Null controllability of a nonlinear age, space and two-sex structured population dynamics model

Deusto CCMSeminar

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Outline

- Motivation and description of two sex structured population dynamics model.

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- Null controllability

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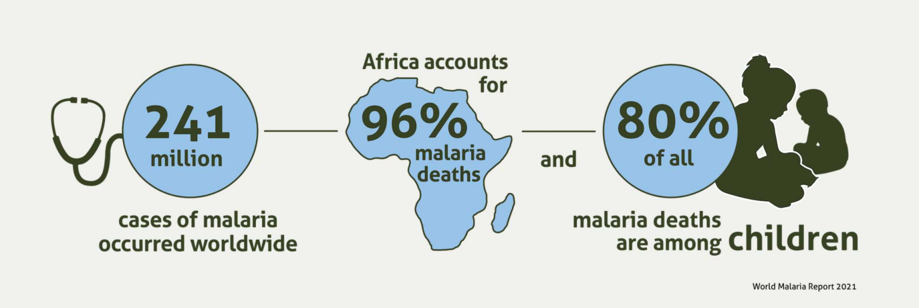
- Null controllability
- Perspectives

Motivation

Malaria is a disease caused by parasites of the genus Plasmodium. According to the WHO, this disease causes approximately one million victims per year worldwide.



(a) Anophel Gambiae

The parasite is transmitted to humans through the bite of an infected mosquito. These mosquitoes, "vectors" of malaria, all belong to the genus Anopheles.

- We have in west Africa "Target Malaria" project underway and which aims to drive the density of wild female mosquitoes to zero in long time horizon.
- In the coming months more than 2 million genetically modified mosquitoes will be released in Florida. The mosquitoes, created by biotech firm Oxitec, will be non-biting Aedes aegypti males engineered to only produce viable male offspring, per the company. Oxitec says the plan will reduce numbers of the invasive Aedes aegypti, which can carry diseases like Zika, yellow fever and dengue.

- In this talk we give mathematically some ideas on the possibility of controlling of mosquitoes population dynamics. For reasons like as the difference in lifespan between male mosquitoes (14 days) and female (30 days) and the difference in mortality functions, we preferred to work with the two-sex model which seems the best fit.
- In the strategies cited, the control methods used seem to be birth control or the combination of birth control and distributed control, in this first work, we will focus on distributed controls.

Description of two sex structured population dynamics model

We denote by $\Xi = \omega \times (a_1, a_2) \times (0, T) \subset Q$ and $\Xi' = \omega' \times (b_1, b_2) \times (0, T) \subset Q$ where $Q = \Omega \times (0, A) \times (0, T)$. We denote also $\Sigma = \partial\Omega \times (0, A) \times (0, T)$, $Q_T = \Omega \times (0, T)$ and $Q_A = \Omega \times (0, A)$. Let (m, f) solution of the following system :

$$\begin{cases} \dfrac{\partial m}{\partial t} + \dfrac{\partial m}{\partial a} - K_m \Delta m + \mu_m m = \chi_{\Xi} v_m & \text{in } Q, \\ \dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial a} - K_f \Delta f + \mu_f f = \chi_{\Xi'} v_f & \text{in } Q, \\ m(\sigma, a, t) = f(\sigma, a, t) = 0 & \text{on } \Sigma, \\ m(x, a, 0) = m_0 \quad f(x, a, 0) = f_0 & \text{in } Q_A, \\ m(x, 0, t) = (1-\gamma) N(x, t), \; f(x, 0, t) = \gamma N(x, t) & \text{in } Q_T, \\ N(x, t) = \int_0^A \beta(a, M) f da; \; M = \int_0^A \lambda(a) m da & \text{in } Q_T. \end{cases} \tag{1}$$

where $m_0 \in L^2(Q_A)$, $f_0 \in L^2(Q_A)$, $v_m \in L^2(Q)$, $v_f \in L^2(Q)$ and $\gamma \in (0, 1)$. The functions, $m(x, a, t)$ and $f(x, a, t)$ represent the density of males and females of age a at time t in position x, respectively. We assume that the fertility functions β, λ and mortality μ_m and μ_f satisfy the following demographic properties:

$$(H_1):\begin{cases} (i)\ \mu_m \geq 0, \quad \mu_f \geq 0 \text{ a.e. in } [0,A], \\ (ii)\ \mu_m \in L^1_{loc}([0,A)), \quad \mu_f \in L^1_{loc}([0,A)), \\ (iii)\ \int_0^A \mu_m(a)da = +\infty, \quad \int_0^A \mu_f(a)da = +\infty. \end{cases}$$

The functions

$$\Pi_m(a) = e^{-\int\limits_0^a \mu_m(s)ds} \text{ and } \Pi_f(a) = e^{-\int\limits_0^a \mu_f(s)ds}$$

denote the probability of survival of male individuals of age a and female individuals of age a, respectively.

$$(H_2): \begin{cases} (i)\ \beta \in C\left([0,A]\times\mathbb{R}\right), \\ (ii)\ \beta(a,p) \geq 0 \text{ for all } (a,p)\in[0,A]\times\mathbb{R}, \\ (iii)\ \beta(a,0) = 0 \text{ in } (0,A). \end{cases}$$

$$(H_3): \begin{cases} \lambda \in C^1\left([0,A]\right), \\ \lambda \geq 0 \text{ for all } a\in[0,A]. \end{cases}$$

Moreover, we suppose that:

$$(H_4): \begin{cases} (i) \text{ there exists } b\in(0,A) \text{ such that } \beta(a,p)=0, \forall (a,p)\in[0,b)\times\mathbb{R}, \\ (ii)\ \text{ there exists } L>0 \text{ such that } |\beta(a,p)-\beta(a,q)| \leq L|p-q| \\ \text{for all } p,q\in\mathbb{R},\ a\in[0,A], \\ (iii) \text{ there exists } \beta_0>0 \text{ such that } 0\leq\beta(a,p)\leq\beta_0,\ \forall(a,p)\in[0,A]\times\mathbb{R}. \end{cases} \tag{2}$$

$$(H_5): \left\{ \lambda\mu_m \in L^1((0,A)). \right.$$

Null controllability: Main result

We have the following

Theorem 1

Suppose that the assumptions $(H_1)-(H_2)-(H_3)-(H_4)-(H_5)$ hold. If $(0,b)\cap(a_1,a_2)\cap(b_1,b_2)\neq\emptyset$, for every time $T>\max\{a_1,b_1\}+\max\{A-a_2,A-b_2\}$ and for every $(m_0,f_0)\in\left(L^2(Q_A)\right)^2$, there exists $(v_m,v_f)\in L^2(\Xi)\times L^2(\Xi')$ such the solution (m,f) of the system (1) verifies:

$$m(x,a,T)=0 \text{ a.e. } x\in\Omega,\ a\in(0,A), \tag{3}$$

$$f(x,a,T)=0 \text{ a.e. } x\in\Omega,\ a\in(0,A). \tag{4}$$

Remark: Notice that $\omega\cap\omega'$ can be empty.

Null controllability of auxiliary system

Let p be a function in $L^2(Q_T)$, we define the auxilliary system given by:

$$\begin{cases} \dfrac{\partial m}{\partial t} + \dfrac{\partial m}{\partial a} - K_m \Delta m + \mu_m m = \chi_\Xi v & \text{in } Q, \\ \dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial a} - K_f \Delta f + \mu_f f = \chi_{\Xi'} u & \text{in } Q, \\ m(\sigma,a,t) = f(\sigma,a,t) = 0 & \text{on } \Sigma, \\ m(x,a,0) = m_0 \quad f(x,a,0) = f_0 & \text{in } Q_A, \\ m(x,0,t) = (1-\gamma)\int_0^A \beta(a,p) f da, & \\ f(x,0,t) = \gamma \int_0^A \beta(a,p) f da & \text{in } Q_T. \end{cases} \tag{5}$$

The system (5) admits a unique solution $(m,f) \in (L^2((0,A)\times(0,T);H_0^1(\Omega)))^2$ and the system (5) is null controllable for every $T > \max\{a_1,b_1\} + \max\{A-a_2, A-b_2\}$. Moreover the null controllability of the system (5) is equivalent of the Observability inequality.

