Research on Control Problems of Several Types of Infinite-Dimensional Systems

Jun Chen

Email: chenjun_bcbm@163.com

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University of Deusto, DeustoTech

CHAIR OF COMPUTATIONAL MATHEMATICS

Outline

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1 Introduction

2 Stabilization

3 Output Tracking

4 Future Plans

The control theory of distributed parameter systems is widely applied in production and real life.

- Stabilization
- Output Tracking/Regulation
- Optimal Control
- Robust Control
- Adaptive Control
- ⋯

Outline

1 Introduction

2 Stabilization

3 Output Tracking

4 Future Plans

Stabilization problem

- How to design a controller to make an originally unstable system stable?
- Or to enable an already stable system to regain stability after being disturbed?

Background

Methods to deal with the uncertainty of the system:

- Adaptive Control
- Sliding Mode Control (SMC)
- **Active Disturbance Rejection Control (ADRC)**
- ……

First proposed: (Han[1],2009)
Progress: (Guo&Jin[2],2013), (Liu&Wang[3],2015), (Zhang et al.[4],2019).

[1] J.Q. Han, From PID to active disturbance rejection control, *IEEE transactions on Industrial Electronics*, 56(3): 900–906, 2009.

[2] B.Z. Guo, F.F. Jin, The Active Disturbance Rejection and Sliding Mode Control Approach to the Stabilization of the Euler–Bernoulli Beam Equation With Boundary Input Disturbance, *Automatica*, 49(9):2911–2918, 2013.

[3] J.J. Liu, J.M. Wang, Active Disturbance Rejection Control and Sliding Mode Control of One-Dimensional Unstable Heat Equation With Boundary Uncertainties, *IMA Journal of Mathematical Control and Information*, 32(1):97–117, 2015.

[4] Y.L. Zhang, M. Zhu, D. Li, J.M. Wang, ADRC Dynamic Stabilization of an Unstable Heat Equation, *IEEE Transactions on Automatic Control*, 65(10):4424–4429,2019

ADRC

The main idea of ADRC:
Regard the effects of all uncertainties—both inside and outside the system (such as unmodeled dynamics, parameter variations, external disturbances, etc.)—as a "**total disturbance**", then estimate it in real time and compensate for it.

Three key parts:

- Tracking differentiator
- Extended state observer (ESO)
- Nonlinear state error feedback control law

Theme

In this talk, we consider the following Timoshenko beam model:

$$\begin{cases} \varepsilon w_{tt}(x,t) = K[w_{xx}(x,t) - \varphi_x(x,t)] \\ I_\varepsilon \varphi_{tt}(x,t) = EI\varphi_{xx}(x,t) + K[w_x(x,t) - \varphi(x,t)] \\ w(0,t) = \varphi(0,t) = 0 \\ K[w_x(1,t) - \varphi(1,t)] = u_1(t) + d_1(t) \\ EI\varphi_x(1,t) = u_2(t) + d_2(t) \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \end{cases} \tag{1}$$

where $x \in (0,1), t > 0$. The function w is the transverse displacement of the beam and φ is the rotation angle of a filament of the beam. The coefficients $\varepsilon, I_\varepsilon, EI$ and K are the mass density, the moment of mass inertia, the rigidity coefficient and the shear modulus of elastic beam, respectively; $u_1(t)$ and $u_2(t)$ denotes the control inputs and $d_1(t)$ and $d_2(t)$ are the unknown disturbances.

Timoshenko beam

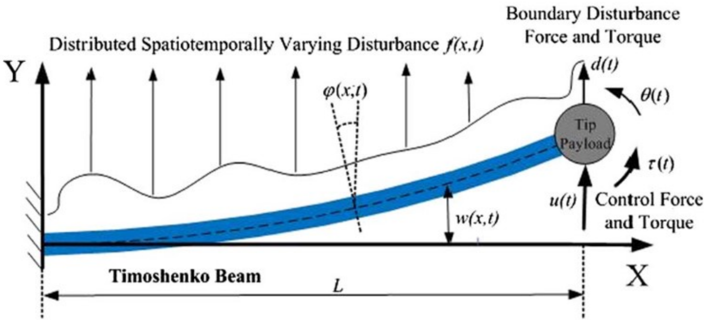

Fig. 1. Typical Timoshenko beam system with a tip payload.

