Optimal control of mixed local-nonlocal parabolic PDE

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Linear optimal control

What is optimal control?

We define optimal control as the active manipulation of dynamical systems to achieve a given engineering goal.

Indeed, starting from a given initial state at time $t = 0$ we want to act on the trajectories through a suitable control in order to match or get close to a desired final state in time $T > 0$.

$$\begin{cases} \frac{d\psi}{dt} = A\psi + Bu, \quad t \in [0, T] \\ \psi(0) = \psi_0 \\ \end{cases}$$

- $A : \mathcal{D}(A) \to H$
- $B \in \mathcal{L}\left(U; \mathcal{D}(A)^*\right)$
- u : control

Here, the operator A can be a local or nonlocal operator.

Why nonlocal operator?

In general relevant models in Continuum Mechanics, Mathematical Physics and Biology are of nonlocal nature:

- Boltzmann equations in gas dynamics;
- Navier-Stokes equations in Fluid Mechanics;
- Keller-Segel model for Chemotaxis.

Here, however, we shall deal with mixed Brownian motion and Lévy path, due to anomalous dispersion and diffusion terms.

In that setting, classical PDE theory fails because of non-locality.

Yet many of the existing techniques can be tuned and adapted, although this is often a delicate matter because modern PDE analysis is based on the use of localisation arguments (test and cut-off functions)

Optimal Control Applications

What are some applications

1. Fluid dynamics: Improve drag reduction, lift increase, and noise reduction in aeronautics.
2. Finance: Maximize profit given a level of risk tolerance.
3. Epidemiology: Effectively suppress a disease with constraints of sensing (blood samples, clinics, etc.) and actuation (vaccines, bed nets, etc.).
4. Industry: Increasing productivity subject to constraints like labor and work safety laws, and environment impact.
5. Autonomy and robotics: self-driving cars and autonomous robots is to achieve a task while interacting safely with a complex environment, including cooperating with human agents.

Goal

Try to develop a systematic analysis of the control theoretical consequences of the possible of mixture of local and non-local terms in the model.

We do it for the following model case of the mixture of Laplace and Fractional Laplace operator of the form:

$$\mathscr{L} := -\Delta + (-\Delta)^s, \qquad 0 < s < 1, \tag{1}$$

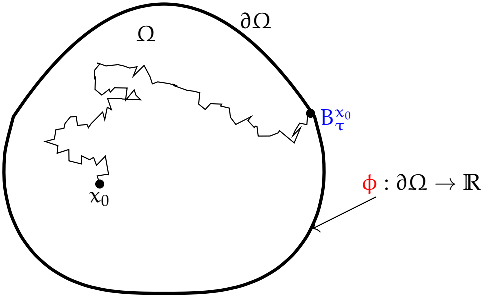
The generator of an N-dimensional Lévy process has the following general structure:

$$\mathscr{L}u = \alpha \sum_{i,j} a_{ij}\partial_{ij}u + \gamma \sum_{j} b_j \partial_j u + \beta \int_{\mathbb{R}^N} (u(x+\xi) - u(x) - \xi \cdot \nabla u(x))\chi_{B_1}(\xi) d\nu(\xi), \tag{2}$$

where ν is the Lévy measure and satisfies $\int_{\mathbb{R}^N} \min\left\{1, |\xi|^2\right\} d\nu(\xi) < +\infty$.

