Almost periodic turnpike phenomenon for time-dependent systems

Sebastián Zamorano

[a]Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Las Sophoras 175, Estación Central, 9170020, Santiago, Chile

Abstract

This paper explores the turnpike behavior of a controlled system with almost periodic inputs and a cost functional based on almost periodic tracking term and periodic observation. Roughly speaking, we establish that the optimal state-control pair for the controlled system is exponentially close to the optimal pair of an almost periodic problem for a significant portion of the time during which the control system evolves. Our approach heavily relies on the thorough investigation of both differential Riccati equations, periodic differential Riccati equations and almost periodic functions.

Keywords: Almost periodic functions, turnpike, dynamic programming, periodic Riccati operator

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1. Introduction

1.1. Motivation

In this article we are interested in the so-called turnpike property, which roughly speaking, describes that the optimal evolutionary solution is made of three acrs: the first and the last being transient short time arcs, and the middle being a long-time arc staying exponentially close to the optimal steady-state, referred to as the turnpike, of the corresponding static optimal control problem. In this scenario, not only the optimal state and control, but also the corresponding adjoint vectors remain exponentially close to the stationary optimal control, state and adjoint vectors for a large enough time-horizon. The optimal control problem investigated within the context of the turnpike phenomenon is a linear quadratic optimal tracking-control problem.

Let us mention some of the principal works, as far as we know, in this direction. In the case of finite dimensional systems [1] is referenced, where the exponential turnpike property has been proven under the Kalman rank condition. For the nonlinear finite dimensional setting we refer to [2]. In [1], a rigorous analysis of the extremal equations has been done for linear infinite dimensional systems under appropriate observability assumptions. These results are extended to semilinear heat equations in [3]. Furthermore, [4] extends these findings for parabolic problems to the Navier-Stokes equations in the two-dimensional setting. All of three works [1, 3, 4] have shown the exponential turnpike property by utilizing the decoupling of extremal equations associated with a suitable minimization problem using the synthesis problem or Dynamic Programming methodology (see, for instance, [5, Chapter IV]), and by employing the algebraic Riccati equation associated with this decoupling.

In [6], the authors establish the exponential turnpike property without relying on the Riccati theory but under assumptions of stabilizability and detectability. All the aforementioned works consider optimal control problems without terminal costs, with the exception of [7], which studies terminal costs. Terminal conditions on the state are considered in [8] under controllability assumptions. Additionally, other contributions, though not exhaustive, focusing on turnpike analysis are presented in [9, 10, 11, 12]. Lastly, [8] provides a turnpike analysis of general evolution equations, and [13] explores the turnpike property in the context of fractional parabolic equations with exterior controls.

Finally, we want to mention the work [14], where the authors have investigated the turnpike property in Hilbert spaces by using general semigroups method with bounded controls and observation operators, and for parabolic equations with control supported on the boundary. They also study the case of a periodic tracking trajectory, which implies that the referred turnpike is not a stationary problem. Instead of a static problem, a periodic optimal control problem is considered. To the best of our knowledge, this phenomenon, as considered in [14], is the first time that an evolutionary problem is regarded as a turnpike.

Inspired by the previous periodic tracking problem, our article aims to investigate the exponential turnpike phenomenon for a class of time-dependent state equations with periodic inputs: periodic control operators and with a linear quadratic cost with periodic observations; and almost periodic controls, source terms and the cost featuring an almost periodic trajectory. Our main result establishes that solutions to the evolutionary extremal equations are exponentially close to their corresponding almost periodic counterparts. In other words, we achieve an exponential turnpike property with a almost periodic referred turnpike problem, allowing for the consideration of more general differential equations, as detailed below. The main assumptions we make revolve around the exponential stabilizability and detectability conditions for both the dynamics and the cost functional.

The proof of our main result relies on a thorough and meticulous analysis of the almost periodic optimal control problem. Specifically, we exploit the uniform asymptotic stability of solutions to the differential Riccati equation associated with the synthesis problem for the extremal equation. Additionally, we utilize the exponential decay rate of the evolution operator generated by a bounded perturbation by the periodic Riccati operator of the original semigroup generator. This remarkable property, crucial for our main result, not only plays a important role in the proof but also distinguishes our work from existing literature on the turnpike phenomenon. The primary reason is that, typically, previous studies have employed the exponential decay rate of the semigroup perturbed by the solution of an algebraic Riccati equation associated with an infinite time horizon optimal control problem, a well-established approach (see, for instance, [15]).

The motivation behind considering periodic control operators stems from well-known results related to the controllability of partial differential equations, where the control region must move to cover the entire domain where the dynamics evolve. This situation arises when a static control region is insufficient to achieve exact or null controllability properties. Examples illustrating this bad control behavior are, for instance, the viscoelasticity system studied in [16], where the authors established a null controllability result by employing a suitable Carleman inequality over the trajectory of the control which can be taken to be determined by the flow generated by some vector field. In addition, this flow need to satisfies certain assumptions in order to the control region can cover all the domain in a finite time. Similar strategy was utilized in [17] in the study of the null controllability of some Sobolev-Galpern equations and in [18] for the Moore-Gibson-Thompson equation. For the improved Boussinesq equation analyzed in [19] the authors proved that the system is null controllable in $H^1(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, in a sufficiently large time. Actually, this control time has to be chosen in such a way that the support of the control, which is moving at constant velocity c, can visit all the domain $\mathbb{T}$. Other examples are the local and nonlocal wave equation with structural damping [20, 21], wave equation with memory term [22], among others.

