Nonlinear Feedback Control Design for Fluid Mixing

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Mixing phenomena

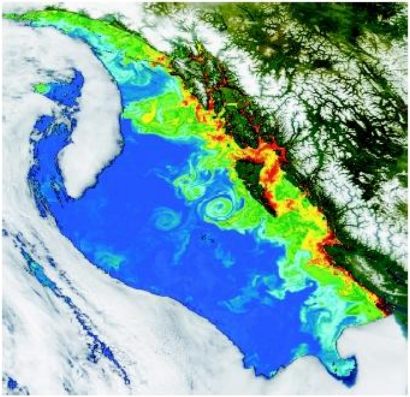
Mixing is to disperse one material or field in another medium. It occurs in many natural phenomena and industrial applications.

Mixing in painting

Mixing in baking

Mixing phenomena

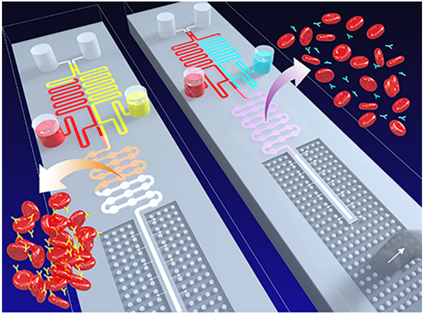
Spreading of a pollutant in the atmosphere



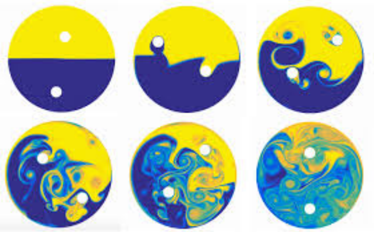
Mixing of temperature, salt, and nutrient in ocean*.

*http://www.waterencyclopedia.com/Mi-Oc/Ocean-Mixing.html

Mixing phenomena



Microfluidic mixing: controllable and fast mixing is critical for practical development of microfluidic and lab-on-chip devices*.



Optimal mixing?

*https://www.elveflow.com/microfluidic-reviews/microfluidic-flow-control/microfluidic-mixers-a-short-review/

Outline

- Feedback control for fluid mixing
  - instantaneous control design (sub-optimal)
- Asymptotic behavior of the nonlinear closed-loop system
- Numerical Implementation

Mixing modeled by transport equation

Consider the transport equation in an open bounded and connected domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, with a regular boundary Γ

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad \theta(0) = \theta_0, \quad x \in \Omega.$$

- θ: mass concentration or density distribution
- v: incompressible velocity field with no-penetration BC, that is,

$$\nabla \cdot v = 0, \quad v \cdot n|_\Gamma = 0.$$

- $\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}, \; p \in [1, \infty], \; t > 0.$

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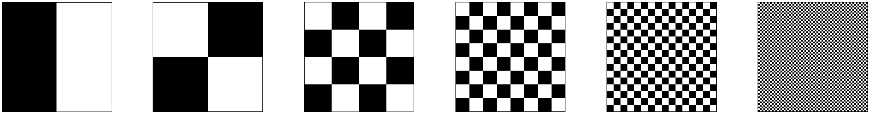

Mix-norm: negative Soblev norm

- **Mix-norm**: consider the 1D periodic interval $[0, L]$. Define

$$d(\theta, x, w) = \frac{1}{w} \int_{x-w/2}^{x+w/2} \theta(y)\, dy$$

for all $x, w \in [0, L]$. The mix-norm $M(\theta)$ is then obtained by averaging d^2 over x and w:

$$M^2(\theta) = \frac{1}{L^2} \int_0^L \int_0^L d^2(\theta, x, w)\, dx\, dw$$

*Mathew-Mezic-Petzold '05, Lin-Thiffeault-Doering '10, Thiffeault '11, etc

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$$M^2(\theta) = \frac{1}{L^2} \int_0^L \int_0^L d^2(\theta, x, w)\, dx\, dw \sim \|\theta\|^2_{H^{-1/2}}$$

In fact, any $H^{-\alpha}$-norm for $\alpha > 0$, which quantifies the weak convergence, can be used as a mix-norm.

