Partially dissipative hyperbolic systems, results and perspectives

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Introduction

Introduction

We consider general n-component nonlinear hyperbolic systems of the form:

$$\frac{\partial V}{\partial t}+\sum_{j=1}^{d}\frac{\partial F_j(V)}{\partial x_j}=\frac{Q(V)}{\varepsilon},\quad \text{where}\quad V:\begin{cases}\mathbb{R}_+\times\mathbb{R}^d \to & \mathcal{O}_V\subset\mathbb{R}^n\\ (t\;\;,\;\;x)\to & V=V(t,x)\end{cases}\qquad (1)$$

and $Q, F_j : \mathcal{O}_V \to \mathbb{R}^n$ are given n-vector valued smooth functions on $\mathcal{O}_V$.

→ Model physical phenomena with finite speed of propagation or conservation laws, such as:

- The compressible Euler equation, Maxwell's equation, Einstein's equation, MHD equations, Yang-Mills equation, etc.
- Numerous applications: non-viscous fluid mechanics, kinetic theory, astrophysics, road traffic modelling, blood vessel circulation, etc.

Natural question: as time goes to infinity, does the solution of the system tend to the corresponding stationary solutions? And if it does, at which rate?

→ There is a surprisingly strong connection between this question and problems related to control theory and Villani's hypocoercivity theory.

Dissipation term

For that issue, the 0-th order term Q, that act as a **dissipation** term, plays a crucial role.

- In the case $Q \equiv 0$, local-in-time strong solutions may develop singularities (shock waves) in finite time (A. Majda, D. Serre).
- If $Q(V)$ acts directly on each component e.g. $Q(V) = -BV$ with $B > 0$

  $\rightarrow$ global existence + convergence to the equilibrium exponentially fast. (T-T. Li)
- In practice, a more reasonable assumption is that the dissipation acts only on certain components of the system.
- This can be written as :

$$Q(V) = \begin{pmatrix} 0_{\mathbb{R}^{n_1}} \\ q(V) \end{pmatrix} \text{ where } q(V) \in \mathbb{R}^{n_2}, n_1, n_2 \in \mathbb{N} \text{ and } n_1 + n_2 = n. \tag{2}$$

- Or, in the linear case,

$$Q(V) = \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix} V \tag{3}$$

The question now is: how can one recovers dissipative properties on all the components if the damping term is only present in some their equations?
→ **Such a coercivity property is necessary to prove decay estimates**

As a toy-model, one may consider the damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + v = 0. \end{cases} \tag{4}$$

For this simple system, considering the following perturbed functional:

$$\mathcal{L}^2 = \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2 + \int_{\mathbb{R}} v \partial_x u,$$

allows to recover dissipation properties on all its components.

Indeed, after basic computations, we obtain

$$\frac{d}{dt}\mathcal{L}^2 + \|v\|_{L^2}^2 + \|(\partial_x u, \partial_x v)\|_{L^2}^2 \leq 0.$$

And since $\mathcal{L}^2 \sim \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2$, we can obtain time-decay estimates.

Note that similar techniques have been used in the study of the compressible Navier-Stokes equations by Raphaël Danchin '00 and for the damped wave equation by Haraux and Zuazua '88.

General case

- In the 80's, Shizuta and Kawashima developed an algebraic criterion to ensure linear stability: the (SK) condition.
- Let's consider the following linear system:

$$\partial_t V + \sum_{j=1}^{d} A\partial_{x_j} V + LV = 0$$

where the A^j and L are constant symmetric matrices.

- Applying the Fourier transform we get

$$\widehat{V}_t + iA(\xi)\widehat{V} + L\widehat{V} = 0 \quad \text{avec} \quad A(\xi) \triangleq \sum_{j=1}^{d} A^j \xi_j.$$

- The (SK) condition is defined as follows:

Definition

$$\forall \xi \in \mathbb{R}^d, \quad \ker L \cap \{\text{eigenvectors of } A(\xi)\} = \{0\}. \tag{SK}$$

- $\rightarrow$ Ensures that **the damping is strong enough** to prevent small solutions (perturbations of constant states) from developing singularities in finite time.