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[5] W. He, S. Zhang, Boundary Output-Feedback Stabilization of a Timoshenko Beam Using Disturbance Observer, *IEEE transactions on Industrial Electronics*, 60(11): 5186–5194, 2013.

Objectives

When the system (1) is undisturbed ($d_1 = d_2 = 0$), it is well-known that the feedback-law[6]

$$u_1(t) := -\alpha_1 w_t(1,t), \quad u_2(t) := -\alpha_2 \varphi_t(1,t), \quad \alpha_1, \alpha_2 > 0, \tag{2}$$

allows to stabilize the system (1)

Objectives:

1. Adopting a new way to estimate disturbances.
2. Designing a controller to stabilize the system (1).

[6] J.U. Kim, Y. Renardy, Boundary control of the Timoshenko beam, *SIAM Journal on Control and Optimization*, 25: 1417–1429, 1987.

Definitions

Definition 2.1

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class $\mathcal{K}$ function if α is nonnegative, increasing and vanishing at 0. It is said to be a class $\mathcal{K}_\infty$ function if moreover it satisfies

$$\lim_{s \to +\infty} \alpha(s) = +\infty.$$

Definition 2.2

A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class $\mathcal{KL}$ function if,

- for each nonnegative value s, the function $r \to \beta(r, s)$ is a class $\mathcal{K}$ function,
- for every positive value r, the function $s \to \beta(r, s)$ is strictly decreasing and satisfies

$$\lim_{s \to +\infty} \beta(r, s) = 0.$$

Estimation of disturbance

Inspired by the paper [7], we choose some observational variables $\eta_0, \eta_1, \eta_2, \gamma_0, \gamma_1$ as follows.

$$\begin{cases} \eta_0(t) := \int_0^1 w_t(x,t)dx, \\ \eta_1(t) := \int_0^1 x[w_x(x,t) - \varphi(x,t)]dx, \\ \eta_2(t) := \int_0^1 x\varphi_t(x,t)dx, \\ \gamma_0(t) := \int_0^1 \varphi_t(x,t)dx, \\ \gamma_1(t) := \int_0^1 w_x(x,t) - \varphi(x,t)dx. \end{cases}$$

[7] I. Balogoun, S. Marx, Y. Orlov, F. Plestan, Active disturbance rejection control for the stabilization of a linear hyperbolic system, *International Journal of Robust and Nonlinear Control*, 35(9): 3691–3699, 2025.

Estimation of disturbance

In order to accurately estimate the two disturbances separately, we consider the following two subsystems successively.

$$\begin{cases} \dot{\gamma}_1(t) = w_t(1,t) - w_t(0,t) - \gamma_0(t) \\ \dot{\gamma}_0(t) = b\left[\frac{u_2(t)+d_2(t)}{EI} - \varphi_x(0,t)\right] + c\gamma_1(t). \end{cases} \tag{3a}$$

$$\begin{cases} \dot{\eta}_2(t) = b\left[\frac{u_2(t)+d_2(t)}{EI} - \varphi(1,t)\right] + c\eta_1(t) \\ \dot{\eta}_1(t) = w_t(1,t) - \eta_0(t) - \eta_2(t) \\ \dot{\eta}_0(t) = \frac{1}{\varepsilon}\left[u_1(t) + d_1(t)\right] - \frac{K}{\varepsilon} w_x(0,t). \end{cases} \tag{3b}$$