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Mixing in Stokes flows

Consider

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad x \in \Omega,$$

where the velocity field is govern by

$$\frac{\partial v}{\partial t} - \nu \Delta v + \nabla p = 0, \quad \nabla \cdot v = 0, \quad v \cdot n|_{\Gamma} = 0.$$

- p: pressure; ν: viscosity

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Motivated by the observation that moving walls accelerate mixing compared to fixed walls with no-slip boundary condition*, we consider the Navier slip boundary control for mixing

$$v \cdot n|_{\Gamma} = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + k v \cdot \tau)|_{\Gamma} = {\color{red} g} \cdot \tau.$$

- n and τ are the outward unit normal and tangential vectors to the boundary Γ
- $\mathbb{D}(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$: deformation tensor
- $k > 0$: coefficient of friction
- ${\color{red} g}$: control input function

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Forward Model Simulations ($\nu = 1$ and $k = 0.5$)

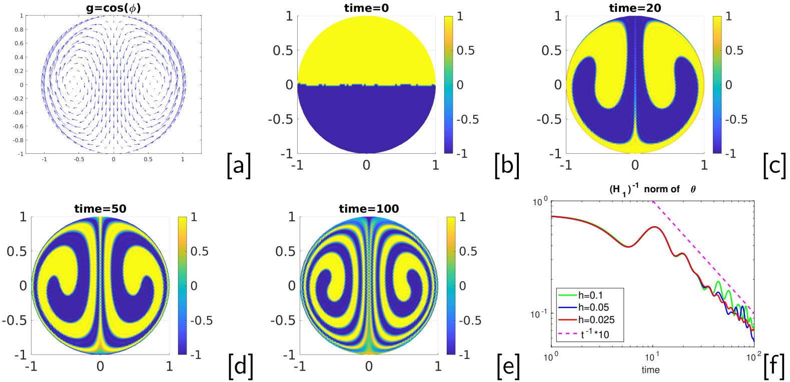

Fig. 1. $\Omega = \{(x, y) \colon x^2 + y^2 < 1\}$ and $g = \cos(\phi)\tau$. [a]: sample of velocity field. The maximum magnitude is roughly 0.4. [b, c, d, e]: θ at t=0, 20, 50, 100. [f]: $(H^1(\Omega))'$ norms of θ in time. All the contour figures of θ are using the data when the mesh size $h = 0.025$.

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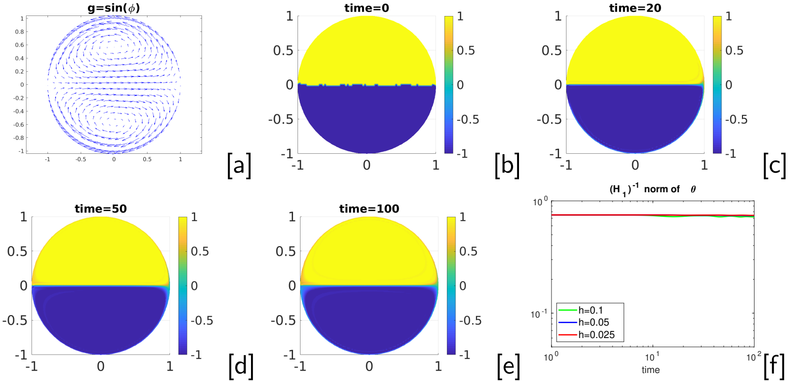

Fig. 2: $g = \sin(\phi)\tau$. [a]: sample of velocity field. The maximum magnitude is roughly 0.4. [b,c,d,e]: θ at t=0, 20, 50, 100. [f]: $(H^1(\Omega))'$ norms of θ in time. All the contour figures of θ are using the data when the mesh size $h = 0.025$.