Brownian motion $\to$ 2nd order PDEs

Expected payoff at $\partial\Omega$



Δ is a local operator, that is: $\text{supp}\,[\Delta u] \subset \text{supp}[u]$

$$u(x) = \mathbb{E}\left(\phi(B_\tau^{x_0})\right) = \mathbb{E}(\text{pay off})$$

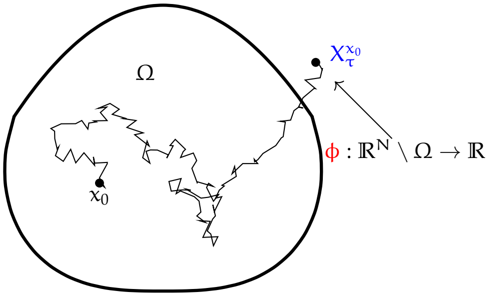
solves:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

- $B_t^{x_0}$: Brownian motion in $\mathbb{R}^N$ starting at x_0;
- τ: stopping time: first time at which $B_t^{x_0}$ hits $\partial\Omega$;

Lévy process → Integro-differential equations

Expected payoff at $\mathbb{R}^N \setminus \Omega$



L is **nonlocal**, that is: $\operatorname{supp}[Lu] \not\subseteq \operatorname{supp}[u]$

$u(x) = \mathbb{E}\left(\phi(X_\tau^{x_0})\right) = \mathbb{E}(\text{pay off})$

solves:

$$\begin{cases} L\,u = 0, & \text{in } \Omega, \\ u = \phi & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

- $X_t^{x_0}$ Lévy process with discontinuous sample paths.
- τ: first time at which $X_t^{x_0}$ is in $\mathbb{R}^N \setminus \Omega$.

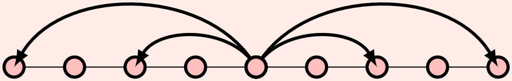
Stable Lévy processes: The fractional Laplacian

A special class of Lévy process when $\mu(y) = \mu(-y)$ is radially symmetric and $\mu(y) = C_{N,s}|y|^{-(N+2s)}$ for $y \neq 0$, $0 < s < 1$ reduces to the so-called Fractional Laplacian

$$(-\Delta)^{s} := C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} \, dy.$$

Remark: other way of deriving the fractional Laplace operator

- Fourier transform : $\widehat{(-\Delta)^s u}(\xi) = c|\xi|^{2s}\widehat{u}(\xi)$
- One can derive through **Long Jump random walks**



and set

$$u(x, t + \tau) = \sum_{k \in \mathbb{Z}^N} \mu(k) u(x + hk, t).$$

- Proceed as in the previous case by setting $\mu(y) = |y|^{-(N+2s)}$ for $y \neq 0$, $0 < s < 1$ and $\tau = h^{2s} \to 0^+$ in the limit, to get

$$u_t(x, t) + (-\Delta)^s = 0.$$

Mixed local-nonlocal problem

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ be a bounded domain with a smooth boundary $\partial\Omega$. For $T > 0$, we consider the minimization problem:

$$\min_{(z_1,z_2)\in\mathcal{Z}_D} J(\psi(z_1,z_2)), \tag{3}$$

subject to the constraints that the state $\psi := \psi(z_1,z_2)$ solves the following initial-boundary-exterior value problem:

$$\begin{cases} \psi_t + \mathscr{L}\psi = 0 & \text{in } Q := \Omega \times (0,T), \\ \psi = z_1 & \text{on } \Gamma := \partial\Omega \times (0,T), \\ \psi = z_2 & \text{in } \Sigma := (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \psi(\cdot,0) = 0, & \text{in } \Omega. \end{cases} \tag{4}$$

- The functional $J : \mathcal{Z}_D \to [0, \infty]$ is weakly lower-semicontinuous (we shall give the precise expression of J later).
- The control $(z_1, z_2) \in \mathcal{Z}_{ad}$ with $\mathcal{Z}_{ad} \subset \mathcal{Z}_D$ being a closed and convex subset, where
$$\mathcal{Z}_D := L^2(\Gamma) \times L^2(\Sigma). \tag{5}$$

The main question

How to formulate the above optimal control problem?

Questions

1. What is the correct Dirichlet boundary condition associated with $\mathscr{L}$?
2. What is the form of normal derivative associated with $\mathscr{L}$?

To obtain an answer to the above question, we need the following notions.

