

Controllability and Synchronization of Coupled Wave Equations

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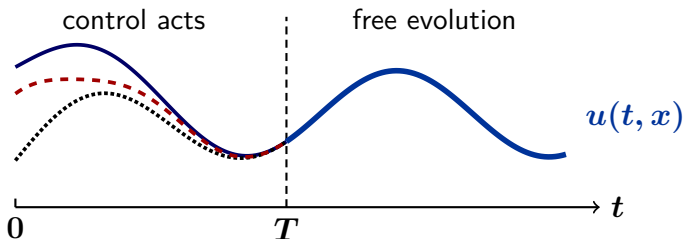
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Outline

1. **Li–Rao's boundary synchronization**
2. **Generalized boundary synchronization**
3. **Research plans at Deusto**

Controllability and synchronization of nonlinear waves.

Finite-time synchronization

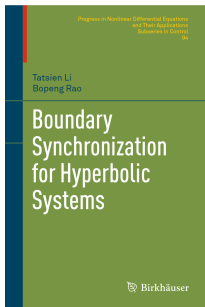


Schematic time traces at a fixed spatial point.

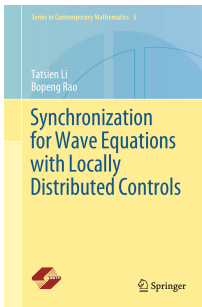
Synchronize at T and remain synchronized.

Li–Rao's synchronization theory

Tatsien Li and Bopeng Rao



Boundary controls



Locally distributed controls

This talk begins with boundary synchronization.

A wave system with boundary controls

$$\begin{cases} U_{tt} - \Delta U + AU = 0 & \text{in } (0, \infty) \times \Omega, \\ U = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ U = DH & \text{on } (0, \infty) \times \Gamma_1. \end{cases}$$

State and coupling

$$U = (u^{(1)}, \dots, u^{(N)})^T$$

$$A \in \mathbb{R}^{N \times N}$$

Boundary actuation

$$H = (h^{(1)}, \dots, h^{(M)})^T$$

$$D \in \mathbb{R}^{N \times M}$$

Initial data: $(U_0, U_1) := (U, U_t)(0)$ in $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$.

Smooth bounded Ω ; $\partial\Omega = \Gamma_0 \cup \Gamma_1$,
 $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, $|\Gamma_1| > 0$.

Linear results: multiplier geometry and sufficiently large T .

Exact boundary synchronization

For every initial state, choose a boundary control

$$H \in L^2((0, T) \times \Gamma_1)^M, \quad H = 0 \text{ for } t > T,$$

such that

$$u^{(1)}(t, \cdot) = \cdots = u^{(N)}(t, \cdot) = \mathbf{u}(t, \cdot) \quad (t \geq T).$$

The synchronizable state $\mathbf{u}$ is not prescribed.

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$$W = CU, \quad C = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}.$$

Synchronization: $W(t) = 0$ for all $t \geq T$.

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Synchronization: $W(t) = 0$ for all $t \geq T$.

At T : $W(T) = W_t(T) = 0$.

Li-Rao: null control of W

Compatibility condition

$$Ae = \alpha e \iff CA = \bar{A}C,$$

where $e = (1, \dots, 1)^T$.

Closed reduced system

$$W_{tt} - \Delta W + \bar{A}W = 0, \quad W|_{\Gamma_1} = CDH.$$

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Under compatibility:

exact synchronization $\iff \text{Rank}(CD) = N - 1$.

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Under compatibility:

exact synchronization $\iff \text{Rank}(CD) = N - 1$.

After synchronization: $u_{tt} - \Delta u + \alpha u = 0$.

Standing geometry and time assumptions apply. Li-Rao (2013); $W|_{\Gamma_0} = 0$.