On the other side, an almost periodic inputs like a tracking term in the cost functional allow us to consider more general applied situations. A typical example is the function $\sin(t) + \cos(\sqrt{2}t)$ which is a sum of two purely periodic functions but is not a periodic function. However, every periodic function is an almost periodic ones. That is, almost periodic function is a generalization of periodicity. The concept of almost periodic function was introduced by the mathematician Harald Bohr in [23] and later continued to be studied by various mathematicians, among them, Salomon Bochner. Almost periodic solutions to boundary value problems for systems of partial differential equations that arise in solving certain problems for inhomogeneous media have been investigated in the research articles [24, 25, 26]. Concerning the existence and uniqueness of almost periodic solutions of the Navier-Stokes-type equations and other important examples, the reader may consult the reference list of [27]. Another application of this kind of solutions can be founded in [28, 29] for applications of almost periodic functions in crystallography.

1.2. Main result

In this part of the article we formulate the main result of this work mathematically. Let X, H be two Hilbert spaces such that $X \subset H \subset X'$ with dense embedding, H being identified with its dual.

Let us recall the well-known concept of almost periodic function introduced in [23].

Definition 1. Let $g : [0, \infty) \to H$ be a continuous function. Given $\varepsilon > 0$, we call π an ε-period for g if and only if

$$\|g(t + \pi) - g(t)\| \leq \varepsilon, \quad t \in [0, \infty). \tag{1}$$

We denote by $\mathcal{V}(g, \varepsilon)$ the set of all ε-periods for g. We will say that g is almost periodic function if and only if the set $\mathcal{V}(g, \varepsilon)$ is relatively dense in $[0, \infty)$. That is, there is a function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that in every subinterval of length $L(\varepsilon)$, there is an ε-translation number π, which means a number π such that (1) holds.

It is crucial to emphasize the frequent occurrence of almost periodic functions. For example, functions of the form $g(t) = \sin(t) + \cos(\sqrt{2}t)$, which is a sum of two purely periodic functions, serve as classical examples of almost periodicity and naturally extend the concept of periodic functions.

We will use the notation $AP([0,\infty); H)$ to represent the Banach space encompassing all continuous almost periodic functions on H with the sup norm. It is worth recalling that every almost periodic function is both bounded and uniformly continuous. Furthermore, the space $AP([0,\infty); H)$ constitutes a Banach algebra. According to [30], for any given $\varepsilon > 0$, there exists a minimum length $L(\varepsilon)$, denoted as $L = \min_{\varepsilon}\{L(\varepsilon)\}$. We refer to L as the modulus of almost periodicity for the function g.

Let $g_1, g_2 \in AP([0,\infty); H)$. Utilizing the fact that H is a Hilbert space, we can define an inner product on $AP([0,\infty); H)$ as follows:

$$\lim_{\Pi\to\infty} \frac{1}{\Pi} \int_0^{\Pi} \langle g_1(t), g_2(t)\rangle_H \, dt.$$

This inner product is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and the corresponding norm is denoted by $\|\cdot\|_{\mathcal{H}}$ (refer to, for instance, [31]). Consequently, let $L^2_{ap}(0,\infty; H)$ be the completion of $AP([0,\infty); H)$ with respect to this inner product. In a similar fashion, we denote by $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ and the corresponding norm $\|\cdot\|_{\mathcal{U}}$ for the space $AP([0,\infty); U)$, and by $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and the corresponding norm $\|\cdot\|_{\mathcal{V}}$ for $AP([0,\infty); V)$.

Let $T >> 1$ and let us consider the optimal control problem that follows

$$\min_{u\in L^2(0,T;U)} J^T(u) = \frac{1}{2}\int_0^T \|C(t)(x(t) - x_d(t))\|_V^2 dt + \frac{1}{2}\int_0^T \|N^{1/2}(t)u(t)\|_U^2 dt, \quad (2)$$

subject to $x = x(t)$ solves

$$\begin{cases} x'(t) = Ax(t) + B(t)u(t) + f(t), & t \in (0,T), \\ x(0) = x_0. \end{cases} \quad (3)$$

Here, $B \in L^\infty((0,\infty); \mathcal{L}(U,H))$, $C \in L^\infty((0,\infty); \mathcal{L}(H,V))$, V is a Hilbert space, and $N \in L^\infty((0,\infty); \mathcal{L}(U,U))$ is an invertible positive definite operator. All three are τ-periodic operators, and $A : D(A) \subset H \to H$ generates an analytic C_0 semigroup $\{S(t)\}_{t\geq 0}$ in H. In this article we consider an external input $f \in AP([0,+\infty); H)$ and $x_d \in L^2_{ap}(0,+\infty; H)$ a given target, both almost periodic functions of modulus L, in the sense of the Definition 1.

