The RBM-MPC Strategy in Linear-Quadratic Optimal Control

Stability and Convergence

Alexandra Borkowski

DyCon Project, Chair of Computational Mathematics, Universidad de Deusto

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Controlling Collective Dynamics: The Herding of Sheep

Introduction of a Semi-Feedback Control Strategy: *Model Predictive Control with Random Batch Methods for a Guiding Problem* by Dongnam Ko and Enrique Zuazua, 2020.

Outline

Linear-Quadratic Optimal Control Problems

Model Predictive Control (MPC)

Random Batch Method (RBM)

Model Predictive Control with Random Batch Method

Numerical Illustration

Perspectives

1 Linear-Quadratic Optimal Control Problems

Linear-Quadratic Optimal Control Problems

We want to find an optimal trajectory $(\boldsymbol{x}_T^*(t), \boldsymbol{u}_T^*(t))$ with

$$\boldsymbol{u}_T^* = \arg \min_{\boldsymbol{u}_T \in L^2(0,T;\mathbb{R}^q)} J_T(\boldsymbol{x}_T(t), \boldsymbol{u}_T(t)) = \frac{1}{2}\int_0^T \left(\boldsymbol{x}_T(t)^T Q \boldsymbol{x}_T(t) + \boldsymbol{u}_T(t)^T R \boldsymbol{u}_T(t) \right) dt,$$

where the state $\boldsymbol{x}_T(t)$ is specified by the linear time invariant system

$$\begin{aligned} \dot{\boldsymbol{x}}_T(t) &= A\boldsymbol{x}_T(t) + B\boldsymbol{u}_T(t), , \quad t \in (0,T] \\ \boldsymbol{x}_T(0) &= \boldsymbol{x}_0, \end{aligned}$$

with $T \in (0, \infty)$.

Linear Quadratic Regulator and the Riccati Equations

It can be shown that the optimal control $\boldsymbol{u}_T^*(t)$ has a feedback form, such that (see Anderson):

$$\begin{aligned}
\dot{\boldsymbol{x}}_T^*(t) &= A\boldsymbol{x}_T^*(t) - B\boldsymbol{u}_T^*(t) \\
&= A\boldsymbol{x}_T^*(t) - BR^{-1}B^T P_T(t)\boldsymbol{x}_T^*(t) \\
&= (A - BR^{-1}B^T P_T(t))\boldsymbol{x}_T^*(t) \\
&:= A_T\boldsymbol{x}_T^*(t).
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\end{aligned}$$

In case of $T < \infty$, $P_T(t)$ solves the Riccati Differential Equation

$$\begin{aligned}
-\dot{P}_T(t) &= A^T P_T(t) + P_T(t)A - P_T(t)BR^{-1}B^T P_T(t) + Q, \\
P_T(T) &= 0.
\end{aligned}$$

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In case of T '=' ∞, $P_\infty(t) = P_\infty$ solves the Algebraic Riccati Equation

$$A^TP_\infty + P_\infty A - P_\infty BR^{-1}B^TP_\infty + Q = 0.$$

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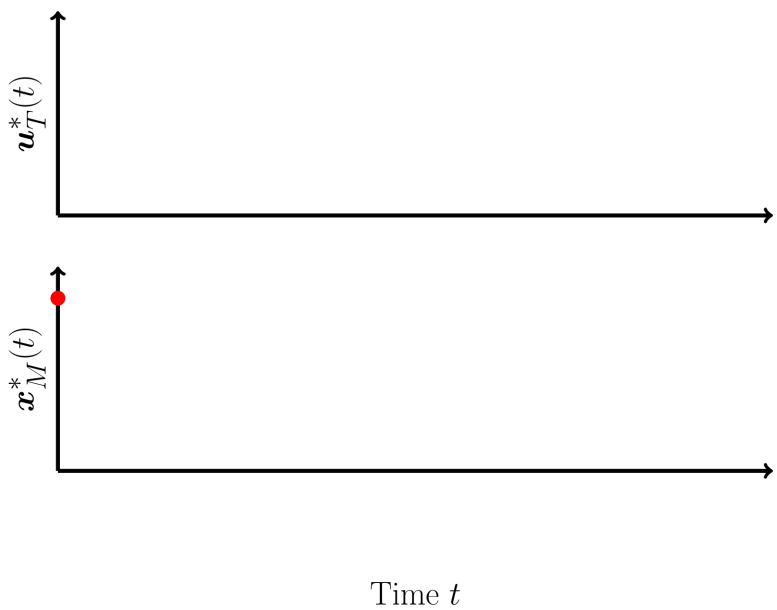
$$A^TP_\infty + P_\infty A - P_\infty BR^{-1}B^TP_\infty + Q = 0.$$

