

Prescribed-Time Boundary Control of PDE-ODE Cascade Systems

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Outline

- 1 Motivation
- 2 State-feedback control
- 3 Output-feedback control
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Prescribed-time stabilization

- **Finite-time stabilization:** the settling time depends on the initial state.
- **Fixed-time stabilization:** the settling time has a uniform upper bound.
- **Prescribed-time stabilization:** the user assigns the convergence time T in advance.

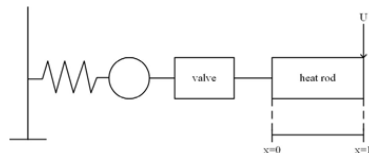
Why the cascade is challenging

The ODE is coupled to one PDE boundary, while the actuator is placed at the opposite boundary. The assigned time T must be independent of the initial state.

Can a boundary actuator regulate the whole cascade before a chosen deadline?

Thermally actuated valve system

- A heated rod drives a valve through thermal expansion.
- The valve connects to a spring-mass-damper assembly.
- The actuator applies temperature at the remote boundary $x = 1$.



Thermo-mechanical PDE-ODE cascade

Control objective

Design $U(t)$ so that the mechanical state and the temperature field converge to zero at a user-selected time T .

The heat equation is both a system to stabilize and the actuator dynamics between $U(t)$ and the valve.

Coupled PDE-ODE model

Plant

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), \\ u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \\ u_x(0,t) = 0, \\ u(1,t) = U(t). \end{cases}$$

Mechanical state

$$X(t) = \begin{bmatrix} s(t) \\ v(t) \end{bmatrix}.$$

- $s(t)$ is the valve displacement.
- $v(t)$ is the valve velocity.
- (A,B) is controllable.

Boundary coupling

- $u(0,t)$ drives the ODE and acts as a virtual input.
- $U(t)$ is the actual input at the opposite boundary.

Design difficulty: the desired ODE feedback must be realized through the PDE dynamics.

Two-step time-varying backstepping

Step 1: embed the desired ODE feedback

$$z(x, t) = u(x, t) - \int_0^x k(x, y, t) u(y, t) dy + \gamma(x, t) X(t).$$

The gain $\gamma(0, t)$ makes the boundary trace act like a prescribed-time ODE feedback.

Step 2: stabilize the remaining PDE dynamics

$$w(x, t) = z(x, t) - \int_0^x p(x, y, t) z(y, t) dy,$$

$$w_t = w_{xx} - c(t)w, \quad w_x(0, t) = 0, \quad w(1, t) = 0.$$

Original cascade $(X, u) \longrightarrow$ intermediate system $(X, z) \longrightarrow$ stable target (X, w)

Time-varying gains near the prescribed time

Scaling function

$$\mu_m(t) = \frac{T^m}{(T-t)^m}, \quad 0 \leq t < T.$$

The gain increases as t approaches T and creates the required terminal decay.

Representative estimate

$$|X(t)| \leq Ce^{-k_1 T(\mu_1(t)-1)} |X(0)|.$$

The exponential decay dominates the polynomial growth of the transformations and their kernels.

- Differential flatness constructs the time-varying ODE gain $\gamma(x, t)$.
- Successive approximations provide the PDE kernel $k(x, y, t)$.
- Inverse transformations transfer target-system stability back to the plant.

The time-varying target creates the decay; backstepping transfers it to the physical plant.

Main result of the first study

Theorem (Prescribed-time stabilization)

For every assigned time $T > 0$, suitable design parameters give

$$|X(t)| \rightarrow 0, \quad \|u(\cdot, t)\|_{L^2(0,1)} \rightarrow 0 \quad \text{as } t \rightarrow T.$$

The boundary input $U(t)$ remains bounded on $[0, T)$ and satisfies $U(t) \rightarrow 0$ as $t \rightarrow T$.

- The result connects a physical valve model with prescribed-time PDE control.
- The proof combines invertible backstepping transformations with a Lyapunov estimate.
- The design assumes available states and accurately known model parameters.

From state feedback to output feedback

The first study assumes full-state information and no unknown external disturbance. The second study removes both assumptions.