Generalized synchronization

Choose a full row-rank $\Theta \in \mathbb{R}^{(N-p) \times N}$, $0 \leq p < N$. For every initial state, seek a control such that

$$\Theta U(t) = 0 \quad (t \geq T).$$

Let $E = (e_1, \dots, e_p)$ be a basis matrix for $\text{Ker } \Theta$:

$$U = Eu, \quad u = (u_1, \dots, u_p)^T \quad (t \geq T).$$

The generalized synchronizable state u is not prescribed.

- Complete synchronization: $\Theta = C$, $p = 1$.
- Synchronization by groups: p profiles.
- Null controllability: $\Theta = I_N$, $p = 0$.

Controlling the reduced system

Compatibility condition

$$A \operatorname{Ker} \Theta \subseteq \operatorname{Ker} \Theta \iff \Theta A = \bar{A} \Theta.$$

Closed reduced system: $W = \Theta U$

$$\begin{cases} W_{tt} - \Delta W + \bar{A}W = 0, \\ W|_{\Gamma_0} = 0, \quad W|_{\Gamma_1} = \Theta DH. \end{cases}$$

Under compatibility:

generalized exact synchronization

$$\iff \operatorname{Rank}(\Theta D) = N - p.$$

Standing geometry and time assumptions apply. Wang (2019, 2020).

An invertible transformation X

Choose $Y = (y_1, \dots, y_p)$ such that $Y^T E = I_p$.

$$X = \begin{pmatrix} \Theta \\ Y^T \end{pmatrix}, \quad XU = \begin{pmatrix} W \\ V \end{pmatrix}.$$

Null controllable part

$$W = \Theta U$$

Synchronizable state part

$$V = Y^T U$$

$$t \geq T : \quad W = 0, \quad V = u.$$

The transformed dynamics

$$XAX^{-1} = \begin{pmatrix} \bar{A} & Z_1 \\ Z_2^T & \tilde{A} \end{pmatrix}.$$

$$W_{tt} - \Delta W + \bar{A}W = -\mathbf{Z}_1 \mathbf{V},$$

$$V_{tt} - \Delta V + \tilde{A}V = -Z_2^T W.$$

$$W|_{\Gamma_1} = \Theta DH, \quad V|_{\Gamma_1} = Y^T DH.$$

**Compatibility means $Z_1 = 0$:
the reduced system closes.**

Both variables vanish on Γ_0 . Initial data transform by X .

Determining the synchronizable state

Strong compatibility

$$AE = E\tilde{A}, \quad A^T Y = Y\tilde{A}^T, \quad Y^T E = I_p.$$

Control subspace

$$\text{Ker } D^T = \text{Im } Y \implies Y^T D = 0.$$

Then V is independent of H for **all** $t \geq 0$:

$$\begin{cases} V_{tt} - \Delta V + \tilde{A}V = 0, \\ V|_{\partial\Omega} = 0, \\ (V, V_t)|_{t=0} = (Y^T U_0, Y^T U_1). \end{cases}$$

After synchronization, $u = V$ for $t \geq T$.

Wang (2019, 2020). See Appendix A1.

Two nonlinear control objectives

$$U_{tt} - \Delta U + F(U) = \chi_\omega DH.$$

$$U|_{\partial\Omega} = 0, \quad H = 0 \text{ for } t > T.$$

Controllability

Steer the full state (U, U_t) to a prescribed target.

Synchronization

Drive $(\Theta U, \Theta U_t)$ to zero.

Can synchronization use fewer controls?

Starting points: Zuazua (1993), Hu–Li–Qu (2016), Zhang (2018). See Appendix A4.

Nonlinear compatibility

1 Preserve synchronization

$$F(\text{Ker } \Theta) \subseteq \text{Ker } \Theta \iff \Theta F(Eu) = 0 \quad \forall u.$$

With controls off, $W(T) = W_t(T) = 0$ implies $W(t) = 0$ for all $t \geq T$.