Estimation of disturbance

Now, the idea is to estimate the disturbance d_2 from the system (3a), which amounts to

$$\dot{\Gamma}(t) = A_1\Gamma(t) + B_1[u_2(t) + d_2(t)] + \Pi(t), \tag{4}$$

and to estimate the disturbance d_1 from the system (3b), which amounts to:

$$\begin{aligned}\dot{H}(t) &= A_2H(t) + B_2[u_1(t) + d_1(t)] + B_{10}[u_2(t) + d_2(t)] + \widetilde{\Upsilon}(t) \\ &= A_2H(t) + B_2[u_1(t) + d_1(t)] + \Upsilon(t),\end{aligned} \tag{5}$$

Control design

In order to make the origin of (1) exponentially stable, the feedback controller u_1, u_2 are designed as follows:

$$u_1(t) := -\alpha_1 w_t(1,t) - \hat{d}_1(t), \quad u_2(t) := -\alpha_2 \varphi_t(1,t) - \hat{d}_2(t). \tag{6}$$

Under the feedback (6), the closed loop system of (1) becomes

$$\begin{cases} \varepsilon w_{tt}(x,t) = K[w_{xx}(x,t) - \varphi_x(x,t)] \\ I_\varepsilon \varphi_{tt}(x,t) = EI\varphi_{xx}(x,t) + K[w_x(x,t) - \varphi(x,t)] \\ w(0,t) = \varphi(0,t) = 0 \\ K[w_x(1,t) - \varphi(1,t)] = -\alpha_1 w_t(1,t) - \hat{d}_1(t) + d_1(t) \\ EI\varphi_x(1,t) = -\alpha_2 \varphi_t(1,t) - \hat{d}_2(t) + d_2(t) \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x) \\ \dot{\overline{Z}}(t) = A_{z0}\overline{Z}(t) + G_z C_z Z(t) + B_z(-\alpha_2\varphi_t(1,t) - \hat{d}_2(t)) + N^{-1}\Gamma(t) \\ \dot{\widehat{Z_1}}(t) = \widehat{Z_2}(t) - g_1\widetilde{Z_1}(t) - Lk_1(\widehat{Z_1}(t) - \widetilde{Z_1}(t)) \\ \dot{\widehat{Z_2}}(t) \in -(c+g_2)\widetilde{Z_1}(t) - L^2k_2(\widehat{Z_1}(t) - \widetilde{Z_1}(t)) \\ \dot{\hat{d_2}}(t) \in -L^2\,\mathrm{sign}(\widehat{Z_1}(t) - \widetilde{Z_1}(t)) \\ \dot{\overline{W}}(t) = A_{w0}\overline{W}(t) + G_w C_w W(t) + B_w(-\alpha_1 w_t(1,t) - \hat{d_1}(t)) + M^{-1}\widetilde{\Gamma}(t) \\ \dot{\widehat{W}}_{11}(t) = \widehat{W}_{12}(t) - \hat{g}_1\widetilde{W}_1(t) - L_1\hat{k}_1(\widehat{W}_{11}(t) - \widetilde{W}_1(t)) \\ \dot{\widehat{W}}_{12}(t) \in \widehat{W}_{23}(t) - \hat{g}_2\widetilde{W}_1(t) - L_1^2\hat{k}_2(\widehat{W}_{11}(t) - \widetilde{W}_1(t)) \\ \dot{\widehat{W}}_{22}(t) = \widehat{W}_{23}(t) - \hat{g}_2\widetilde{W}_1(t) - L_2\hat{k}_1(\widehat{W}_{22}(t) - \widehat{W}_{12}(t)) \\ \dot{\widehat{W}}_{23}(t) \in \widehat{W}_{34}(t) - c\widetilde{W}_2(t) - \hat{g}_3\widetilde{W}_1(t) - L_2^2\hat{k}_2(\widehat{W}_{22}(t) - \widehat{W}_{12}(t)) \\ \dot{\hat{d_1}}(t) \in -L_2^2\,\mathrm{sign}(\widehat{W}_{22}(t) - \widehat{W}_{12}(t)). \end{cases} \tag{7}$$

