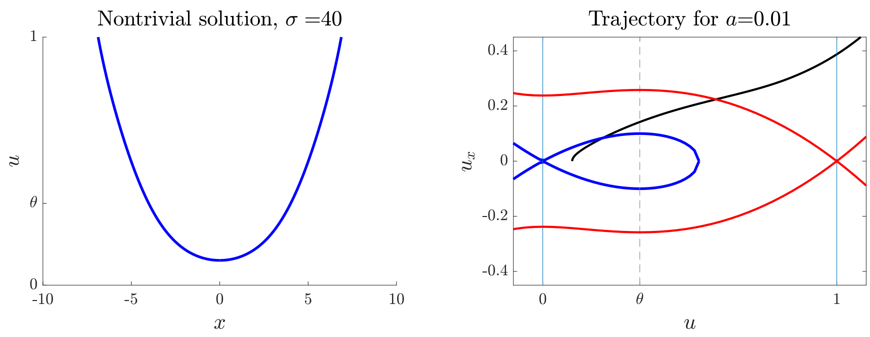

FIGURE 10.7. (Left) Upper barrier, solution of (10.3.3) for $\sigma = 40$. (Right) Sketch of the phase-plane analysis for the trajectory leading to a solution of (10.3.3)

Therefore, for big domains, one cannot guarantee the controllability to any of the constant steady-states.

11. Perspectives

To establish the perspectives, we will first enhance the limitations of the methods applied here. One of the most illustrative limitations is seen when nonautonomous dynamics are considered.

1) Nonautonomous dynamics and the need of a new method. Let us consider the system

$$\begin{cases} \partial_t u - \mu(t)\Delta u = f(u) & (x,t) \in \Omega \times (0,T) \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ 0 \leq u(x,0) \leq 1 & x \in \Omega \end{cases} \tag{11.1}$$

where $a \in L^\infty((0,T) \times \partial\Omega; [0,1])$, $\mu \in C^1((0,T);\mathbb{R}^+)$ and $f : \mathbb{R} \to \mathbb{R}$ is bistable. Note that the only possible steady-states are the constant steady-states

$$w \equiv 0, \quad w \equiv \theta, \quad w \equiv 1.$$

Therefore, there cannot exist any path of steady-states connecting any pair of steady-states. Moreover, observe that if μ is a decreasing function, i.e. $\mu' < 0$, then the steady-state $w \equiv \theta$ can become, loosely speaking, *more unstable* as time advances. Linearizing around $w \equiv \theta$, keeping the control $a \equiv \theta$, we obtain:

$$\begin{cases} \partial_t v - \mu(t)\Delta v = f'(\theta)v & (x,t) \in \Omega \times (0,T) \\ v(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T) \\ v(x,t=0) = v_0. \end{cases} \tag{11.2}$$

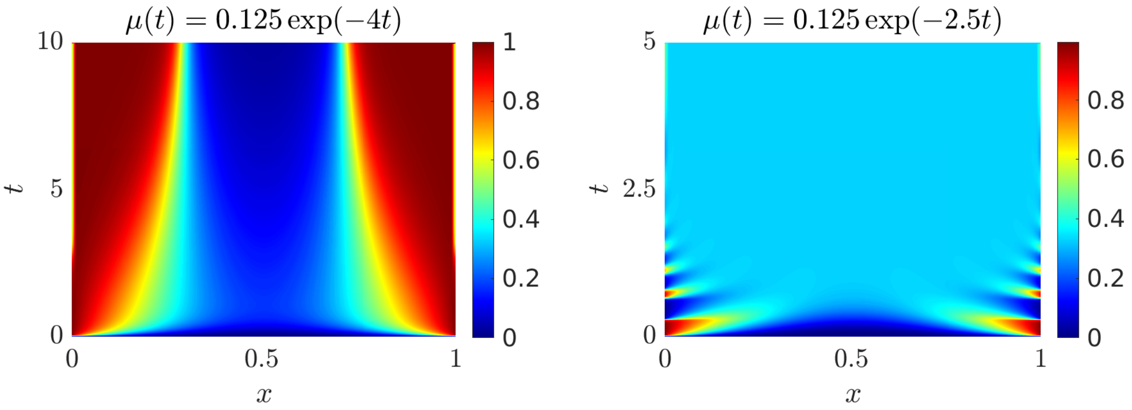

FIGURE 11.1. Space-time representation of the optimal control to $w \equiv \theta$ with initial data $w \equiv 0$. The nonlinearity is $f(s) = s(s-\theta)(1-s)$ with $\theta = 0.33$ (therefore $\int_0^1 f(s) > 0$) and the controls have been limited to take values in $[0,1]$. The optimal control problem is to minimize the L^2 distance with respect $w \equiv \theta$. At the left $\mu(t) = 0.125\text{exp}(-4t)$, at the right $\mu(t) = 0.125\text{exp}(-2.5t)$.