Plant and measurements

$$\begin{cases} \dot{X}(t) = AX(t) + B[u(0,t) + d(t)], \\ u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \\ u_x(0,t) = 0, \quad u(1,t) = U(t), \\ y_{\text{ode}}(t) = CX(t), \quad y_{\text{pde}}(t) = u_x(1,t). \end{cases}$$

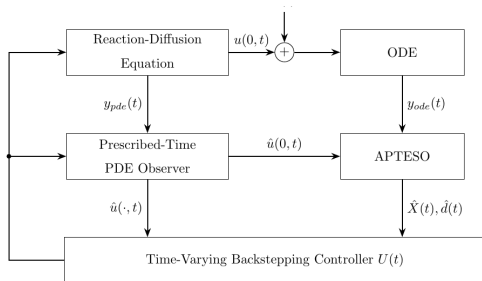
Unavailable quantities

- full ODE state $X(t)$
- distributed PDE state $u(x,t)$
- persistent disturbance $d(t)$

Available signals

- one ODE output $CX(t)$
- one boundary flux $u_x(1,t)$
- boundary actuation $U(t)$

Dual-observer output-feedback architecture



- A boundary observer reconstructs the PDE state.
- An adaptive prescribed-time extended state observer estimates $X(t)$ and $d(t)$.
- The controller uses only the estimated quantities.

Timing order

$$T_{o2} < T_{o1} < T_c.$$

The PDE estimate settles first, followed by the ODE and disturbance estimates, then the plant regulation.

Regularized scaling and global well-posedness

Classical prescribed-time gain

- grows without bound as $t \rightarrow T_c$
- may produce singular kernels and control inputs
- gives well-posedness only before the terminal time

Analytic regularized gain $\mu_c(t)$

- remains finite for every $t \geq 0$
- preserves the regularity required by the kernel equations
- supports a global classical solution

$$1 \leq \mu_c(t) \leq \mu_{c,\max} = T_c^3 / \varepsilon_c^3$$

Design consequence

The regularization replaces exact terminal convergence under persistent disturbances with convergence to a user-selected small neighborhood.

Practical prescribed-time stability

Composite Lyapunov inequality

$$\dot{V}(t) \leq -\kappa\mu_c(t)V(t) + \sigma_{\text{total}}, \quad t \geq T_{o1}.$$

Theorem (Main closed-loop properties)

For any $T_c > 0$ and prescribed accuracy $\varepsilon_{\text{target}} > 0$, a suitable regularization parameter ε_c ensures that

- the ODE state enters the required neighborhood by T_c and remains there,
- the PDE state remains globally bounded while compensating for $d(t)$,
- the boundary input $U(t)$ remains bounded for all $t \geq 0$.

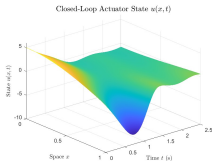
Physical meaning

A non-vanishing $d(t)$ requires a bounded PDE compensation profile. Exact PDE convergence is not expected.

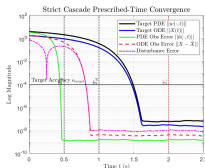
Trade-off

Smaller target accuracy requires a larger peak gain and greater actuator effort.

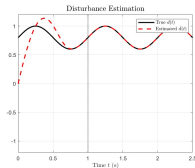
Numerical validation



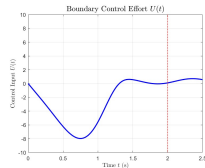
Closed-loop PDE state



State and estimation errors



Disturbance and its estimate



Boundary control input

Data-assisted control under parameter uncertainty

Initial research question

Can sparse measurements support both parameter identification and reliable prescribed-time boundary control?

- 1 Use a physical PDE solver and a data-driven model to estimate an unknown reaction or coupling parameter.
- 2 Start with offline identification, then insert the estimate into the existing controller.
- 3 Quantify how parameter and state-estimation errors affect the accuracy reached at T_c .

Possible later extension: fragmented sensors through Bi-HYCO.

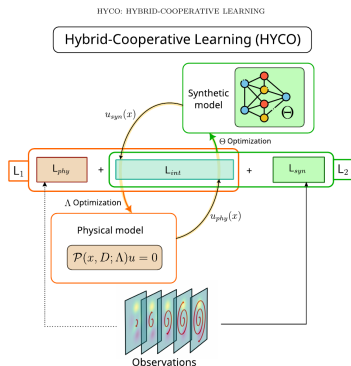


FIGURE 1.1. Schematic representation of the HYCO strategy. We train two models, alternating an optimizer

Summary

- **State-feedback study:** time-varying backstepping achieves prescribed-time stabilization of a thermo-mechanical PDE-ODE cascade.
- **Output-feedback study:** dual observers and analytic regularization handle unmeasured states and persistent disturbances without a terminal singularity.
- **Next step:** begin with an offline study of parameter identification and examine its effect on prescribed-time accuracy.

Goal: prescribed-time control that remains reliable with limited measurements and uncertain models.

Thanks

Thanks for listening!

Questions and comments are welcome.