Well-posedness

In this section, we discuss the well-posedness of the closed-loop system (7). For this, let $H_E^k(0,1) = \{f \in H^k(0,1) | f(0) = 0\}, k = 1, 2$, where $H^k(0,1)$ is the usual Sobolev space, and we take the state space as follows

$$\mathcal{H} = H_E^1(0,1) \times L^2(0,1) \times H_E^1(0,1) \times L^2(0,1),$$
$$\mathcal{J} = \mathcal{H} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^4,$$

Theorem 2.3

Let parameters (κ, L) be fixed as in Proposition 2.1 and (ρ, L_1, L_2) be fixed as in Proposition 2.2. Then for any initial state $(w(\cdot,0), w_t(\cdot,0), \varphi(\cdot,0), \varphi_t(\cdot,0), \overline{Z}(0), \widehat{Z}(0), \overline{W}(0), \widehat{W}(0)) \in \mathcal{J}$, the closed-loop system (7) admits a mild solution $(w, w_t, \varphi, \varphi_t, \overline{Z}, \widehat{Z}, \overline{W}, \widehat{W}) \in C(0, \infty; \mathcal{J})$.

Stability

Theorem 2.4

Let parameters (κ, L) be fixed as in Proposition 2.1 and (ρ, L_1, L_2) be fixed as in Proposition 2.2. Then, there exists a $\mathcal{KL}$ function β such that , the following inequality holds

$$\left\| \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \\ \varphi(\cdot,t) \\ \varphi_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} \leq \beta\left(\left\| \begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \\ \varphi(\cdot,0) \\ \varphi_t(\cdot,0) \end{pmatrix} \right\|_{\mathcal{H}} + |\xi(0)|_{\mathbb{R}^6}, t \right) \tag{8}$$

for any initial state $(w(\cdot,0), w_t(\cdot,0), \varphi(\cdot,0), \varphi_t(\cdot,0), \overline{Z}(0), \widehat{Z}(0), \overline{W}(0), \widehat{W}(0)) \in \mathcal{J}$, *for all* $t \geq 0$ *and for all solutions* $(w, w_t, \varphi, \varphi_t, \overline{Z}, \widehat{Z}, \overline{W}, \widehat{W})$ *of* (7), *where* ε *is the vector of component* $\xi_0 = (E_1, E_2)$ *and* $\xi_i = (E_{ii}, E_{i(i+1)})$ *with* $i = 1, 2$.

Numerical Simulations

In this section, we give some numerical simulations to show the effectiveness of the proposed controller, which are carried out by the finite difference method with time interval $dt = 6 \times 10^{-5}$ and space interval $dx = 0.004$, respectively. The system parameters in system (1) are chosen as $\varepsilon = 1, I_\varepsilon = 2, K = 16, EI = 8, w_0(x) = x(1-x), w_1(x) = 0, \varphi_0(x) = 2\sin(x), \varphi_1(x) = 0$. The parameters $\alpha_1 = 2, \alpha_2 = 2$. The external disturbances $d_1(t)$ and $d_2(t)$ are taken as
$d_1(t) = 5\sin(t), d_2(t) = \cos(2\pi t)$.

Numerical Simulations

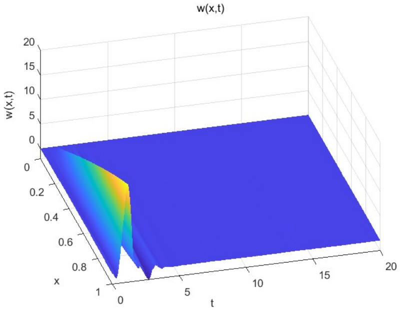

Figure 1: State of closed-loop system (1)

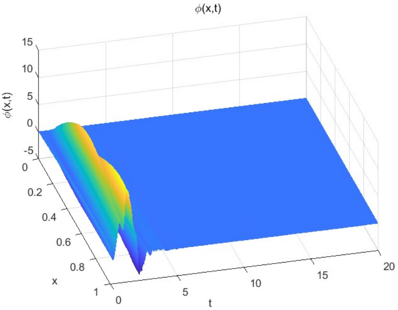

Figure 2: State of closed-loop system (1)