- Under this condition, they are able to prove the following decay estimates:

$$\|V^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda t}\|V_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)},$$
$$\|V^\ell(t)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \leq Ct^{-\frac{d}{2}}\|V_0\|_{L^1(\mathbb{R}^d,\mathbb{R}^n)} \tag{5}$$

  where V^h and V^ℓ correspond, respectively, to the high and low frequencies of the solution.
- Many results with this condition:
  - Hanouzet and Natalini '03, Ruggeri and Serre '04, **Yong '04**, Kawashima and Yong '04, '09 and Kawashima and Xu '14
  - **Bianchini, Hanouzet and Natalini '07** and Kawashima and Xu '15.
- But the methods developed above do not tell us what is the appropriate Lyapunov functional to recover decay rates explicitly.

Beauchard and Zuazua's method

Another point of view

Their paper (ARMA 2011) is based on the following classical proposition.

Proposition

Let A and L be two real matrices $n \times n$. The followings properties are equivalent:

1. *the couple (A, L) satisfies the (SK) condition;*
2. *the couple (A, L) satisfies the Kalman rank condition : the rank of $\begin{pmatrix} L & AL & \dots & A^{n-1}L \end{pmatrix}$ is n;*

$\rightarrow$ link between the stability of these systems, control theory and Villani's hypocoercivity theory.

Inspired by this result, Beauchard and Zuazua constructed the following Lyapunov functional:

$$\mathcal{L}^2 \triangleq \|V\|_{L^2}^2 + \int_{\mathbb{R}^d} \min(\rho, \rho^{-1})\mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(LA_\omega^{k-1}\widehat{V} \cdot LA_\omega^k \widehat{V}\big)$$

for $n-1$ positives parameters $\varepsilon_1, \cdots \varepsilon_{n-1}$.

Differentiating in time $\mathcal{L}$ and adjusting the coefficients ε_k, we obtain:

$$\frac{d}{dt}\mathcal{L}^2 + \mathcal{H} \leq 0 \quad \text{where} \quad \mathcal{H} \triangleq \frac{1}{2}\int_{\mathbb{R}^d} \sum_{k=0}^{n-1} \varepsilon_k \min(1, \rho^2)|LA_\omega^k \widehat{V}(\xi)|^2 \, d\xi.$$

- If $\varepsilon_1, \cdots, \varepsilon_{n-1}$ are chosen small enough, then $\mathcal{L} \sim \|V\|_{L^2}^2$
- And, if the (SK) condition is satisfied then

$$\mathcal{H} \geq \kappa \min(1, |\xi|^2)\mathcal{L},$$

which brings us to

$$\frac{d}{dt}\mathcal{L} + \kappa \min(1, |\xi|^2)\mathcal{L} \leq 0$$

- At the linear level and under the condition (SK), this method allows to recover Kawashima's decomposition in a simple way.
- More explicit method than Kawashima's $\to$ very useful to study the high frequencies of non-linear systems.
- Their method covers some cases where condition (SK) is not verified.
- However, it does not tell the full story about the low frequencies, which turns out to be essential to study the relaxation limit issue.

Thesis' results

Purpose of my work

- Work in the framework of *hybrid* and *critical* homogeneous Besov spaces
  - *Hybrid* norm := norm with different regularity and/or Lebesgue indices in high and low frequencies.
  - *Critical* := optimal conditions of smallness allowing to justify the global well-posedness of the system. $\rightarrow$ Corresponds to the largest space in which one can obtain the uniqueness of strong solutions.
- $\rightarrow$ More accurate information on the qualitative properties of the constructed solutions and a larger class of initial data under control.
- To have a better understanding of where our improvement fits in, we need to keep in mind the following embeddings:

$$H^s(s > \frac{d}{2}+1) \hookrightarrow B^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{p}}_{p,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1}(p>2) \hookrightarrow \mathcal{C}^1_b.$$

- Moreover, our method allows to obtain new results concerning the relaxation limit issue and to obtain explicit convergence rates.