The direct method o the calculus of variations gives the existence of a minimizer u^T of J^T and an optimal state x^T. Besides, the first order optimality condition ensures the existence of $p^T \in C([0,T]; H)$ such that

$$\begin{cases} (x^T)'(t) = Ax^T(t) + B(t)u^T(t) + f(t), \\ (p^T)'(t) = -A^* p^T(t) - C^*(t)C(t)(x^T(t) - x_d(t)), \\ x^T(0) = x_0, \\ p^T(T) = 0, \\ u^T = -N^{-1}B^* p^T. \end{cases} \quad (4)$$

As mentioned in the introduction, our reference turnpike problem is now an evolutionary problem since our cost functional involve several time-dependent terms. In this case, as the tracking term is an almost periodic function, we consider as a turnpike an almost periodic problem. Namely, we consider the almost periodic optimal control problem of modulus L:

$$\min_{u\in L^2_{ap}(0,\infty;U)} J^L(u) = \frac{1}{2}\|C(x-x_d)\|^2_{\mathcal{V}} + \frac{1}{2}\|N^{1/2}u\|^2_{\mathcal{U}}, \quad (5)$$

subject to the almost periodic state $x = x(t) \in AP([0,\infty);H)$ solves the following system:

$$x'(t) = Ax(t) + B(t)u(t) + f(t), \quad t \in (0,\infty), \quad (6)$$

Determining the existence of an admissible solution $u \in L^2_{ap}(0,\infty;U)$ for this problem is not immediate. A classical approach (see, for instance, [32]) to obtaining an admissible almost periodic control assumes that the operator $A - BG$, for some $G \in C(\mathbb{R};\mathcal{L}(H,U))$, generates an exponentially stable evolution operator $\{S_{A-BG}(t)\}_{t\geq 0}$. In this case, the feedback law

$$u = -Gx + v, \quad v \in L^2_{ap}(0,\infty;U),$$

is admissible, and furthermore,

$$x(t) = \int_{-\infty}^{t} S_{A-BG}(t-s)\Big(B(s)v(s) + f(s)\Big)\,ds,$$

represents the unique mild solution of (6) in $AP([0,\infty);H)$. Given the almost periodic functions f and x_d, it is reasonable to assume the existence of such an operator G.

The main assumptions, which align with the above, for our problem are as follows:

(H1) The pair (A,B) is exponentially stabilizable, meaning that there exists a τ-periodic operator $K \in C(\mathbb{R};\mathcal{L}(H,U))$ such that the evolution operator generated by $A_K(t) := A - B(t)K(t)$ is exponentially stable on H.

(H2) The pair (A,C) is exponentially detectable, indicating that there exists a τ-periodic operator $R \in C(\mathbb{R};\mathcal{L}(H,U))$ such that the evolution operator generated by $A_R(t) := A^* - C^*(t)R^*(t)$ is exponentially stable on H.

Under the preceding hypotheses, as elaborated in the next section, the almost periodic optimization problem has an optimal almost periodic control u^L minimizing the functional J^L, and there exists x^L, the associated optimal state.

Our focus is on analyzing the relationship between problems (2)-(3) and (5)-(6) in the following sense:

Is it possible for the optimal solutions of the two problems to be close enough for a long time?

We can now present our main result, commonly known in the literature as the exponential turnpike property. The proof draws inspiration from the results in [1].

Theorem 1. Let $T >> 1$ and $\tau > 0$ such that $T > \tau$. Suppose that **(H1)** and **(H2)** holds. Let (x^T, u^T) and (x^L, u^L) be the optimal solutions of problems (2) and (5), respectively. Then, there exist positive constants $C > 0$ and $\mu > 0$, such that for any $T >> 1$

$$\|x^T(t) - x^L(t)\|_H + \|u^T(t) - u^L(t)\|_U \leq C\Big(e^{-\mu t}\|x_0 - x^L(0)\|_H + e^{-\mu(T-t)}\|p^L(T)\|_H\Big), \quad (7)$$

for every $t \in [0,T]$.

The remainder of the article is structured as follows. Section 2 provides a detailed exploration of the almost periodic cell, specifically addressing problem (5)-(6). The proof of our main result, Theorem 1, is presented in Section 3.

2. The Almost Periodic Cell

In this section, we delve into the examination of almost periodic optimal control problem of modulus L given in the previous section. That is, we consider

$$\min_{u\in L^2_{ap}(0,\infty;U)} J^L(u) = \frac{1}{2}\|C(x-x_d)\|^2_{\mathcal{V}} + \frac{1}{2}\|N^{1/2}u\|^2_{\mathcal{U}}, \quad (8)$$

subject to the almost periodic solution $x = x(t) \in AP([0,\infty);H)$ of the following system:

$$x'(t) = Ax(t) + B(t)u(t) + f(t), \quad t \in (0,\infty). \quad (9)$$

Here, $B \in L^\infty((0,\infty);\mathcal{L}(U,H))$, $C \in L^\infty((0,\infty);\mathcal{L}(H,V))$, V is a Hilbert space, and $N \in L^\infty((0,\infty);\mathcal{L}(U,U))$ is an invertible positive definite operator. All three are τ-periodic operators, and $f \in AP([0,+\infty);H)$ and $x_d \in AP([0,+\infty);H)$ are given almost periodic functions of modulus L (see Definition 1). Additionally, $A : D(A) \subset H \to H$ generates an analytic C_0 semigroup $\{S(t)\}_{t\geq 0}$ in H.