Observability Inequality: adjoint system

Let (n, l) be the solution of the following adjoint system to the auxilliary system (5)

$$\begin{cases} -\dfrac{\partial n}{\partial t} - \dfrac{\partial n}{\partial a} - K_m \Delta n + \mu_m n = 0 & \text{in } Q, \\ -\dfrac{\partial l}{\partial t} - \dfrac{\partial l}{\partial a} - K_f \Delta l + \mu_f l = (1-\gamma)\beta(a,p)n(x,0,t) + \gamma\beta(a,p)l(x,0,t) & \text{in } Q, \\ n(\sigma,a,t) = l(\sigma,a,t) = 0 & \text{on } \Sigma, \\ n(x,a,T) = n_T \quad l(x,a,T) = l_T & \text{in } Q_A, \\ n(x,A,t) = 0, \quad l(x,A,t) = 0 & \text{in } Q_T. \end{cases} \tag{6}$$

Under the assumptions on the time T, we have the following:

Observability Inequality

Theorem 2

Under the assumptions of Theorem 1, for every $T > \max\{a_1, b_1\} + \max\{A - a_2, A - b_2\}$, there exists a constant $C_T > 0$ independent of p such that the solution (n, l) of the system (6) verifies:

$$\int_0^A \int_\Omega n^2(x,a,0)\,dxda + \int_0^A \int_\Omega l^2(x,a,0)\,dxda$$

$$\leq C_T \left(\int_\Xi n^2(x,a,t)\,dxdadt + \int_{\Xi'} l^2(x,a,t)\,dxdadt \right).$$

Representation of the solution of adjoint system

The idea to establish the observability inequality is the estimation of the non local terms of the adjoint system. For this reason, we first begun to formulating a representation of the solution of cascade adjoint system by caracteristics method and semigroup.

For $(n_T, I_T) \in (L^2(Q_A))^2$, under the assumptions (H_1) and (H_2), the cascade system (6) admits a unique solution (n, I). Moreover, integrating along the characteristics line the solution (n, I) of (6) is given by:

$$n(t) = \begin{cases} \dfrac{\pi_1(a+T-t)}{\pi_1(a)} e^{(T-t)K_m\Delta} n_T(x, a+T-t) \text{ if } T-t \leq A-a, \\ 0 \text{ if } A-a < T-t, \end{cases} \tag{7}$$

and

$$I(t)=\begin{cases}\frac{\pi_2(a+T-t)}{\pi_2(a)}e^{(T-t)K_f\Delta}I_T(x,a+t-T)\\ +\int_t^T\frac{\pi_2(a+s-t)}{\pi_2(a)}e^{(s-t)K_f\Delta}\beta(a+s-t,p(x,s))((1-\gamma)n(x,0,s)+\gamma I(x,0,s))ds \text{ in } D_1,\\ \int_t^{t+A-a}\frac{\pi_2(a+s-t)}{\pi_2(a)}e^{(s-t)K_f\Delta}\beta(a+s-t,p(x,s))((1-\gamma)n(x,0,s)+\gamma I(x,0,s))ds \text{ in } D_2,\end{cases} \tag{8}$$

where $\pi_1(a)=e^{-\int_0^a\mu_m(r)dr}$, $\pi_2(a)=e^{-\int_0^a\mu_f(r)dr}$, $e^{tK_m\Delta}$ is the semigroup of $-K_m\Delta$ with the Dirichlet boundary condition and

$$D_1=\{(a,t)\in(0,A)\times(0,T) \text{ such that } T-t\le A-a\},$$

$$D_2=\{(a,t)\in(0,A)\times(0,T) \text{ such that } T-t> A-a\}.$$

Using the fact that $\beta(a,p)=0$ for all $a\in[0,b)$. We establish the following:

Estimation of the non local terms

Proposition 2

Under the assumptions of Theorem 1, for every η satisfying $a_1 < \eta < T$, there exists $C > 0$ such that the following inequality

$$\int_0^{T-\eta} \int_\Omega n^2(x,0,t)\,dxdt \le C \int_0^T \int_{a_1}^{a_2} \int_\omega n^2(x,a,t)\,dxdadt \tag{9}$$

holds. For every η verifying $b_1 < \eta < T$, there exists $C > 0$ such that the following inequality

$$\int_0^{T-\eta} \int_\Omega l^2(x,0,t)\,dxdt \le C \int_{\Xi'} l^2(x,a,t)\,dxdadt \tag{10}$$

holds.

First we recall the observability inequality for the parabolics equations:

Proposition 3

Let $T > 0$, t_0 and t_1 such that $0 < t_0 < t_1 < T$. Therefore, for all $w_0 \in L^2(\Omega)$, the solution w of the system:

$$\begin{cases} \dfrac{\partial w(x,\lambda)}{\partial \lambda} - K_m \Delta w(x,\lambda) = 0 & \text{in } (t_0, T) \times \Omega, \\ w = 0 & \text{on } (t_0, T) \times \partial\Omega, \\ w(x, t_0) = w_0(x) & \text{in } \Omega, \end{cases} \tag{11}$$

verifies the following estimates

$$\int_\Omega w^2(T,x)dx \leq \int_\Omega w^2(x,t_1)dx \leq c_1 e^{\dfrac{c_2}{t_1 - t_0}} \int_{t_0}^{t_1} \int_\omega w^2(x,\lambda)dxd\lambda, \tag{12}$$

where the constants c_1 and c_2 depend of T and Ω.

An idea of the proof of the Proposition 2

Let $\tilde{n}(x,a,t) = n(x,a,t)e^{-\int_0^a \mu(\alpha)d\alpha}$. Then $\tilde{n}$ satisfies

$$\begin{cases} \dfrac{\partial \tilde{n}}{\partial t} + \dfrac{\partial \tilde{n}}{\partial a} + K_m \Delta \tilde{n} = 0 & \text{in } \Omega \times (0,a_2) \times (0,T), \\ \hat{n} = 0 & \text{on } \partial\Omega \times (0,a_2) \times (0,T), \\ \hat{n}(.,.,T) = n_T e^{-\int_0^a \mu_m(\alpha)d\alpha} & \text{in } \Omega \times (0,A). \end{cases} \quad (13)$$

Proving the inequality (9) leads also to show that, there exits a constant $C > 0$ such that the solution $\tilde{n}$ of (13) satisfies

$$\int_0^{T-\eta} \int_\Omega \tilde{n}^2(x,0,t)dxdt \leq C \int_0^T \int_{a_1}^{a_2} \int_\omega \tilde{n}(x,a,t)dxdadt. \quad (14)$$

Let:

$$w(\lambda) = \tilde{n}(x, T-\lambda, T+t-\lambda) \; ; (\lambda \in (T-a_2, T) \text{ and } x \in \Omega).$$

Then, w verifies the following system:

$$\begin{cases} \dfrac{\partial w(\lambda)}{\partial \lambda} - k_m \Delta w(\lambda) = 0 & \text{in } \Omega \times (T-a_2, T), \\ w = 0 & \text{on } \partial\Omega \times (T-a_2, T), \\ w(0) = \tilde{n}(x, T, T+t) & \text{in } \Omega. \end{cases} \quad (15)$$

Using the Proposition 2.3 with $T - a_2 < t_0 < t_1 < T$ we obtain:

$$\int_\Omega w^2(T)dx \leq \int_\Omega w^2(t_1)dx \leq c_1 e^{\frac{c_2}{t_1 - t_0}} \int_{t_0}^{t_1} \int_\Omega w^2(\lambda) dx d\lambda.$$

