A Fisher-KPP model with a fast diffusion line in periodic media

Elisa Affili

University of Deusto, Chair of Computational Mathematics
Fundacion Deusto, Bilbao, Spain

Abstract

We treat a model of *population dynamics* in a periodic environment presenting a fast diffusion line. We call the fast diffusion line the *road* and the surrounding environment the *field*. The "road-field" model [4] is a system of coupled reaction-diffusion equations set in domains of different dimensions. Here, we consider the case of a reaction term depending on a spatial variable in a periodic fashion, which is of great interest for both its mathematical difficulties and for its applications. We study asymptotic behaviour and the influence of the road on it.

The importance of roads

In Western Canadian Forest, GPS observations on wolves proved that the animals exploit seismic lines, that are straight roads used by the oil companies to test reservoirs, to move faster and therefore to increase their probability of meeting a prey (McKenzie et al, 2012). This in not the only example of ecological diffusion acceleration by fast diffusion lines. For example, motorways have accelerated the diffusion of Covid-19 in Italy during March 2020 (Gatto et al, 2020).

The road-field model

We take an environment with a straight fast diffusion line. The idea (Berestycki, Roquejoffre and Rossi, [4]) is to split the population into two groups:

- on the road, that is the $x-$axis, the density is $u(t, x)$;
- on the rest of the environment (*the field*), that is up to symmetrisation $\Omega = \mathbb{R} \times (0, +\infty)$, the density is $v(t, x, y)$.

These two groups continuously exchange along the road. The diffusivity is different in the two environments. The resulting system, called *road-field model*, reads:

$$\begin{cases} \partial_t u(x,t) - D\partial^2_{xx} u(x,t) = \nu v(x,0,t) - \mu u(x,t), & x \in \mathbb{R},\ t > 0, \\ \partial_t v(x,y,t) - d\Delta v(x,y,t) = f(v), & (x,y) \in \Omega,\ t > 0, \\ -d\partial_y v(x,0,t) = -\nu v(x,0,t) + \mu u(x,t), & x \in \mathbb{R},\ t > 0. \end{cases}$$

In [4], the reproduction function satisfies $f(0) = f(1) = 0$, and the Fisher-KPP hypothesis

$$0 < f(s) < f'(0)s \qquad \text{for } s \in (0,1).$$

The positivity of $f'(0)$ assures the survival of the population.

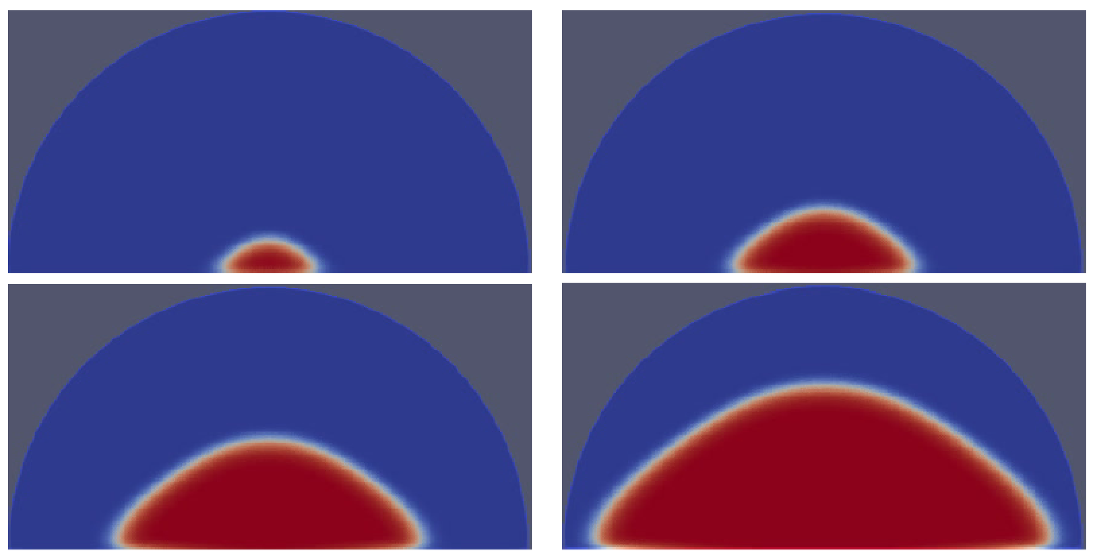


Figure 1: Image by Romain Ducasse

The problem

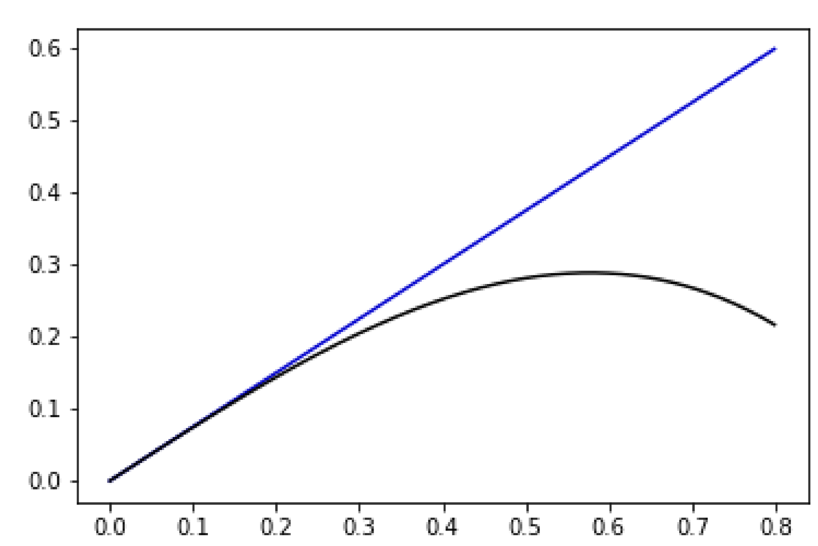
Here, we study the road-field system in a periodic medium:

$$\begin{cases} \partial_t u(x,t) - D\partial^2_{xx} u(x,t) = \nu v(x,0,t) - \mu u(x,t), & x \in \mathbb{R},\ t > 0, \\ \partial_t v(x,y,t) - d\Delta v(x,y,t) = f(x,v), & (x,y) \in \Omega,\ t > 0, \\ -d\partial_y v(x,0,t) = -\nu v(x,0,t) + \mu u(x,t), & x \in \mathbb{R},\ t > 0. \end{cases}$$

with the following hypothesis on f:

- the reproduction depends on space periodically: $f = f(v, x)$, $f(v, x) = f(v, x + \ell)$;
- the dependence is only on the spatial variable in the direction of the road;
- $f(x, 0) = 0$, $\exists M > 0$ (a saturation level) such that $f(x, v) < 0$ for all $v > M$;
- a Fisher-KPP hypothesis without sign requirements:

$$\frac{f(x,s_2)}{s_2} < \frac{f(x,s_1)}{s_1} \qquad \text{for } 0 < s_1 < s_2,\ x \in [0, \ell].$$



Asymptotic behaviour

The main issue is to find conditions entailing one of these two scenarios:

- *extinction*, that is $(u, v) \overset{t\to\infty}{\longrightarrow} (0, 0)$ uniformly in space;
- *persistence*, that is (u, v) tends to some positive stationary solution as $t \to \infty$.

A useful eigenvalue

The idea is that the stability of the equilibrium $(0, 0)$ is crucial to determine the asymptotic beaviour of the system. Thus, we are interested in the first eigenvalue, which has the property of being associated to a positive eigenfunction. But since the domain we work on are unbounded, we deal with a generalised version (Berestycki, Ducasse and Rossi, 2020):

$$\lambda_1 = \sup\{\lambda \in \mathbb{R} \ :\ \exists (u, v) \text{ super-eigenfun. to the linearised system at } (0, 0)\}.$$

This quantity has also the property of being the limit of eigenvalues on invading bounded domains.

What happens to the population?

Theorem 1 (A. 2020)**.** *For any non zero, non negative bounded initial datum and $f = f(v, x)$ KPP, periodic in x:*

1. *If $(0, 0)$ is stable ($\lambda_1 \geq 0$), then extinction occurs.*
2. *If $(0, 0)$ is unstable ($\lambda_1 < 0$), then persistence occurs and all solutions converge to the unique stationary solution, which is periodic.*

What happens without the road?

Diffusion on periodic media was studied by Berestycki, Hamel and Roques in 2012. Without the road, the system boils up to

$$\partial_t v - d\Delta v = f(v, x),\ x \in \mathbb{R}^n$$

with f periodic in x and KPP. They used the periodic eigenvalue λ_p connected to the first periodic eigenfunction and found:

Theorem 2 (Berestycki, Hamel and Roques, 2012)**.** *For any non zero, non negative bounded initial datum and $f = f(v, x)$ KPP, periodic in x:*

1. *If 0 is stable ($\lambda_p \geq 0$): extinction occurs.*
2. *If 0 is unstable ($\lambda_p < 0$): persistence occurs and all solutions converge to the unique (periodic) stationary solution.*

Influence of the road

Theorem 3 (A. 2020)**.** *For $f = f(v, x)$ periodic, KPP:*

1. *If $\lambda_p < 0$, then $\lambda_1 < 0$, that is, if persistence occurs for the equation "without the road", then it occurs also for system "with the road".*
2. *If $\lambda_p \geq 0$, then $\lambda_1 \geq 0$, that is, if extinction occurs for the equation "without the road", then it occurs also for system "with the road".*

Hence, we find that the road has no impact on the survival chances of the population in periodic media. Notice that in other cases such as that of an ecologial niche (Berestycki, Ducasse and Rossi, 2020) the road is found to have a deleterious effect on the survival.

Effect of fragmentation

From [3] we can derive the following dependence on the fragmentation of the environment. Call

$$\tilde{f}(x, v) = \alpha f(x, v)$$

and $\lambda_1(\Omega, \alpha)$ the generalised principal eigenvalue corresponding to $\tilde{f}$. Then:

Corollary 4 (A. 2020)**.** 1. *If $\int_0^\ell f_v(x, 0) > 0$, or if $\int_0^\ell f_v(x, 0) = 0$ and $f \not\equiv 0$, then for all $\alpha > 0$ we have $\lambda_1(\Omega, \alpha) < 0$.*

2. *If $\int_0^\ell f_v(x, 0) < 0$, then $\lambda_1(\Omega, \alpha) > 0$ for α small enough; if moreover there exists $x_0 \in [0, \ell]$ such that $f_v(x_0, 0) > 0$, then for all α large enough $\lambda_1(\Omega, \alpha) < 0$.*

Bibliography

References

[1] E.A. *A Fisher-KPP model with a fast diffusion line in periodic media.* arXiv preprint arXiv:2009.14760, 2020.

[2] H.Berestycki, R.Ducasse, L.Rossi. *Generalized principal eigenvalues for heterogeneous road-field systems.* Communications in Contemporary Mathematics, 22(01):1950013, 2020.

[3] H.Berestycki, F.Hamel, L.Roques. *Analysis of the periodically fragmented environment model: I-species persistence.* Journal of Mathematical Biology, 51(1):75-113, 2005.

[4] H.Berestycki, J.M.Roquejoffre, L.Rossi. *The influence of a line with fast diffusion on Fisher-KPP propagation.* Journal of mathematical biology, 66(4):743-766, 2013.

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