2 Close the reduced system (stronger)

$$\exists G : \quad \Theta F(U) = G(\Theta U) \quad \forall U, \quad G(0) = 0.$$

$$W_{tt} - \Delta W + G(W) = \chi_\omega \Theta DH.$$

Sine model: closure. Cubic model: invariance only.

$W = \Theta U$, $\text{Ker } \Theta = \text{Im } E$. Persistence assumes well-posedness and uniqueness.

A model with sinusoidal coupling

Two waves, one control

$$\begin{cases} u_{tt}^{(1)} - \Delta u^{(1)} = \sin(u^{(1)} - u^{(2)}) + \chi_\omega h, \\ u_{tt}^{(2)} - \Delta u^{(2)} = \sin(u^{(2)} - u^{(1)}). \end{cases}$$

A closed reduced system

Let $w = u^{(1)} - u^{(2)}$. Subtracting gives

$$w_{tt} - \Delta w = 2 \sin w + \chi_\omega h.$$

Target: $w(T) = w_t(T) = 0$.

Synchronizable state: $u_{tt} - \Delta u = 0$ for $t \geq T$.

In the form $U_{tt} - \Delta U + F(U) = \chi_\omega DH$, $G(w) = -2 \sin w$. Homogeneous Dirichlet data; $h = 0$ after T .

A model with cubic coupling

Let $U = (u^{(1)}, u^{(2)})^T$ and $F(U) = |U|^2 U$.

$$\begin{cases} u_{tt}^{(1)} - \Delta u^{(1)} + |U|^2 u^{(1)} = \chi_\omega h, \\ u_{tt}^{(2)} - \Delta u^{(2)} + |U|^2 u^{(2)} = 0. \end{cases}$$

$$w = u^{(1)} - u^{(2)}, \quad q = \frac{1}{2}(u^{(1)} + u^{(2)}).$$

$$w_{tt} - \Delta w + \left(2q^2 + \frac{1}{2}w^2\right) w = \chi_\omega h.$$

Target: $w(T) = w_t(T) = 0$.

With $h = 0$ after T : $u^{(1)} = u^{(2)} = u$.

Proposed model: a smooth bounded domain in 3D, homogeneous Dirichlet data and a control region satisfying the geometric control condition.

Work programme at Deusto

1 Local theory

Local synchronization for the cubic model.

2 Fewer controls

Full controllability versus synchronization.
Indirect controllability.

3 Extensions

Boundary actuation and larger data where estimates permit.
Related problems.

Invariance, observability and nonlinear estimates.

Thank you.

A1. Strong compatibility

Necessity

Exact generalized synchronization and control-independent $V = Y^T U$ throughout the evolution imply

$$\begin{aligned} A \operatorname{Im} E &\subseteq \operatorname{Im} E, & A^T \operatorname{Im} Y &\subseteq \operatorname{Im} Y, \\ Y^T E &= I_p, & \operatorname{Ker} D^T &= \operatorname{Im} Y. \end{aligned}$$

Sufficiency

Under strong compatibility, choose $\operatorname{Ker} D^T = \operatorname{Im} Y$.

$$XAX^{-1} = \begin{pmatrix} \bar{A} & 0 \\ 0 & \tilde{A} \end{pmatrix}, \quad \operatorname{Rank}(\Theta D) = N - p.$$

Under the standing assumptions, W is null controllable and V is autonomous for $t \geq 0$. After synchronization, $u = V$ for $t \geq T$.

Independence concerns the whole evolution of V . Wang (2019, 2020).

A2. Li–Rao's framework

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A3. Generalized synchronization

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A4. Nonlinear starting points

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[Numdam full text](#)

L. Hu, T. Li and P. Qu (2016). Exact boundary synchronization for a coupled system of 1-D quasilinear wave equations.

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C. Zhang (2018). Internal controllability of systems of semilinear coupled one-dimensional wave equations with one control.

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[doi:10.1137/17M1128885](https://doi.org/10.1137/17M1128885)