- Appropriate Sobolev spaces.
- An integration by parts formula for $\mathscr{L}$. That is, an appropriate Green type formula for $\mathscr{L}$.

To provide a flavour to the answer to this question, we start by given the notion of existence of solution and uniqueness of solution by passing through some functional setting.

Existing literature in the case of fractional Laplacian

H. Antil, R. Khatri, and M. Warma. *External optimal control of nonlocal PDEs*. Inverse Problems, 35(8):084003, 35, 2019.

H. Antil and D. Verma and M. Warma. *External optimal control of fractional parabolic PDEs*. ESAIM Control Optim. Calc. Var., 26:Paper No. 20, 33, 2020.

U. Biccari and M. Warma and E. Zuazua. *Local regularity for fractional heat equations*. In Recent advances in PDEs: analysis, numerics and control, volume 17 of SEMA SIMAI Springer Ser., pages 233–249. Springer, Cham, 2018.

M. Warma and S. Zamorano. *Exponential turnpike property for the fractional parabolic equations with non-zero exterior data*. ESAIM: COCV 27 (2021).

Sobolev Spaces

We introduce the classical first order Sobolev space

$$H^1(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega |\nabla u|^2 \, dx < \infty \right\}$$

which is endowed with the norm defined by

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

$$H_0^1(\Omega) := \left\{ w \in H^1(\mathbb{R}^N) : w \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}, \tag{6}$$

is a (real) Hilbert space endowed with the scalar product and associated norm

$$\int_\Omega \nabla w \cdot \nabla \varphi \, dx, \quad \|\varphi\|_{H_0^1(\Omega)} := \|\nabla \varphi\|_{L^2(\Omega)}. \tag{7}$$

Fractional order Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set and $0 < s < 1$.

- We denote

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \right\}$$

  and we endow it with the norm defined by

$$\|u\|_{H^s(\Omega)} = \left(\int_\Omega |u|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.$$

- Then, $H^s(\Omega)$ is a Hilbert space.

We let $H_0^s(\Omega) = \left\{u \in H^s\left(\mathbb{R}^N\right) : u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega\right\}$ and notice that

$$H_0^1(\Omega) \hookrightarrow H_0^s(\Omega). \tag{8}$$

Define the bilinear form $\mathcal{F} : H_0^s(\Omega) \times H_0^s(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}(\varphi, \psi) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} \, dx \, dy. \tag{9}$$

In view of (7) and (8), we can deduce that

$$(\varphi, \psi)_{H_0^1(\Omega)} := \mathcal{F}(\varphi, \psi) + \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx \tag{10}$$

defines a scalar product on $H_0^1(\Omega)$ with associated norm

$$\|\varphi\|_{H_0^1(\Omega)} := \left(\mathcal{F}(\varphi, \varphi) + \int_\Omega |\nabla\varphi|^2 \, dx\right)^{\frac{1}{2}}. \tag{11}$$

The norm given in (11) is equivalent to the one given in (7).

The zero Dirichlet boundary-exterior condition (BC) for $\mathscr{L}$

1. The zero Dirichlet BC for Δ is given by $u = 0$ on $\partial\Omega$.
2. The zero Dirichlet BC for $(-\Delta)^s$ is given by $u = 0$ on $\mathbb{R}^N \setminus \Omega$.
3. Let $\mathbb{A}$ be the operator on $L^2(\Omega)$ given by

$$D(\mathbb{A}) = \mathbb{V} := \{u \in H_0^1(\Omega) : (\mathscr{L}u)|_\Omega \in L^2(\Omega)\}, \quad \mathbb{A}u = (\mathscr{L}u)|_\Omega \text{ in } \Omega.$$

Then, $\mathbb{A}$ be the realization of $\mathscr{L}$ with the zero Dirichlet exterior condition. Here, the Dirichlet BC is characterized by $u = 0$ on $\partial\Omega$ and $u = 0$ in $\mathbb{R}^N \setminus \Omega$.