That is equivalent to

$$\int_\Omega \tilde{n}^2(x,0,t)dx \leq c_1 e^{\frac{c_2}{t_1 - t_0}} \int_{t_0}^{t_1} \int_\Omega \tilde{n}^2(x, T-\lambda, t+T-\lambda) dx d\lambda$$

$$C \int_{T-t_1}^{T-t_0} \int_\Omega \tilde{n}^2(x,a,t+a) dx da.$$

Illustration of the estimations of non local terms

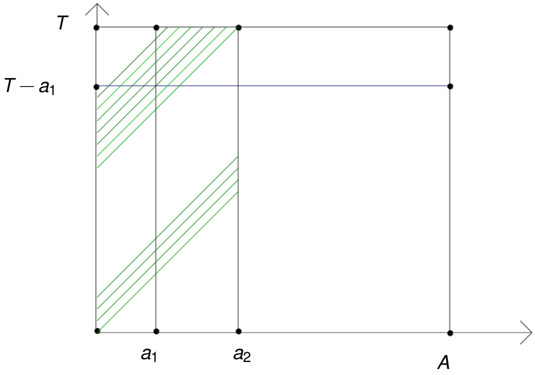


Figure: Estimation of $n(x,0,t)$ and $I(x,0,t)$.

Estimations of $l(x,a,0)$ and $n(x,a,0)$ in $\Omega\times(0,a_0)$

We state these two propositions necessary for the proof of the inequality:

Proposition 4

Under the assumptions $(H_1)-(H3)$, for all $T>\max\{a_1, A-a_2\}$, there exists $C_T>0$ such that the solution (n,l) of the system (6) verifies the following inequality:

$$\int_0^A\int_\Omega n^2(x,a,0)dxda \le C_T\int_\Xi n^2(x,a,t)dxdadt. \tag{16}$$

Note that here we first show that $n(x,a,0)=0$ in (a_0,A) and we use the same technique as in the Proposition 2 to estimate $n(x,a,0)$ in $(0,a_0)$

Proposition 5

Under the assumptions $(H_1)-(H_2)$ and the hypothesis $b_1<a_0<b_2$ and $T>b_1$. There exists $C_T>0$ such that the solution (n,l) of the system (6) verifies the following inequality:

$$\int_0^{a_0}\int_\Omega l^2(x,a,0)dxda \le C_T\int_{\Xi'} l^2(x,a,t)dxdadt. \tag{17}$$

An idea of the proof of the observability inequality

Lemma

Suppose that $(0,b)\cap(a_1,a_2)\cap(b_1,b_2)\neq\emptyset$. For all time $T>\max\{a_1,b_1\}+\max\{A-a_2,A-b_2\}$ there exists $a_0\in(a_1,a_2)\cap(b_1,b_2)$ and $\kappa>0$ such that

$$T>T-(\max\{a_1,b_1\}+\kappa)>A-a_0>A-a \text{ for all } a\in(a_0,A). \tag{18}$$

Moreover,

$$I(x,a,0)=\int_0^{A-a}\frac{\pi(a+s)}{\pi(a)}\left(e^{s\Delta}\beta(a+s,p(x,s))I(x,0,s)+e^{s\Delta}\beta(a+s,p(x,s))n(x,0,s)\right)ds \tag{19}$$

$$\text{in } (x,a)\in\Omega\times(a_0,A).$$

According to the Lemma, on (a_0,A), $I(x,a,0)$ depends mainly on the non-local terms. Moreover if we consider $\eta=\max\{a_1,b_1\}+\kappa$ as in the Proposition 2, and as $\max\{a_1,b_1\}<\max\{a_1,b_1\}+\kappa<T$, we have the estimation of non local term between 0 and $T-(\max\{a_1,b_1\}+\kappa)$.