Solving (11.2) one obtains

$$v(x,t) = \sum_{n=1}^{\infty} c_n(0) \exp\left\{ f'(\theta)t - \lambda_n \int_0^t \mu(s)ds \right\} e_n(x)$$

where e_n is the n-th eigenfunction of the Dirichlet Laplacian and λ_n its eigenvalue $-\Delta e_n = \lambda_n e_n$, moreover $c_n(0) = \langle v_0, e_n \rangle_{L^2(\Omega)}$. Note that if, for instance, $\int_0^\infty \mu(s)ds$ is finite, then since $f'(\theta) > 0$, the solution will grow exponentially after a certain critical time t^*. Conversely, if $\lim_{t\to\infty} \frac{1}{t}\int_0^t \mu(s)ds = +\infty$ then every eigenmode will become stable after a certain t_n^{**}.

There are two main questions to address:

- **Can the time dependence of μ create an obstruction to the controllability?**
  If there there is an initial datum $u_0 \in L^\infty(\Omega; [0,1])$ such that the solution u of the problem

$$\begin{cases} \partial_t u - \mu(t)\Delta u = f(u) & (x,t) \in \Omega \times \mathbb{R}^+ \\ u = 1 & (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ u(\cdot, t = 0) = u_0 \end{cases} \tag{11.3}$$

  never approaches $w \equiv 1$ as $t \to +\infty$, or equivalently, for every $t \in \mathbb{R}^+$ there exists an open set $\omega(t) \subset \Omega$ such that:

$$u(x,t) < \theta \text{ if } x \in \omega(t).$$

  If this property holds for bistable nonlinearities of the type $\int_0^1 f(s)ds \geq 0$, then we would have a fundamental obstruction to the controllability to $w \equiv \theta$ purely governed by nonautonomous phenomenology.

  A further intuition that this situation is likely to occur can be seen in the following example: consider μ such that there exists t^* for which $\mu(t^*) = 0$, then the equation becomes uncontrollable due to the lack of diffusion. When constraints are present, one should expect that a nonautonomous obstruction arises since the controllability time is not arbitrarily small.
- **How can we ensure in which situations we can control?** The question above was addressing the possibility of a new type of fundamental obstructions. However, we lack a method (so far) to guarantee controllability under state constraints. Existing methods are based either on dissipation plus local controllability or the staircase method that relies on steady-states' paths. In the simple dynamics presented above, there cannot exist any steady-state path; furthermore, there is no dissipativity towards $w \equiv \theta$.

In Figure 11.1 we show two simulations with different decays of $\mu(\cdot)$. The simulation points out that if $\mu' << -1$ one loses the controllability while if μ' is small the controllability can still be achieved.

The dynamics 11.1 has served us as a simple example, however, nonautonous dynamics are of paramount importance in mathematical biology. One of the simplest examples are reaction-diffusion equations in growing domains [21]

$$\begin{cases} \partial_t u - \mu \partial_{xx} u = f(u) & (x,t) \in (0, l(t)) \times (0,T) \\ u(x,t) = a(x,t) & (x,t) \in \{0, l(t)\} \times (0,T) \\ 0 \leq u(x,0) \leq 1 \end{cases} \tag{11.4}$$

where $a \in L^\infty((0,T) \times \partial\Omega; [0,1])$ and $l(t) \in C^1((0,T); \mathbb{R}^+)$. As a more general mathematical paradigm, one may think on methods on how to ensure the controllability (and main obstructions to the controllability) of systems such as

$$\begin{cases} \partial_t u - \mu \Delta u = f(u,t) & (x,t) \in \Omega \times (0,T) \\ u(x,t) = a(x,t) & (x,t) \in \partial\Omega \times (0,T) \\ 0 \leq u(x,0) \leq 1 \end{cases} \tag{11.5}$$

where $a \in L^\infty((0,T) \times \partial\Omega; [0,1])$. Moreover, one cannot ignore the treatment of the controllability of free boundary problems which arise naturally in many physical and biological phenomena. In this type of problems, the domain itself is part of the unknowns of the system and the evolution of the boundary is coupled with the state, for instance such as

$$\begin{cases} \partial_t u - \mu \partial_{xx} u = f(u) & (x,t) \in (0, l(t)) \times (0,T) \\ u(0,t) = a(t) & t \in (0,T) \\ l'(t) = g(l(t), u(1,t), \partial_x u(1,t)) & t \in (0,T) \\ 0 \leq u(x,0) \leq 1. \end{cases} \tag{11.6}$$

In such setting, even the controllability without state constraints is a challenging question [44].