Remark 1

We notice that in the situation of zero Dirichlet exterior condition the system

$$\mathscr{L}w = f \text{ in } \Omega, \quad w = g \text{ in } \mathbb{R}^N \setminus \Omega,$$

is a well-posed problem. That is, the condition on $\partial\Omega$ is not needed.

How can we define a "fractional" normal derivative?

- Recall that if u is a smooth function defined on a smooth open set Ω, then the normal derivative of u is given by

$$\frac{\partial u}{\partial \nu} := \nabla u \cdot \vec{\nu},$$

  where $\vec{\nu}$ is the normal vector at the boundary $\partial\Omega$.
- For $0 < s < 1$ and a function u defined on $\mathbb{R}^N$ we let

$$\mathcal{N}_s u(x) = C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N \backslash \overline{\Omega},$$

  provided that the integral exists. This is clearly a nonlocal operator.
- $\mathcal{N}_s$ is well-defined and continuous from $H^s\left(\mathbb{R}^N\right)$ into $L^2\left(\mathbb{R}^N \backslash \Omega\right)$.
- We call $\mathcal{N}_s u$ the nonlocal normal derivative of u.

Local & nonlocal normal derivative

- Recall the divergence theorem:

$$\int_{\Omega} \Delta u \, dx = \int_{\Omega} \operatorname{div}(\nabla u) \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma, \quad \forall u \in C^2(\overline{\Omega})$$

- For $(-\Delta)^s$ we have the following:

$$\int_{\Omega} (-\Delta)^s u \, dx = -\int_{\mathbb{R}^N \setminus \Omega} \mathcal{N}_s u \, dx, \quad \forall u \in C_0^2\left(\mathbb{R}^N\right).$$

Local & nonlocal normal derivative

- Green Formula: $\forall u \in C^2(\overline{\Omega})$ and $\forall v \in C^1(\overline{\Omega})$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega v \Delta u \, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} d\sigma.$$

- For $(-\Delta)^s$ we have the following: $\forall u \in C_0^2\left(\mathbb{R}^N\right)$ and $v \in C_0^1\left(\mathbb{R}^N\right)$

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy$$

$$= \int_\Omega v(-\Delta)^s u \, dx + \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_s u \, dx.$$

$$\mathbb{R}^{2N} \setminus \left(\mathbb{R}^N \setminus \Omega\right)^2 := (\Omega \times \Omega) \cup \left(\Omega \times \left(\mathbb{R}^N \setminus \Omega\right)\right) \cup \left(\left(\mathbb{R}^N \setminus \Omega\right) \times \Omega\right).$$

Formulation: Mixed Dirichlet boundary-exterior control problem

Recall that $\mathcal{Z}_D := L^2(\Gamma) \times L^2(\Sigma)$ is endowed with the norm given by

$$\|(u_1, u_2)\|_{\mathcal{Z}_D} = \left(\|u_1\|^2_{L^2(\Gamma)} + \|u_2\|^2_{L^2(\Sigma)}\right)^{\frac{1}{2}}. \tag{12a}$$

. We consider the minimization problem

$$\min_{\psi \in L^2(Q), (z_1, z_2) \in L^2(\mathcal{Z}_D)} J(\psi) + \frac{\alpha}{2}\|(z_1, z_2)\|^2_{\mathcal{Z}_D}, \tag{12b}$$

subject to the Dirichlet boundary-exterior value problem: Find $\psi \in L^2\left((0,T) \times \mathbb{R}^N\right)$ solving

$$\begin{cases} \psi_t + \mathscr{L}\psi = 0 & \text{in } Q, \\ \psi = z_1 & \text{on } \Gamma, \\ \psi = z_2 & \text{in } \Sigma, \\ \psi(\cdot, 0) = 0, & \text{in } \Omega, \end{cases} \tag{12c}$$