Littlewood-Paley decomposition

- We define $\dot{\Delta}_j$ as dyadic blocks such that $f \in \mathcal{S}'(\mathbb{R}^d)$

$$f = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f \quad \text{et} \quad \text{supp}(\widehat{\dot{\Delta}_j f}) \subset \{\xi \in \mathbb{R}^d \text{ t.q. } \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}.$$

- The main motivation behind this decomposition is the following Bernstein inequality: for all $k \in \mathbb{Z}$

$$c2^{jk}\|\dot{\Delta}_j f\|_{L^p} \leq \|D^k\dot{\Delta}_j f\|_{L^p} \leq C2^{jk}\|\dot{\Delta}_j f\|_{L^p}.$$

- The homogeneous Besov semi-norms are defined as follows:

$$\|f\|_{\dot{\mathbb{B}}^s_{p,1}} \triangleq \sum_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^p}.$$

- For a threshold $J_0 \in \mathbb{Z}$ and $s, s' \in \mathbb{R}$, we define:
- High-frequency restricted norms : $\|f\|^h_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{j\geq J_0} 2^{js}\|\dot{\Delta}_j f\|_{L^2}$,
- Low-frequency restricted norms : $\|f\|^\ell_{\dot{\mathbb{B}}^{s'}_{p,1}} \triangleq \sum_{j\leq J_0} 2^{js'}\|\dot{\Delta}_j f\|_{L^p}$.

Crucial observations

- In the previous methods, the analysis of the low frequencies behaviour was not complete.
- Back to the damped p-system

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \dfrac{v}{\varepsilon} = 0. \end{cases} \tag{6}$$

A spectral analysis of the matrix

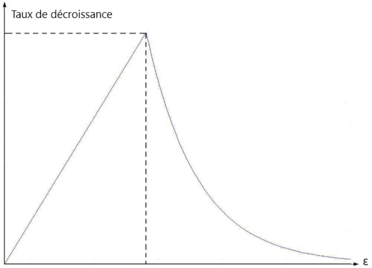
$$\begin{pmatrix} 0 & i\xi \\ i\xi & \dfrac{1}{\varepsilon} \end{pmatrix}.$$

shows that:

- the threshold between low and high frequencies is at $\dfrac{1}{2\varepsilon}$.
- In low frequencies (i.e. $|\xi| \ll \varepsilon^{-1}$), this matrix has two real eigenvalues asymptotically equal to $\dfrac{1}{\varepsilon}$ and $\varepsilon\xi^2$ for ξ close to 0;
- In high frequencies (i.e. $|\xi| \gg \varepsilon^{-1}$), two complex conjugate eigenvalues coexist, whose real parts are asymptotically equal to $\dfrac{1}{2\varepsilon}$.

This shows us two things:

- There should exist a damped mode in the low frequencies regime associated to the eigenvalue $1/\varepsilon$.
- The asymptotic behaviour of the solution when $\varepsilon \to 0$ is not so intuitive in reality.
  - Naively, we expect that as the damping coefficient becomes larger the dissipation becomes more dominant.
  - However, the so-called *overdamping* effect occurs: the decay rate behaves like $(\varepsilon, 1/\varepsilon)$.



- This phenomena makes things difficult when trying to study the relaxation limit of concrete systems.

$\rightarrow$ To "overcome" this phenomenon, we choose a threshold between low and high frequencies that depends on the relaxation parameter ε.

- We define $J_\varepsilon = \lfloor -\log_2 \varepsilon \rfloor$ to have a better tracking of the relaxation parameter.
- In the previous approaches, all the frequencies were "mixed" $\rightarrow$ the dissipative aspect in low frequencies was not fully exploited because the overall behaviour that emerges from a mixing of frequencies is always the "least good".
- Let us now see how to transcribe the spectral analysis into concrete properties.