Utilizing the fact that A is the generator of the semigroup $\{S(t)\}_{t\geq 0}$, it is well-known that system (9) has a unique mild solution in $AP([0,\infty);H)$ given by

$$x(t) = S(t-s)x(s) + \int_s^t S(t-r)(B(r)u(r) + f(r))dr, \quad (10)$$

for every $t \geq s$. Let us observe that a easily computation shows that if the semigroup $\{S(t)\}_{t\geq 0}$ is exponentially stable, then the almost periodic solution is given by

$$x(t) = \int_{-\infty}^t S(t-s)(B(s)u(s) + f(s))ds. \quad (11)$$

It is important to recall that more general condition on A guarantee the existence of a unique almost periodic solution of (9), as the case if $\{S(t)\}_{t\geq 0}$ satisfies an exponential dichotomy (see, for instance, [33])

Moving on, under hypotheses (**H1**) and (**H2**) the existence of an optimal almost periodic pair (x^L, u^L) for the optimal control problem is direct, where J^L attains its minimum at u^L. This is because minimizing the J^L-functional over almost periodic controls is not straightforward without the aforementioned assumptions. These directly imply that feedback control given by

$$u(t) = -K(t)x(t) + v(t), \quad v \in L^2_{ap}(0,\infty;U),$$

is admissible and in this case the unique almost solution of (9) can be expressed as

$$x(t) = \int_{-\infty}^t S_{A-BK}(t-s)(B(s)v(s) + f(s))ds,$$

where $\{S_{A-BK}(t)\}_{t\geq 0}$ is the exponentially stable evolution operator generated by the operator $A_K = A - BK$.

Furthermore, the first-order optimality conditions are satisfied, implying the existence of almost periodic function p^L such that:

$$\begin{cases} (x^L)'(t) = Ax^L(t) + B(t)u^L(t) + f(t), & t \in (0,\infty), \\ (p^L)'(t) = -A^* p^L(t) - C^*C(x^L(t) - x_d(t)), & t \in (0,\infty), \\ u^L = -N^{-1}B^* p^L. \end{cases} \quad (12)$$

However, in this work we will need a closed loop representation of the optimal solution of the almost periodic system. In other words, we want to obtain a feedback control $u^L = Dx^L$, with some appropriate operator D, such that the optimal state x^L can be write as the solution of the closed loop equation

$$(x^L)'(t) = (A-D)x^L(t) + f(t), \quad t \in (0,\infty).$$

In order to realize that, we follow the Dynamic Programming approach. That is, first we look for a

τ-periodic solution Q of the following differential Riccati equation

$$\begin{cases} Q' + A^*Q + QA - QBN^{-1}B^*Q + C^*C = 0, \\ Q(0) = Q(\tau). \end{cases} \tag{13}$$

We remark that in our case since we are considering time-dependent operators in the functional to minimize, the Riccati equation is a differential equation, contrary to the case of autonomous system where a solution to Riccati's algebraic equation is sought.

Then, the optimal control is given by the feedback formula (synthesis problem)

$$u^L(t) = -N^{-1}(t)B^*(t)\Big(Q(t)x^L(t) + r(t)\Big),$$

where $r = r(t)$ is the almost periodic mild solution of the backward problem

$$r' + (A^* - QBN^{-1}B^*)r + Qf - C^*Cx_d = 0.$$

Let us start analysing the meaning of solution for the τ-periodic Riccati equation. Let $C_s([0,\tau];\mathcal{L}(H,H))$ be the set of all mappings $T : [0,\tau] \to \mathcal{L}(H,H)$ such that $T(\cdot)x$ is continuous for any $x \in H$. Let $\Sigma(H)$ be the set of all linear bounded self-adjoint operator on H, that is,

$$\Sigma(H) := \{T \in \mathcal{L}(H,H) \ : \ T = T^*\}.$$

And, by $\Sigma^+(H)$ we denote the set of all linear bounded self-adjoint nonnegative operators. Namely,

$$\Sigma^+(H) := \{T \in \Sigma(H) \ : \ T \geq 0\}.$$

Definition 2. We will say that $Q \in C_s([0,\tau];\Sigma^+(H))$ is a τ-periodic solution of (13) if Q satisfies the following identity

$$\begin{aligned} Q(t)x = S^*(\tau - t)Q(0)S(\tau - t)x \\ - \int_t^\tau S^*(s-t)[Q(s)B(s)N^{-1}(s)B^*(s)Q(s) \\ + C^*(s)C(s)]S(s-t)x\,ds, \ x \in H, \end{aligned} \tag{14}$$

with $Q(0) = Q(\tau)$.

Under the hypotheses (**H1**) and (**H2**), the following result was proved in [34].

Theorem 2. [34, Theorem 2.1] Suppose that hypotheses (**H1**) and (**H2**) holds. Then there exists at most one τ-periodic bounded nonnegative solution $Q \in C_s([0,\tau];\Sigma^+(H)$ of the differential Riccati equation (13). Moreover, the evolution operator $\{S_F(t)\}_{t\geq 0}$ generated by the operator $F := A - BN^{-1}B^*Q$ is exponentially stable. That is, there exist constants $C > 0$ and $\mu > 0$ such that:

$$\|S_F(t)\|_{\mathcal{L}(H,H)} \leq Ce^{-\mu t}, \quad \forall t \geq 0. \tag{15}$$

Even more, an important point in the proof of our main finding is that the τ-periodic solution Q is stable in the following sense.