Elliptic problem

Consider the following non-homogeneous Dirichlet problem associated with the operator $\mathscr{L}$, that is

$$\begin{cases} \mathscr{L}w = f & \text{in } \Omega, \\ w = g_1 & \text{on } \partial\Omega, \\ w = g_2 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{13}$$

Definition 2 (Very-weak solution)

Let $f \in H^{-1}(\Omega)$, $g_1 \in L^2(\partial\Omega)$, and $g_2 \in L^2(\mathbb{R}^N \setminus \Omega)$. A function $w \in L^2(\mathbb{R}^N)$ is called a very-weak solution (or a solution by transposition) of (13), if the identity

$$\int_\Omega w \mathscr{L}\varphi \, dx = \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_{\partial\Omega} g_1 \partial_\nu \varphi \, d\sigma - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g_2 \mathcal{N}_s \varphi \, dx \tag{14}$$

holds, for every $\varphi \in \mathbb{V} := \left\{ \varphi \in H_0^1(\Omega) : \mathscr{L}\varphi \in L^2(\Omega) \right\}$.

Remark 3

We notice that Definition 2 of very-weak solutions makes sense if every function $\varphi \in \mathbb{V}$ satisfies $\partial_\nu \varphi \in L^2(\partial\Omega)$, and $\mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \overline{\Omega})$.

Very-weak solutions

Let $0 < s \leqslant 3/4$. Then for every $f \in H^{-1}(\Omega)$, $g_1 \in L^2(\partial\Omega)$ and $g_2 \in L^2(\mathbb{R}^N \setminus \Omega)$, the system (13) has a unique very-weak solution $w \in L^2(\mathbb{R}^N)$ in the sense of Definition 2, and there is a constant $C > 0$ such that

$$\|w\|_{L^2(\mathbb{R}^N)} \leqslant C\left(\|f\|_{H^{-1}(\Omega)} + \|g_1\|_{L^2(\partial\Omega)} + \|g_2\|_{L^2(\mathbb{R}^N\setminus\Omega)}\right). \tag{15}$$

In addition, if g_1 and g_2 are as in Definition **??**, then the following assertions hold.

1. Every weak solution of (13) is also a very-weak solution.
2. Every very-weak solution of (13) that belongs to $H^1(\mathbb{R}^N)$ is also a weak solution.

Proof

Step 1: Let $\mathbb{A}$ be the realization of $\mathscr{L}$ in $L^2(\Omega)$ with zero Dirichlet exterior condition, that is,

$$D(\mathbb{A}) = \mathbb{V} := \{u \in H_0^1(\Omega) : \ (\mathscr{L}u)|_\Omega \in L^2(\Omega)\}, \ \ \mathbb{A}u = (\mathscr{L}u)|_\Omega \ \text{ in } \ \Omega. \quad (16)$$

Notice that $\|v\|_{\mathbb{V}} := \|\mathscr{L}v\|_{L^2(\Omega)}$ defines an equivalent norm on $\mathbb{V}$. This follows from the fact that the operator $\mathbb{A}$ is invertible, has a compact resolvent, and its first eigenvalue is strictly positive.

We claim that

$$\mathbb{V} = \left\{\varphi \in H_0^1(\Omega) : \ \mathscr{L}\varphi \in L^2(\Omega), \ \ \partial_\nu\varphi \in L^2(\partial\Omega) \text{ and } \ \mathcal{N}_s\varphi \in L^2(\mathbb{R}^N \setminus \overline{\Omega})\right\}.$$

It suffices to show that $\partial_\nu\varphi \in L^2(\partial\Omega)$ and $\mathcal{N}_s\varphi \in L^2(\mathbb{R}^N \setminus \overline{\Omega})$ for every $\varphi \in \mathbb{V}$. Indeed, let $\varphi \in \mathbb{V}$. We have two cases.