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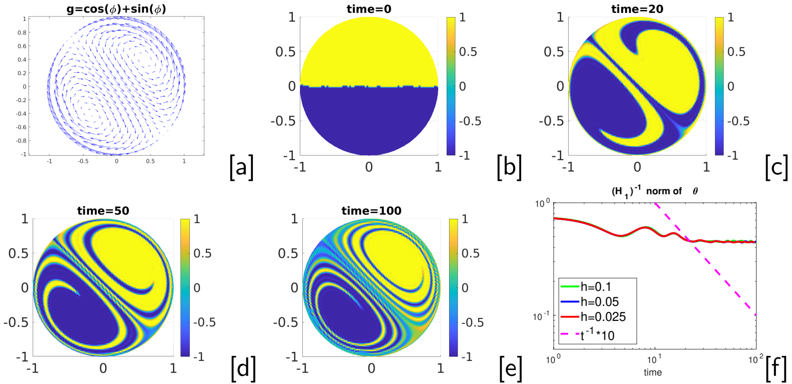

Fig. 3: $g = \cos(\phi)\tau + \sin(\phi)\tau$. [a]: sample of velocity field. The maximum magnitude is roughly 0.4. [b,c,d,e]: θ at t=0, 20, 50, 100. [f]: $(H^1(\Omega))'$ norms of θ in time. All the contour figures of θ are using the data when the mesh size $h = 0.025$.

Problem formulation: optimal bilinear control

- **Minimize**

$$J(g) = \frac{1}{2}\|\theta(T)\|^2_{(H^1(\Omega))'} + \frac{\gamma}{2}\|g\|^2_{U_{ad}}, \quad \gamma > 0, \quad (P),$$

for a given $T > 0$, subject to

$$\begin{cases} \frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \\ \frac{\partial v}{\partial t} - \nu \Delta v + \nabla p = 0, \\ \nabla \cdot v = 0, \end{cases}$$

with Navier slip boundary control:

$$v \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + kv \cdot \tau)|_\Gamma = g \cdot \tau,$$

and initial conditions $\theta(0) = \theta_0$ and $v(0) = v_0$. Here $\gamma > 0$ is the control weight and U_{ad} stands for the set of admissible controls.

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- **Procedures**: (1) Prove the well-posedness of problem (P); (2) Identify the set of admissible controls; (3) Prove the existence of an optimal control and establish the optimality conditions.

Challenges in analysis and computation

- *Nonlinearity:* The nonlinear coupling due to advection essentially leads to a nonlinear control and non-convex optimization problem.
- *Zero diffusivity:* Differentiability leads to a high-order regularity required for the velocity field.
- *Boundary Control:*
  1. Creation of vorticity on the domain boundary;
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- *Boundary Control:*
  1. Creation of vorticity on the domain boundary;
  2. Compatibility conditions may come into play even in the case of non-smooth solutions.
- *Computation:*
  1. Mass conservation of scalar transport in incompressible flows;
  2. Small-scale structures and large gradients of the scalar field will develop in the mixing process.

Well-posedness of problem (P)

- Cost functional:
$$J(g) = \frac{1}{2}\|\theta(T)\|^2_{(H^1(\Omega))'} + \frac{\gamma}{2}\|g\|^2_{U_{ad}}, \quad \gamma > 0. \quad (P)$$
Consider
$$(-\Delta + I)\eta = \theta, \quad \frac{\partial \eta}{\partial n}|_\Gamma = 0.$$
Let $\Lambda = (-\Delta + I)^{1/2}$. Then
$$\|\theta\|_{(H^1(\Omega))'} = \|\Lambda^{-1}\theta\|_{L^2(\Omega)} = \|\Lambda\eta\|_{L^2(\Omega)} = \|\eta\|_{H^1(\Omega)}.$$

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$$\|\theta\|_{(H^1(\Omega))'} = \|\Lambda^{-1}\theta\|_{L^2(\Omega)} = \|\Lambda\eta\|_{L^2(\Omega)} = \|\eta\|_{H^1(\Omega)}.$$
- Define
$$V_n^s(\Omega) = \{v \in H^s(\Omega) : \text{div } v = 0, \ v \cdot n|_\Gamma = 0\}, \quad s \geq 0,$$
$$V_n^s(\Gamma) = \{g \in H^s(\Gamma) : g \cdot n|_\Gamma = 0\}, \quad s \geq 0.$$
- For $(\theta_0, v_0) \in L^\infty(\Omega) \times V_n^0(\Omega)$, there exists $g \in L^2(0, T; V_n^0(\Gamma))$ such that J is finite.

Theorem (Existence, H., AMO (2018))

Assume that $(\theta_0, v_0) \in L^\infty(\Omega) \times V_n^0(\Omega)$. There exists an optimal solution $g \in L^2(0, T; V_n^0(\Gamma))$ to problem (P).