Numerical Simulations

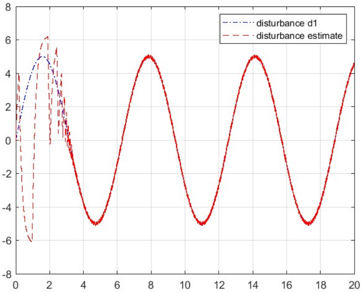

Figure 3: Disturbance estimator d1

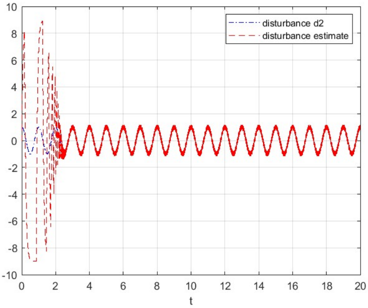

Figure 4: Disturbance estimator d2

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2 Stabilization

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4 Future Plans

Output Tracking

Output tracking is one of the fundamental issues in control theory, which aims to find a control law such that the output of the system can track the given reference signal. These problems can be broadly classified into two categories depending on prior knowledge of the tracking signal.

- the tracking signal is known;
- the tracking signal is unknown and only the tracking error is measurable, the output tracking problem is also known as the output regulation problem.

Output Tracking

We consider the following unstable heat equation with unknown external disturbances entering all possible channels in the boundary inputs as follows:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + g(x)p_1(t), & x \in (0,1),\ t > 0, \\ w_x(0,t) = -qw(0,t) + p_2(t), & t \geq 0, \\ w_x(1,t) = u(t) + p_3(t), & t \geq 0, \\ w(x,0) = w_0(x), & x \in [0,1], \\ Y_{\text{out}}(t) = w(0,t), & t \geq 0, \end{cases} \tag{9}$$

$$p_k(t) = \sum_{j=1}^{n} p_{k_j}(t) := \sum_{j=1}^{n} \left[A_{k_j} \sin\left(\omega_{k_j} t + \bar{b}_{k_j}\right) + B_{k_j} \cos\left(\omega_{k_j} t + \bar{b}_{k_j}\right)\right],\ k = 1, 2, 3$$

$$r(t) = \sum_{j=1}^{n} r_j(t) := \sum_{j=1}^{n} \left[A_{4_j} \sin\left(\omega_{4_j} t + \bar{b}_{4_j}\right) + B_{4_j} \cos\left(\omega_{4_j} t + \bar{b}_{4_j}\right)\right]. \tag{10}$$

Output Tracking

The objective of this paper is to design an adaptive control law for system (10) such that the following statements hold:

- the resulting closed-loop system disconnected with disturbances and reference signals will be exponentially stable;
- the output Y_{out} tracks the given reference signal r in the presence of the external disturbances, i.e.,

$$\lim_{t\to\infty} \mathrm{e}(t) = \lim_{t\to\infty} Y_{\text{out}}(t) - r(t) = 0. \tag{11}$$

Output Tracking

We first consider the following auxiliary system:

$$\begin{cases} \hat{w}_t(x,t) = \hat{w}_{xx}(x,t), & x \in (0,1),\ t > 0, \\ \hat{w}_x(0,t) = -(q+c)\mathrm{e}(t) + c\hat{w}(0,t), & t \geq 0, \\ \hat{w}_x(1,t) = u(t), & t \geq 0, \\ \hat{w}(x,0) = \hat{w}_0(x), & x \in [0,1], \end{cases} \tag{12}$$

Let

$$\tilde{w}(x,t) = w(x,t) - \hat{w}(x,t) - \phi(x,t), \tag{13}$$

Output Tracking

where ϕ is a suitable trajectory generated by both the disturbances and reference signal, and it takes the following form:

$$\phi(x,t) = \sum_{j=1}^{n} \left[\sum_{k=1}^{3} \Gamma_{k_j}(x)(p_{k_j}(t), \dot{p}_{k_j}(t))^\top + \Gamma_{4_j}(x)(r_j(t), \dot{r}_j(t))^\top \right]. \quad (14)$$