Finally, one of the paradigms that has not been addressed is the controllability to specific trajectories of the control system. Let us restrict, for instance to a periodic (in time) trajectory as a target. In these notes we have mainly considered steady-states as targets. How can we control to these trajectories? The staircase method is also limited for achieving this purpose.

As we have seen with the case of the nonautonomous dynamics, there is a need for a method that is not steady-state reliant to guarantee controllability. Even starting with the autonomous case, it is crucial to have a way, in particular, to guarantee that a path exists without passing through an explicit construction or providing the controllability directly without relying on steady-states. Furthermore, we also pointed out that, already in the scalar case, there are uncontrollability phenomena intrinsic from nonautonomous dynamics that is not completely understood yet.

This understanding of scalar equations is a must for later going to more difficult and challenging problems such as the controllability of reaction-diffusion systems with state constraints. At least one will be able to encounter the difficulties that scalar equations present yet remain unsolved. Moreover, the study of control properties of free boundary problems with (and without) constraints is an important perspective to consider.

2) The construction of paths of steady-states. The construction of paths of steady-states is by now an artisanal work. For instance if one seeks to guarantee the controllabiiity of

$$\begin{cases} \partial_t u - \operatorname{div}(A(x)\nabla u) + \langle b(x), \nabla u \rangle = f(u,x) & (x,t) \in \Omega \times (0,T), \\ u = a(x,t) & (x,t) \in \partial\Omega \times (0,T), \\ 0 \leq u(x,0) \leq 1 & x \in \Omega, \end{cases}$$

for a general f, A and b, the construction of the paths and the conditions for when they do exist is not trivial. However, heterogeneities are essential in practice, since the environment is, in general diverse and space-dependant. In some places, the growth of the population can be bigger due to a higher capacity, or the diffusion be smaller because the terrain is abrupt. In [82], a particular type of heterogeneities are considered.

From the other side, we can also think on the control of a multidimensional semilinear equation with a control acting on part of the boundary. In [82, 104], the authors considered a control acting in the whole boundary. How can the paths be constructed when one acts only on a portion of the boundary? How is the structure of the set of the steady-states? Let $\eta \subset \partial\Omega$ and consider:

$$\begin{cases} \partial_t u - \Delta u = f(u) & (x,t) \in \Omega \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \eta \times (0,T), \\ \frac{\partial}{\partial \nu} u(x,t) = 0 & (x,t) \in \partial\Omega \backslash \eta \times (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in \Omega \times [0,T]. \end{cases}$$

As a general purpose, the interesting question to be able to answer is the following

Given two steady-states, can we guarantee the existence (or not) of a path of steady-states between the two steady-states without explicitly constructing a path?

A partial answer to the negative question is given in [104], where, using the comparison principle the authors establish a necessary condition for the existence of an admissible path between two steady-states. Understanding the steady-state structure and its connectivity is still of independent mathematical interest with applications aside from control.

3) Other types of diffusions and systems. In the lecture notes, we have dealt with the Laplacian for modelling a diffusion process. However, there are plenty of other linear and nonlinear diffusion operators that are relevant and whose controllability with constraints or the construction of paths of steady states will be more challenging. This is the case, for instance, of the porous medium equation [19, 43, 85, 117]:

$$\begin{cases} \partial_t u - \partial_{xx}(u^m) = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = a_1(t), \quad u(L,t) = a_2(t) & t \in (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in (0,L) \times [0,T]. \end{cases}$$

or other nonlinear problems [3,4]. A specific analysis also has to be done for dealing with other diffusions such as the fractional diffusion [2, 12, 47] or even non-local reaction terms [52] for which the dynamical system theory for generating paths of steady-states does not directly apply.