Proof

- Case $0 < s < 1/2$. Since $\varphi \in H_0^1(\Omega) \hookrightarrow H_0^s(\Omega)$, it follows from the regularity result that $(-\Delta)^s \varphi \in L^2(\Omega)$.

  Thus, $\mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \overline{\Omega})$. As $\mathscr{L} u \in L^2(\Omega)$, this also implies that $\Delta\varphi \in L^2(\Omega)$.

  Since Ω is assumed to be smooth, using the well-known elliptic regularity results for the Laplace operator, we have that $\varphi \in H^2(\Omega)$. Thus, $\partial_\nu \varphi \in H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.

- Case $1/2 \leqslant s < 1$. Since $\varphi \in H_0^1(\Omega)$, it follows that $(-\Delta)^s \varphi \in H^{1-2s}(\Omega)$. Hence, $\Delta\varphi \in H^{1-2s}(\Omega)$ and this implies that $\varphi \in H^{3-2s}(\Omega)$.

  Thus, $\partial_\nu \varphi \in H^{3/2-2s}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ if $3/2 - 2s \geqslant 0$.

Since $\varphi \in H^{3-2s}(\Omega)$, it follows that if $3 - 4s \geqslant 0$, then $(-\Delta)^s \varphi \in L^2()$.

Since $\mathscr{L}\varphi \in L^2(\Omega)$, we can deduce that $\Delta\varphi \in L^2(\Omega)$. From elliptic regularity for the Laplace operator operator we can conclude that $\varphi \in H^2(\Omega)$.

Thus, $\partial_\nu \varphi \in H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$.

Since we have assumed that $0 < s \leqslant 3/4$, it follows that $3/2 - 2s \geqslant 0$. and $3 - 4s \geqslant 0$.

Thus, we can conclude that $\partial_\nu \varphi \in L^2(\partial\Omega)$ and $\mathcal{N}_s \varphi \in L^2(\mathbb{R}^N \setminus \overline{\Omega})$. The claim is proved.

Step 2: Apply the Babuška-Lax-Milgram theorem to finish the proof.

Definition 4 (Weak solution to the mixed local-nonlocal parabolic problem)

Let $z_2 \in H^1((0,T); H^1(\mathbb{R}^N \setminus \overline{\Omega})$ and $z_1 \in L^2((0,T); H^{1/2}(\partial\Omega))$ be such that $z_2|_\Gamma = z_1$. Let $\tilde{z} \in H^1((0,T); H^1(\mathbb{R}^N)$ be such that $\tilde{z} = z_2$ in Σ. A function

$\psi \in L^2((0,T); H^1(\mathbb{R}^N)) \cap H^1((0,T); H^{-1}(\Omega))$ is said to be a weak solution of the system (12c) if $\psi - \tilde{z} \in L^2((0,T); H_0^1(\Omega)) \cap H^1((0,T); H^{-1}(\Omega))$ and the identity

$$\langle \psi_t, \zeta \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_\Omega \nabla\psi \cdot \nabla\zeta \, dx + \mathcal{F}(\psi, \zeta) = 0 \tag{17}$$

holds, for every $\zeta \in H_0^1(\Omega)$ and almost every $t \in (0,T)$.

Weak solution to the mixed local-nonlocal Dirichlet problem

Let z_2 and z_1 be as in Definition 4. Then, the system (12c) has a unique weak solution $\psi \in L^2((0,T);H^1(\mathbb{R}^N)) \cap H^1((0,T);H^{-1}(\Omega))$ in the sense of Definition 4. In addition, there is a constant $C > 0$ such that

$$\|\psi\|_{L^2((0,T);H^1(\mathbb{R}^N))\cap H^1((0,T);H^{-1}(\Omega))} \leqslant C\|z_2\|_{H^1((0,T);H^1(\mathbb{R}^N\setminus\overline{\Omega}))} \tag{18}$$

Remark 5

- It is worthwhile noticing that the regularity of $(z_1, z_2) \in L^2(\Gamma) \times L^2(\Sigma)$ is not enough to have a weak solution in the sense of Definition 4.
- Therefore we need a new notion of solutions.