Proof of the Observability inequality

We already have the estimate of $n(x,a,0)$ on $(0,A)$ and the estimate of $I(x,a,0)$ on $(0,a_0)$. So we split $\int_0^A \int_\Omega I^2(x,a,0)dxda$ as the following

$$\int_0^A \int_\Omega I^2(x,a,0)dxda = \int_0^{a_0} \int_\Omega I^2(x,a,0)dxda + \int_{a_0}^A \int_\Omega I^2(x,a,0)dxda. \tag{20}$$

Using the assumptions of Theorem 1 and the result of Lemma, we show the existence $K_T > 0$ independent of p such that:

$$\int_{a_0}^{A}\int_{\Omega} l^2(x,a,0)\,dxda$$

$$\leq K_T\left(\int_0^{T-(\max\{a_1,b_1\}+\kappa)}\int_{\Omega} n^2(x,0,t)\,dxdt+\int_0^{T-(\max\{a_1,b_1\}+\kappa)}\int_{\Omega} l^2(x,0,t)\,dxdt\right).$$

$$\int_0^{T-(\max\{a_1,b_1\}+\kappa)}\int_{\Omega} n^2(x,0,t)\,dxdt \tag{21}$$

By combining (20) and the results of Propositions 2, 4 and 5

$$\int_0^A\int_{\Omega} n^2(x,a,0)\,dxda+\int_0^A\int_{\Omega} l^2(x,a,0)\,dxda\leq C_T\left(\int_{\Xi} n^2(x,a,t)\,dxdadt+\int_{\Xi'} l^2(x,a,t)\,dxdadt\right).$$

Illustration of the observability inequality and estimation of the non local terms

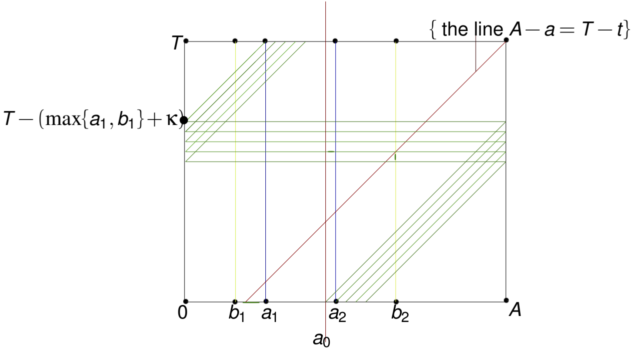


Figure: The backward characteristics starting from $(a,0)$ with $a \in (a_0,A)$ (green lines) hits the boundary $(a=A)$, gets renewed by the renewal condition $(1-\gamma)\beta(a,p)n(x,0,t)+\gamma\beta(a,p)I(x,0,t)$ and then enters the observation domain (green lines). So, with the conditions $T > \max\{a_1,b_1\}+\max\{A-a_2,A-b_2\}$ all the characteristics starting at $(a,0)$ with $a \in (a_0,A)$ get renewed by the renewal condition $(1-\gamma)\beta(a,p)n(x,0,t)+\gamma\beta(a,p)I(x,0,t)$ with $t < T-\max\{a_1,b_1\}$.

Proof of Theorem 1

Let Λ be a operator define as follow:

$$\Lambda : L^2(Q_T) \longrightarrow L^2(Q_T) \quad p \longmapsto \int_0^A \lambda(a) m(p) da \tag{22}$$

where the couple $(m(p), f(p))$ is the solution of the following auxilliary system verifying

$$m(x,a,T) = 0 \text{ a.e. } x \in \Omega \; a \in (0,A), \tag{23}$$

$$f(x,a,T) = 0 \text{ a.e. } x \in \Omega \; a \in (0,A). \tag{24}$$

Under the assumptions of Theorem 1.1, we can show that the operator Λ is continuous, and the set $\Lambda(L^2(Q_T))$ is relatively compact in $L^2(Q_T)$. By Schauder's fixed point theorem Λ admits a fixed point and we get the reult of the Theorem 1.

Perspectives

- Null controllability with positivity constraints

Perspectives

- Null controllability with positivity constraints
- Null controllability with birth control

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