Low frequencies in a simple case

Let us first see how to proceed in a simple case, for the damped p-system:

$$\begin{cases} \partial_t a + \partial_x u = 0 \\ \partial_t u + \partial_x a + u = 0, \end{cases}$$

Defining the damped mode $z = u + \partial_x a$, the system can be rewritten

$$\begin{cases} \partial_t a - \partial_{xx}^2 a = -\partial_x z \\ \partial_t z + z = -\partial_{xx}^2 u. \end{cases}$$

$\rightarrow$ We find the two behaviours observed by the spectral analysis.
$\rightarrow$ It is possible to study the two equations in a decoupled way as the source terms can be absorbed in the low frequency regime.

$\rightarrow$ This allows to prove a well-posedness result with the low frequencies of the solution being bounded in L^p spaces, with $2 \leq p \leq 4$.

General case

In the general case, the system can be rewritten as follows:

$$\begin{cases} \partial_t Z_1 + \sum_{k=1}^{d} \Big(A_{1,1}^k(V)\partial_k Z_1 + A_{1,2}^k(V)\partial_k Z_2 \Big) = 0, \\ \partial_t Z_2 + \sum_{k=1}^{d} \Big(A_{2,1}^k(V)\partial_k Z_1 + A_{2,2}^k(V)\partial_k Z_2 \Big) + \dfrac{L_2 Z_2}{\varepsilon} = 0. \end{cases} \tag{7}$$

Defining the damped mode :

$$W \triangleq Z_2 + \sum_{k=1}^{d} L_2^{-1}(A_{2,1}^k(V)\partial_k Z_1 + A_{2,2}^k(V)\partial_k Z_2) = -{L_2}^{-1}\partial_t Z_2$$

the system can be rewritten

$$\begin{cases} \partial_t W + L_2 W = g \\ \partial_t Z_1 - \sum_{k=1}^{d}\sum_{\ell=1}^{d} \bar{A}_{1,2}^k L_2^{-1} \bar{A}_{2,1}^\ell \partial_k \partial_\ell Z_1 = f \end{cases} \tag{8}$$

where f and g are controllable in the low-frequency regime.

General case

To study the equation of Z_1, we have the following property

Lemma

Assume that $\forall\ k \in \{1, \cdots, d\},\ \bar{A}^k_{1,1} = 0$. *The following assertions are equivalent:*

- *the system satisfy the (SK) condition at* $\bar{V}$;
- *the operator* $\mathcal{A} := \sum_{k=1}^{d} \sum_{\ell=1}^{d} \bar{A}^k_{1,2} L_2^{-1} \bar{A}^\ell_{2,1} \partial_k \partial_\ell$ *is strongly elliptic.*

$\rightarrow$ We may study the equations of W and Z_1 separately, the former as a damped equation and the latter as a heat equation.

This leads to the following result.

Well-posedness result

Theorem (Danchin, C-B '21)

Let $d \geq 1$, $p \in [2,4]$ et $\varepsilon > 0$. There exists $k_p \in \mathbb{Z}$ et $c_0 = c_0(p) > 0$ such that for all $J_\varepsilon \triangleq \lfloor -\log_2 \varepsilon \rfloor + k_p$, if we assume

$$\|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

then the system admits a unique solution Z satisfying

$$X_{p,\varepsilon}(t) \lesssim \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \quad \text{for all } t \geq 0, \text{ and where}$$

$$X_{p,\varepsilon}(t) \triangleq \varepsilon \|Z\|^{h}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z\|^{h}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \varepsilon^{-\frac{1}{2}} \|Z_2\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}$$
$$+ \|Z\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon \|Z_1\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \|W\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}.$$

Relaxation limit $\varepsilon \to 0$

As a direct application of our result, we can treat the relaxation limit of the compressible Euler equations.