Lemma 3. Let P be a mild solution of the Riccati equation

$$\begin{cases} P' + A^*P + PA - PBN^{-1}B^*P + C^*C = 0, \quad t \leq 0 \\ P(T) = P_0, \end{cases} \tag{16}$$

where $P_0 \in \Sigma^+$. Then:

$$\lim_{\|P_0\|\to 0} \|Q(t) - P(t)\| = 0, \quad \text{uniformly in } t. \tag{17}$$

Proof. Let $Y = P - Q$ and denote again by $F = A - BN^{-1}B^*Q$. Then, a straightforward calculation tells us that Y is a mild solution of

$$Y' + L^*Y + YL - YBN^{-1}B^*Y = 0,\ \ Y(T) = Y_0.$$

Then, since the evolution operator $\{S_F(t)\}_{t\geq 0}$ is exponentially stable, a linearisation argument show that Q is uniformly asymptotic stable (see for instance [35, Chapter 2]) and the proof is finished. □

Now, following the ideas from [5, Chapter IV], we want to decouple the system (12) by the use of Q, the unique τ-periodic solution of (13). Let us write:

$$p^L = Qx^L + r. \tag{18}$$

A formal computation shows us that Q is the unique τ-periodic solution of (13) and r satisfies:

$$r' + (A^* - QBN^{-1}B^*)r + Qf - C^*Cx_d = 0. \tag{19}$$

Then, the following result is classical in the context of optimal control (see [32]).

Theorem 4. Let us assume that **(H1)** and **(H2)** hold. Then, for every $f \in AP([0, +\infty); H)$ and $x_d \in AP([0, +\infty); H)$ there exists a unique $r \in AP([0, +\infty); H)$ solution of (19). Moreover, r is given by:

$$r(t) = \int_t^\infty S_F^*(s-t)\Big(Q(s)f(s) - C^*(s)C(s)x_d(s)\Big)ds, \tag{20}$$

where S_F is the evolution operator generated by the operator $A - BN^{-1}B^*Q$.

Collecting all the previous content, we finally get that the optimal control of (8) is given by:

$$u^L = -N^{-1}B^*(Qx^L + r), \tag{21}$$

where Q is the unique bounded nonnegative τ-periodic solution of (13) and r is the unique almost periodic solution of (19) and

$$J(u^L) = 2\langle r, f\rangle_{\mathcal{H}} - \|N^{-1}B^*r\|^2_{\mathcal{H}}. \tag{22}$$

In addition, the closed-loop system corresponding to the optimal control problem (8) is given by:

$$(x^L)' = (A - BN^{-1}B^*Q)x^L + f - BN^{-1}B^*r, \quad t \in (0, \infty). \tag{23}$$

Since the evolution operator generated by F is exponential stable, we finally get that x^L is given by

$$x^L(t) = \int_{-\infty}^t S_F(t-s)\Big(f(s) - B(s)N^{-1}(s)B^*(s)r(s)\Big)ds.$$

We refer to [15, Part V, Chapter 1] and [5, Chapter IV] for a comprehensive treatment of this topic.

3. Proof of Theorem 1

Proof. We will divide the proof into several steps for a better presentation of that.

Step 1: First, let us recollect some well-known consequence of hypotheses **(H1)** and **(H2)** for the optimal control problem (2). Under that hypotheses, we know that there exists a unique bounded nonnegative operator Q^T solution of the following differential Riccati equation.

$$\begin{cases} Q' + A^*Q + QA - QBN^{-1}B^*Q + C^*C = 0, & t \in (0, T) \\ Q(T) = 0. \end{cases} \tag{24}$$

By Theorem 7.1, Part IV in [15] we obtain the following closed loop-system for x^T:

$$\begin{cases} (x^T)' = (A - BN^{-1}B^*Q^T)x^T + f - BN^{-1}B^*r^T, \ t \in (0, T) \\ x^T(0) = x_0, \end{cases}$$

where r^T is the mild solution of

$$\begin{cases} (r^T)' + (A^* - Q^T BN^{-1}B^*)r^T + Q^T f - C^*Cx_d = 0, \\ r^T(T) = 0. \end{cases} \tag{25}$$

Moreover, the optimal control is given by the feedback formula $u^T = -N^{-1}B^*(Q^T x^T + r^T)$.

Step 2: Let (x^T, u^T, p^T) and (x^L, u^L, p^L) be the optimal solutions of problems(4) and (12), respectively. Then,

$$(x^T - x^L)' = A(x^T - x^L) - BN^{-1}B^*(p^T - p^L).$$

Since the operator $F = A - BN^{-1}B^*Q$, where Q is the unique τ-periodic nonnegative bounded operator solution of (13), is exponentially stable with rate $\mu > 0$, we rewrite the previous equations as follows

$$\begin{aligned}(x^T - x^L)' = F(x^T - x^L) + BN^{-1}B^*\Big(Q(x^T - x^L) \\ - (p^T - p^L)\Big).\end{aligned}$$

Drawing inspiration from the decoupling method presented in the previous section for the periodic problem, let us introduce the new variable

$$h(t) + Q^T(t)(x^T(t) - x^L(t)) = p^T(t) - p^L(t), \quad t > 0. \tag{26}$$

Therefore, we get

$$\begin{aligned}(x^T - x^L)' = F(x^T - x^L) - BN^{-1}B^*(Q - Q^T)(x^T - x^L) \\ + BN^{-1}B^*h \end{aligned} \tag{27}$$

and h is the unique solution of the backward problem

$$\begin{cases} h' = -(A^* - Q^T BN^{-1}B^*)h, \quad t \in (0, T) \\ h(T) = -p^L(T). \end{cases} \tag{28}$$

We observe that in order to get the exponential estimate for $x^T - x^L$, we need to obtain a similar estimate for the difference $Q - Q^T$ and for h. These will be the next two steps in our proof and the most important key tool is the exponentially stability of the evolution operator generated by F.