Definition 6 (Very-weak solution to the Dirichlet problem)

Let $z_1 \in L^2(\Gamma)$ and $z_2 \in L^2(\Sigma)$. A function $\psi \in L^2((0,T) \times \mathbb{R}^N)$ is said to be a very weak-solution (or a solution by transposition) to the system (12c) if the identity

$$\int_Q \psi\Big(-\varphi_t + \mathscr{L}\varphi\Big)\,dx\,dt = -\int_\Gamma z_1 \partial_\nu \varphi \,d\sigma\,dt - \int_\Sigma z_2 \mathcal{N}_s \varphi\,dx\,dt, \tag{19}$$

holds, for every $\varphi \in L^2((0,T),\mathbb{V}) \cap H^1((0,T);L^2(\Omega))$ with $\varphi(\cdot,T) = 0$ a.e. in Ω.

Existence of very-weak solutions

Let $0 < s \leqslant 3/4$, $z_1 \in L^2(\Gamma)$, and $z_2 \in L^2(\Sigma)$. Then, there exists a unique very-weak solution $\psi \in L^2((0,T) \times \mathbb{R}^N)$ to (12c) according to Definition 6, and there is a constant $C > 0$ such that

$$\|\psi\|_{L^2((0,T)\times\mathbb{R}^N)} \leqslant C\left(\|z_1\|_{L^2(\Gamma)} + \|z_2\|_{L^2(\Sigma)}\right). \tag{20}$$

Moreover, if z_1 and z_2 are as in Definition 4, then the following assertions hold.

1. Every weak solution of (12c) is also a very-weak solution.
2. Every very-weak solution of (12c) that belongs to $L^2((0,T);H^1(\mathbb{R}^N)) \cap H^1((0,T);H^{-1}(\Omega))$ is also a weak solution.

Dirichlet boundary-exterior control problem

In view of the above existence result, the following (solution-map) control-to-state map is well-defined

$$S : L^2(\Gamma) \times L^2(\Sigma) \to L^2(Q), \quad (z_1, z_2) \quad \mapsto S(z_1, z_2) = \psi$$

and is linear and continuous. We also notice that for $(z_1, z_2) \in \mathcal{Z}_D := L^2(\Gamma) \times L^2(\Sigma)$, we have that $\psi := S(z_1, z_2) \in L^2\left((0,T) \times \mathbb{R}^N\right)$. As a result we can write the reduced Dirichlet boundary-exterior control problem

$$\min_{(z_1,z_2)\in Z_{ad}} \mathcal{J}((z_1, z_2)) := \min_{(z_1,z_2)\in Z_{ad}} \left(J(S(z_1, z_2)) + \frac{1}{2}\|(z_1, z_2)\|^2_{\mathcal{Z}_D} \right) \tag{21}$$

with

$$J(S(z_1, z_2)) := \frac{1}{2}\|\psi((z_1, z_2)) - z_d^1\|^2_{L^2(Q)}.$$

Existence: Dirichlet boundary-exterior control problem

Let $0 < s \leqslant 3/4$, $z_1 \in L^2(\Gamma)$, and $z_2 \in L^2(\Sigma)$. Let $\mathcal{Z}_{ad}$ be a closed and convex subset of $\mathcal{Z}_D$, and let $\psi = \psi(z_1, z_2)$ satisfy

$$\begin{cases} \psi_t + \mathscr{L}\psi = 0 & \text{in } Q, \\ \psi = z_1 & \text{on } \Gamma, \\ \psi = z_2 & \text{in } \Sigma, \\ \psi(\cdot, 0) = 0, & \text{in } \Omega, \end{cases} \tag{22}$$

in the very-weak sense. Then, there exists a unique control $(\bar{z}_1, \bar{z}_2) \in \mathcal{Z}_{ad}$