Introducing the slow-time variable $\tau = \varepsilon t$ and using the diffusive change of variable:

$$(\tilde{\rho}_\varepsilon, \tilde{J}_\varepsilon)(\tau, x) = (\rho, \frac{\rho u}{\varepsilon})(t, x). \tag{9}$$

We get that $(\tilde{\rho}_\varepsilon, \tilde{J}_\varepsilon)$ satisfies:

$$\begin{cases} \partial_\tau \tilde{\rho}_\varepsilon + \nabla \cdot \widetilde{J}_\varepsilon = 0, \\ \varepsilon^2 \left(\partial_\tau \widetilde{J}_\varepsilon + \mathrm{div} \left(\dfrac{\widetilde{J}_\varepsilon \otimes \widetilde{J}_\varepsilon}{\widetilde{\rho}_\varepsilon} \right) \right) + \nabla P(\widetilde{\rho}_\varepsilon) + \widetilde{J}_\varepsilon = 0. \end{cases} \tag{10}$$

Then, formally, we have

$$-\tilde{J}_\varepsilon - \nabla P(\widetilde{\rho}_\varepsilon) \overset{*}{\rightharpoonup} 0 \quad \text{et donc} \quad \partial_\tau \tilde{\rho}_\varepsilon - \Delta P(\tilde{\rho}_\varepsilon) \overset{*}{\rightharpoonup} 0.$$

And therefore, we expect $\widetilde{\rho}_\varepsilon$ to converge to $\mathcal{N}$, the solution of the porous media equation:

$$\begin{cases} \partial_\tau \mathcal{N} - \Delta P(\mathcal{N}) = 0, \\ \mathcal{N}_{\tau=0} = \rho_0 \end{cases} \tag{11}$$

Relaxation result

Theorem (Danchin, C-B '21)

Let $d \geq 1$, $p \in [2,4]$ and $\varepsilon > 0$. Let $\bar{\rho}$ be a strictly positive constant and $(\rho - \bar{\rho}, v)$ be the solution obtained with the previous theorem.

Let the positive function $\mathcal{N}_0$ such that $\mathcal{N}_0 - \bar{\rho}$ is small enough in $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$, and let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$ be the unique solution associated to the Cauchy problem:

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \end{cases}$$

If we assume that

$$\|\widetilde{\rho}_0^{\varepsilon} - \mathcal{N}_0\|_{\mathbb{B}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^{\varepsilon})}{\widetilde{\rho}^{\varepsilon}} + \widetilde{v}^{\varepsilon} \right\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

Extensions

Extensions

Slightly extending our method, we are able (or expect to be able) to treat the following cases:

- Damped Baer-Nunziato system (compressible multi-fluid system). Joint work with C.Burtea and J. Tan.
- The hyperbolic-parabolic chemotaxis system. Joint work with Q. He and L-Y. Shou.
- Jin-Xin System
- Anisotropic systems e.g. 2D Boussinesq
- Relaxation limit of the damped Euler-Riesz system to the fractional porous media equation

Postdoc perspectives

Postdoctoral researches

Damping active outside of a ball

We consider the linear system

$$\begin{cases} \partial_t U + A\partial_x U = B(x)U, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ U(0,x) = U_0(x), & x \in \mathbb{R}, \end{cases}$$

where $U = (u_1, u_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

We assume that A is a *strictly hyperbolic matrix*, i.e. A has n real distinct eigenvalues

$$\lambda_1 < \cdots < \lambda_n.$$

We assume that $B \geq 0$ and satisfies

$$B(x) > 0, \quad x \in \omega, \qquad \text{and} \qquad B(x) = 0, \quad x \notin \omega,$$
$$B(x) := \begin{pmatrix} 0_{n_1 \times n_2} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & \chi_\omega(x) D \end{pmatrix},$$

where

$$\omega := \mathbb{R} \setminus B_R(0) = \{x \in \mathbb{R} : \|x\| \geq R\} \quad \text{for a fixed } R > 0, .$$

In other words: **the damping term acts only on n_2 components of the system and is effective only in the region ω.**

The choice of ω as an exterior domain is motivated by a *geometric control condition*: the ray of geometric optics may escape the damping effect if the inclusion $\{\|x\| \geq r\} \subset \omega$ is not satisfied for some $r > 0$.

Objective: Quantify the decay as in the classical case.

Main issues:

- Previous arguments fail to work because applying the Fourier transform on $B(x)$ engenders some convolution term that mixes the frequencies too much.
- As for the damped wave equation, the perturbed functional method does not seem to be able to give us the desired result.