Step 3: In this step we will show that the τ-periodic solution Q of the Riccati equation (13) is exponentially near to the solution Q^T of the differential Riccati equation (24).

Let us consider the Banach space

$$\begin{aligned}\mathcal{J} &:= C_\mu([0, \infty); \Sigma(H)) \\ &= \{Y \in C_s([0, \infty); \Sigma(H)) : \sup_{t \geq 0} \|e^{2\mu t}Y(t)\| < \infty\},\end{aligned} \tag{29}$$

with its norm denote by $\|\cdot\|_\mu$. In addition, for every $r > 0$ let $\mathcal{B}_r$ the ball of radius $r > 0$ in $\mathcal{J}$, that is,

$$\mathcal{B}_r := \{Y \in C_\mu([0, \infty) : \Sigma(H)) : \sup_{t \geq 0} \|e^{2\mu t}Y(t)\| \leq r\}. \tag{30}$$

Now, set $Z \in \mathcal{B}_r$. Then, it is immediate that $Z = Z(t)$ is the mild solution of the following differential equation

$$\begin{cases} Z' + F^*Z + ZF - ZBN^{-1}B^*Z = 0, \quad t \in (0, T) \\ Z(T) = Z_0 \end{cases} \tag{31}$$

if and only if, the application $\mathcal{T} : \mathcal{B}_r \to \mathcal{B}_r$ defined by

$$\mathcal{T}(Z)(t)x = S_F^*(T - t)Z_0 S_F(T - t)x$$
$$- \int_t^T S_F^*(s-t)Z(s)B(s)N^{-1}(s)B^*(s)Z(s)S_F(s-t)x ds,$$

for every $x \in H$, has a fixed point in $\mathcal{B}_r$. Let us prove that there exists a fixed point for $\mathcal{T}$. Indeed, for $Z \in \mathcal{B}_r$ we get

$$\|\mathcal{T}(Z)\|_\mu \leq \|Z_0\|_\mu + C\|Z\|_\mu^2. \tag{32}$$

Here we use in a strong way that the evolution operator $S_F(t)_{t\geq 0}$ generated by F is exponential stable with rate $\mu > 0$.

Chosen $r > 0$ sufficiently small in the sense that $\|Z_0\|_\mu < \frac{r}{2}$ and $\frac{r}{2} + Cr^2 \leq r$, we obtain that $\|\mathcal{T}(Z)\|_\mu \leq r$. Namely, the set $\mathcal{B}_r$ is invariant with respect to the application $\mathcal{T}$. From the definition of $\mathcal{T}$ is direct that this operator is continuous and compact. Therefore, there exists a fixed point $Z \in \mathcal{B}_r$ of $\mathcal{T}$.

Observe the following fact. Since the operator Q is stable in the sense of (17), we know that there exists $t_0 > 0$ such that we can chose $Z_0 = Q(T-t_0) - Q^T(T-t_0)$ and $\|Q(T-t_0) - Q^T(T-t_0)\| < \frac{r}{2}$. Then, the fixed point satisfies

$$\begin{cases} Z' + F^*Z + ZF - ZBN^{-1}B^*Z = 0, & t \in (0,T) \\ Z(T) = Q(T-t_0) - Q^T(T-t_0). \end{cases} \tag{33}$$

Using the previous result, we are able to prove the following estimate

$$\|Q(t) - Q^T(t)\| \leq Ce^{-2\mu(T-t)}, \quad \forall t > 0. \tag{34}$$

Indeed, by performing an argument similar to the one performed to determine (27), we obtain

$$\begin{aligned} &(Q-Q^T)' + F^*(Q-Q^T) + (Q-Q^T)F \\ &- (Q-Q^T)BN^{-1}B^*(Q-Q^T) = 0, \quad t \in (0,T). \end{aligned} \tag{35}$$

Thus, the fixed point Z satisfies the same equation of $Q - Q^T$. That is, we can deduce that $Z(t) = Q(t-t_0) - Q^T(t-t_0)$. As we have $Z \in \mathcal{B}_r$, we finally get

$$\|Q(t) - Q^T(t)\| \leq Ce^{-2\mu(T-t)}, \quad \forall t \in (0,T).$$

Step 4: Now it is time to prove a similar result for h. As in the previous step, it is possible to rewrite the system (28) as follows

$$\begin{cases} h' = -(A - B^*N^{-1}BQ)^*h + (Q^T - Q)BN^{-1}B^*h, \ t \in (0,T) \\ h(T) = -p^L(T). \end{cases} \tag{36}$$

Moreover, we know that the mild solution $h = h(t)$ of (36) is given by

$$\begin{aligned} &h(t) = -S_F^*(T-t)p^L(T) \\ &+ \int_t^T S_F^*(s-t)(Q^T(s) - Q(s))B(s)N^{-1}(s)B^*(s)h(s)ds. \end{aligned} \tag{37}$$