The other aspect that needs a more carfully detailed analysis is the case of systems. For instance, in the context of evolutionary game theory [53–55], typically one deals with more than two strategies. In this text, we motivated the problem in this context with a two strategy game that was reduced to a single evolution equation. However, if we deal with more strategies, the reduction to a scalar equation is not possible:

$$\begin{cases} \partial_t u_1 - \mu_1 \Delta u_1 = f_1(u_1,u_2,u_3) & (x,t) \in \Omega \times (0,T), \\ \partial_t u_2 - \mu_2 \Delta u_2 = f_2(u_1,u_2,u_3) & (x,t) \in \Omega \times (0,T), \\ \partial_t u_3 - \mu_3 \Delta u_3 = f_3(u_1,u_2,u_3) & (x,t) \in \Omega \times (0,T), \\ u_1 = a(x,t) \in [0,1] & (x,t) \in \partial\Omega \times (0,T), \\ \frac{\partial}{\partial \nu} u_j = 0 & (x,t) \in \partial\Omega \times (0,T) \quad j = 2,3, \\ 0 \leq u_i(x,0) \leq 1 & i = 1,2,3. \end{cases}$$

Constrained controllability of linear parabolic systems has been implemented in [69].Note that a phase plane analysis will be more intricate as the complexity of the nonlinear ODE system increases significantly with the dimension.

4) Optimal constrained controls

We have seen how the length of the domain is a crucial parameter that determines the controllability of the equation. This can be also linked with the *optimal placement of controls*. For big domains, the boundary control cannot be effective. However, the situation might be different if we consider interior control. If we set Neumann boundary conditions and a pointwise control in the middle of the interval, by symmetrization, we also observe the existence of fundamental obstructions as the domain grows. On the other hand, if we allow ourselves to choose in which region are we placing our control, one can clearly split the control region into small pieces distributed over the domain so that a barrier functions cannot exist.

For the one-dimensional homogeneous reaction-diffusion equation, this placement is an easy problem. However, if we consider a multidimensional problem with spatial heterogeneities, to determine where is the "best" control region with a fixed measure is a nontrivial problem with high relevance in applications.

For related literature on the placement of sensors and actuators we refer to [97–99].

On the other hand, the existence of a minimal controllability time and the controls in the minimal time are also a perpective to further understand. Simulations suggest that a bang-bang control might be possible for the minimal time control. How is the control in minimal time? Can we develop precise estimates or a formula for the minimal controllability time?

One possible way to attack these questions would be to make use of the Pontryagin maximum principle (see [72] for similar questions).

The starcase method gives a way to control in which the trajectory will be always inside a tubular neighborhood of the path of steady-states. Knowing that generally speaking, there is not a unique way to reach a specific configuration. How can we distinguish among different control possibilities? What is the best control (in terms of minimal L^2 norm or minimal flow, for example) for going from 0 to θ, for instance?

5) Other modellings and hyperbolic problems. Here we have been addressing parabolic equations where the comparison principle played a very strong role. However, the controllability with bounded controls is also relevant for hyperbolic models, for instance, for the semilinear wave equation:

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = f(u) & (x,t) \in (0,L) \times (0,T), \\ u(x,t) = a(x,t) & (x,t) \in \{0,L\} \times (0,T), \\ 0 \leq u(x,t) \leq 1 & (x,t) \in (0,L) \times [0,T). \end{cases}$$

The equation above has the same steady-states than the semilinear heat equation discussed in this text. This means that the paths of steady-states and nontrivial solutions are the same. However, for the wave equation, we do not have a maximum principle, this means that barriers might be avoidable, see [92].

On the other hand, the damped semilinear wave equation (telegraph equation) might present under certain conditions a maximum principle (see [42]). There is more variety of systems where the positivity of the state should be considered, for instance in thermoelastic systems [123].

One important remark is that control theory should be intimately related to the modeling. The natural constraints on the modeling give rise to new mathematical challenges in control theory. In the context of game theory, we modeled in Section 2 two strategies and an evolution of the proportion by the replicator dynamics. However, in practical situations makes sense to consider a continuum of strategies. For instance, in linguistics, most of the individuals are not perfectly bilingual, and their ability in the minority language can range in a continuum spectra. Interactions with other individuals can create an increase or decrease in this trait. For the modeling of this situation, we cite [7]. Moreover, diffusion models are more appropriate for simple living species or chemicals. When dealing with intelligent animals, the understanding of their way of moving in the environment is essential. In this situation also finite-dimensional models are appropriate, and the role of the network in which individuals move plays a crucial role in the dynamics [88, 111].

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