solution of

$$\inf_{(z_1, z_2) \in \mathcal{Z}_D} \mathcal{J}((z_1, z_2)). \tag{23}$$

The adjoint operator S^*

The adjoint operator $S^* : L^2(Q) \to L^2(\Gamma) \times L^2(\Sigma)$ of the control-to-state map S (solution-map) is given by

$$\eta \mapsto S^*\eta = (-\partial_\nu p, -\mathcal{N}_s p).$$

where $p \in L^2\left((0,T); H_0^1(\Omega)\right) \cap H^1\left((0,T); L^2(\Omega)\right)$ is the weak solution to the parabolic problem

$$\begin{cases} -p_t + \mathscr{L} p = \eta & \text{in } Q, \\ p = 0 & \text{in } \Gamma, \\ p = 0 & \text{in } \Sigma, \\ p(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{24}$$

Sketch of the proof.

According to the definition of S^* we have $\forall\eta \in L^2(Q)$ and $\forall(z_1, z_2) \in L^2(\Gamma) \times L^2(\Sigma)$,

$$(\eta, Sz)_{L^2(\Omega)} = (S^*\eta, z)_{L^2(\mathbb{R}^N\setminus\Omega)}.$$

Testing (24) with Sz and using the fact that Sz is a very-weak solution of (12c), we arrive at

$$\begin{aligned}(\eta, \phi)_{L^2(Q)} = (\phi, -p_t + \mathscr{L}p)_{L^2(Q)} = (S^*z, \eta)_{L^2(Q)} \\ =(z_1, \partial_\nu p)_{L^2(\Gamma)} + (-z_2, \mathcal{N}_s p)_{L^2(\Sigma)}.\end{aligned}$$

This yields the asserted result. □

First Order Necessary Optimality Conditions for (21)

Let $0 < s \leqslant 3/4$. Let $\mathcal{Z}_D$ be an open set in $L^2(\Sigma)$ such that $Z_{ad} \subset \mathcal{Z}_D$. Let $\psi \mapsto J(\psi) : L^2(Q) \to \mathbb{R}$ be Fréchet differentiable with $J'(\psi) \in L^2(Q)$. Let $(\bar{z}_1, \bar{z}_2)$ be a minimizer of (21). Then,

$$(\partial_\nu \overline{p} + \xi \bar{z}_1, z_1 - \bar{z}_1)_{L^2(\Gamma)} + (\mathcal{N}_s \overline{p} + \xi \bar{z}_2, z_2 - \bar{z}_2)_{L^2(\Sigma)} \geqslant 0, \quad \forall (z_1, z_2) \in Z_{ad}, \tag{25}$$

where $\overline{p} \in L^2\left((0,T); H_0^1(\Omega)\right) \cap H^1\left((0,T); H^{-1}(\Omega)\right)$ solves the adjoint equation

$$-\overline{p}_t + \mathscr{L}\overline{p} = J'(\overline{\psi}) \quad \text{in } Q, \quad \overline{p} = 0 \ \text{ in } \ \Sigma, \quad \overline{p}(\cdot, T) = 0 \ \text{ in } \ \Omega. \tag{26}$$

Equivalently we can write (25) as

$$(\bar{z}_1, \bar{z}_2) = \mathbb{P}_{Z_{ad}}\left(-\frac{1}{\xi}\partial_\nu \bar{p}, -\frac{1}{\xi}\mathcal{N}_s \bar{p}\right) \tag{27}$$

where $\mathbb{P}_{Z_{ad}}$ is the projection onto the set Z_{ad}. If J is convex then (25) is also a sufficient condition.

Future perspectives:

- Numerical approximations and simulation of the optimal control problem.
- Question of null and approximate controllability of the problem with Dirichlet, Neumann and Robin type boundary conditions.
- Investigate Turnpike property for the mixed local-nonlocal problem.

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Thank you !