Here, $\Gamma_{k_j}(x) = \big(a_{k_j}(x), \quad b_{k_j}(x)\big) \in \mathbb{R}^{1\times 2}$ for $k \in I$ and $j \in J$ will be determined such that ϕ satisfies the following system:

$$\begin{cases} \phi_t(x,t) = \phi_{xx}(x,t) + g(x)p_1(t), & x \in (0,1),\ t > 0, \\ \phi_x(0,t) = c\phi(0,t) + p_2(t) - (c+q)r(t), & t \geq 0, \\ \phi_x(1,t) = p_3(t), & t \geq 0, \\ \phi(x,0) = \phi_0(x), & x \in [0,1], \end{cases} \quad (15)$$

Control Design

$$\hat{\mathrm{e}}(t) = \mathrm{e}(t) - \frac{1}{\alpha^{8n}}\mu(t),$$

We proved that $\mu(t) \to 0$. The goal: $\hat{\mathrm{e}}(t) \to 0$,
First, we introduce the transformation

$$\varepsilon(x,t) = \hat{w}(x,t) - \Xi(x)\hat{V}(t), \tag{16}$$

we get the following $(\varepsilon - \hat{V})$ system, turns to stabilization problem:

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t) - \Xi(x)K_1\mu(t), \\ \varepsilon_x(0,t) = -q\varepsilon(0,t) - \frac{q+c}{\alpha^{8n}}\mu(t), \\ \varepsilon_x(1,t) = u(t) - \Xi_x(1)\hat{V}(t), \\ \dot{\hat{V}}(t) = Z\hat{V}(t) + K_1\mu(t), \\ \hat{\mathrm{e}}(t) = \varepsilon(0,t). \end{cases} \tag{17}$$

Control Design

By using backstepping approach, we design the control law:

$$u(t) = -(q+\beta)\varepsilon(1,t) - \beta \int_0^1 q e^{q(1-y)} \varepsilon(y,t) dy + \Xi_x(1)\hat{V}(t), \tag{18}$$

Control Design

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + g(x)p_1(t), \\ w_x(0,t) = -qw(0,t) + p_2(t), \\ w_x(1,t) = u(t) + p_3(t), \\ w(x,0) = w_0(x), \end{cases} \tag{19a}$$

$$\begin{cases} \hat{w}_t(x,t) = \hat{w}_{xx}(x,t), \\ \hat{w}_x(0,t) = -(q+c)\mathrm{e}(t) + c\hat{w}(0,t), \\ \hat{w}_x(1,t) = u(t), \end{cases} \tag{19b}$$

$$\begin{cases} \dot{\hat{V}}(t) = \hat{Z}\hat{V}(t) + K\left[\mathrm{e}(t) - \hat{w}(0,t)\right], \\ \dot{\hat{\Theta}}(t) = \Lambda\Omega(t)\left(F\hat{V}(t) - \alpha^{8n}\left[\mathrm{e}(t) - \hat{w}(0,t)\right] - \Omega(t)^\top\hat{\Theta}(t)\right), \end{cases} \tag{19c}$$

$$\begin{cases} \mathrm{e}(t) = w(0,t) - r(t), \\ u(t) = -(q+\beta)\hat{w}(1,t) - \beta q\int_0^1 e^{q(1-x)}\hat{w}(x,t)dx + \Psi[\hat{\Theta}](t)\hat{V}(t). \end{cases} \tag{19d}$$