Using the exponential decay of the evolution operator $\{S_F(t)\}_{t\geq 0}$ generated by $F = A - B^*N^{-1}BQ$ and (34), we deduce

$$\begin{aligned} \|h(t)\| &\leq e^{-\mu(T-t)}\|p^L(T)\| + C\int_t^T e^{-\mu(s-t)}e^{-2\mu(T-s)}\|h(s)\|ds \\ &= e^{-\mu(T-t)}\|p^L(T)\| + Ce^{-\mu(T-t)}\int_t^T e^{-\mu(T-s)}\|h(s)\|ds. \end{aligned}$$

A direct consequence of the Gronwall Lemma gives us

$$\|h(t)\| \leq Ce^{-\mu(T-t)}\|p^L(T)\|.$$

Step 5: In this last step, we can prove the turnpike property for our problem. Returning to the equation (27), the previous two steps allows us to obtain the exponential estimate. Indeed, since

$$\begin{aligned} (x^T - x^L)' = F(x^T - x^L) - BN^{-1}B^*(Q-Q^T)(x^T - x^L) \\ + BN^{-1}B^*h, \end{aligned}$$

we have that

$$\begin{aligned} x^T(t) - x^\tau(t) &= S_F(t)(x_0 - x^L(0)) \\ &- \int_0^t S_F(t-s)\Big(B(s)N^{-1}(s)B^*(s)(Q(s) \\ &- Q^T(s))(x^T(s) - x^L(s)) - B(s)N^{-1}(s)B^*(s)h(s)\Big)ds. \end{aligned}$$

Therefore, using the exponential bounds obtained in steps three and four, the following estimate can be done

$$\begin{aligned}\|x^T(t)-x^\tau(t)\| &\le \|x_0-x^L(0)\|e^{-\mu t}\\ &\quad + C\int_0^t e^{-\mu(t-s)}\Big(e^{-2\mu(T-s)}\|x^T(s)-x^L(s)\|\\ &\quad + e^{-\mu(T-s)}\|p^L(T)\|\Big)ds\\ &\le \|x_0-x^L(0)\|e^{-\mu t}+C\|p^L(T)\|e^{-\mu(T-t)}\\ &\quad + Ce^{-\mu(T-t)}\int_0^t e^{-\mu(2T-t)}e^{3\mu s}\|x^T(s)-x^\tau(s)\|ds.\end{aligned}$$

Applying Gronwall inequality one more time, we get the desired estimate for the optimal state

$$\|x^T(t)-x^\tau(t)\|\le C\Big(\|x_0-x^L(0)\|e^{-\mu t}+\|p^L(T)\|e^{-\mu(T-t)}\Big),$$

for every $t\in(0,T)$.

Finally, the desired estimate for the control is easily obtained from the identity (26). Indeed, since $h(t)+Q^T(t)(x^T(t)-x^L(t)) = p^T(t)-p^L(t)$ and using the fact that Q^T is bounded, we get

$$\begin{aligned}\|u^T(t)-u^L(t)\| &\le C\|p^T(t)-p^L(t)\|\\ &\le C\|h(t)\|+C\|x^T(t)-x^L(t)\|\\ &\le C\Big(\|x_0-x^L(0)\|e^{-\mu t}+\|p^L(T)\|e^{-\mu(T-t)}\Big),\end{aligned}$$

for every $t\in(0,T)$. □

Remark 1. Let us mention some comments regarding our work:

- Let us observe that the key tool for obtaining the exponential estimate for $Q-Q^T$ and h, solutions of (35) and (36), respectively, is the exponential stability of the evolution operator generated by $F=A-BN^{-1}B^*Q$. In contrast to previous literature, as seen in [7, 1, 3, 14, 4], where the solution of an algebraic Riccati equation for the infinite horizon LQ problem was considered, in our case, this operator is given by the τ-periodic solution of a differential Riccati equation.

- In the previous work [14], the authors have studied the exponential turnpike property when the desired target, namely x_d, is a periodic function. They proved the turnpike using the Riccati's theory as well. However, it is worth mentioning some differences with respect to our paper. The first is related to the operators involved in the state-equation and the cost functional. We do not only take into account the almost periodic target, but we can also consider periodic control and observation operators. The second has to do with the use of the Riccati differential equation, both for the finite and infinite horizon problem.

- Additionally, in [36], the authors studied the turnpike property for periodic systems. Here, we move one step further by considering almost periodic systems and also provide a different proof of the turnpike result.

- Our referenced turnpike system is an almost periodic control problem in infinite horizon time. As far as we know, this is the first work considering this type of turnpike. In the same line, we can mention that if the target function x_d and the source term f are both τ-periodic, then we can consider a τ-optimal control problem as a turnpike. That is,

$$\min_{u\in L^2(0,\tau;U)} J^\tau(u)=\frac{1}{2}\int_0^\tau \|C(t)(x(t)-x_d(t))\|_V^2dt + \frac{1}{2}\int_0^\tau \|N^{1/2}(t)u(t)\|_U^2dt,$$

subject to the periodic solution $x = x(t)$ of the following system:

$$\begin{cases} x'(t) = Ax(t) + B(t)u(t) + f(t), \quad t \in (0, \tau), \\ x(0) = x(\tau), \end{cases}$$

and then extend both the control and state by periodicity to the interval $[0, \infty)$.