First-order optimality system via an approximating control approach

- To summarize, if $(g^{opt}, v^{opt}, \theta^{opt})$ is the optimal solution, then it satisfies

State Equations
$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta = 0, \quad \theta(0) = \theta_0, \\ \partial_t v - \nu \Delta v + \nabla p = 0, \quad \nabla \cdot v = 0, \quad v(0) = v_0, \\ v \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + kv \cdot \tau)|_\Gamma = g \cdot \tau, \end{cases}$$

Adjoint Equations
$$\begin{cases} -\partial_t \rho - v \cdot \nabla \rho = 0, \quad \rho(T) = \Lambda^{-2}\theta(T), \\ -\partial_t w - \nu \Delta w + \nabla q = \theta \nabla \rho, \quad \nabla \cdot w = 0, \quad w(T) = 0, \\ w \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(w) \cdot \tau + kw \cdot \tau)|_\Gamma = 0, \end{cases}$$

Optimality Condition: $\boxed{g = \frac{1}{\gamma} w|_\Gamma.}$

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State Equations
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Theorem (Uniqueness, H. (AMO, 2019))

For $d=2$ and $\gamma>0$ sufficiently large, there exists at most one optimal controller to problem (P).

- Numerical results can be found in https://arxiv.org/abs/2108.09533 (H.-Zheng '21)

Construction of feedback laws

- Recall that

$$\textbf{State equations} \qquad \begin{cases} \partial_t \theta = -v \cdot \nabla \theta, & \theta(0) = \theta_0, \\ \partial_t v = Av + Bg, & v(0) = v_0, \end{cases}$$

where $A = \mathbb{P}\Delta$ and B is the control input operator.

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where $A = \mathbb{P}\Delta$ and B is the control input operator.

- **Instantaneous control design:** consider a uniform partition of $[0, T]$ and let $\delta = \frac{T}{n}$ for $n \in \mathbb{N}$. Using Euler's semi-implicit in time for discretizing the state equations in t gives

$$\begin{cases} \theta^{i+1} = \theta^i - \delta v^{i+1} \cdot \nabla \theta^i, \\ (-\Delta + I)\eta^{i+1} = \theta^{i+1}, \quad \frac{\partial \eta^{i+1}}{\partial n}|_\Gamma = 0, \\ v^{i+1} = v^i + \delta A v^{i+1} + B g^{i+1}. \end{cases} \tag{1}$$

- Consider now the cost functional for one time step

$$J(g^{i+1}) = \frac{1}{2}\|\Lambda^{-1}\theta^{i+1}\|_{L^2}^2 + \frac{\gamma}{2}\|g^{i+1}\|_{U_{ad}}^2.$$

- This method is closely tied to receding horizon control (RHC) or model predictive control (MPC) with finite time horizon (cf. Hinze-Kunisch '97, Hinze-Volkwein '02).

Construction of feedback laws (cont'd)

- Let (ρ^{i+1}, w^{i+1}) be the adjoint state of (θ^{i+1}, v^{i+1}). Applying the Euler-Lagrange method leads to

$$\rho^{i+1} = \eta^{i+1}, \quad (I - \delta A)w^{i+1} = \delta \mathbb{P}(\theta^i \nabla \rho^{i+1}), \tag{2}$$

and the optimality condition

$$\gamma g^{i+1} + B^* w^{i+1} = 0. \tag{3}$$

The optimality system (1)–(3) admits a unique solution due to the quadratic cost functional and the uniqueness of (1).

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- Compute $(g^{i+1}, v^{i+1}, \theta^{i+1})$ recursively by setting $g_0^i = 0$, which turns out to be the semi-implict time discretization of the closed-loop system

$$\begin{cases} \partial_t \theta = -v \cdot \nabla \theta, \quad \theta(0) = \theta_0, \\ \partial_t v = Av + Bg, \quad v(0) = v_0, \\ g = -\gamma^{-1} \delta B^* (I - \delta A)^{-1} \mathbb{P}(\theta \nabla \eta) \quad \text{(sub-optimal)}. \end{cases}$$

Well-posedness and stability of the closed-loop system

- The closed-loop system reads

$$\begin{cases} \partial_t \theta = -v \cdot \nabla \theta, \quad \theta(0) = \theta_0, \\ \partial_t v = Av + Bg, \quad v(0) = v_0, \\ g = -\gamma^{-1} \delta B^* (I - \delta A)^{-1} \mathbb{P}(\theta \nabla \eta) \quad \text{(sub-optimal)}, \end{cases}$$

  where $\eta = (I - \Delta)^{-1} \theta$, γ and δ are the fixed parameters.