Idea of solution

As the matrix A is symmetric with n real distinct eigenvalues, there exists a matrix $P \in O(n, \mathbb{R})$ such that

$$P^{-1}AP = D \quad \text{where} \quad D = \text{diag}(\lambda_1, ..., \lambda_n).$$

Setting $V = P^{-1}U$, the system can be reformulated into

$$\begin{cases} \partial_t V + D\partial_x V = P^{-1}B(x)PV, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ V(0,x) = V_0(x), & x \in \mathbb{R}, \end{cases} \tag{12}$$

Decomposing $V = (v_1, \dots, v_n)$, the system is equivalent to the following system of coupled transport equations:

$$\begin{cases} \partial_t v_1 + \lambda_1 \partial_x v_1 & = \sum_{j=1}^n b_{1,j} v_j \, a(x) \\ & \vdots \\ \partial_t v_n + \lambda_n \partial_x v_n & = \sum_{j=1}^n b_{n,j} v_j \, a(x) \end{cases}$$

For all $i = 1, \dots, n$, we define

- τ_i as the time that the component v_i spends in the undamped region ω^c.

Lemma (Propagation times)

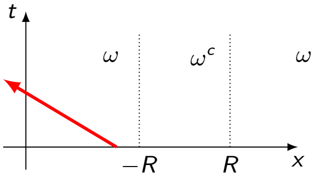
Assume that A has no zero eigenvalues and let V be the solution of (12) associated with the initial data $V_0 \in L^1 \cap L^2(\mathbb{R}, \mathbb{R}^n)$. For all $i = 1, \dots, n$, the time τ_i satisfies

$$0 \leq \tau_i \leq \frac{2R}{\lambda_i}.$$

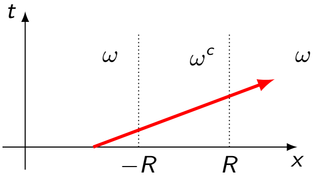
We define the characteristic line X_i of each equations passing through the point $(t_0, x_0) \in [0, T] \times \mathbb{R}$ as

$$X_i\left(t, t_0, x_0\right) := \lambda_i(t - t_0) + x_0, \quad t \in [0, T]$$

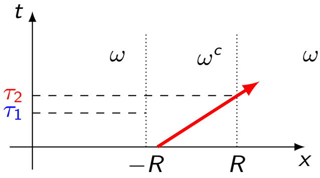
Propagation of characteristics and their location with respect to the region $\omega = \mathbb{R} \setminus B_R$ where the damping is active.



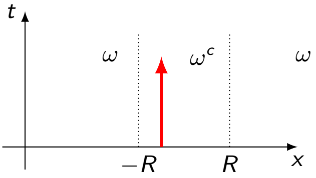
(a) **Case 1:** The initial support is in the damped region and the characteristics are going away from the un-damped region.



(b) **Case 2:** The initial support is in the damped region and the characteristics cross the un-damped region



(c) **Case 3:** The initial support is in the un-damped region



(d) **Case 4:** There is one zero eigenvalue. Standing wave

Expected result

Theorem

Assume that A has no zero eigenvalues and that the (SK) condition is satisfied. Let U be the solution of the system associated with the initial data $U_0 \in L^1 \cap L^2(\mathbb{R}, \mathbb{R}^n)$. There exist a finite time $\tau > 0$ such that for all $t \geq \tau$, we have

$$\|U_h(t,\cdot)\|_{L^2(\mathbb{R},\mathbb{R}^n)} \leq Ce^{-\gamma(t-\tau)}\|U_0\|_{L^2(\mathbb{R},\mathbb{R}^n)},$$
$$\|U_\ell(t,\cdot)\|_{L^\infty(\mathbb{R},\mathbb{R}^n)} \leq C(t-\tau)^{-1/2}\|U_0\|_{L^1(\mathbb{R},\mathbb{R}^n)},$$

and $U = U_h + U_\ell$. The time τ depends on the time each components spend into ω^c.

Other perspectives

Other perspectives

- The well-posedness issue for partially dissipative hyperbolic systems on networks.
- A better understanding of the (SK)/Kalman condition.

Merci !