Main Results

Theorem 3.1

Suppose that $c > \frac{1}{2}, q > 0, \beta > 0$, r *and* p_k *for* $k = 1, 2, 3$ *satisfy* (10), *and the matrix* Λ *is positive definite. Then, for any initial state*

$$\Big(w(\cdot, 0), \hat{w}(\cdot, 0), \hat{V}(0), \hat{\Theta}(0)\Big) \in \mathcal{J}, \tag{20}$$

the closed-loop system (19) *admits a unique solution such that*

$$(w, \hat{w}, \hat{V}, \hat{\Theta}) \in C([0, \infty); \mathcal{J}), \tag{21}$$

satisfying

$$\lim_{t \to \infty} |e(t)| = 0.$$

Simulations

As an illustrating example, we consider the following system:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), & x \in (0,1),\ t>0, \\ w_x(0,t) = -0.1w(0,t) + \sin(t), & t>0, \\ w_x(1,t) = u(t), & t \geq 0, \\ w(x,0) = 2\cos(\pi x) - 3, & x \in [0,1], \\ Y_{\text{out}}(t) = w(0,t), & t \geq 0, \end{cases} \tag{22}$$

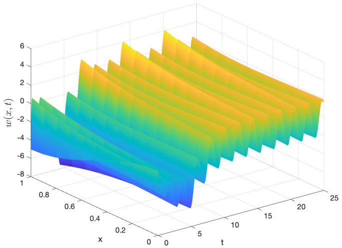
The initial conditions are specified as follows:

$$\hat{w}(x,0) = 0,\ r(t) = 2\cos(3t+1),\ \hat{V}(0) = 0,\ \hat{\Theta}(0) = 0. \tag{23}$$

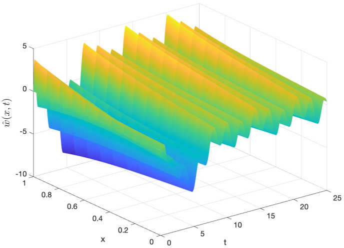
The parameters selected for the simulation are:

$$q = 0.1,\ \alpha = 1.0,\ c = 1.1,\ \beta = 3.0,\ \Lambda = \begin{pmatrix} 25.0 & 0 \\ 0 & 25.0 \end{pmatrix}. \tag{24}$$

Simulations



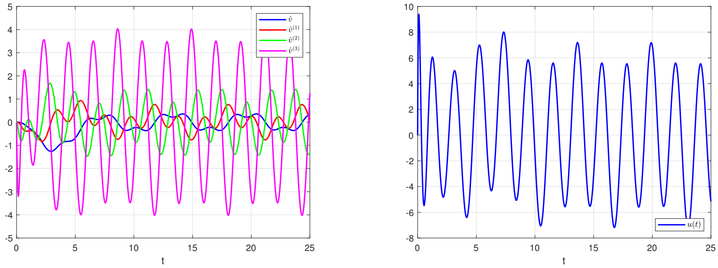
(a) $w(x,t)$ in system (19a).



(b) $\hat{w}(x,t)$ in system (19b).

Figure 5: Solution of the closed-loop system (19).

Simulations

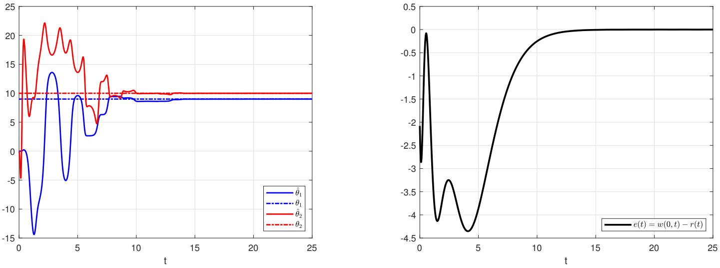


(a) $\hat{V}(t)$ in system (19c).

(b) $u(t)$ in system (19d).

Figure 6: Internal mode dynamic $\hat{V}(t)$ and the controller $u(t)$.

Simulations



(a) $\hat{\Theta}(t)$ in system (19c). (b) e(t) in system (19d).

Figure 7: The parameters estimation $\hat{\Theta}(t)$ and the tracking error e(t).

Outline

In the future

- ⇒ Control with Machine Learning
- ⇒ Optimal Control with DeepONet

Thank You!

chenjun_bcbm@163.com