- Let $B = \mathbb{P}$ (internal control). Then

$$\partial_t v = Av - \gamma^{-1} \delta (I - \delta A)^{-1} \mathbb{P}(\theta \nabla \eta).$$

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$$\partial_t v = Av - \gamma^{-1}\delta(I-\delta A)^{-1}\mathbb{P}(\theta\nabla\eta).$$

- Applying energy estimates yields

$$\frac{d}{dt}\text{Total Energy} = \frac{d}{dt}\|\theta\|^2_{(H^1(\Omega))'} + \frac{\gamma}{\delta}\frac{d}{dt}\|v\|^2_{H^1(\Omega)} \leq -C\|v\|^2_{H^1(\Omega)} < 0.$$

Well-posedness and stability (cont'd)

- **Well-posedness:** For $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^2(\Omega)$, there exists a unique solution to the closed-loop system.
- **Convergence results:**
  1. $\|v\|_{L^2}, \|\nabla v\|_{L^2}, \|Av\|_{L^2}, \|\partial_t v\|_{L^2} \to 0$ as $t \to +\infty$;
  2. $\|\theta\|_{(H^1(\Omega))'} \to c_0$ as $t \to \infty$, and $c_0 < \sqrt{\frac{\gamma}{\delta}\|v_0\|_{L^2}^2 + \|\theta_0\|_{(H^1(\Omega))'}^2}$;
  3. $\|g\|_{L^2} \to 0$ as $t \to +\infty$;
  4. $\|\theta\nabla\eta - \nabla p\|_{(H^2(\Omega))'} \to 0$ as $t \to +\infty$.

Numerical simulation of the closed-loop system

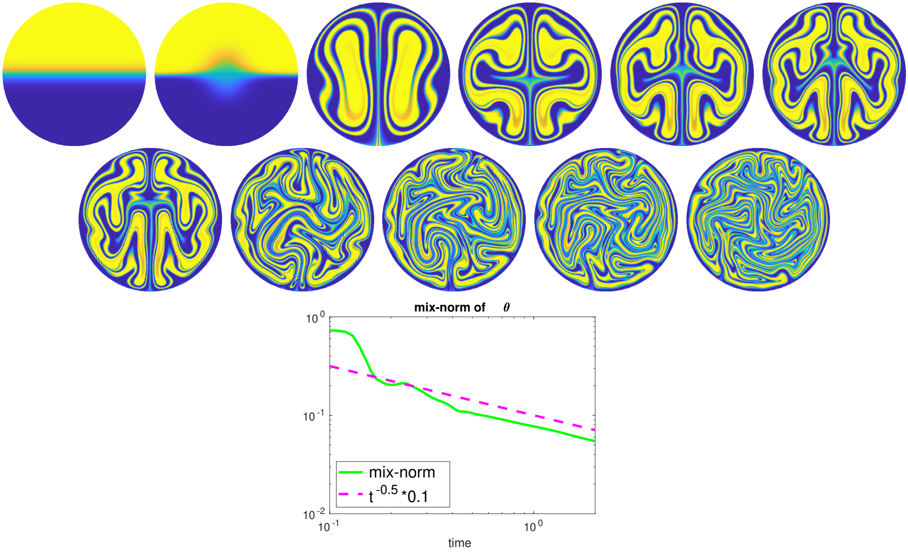

Fig. 5. $\theta_0 = \tanh(y/0.1)$. Density evolution for $t \in [0, 2]$, $h = 0.0125, \delta = 0.1, \gamma =$1e-6

Ongoing work

- Investigate the optimality of the feedback laws
- Justify the polynomial decay rate of the mix-norm $\|\theta\|_{(H^1(\Omega))'}$ in time and its relation to the control actuation
- Analyze the asymptotic behavior of the closed-loop system by localized internal control and Navior slip boundary control

Thank you for your attention!

Questions?