- An important result to highlight is that our exponential estimate for the difference $Q - Q^T$ given in (34) is of order 2μ, while in [1, 36] a rate equal to μ is established for parabolic problems and periodic problems, respectively.
- It is important to mention that also our turnpike estimate has an explicit dependence on the terms $\|x_0 - x^L(0)\|$ and $\|p^L(T)\|$.
- The assumption on the control operator belonging to the space $L^\infty((0, \infty); \mathcal{L}(U, H))$ is not restrictive. This is to avoid technicalities when considering the control operator in $L^\infty((0, \infty); \mathcal{L}(U, X'))$. If we assume only the last regularity on our control operator, the turnpike result could only be obtained for the adjoint difference, that is, for $\|p^T(t) - p^L(t)\|$ instead of the controls $\|u^T(t) - u^L(t)\|$.

Finally, let us consider the following example.

Example 1. Let $T > 0$ and Ω denote a bounded open subset of $\mathbb{R}^N$ with a C^2 boundary. We introduce the optimal control represented by

$$\inf_{u \in L^2(\Omega \times (0,T))} J^T(u) = \frac{1}{2} \int_0^T \Big(\|y(x, t) - y_d(x, t)\|^2_{L^2(\Omega)} + \|u(x, t)\|^2_{L^2(\omega(t))} \Big) dt, \quad (38)$$

where $y_d \in C([0, \infty[; L^2(\Omega))$ is a almost periodic function, subject to $y \in C([0, T]; H^1_0(\Omega)) \cap C^1(]0, T]; L^2(\Omega))$ satisfying

$$\begin{cases} y_{tt} - \Delta y - \Delta y_t + b(x) y_t = \chi_{\omega(t)} u(x, t), & \text{in } \Omega \times (0, T), \\ y = 0, & \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), & x \in \Omega, \end{cases} \quad (39)$$

where $b \in L^\infty(\Omega)$ is a given function representing frictional damping, and $(y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)$ are the initial conditions.

We apply Theorem 1 to this system. To do so, we rewrite the viscoelasticity system (39) as a first-order Cauchy problem. Let $H = H^1_0(\Omega) \times L^2(\Omega)$ and $U = V = L^2(\Omega)$. Then,

$$\begin{cases} Y_t + AY = B(t)U, \quad \text{in } (0, T), \\ Y(0) = (y_0, y_1)^\mathsf{T}, \end{cases}$$

where $Y = (y, y_t)^\mathsf{T}$, $U = (0, u)^\mathsf{T}$, and the operators A and $B = B(t)$ are given by

$$A = \begin{pmatrix} 0 & -I \\ -\Delta & -\Delta + bI \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ \chi_{\omega(t)} \end{pmatrix}.$$

Note that A can be seen as a bounded perturbation of A_0, i.e., $A = A_0 + P$ where

$$A_0 = \begin{pmatrix} 0 & -I \\ -\Delta & -\Delta \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ 0 & bI \end{pmatrix}.$$

The operator A_0 is equipped with the natural domain

$$\begin{cases} D(A_0) = (u, v) \in H^1_0(\Omega) \times H^1_0(\Omega) : \ u + v \in H^2(\Omega), \\ A_0(u, v) = (-v, -\Delta(u + v)) \in H^1_0(\Omega) \times L^2(\Omega). \end{cases}$$

Observe that P is a bounded operator, as $b \in L^\infty(\Omega)$. Therefore, A_0 is a m-dissipative operator on $H^1_0(\Omega) \times L^2(\Omega)$. The well-posedness of system (39) on $H^1_0(\Omega) \times L^2(\Omega)$ is an immediate consequence of the fact that A is a bounded perturbation of a m-dissipative operator A_0. Standard semigroup theory implies that A itself generates a strongly continuous semigroup on $H_0^1(\Omega) \times L^2(\Omega)$.

In the study conducted by Chavez et al. [16], the authors demonstrated the achievability of null controllability for the system (39) with an interior control by employing a moving control region that covers the entire domain where the dynamics evolve. Consequently, the function $\omega(t)$ can be considered as a τ-periodic function, where $\tau > 0$ represents the minimal time required to cover the entire domain with $\omega(t)$.

Assuming that $(\Omega, \omega(t))$ satisfies a set of geometric properties outlined in [16] (refer to equations (1.11)-(1.17)), the authors established in the same work that there exists a time $T_0 > 0$ such that for every $T > T_0$, the null controllability property holds for the solution of (39). Consequently, we can deduce that the system is exponentially stabilizable with respect to (A, B) and exponentially detectable with respect to (A, C). According to Theorem 1, we can assert that the almost periodic turnpike property holds for the viscoelastic system when $T > \tau$.

Applying a similar argument, we can derive a periodic turnpike property for the Barenblatt-Zheltov-Kochina and Benjamin-Bona-Mahony equations (both Sobolev-Galpern type equations) presented in [17]. In this case, a moving control domain was employed to achieve a null controllability result. This approach is also applicable to the Moore-Gibson-Thompson equation, a third-order partial differential equation studied in [18], where a moving control domain was also utilized. In both articles, the authors referred to the list of geometrical properties satisfied by the control region introduced in [16] as the "Moving Geometrical Control Condition."

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