Exponential stability concerning the matrix $A_\infty = A - R^{-1}B^TP_\infty$ can be proven:

$$\exists\, \mu_\infty, M_\infty > 0 : ||e^{A_\infty t}|| \leq M_\infty e^{-\mu_\infty t}.$$

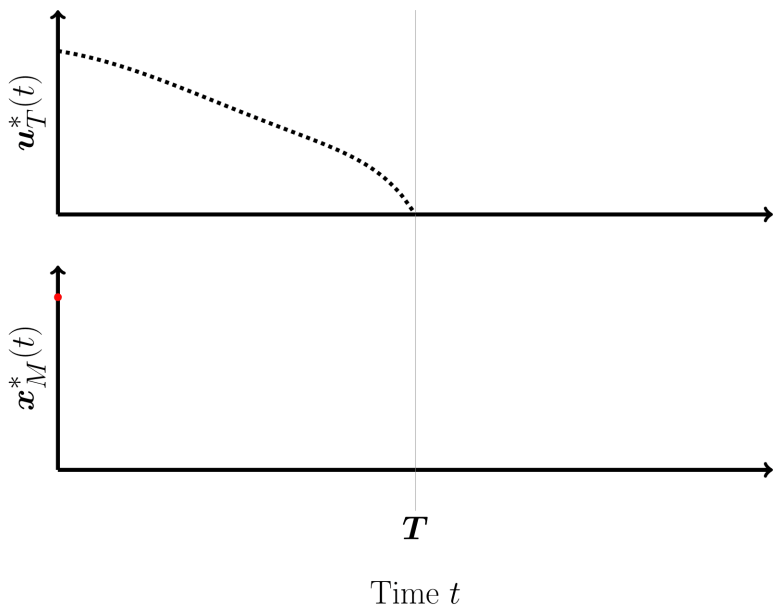
2 Model Predictive Control (MPC)

Model Predictive Control in Linear-Quadratic Optimal Control



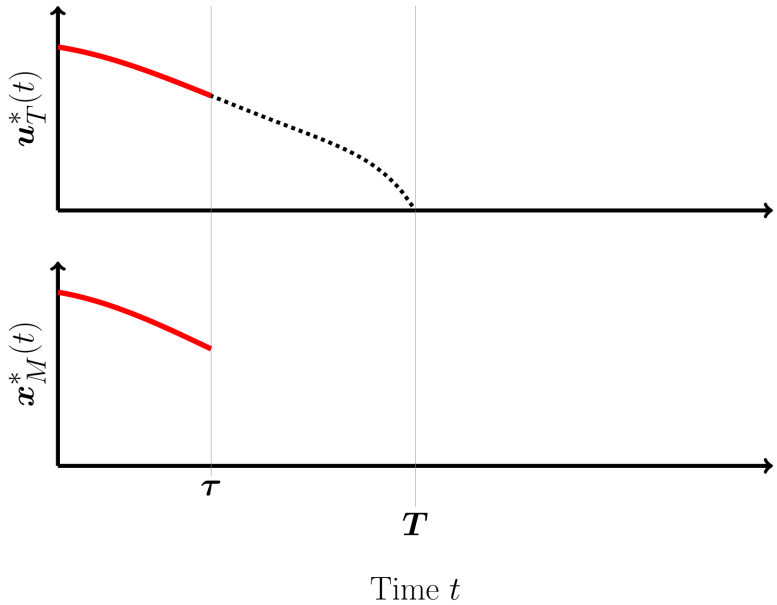
<u>Main Idea:</u> Approximate $\boldsymbol{x}_\infty^*$ recursively by finite-time horizon solutions $\boldsymbol{x}_T^*$.

Model Predictive Control in Linear-Quadratic Optimal Control



Compute $\boldsymbol{u}_T^*(t, \boldsymbol{x}(0))$ on $[0, T]$.

Model Predictive Control in Linear-Quadratic Optimal Control

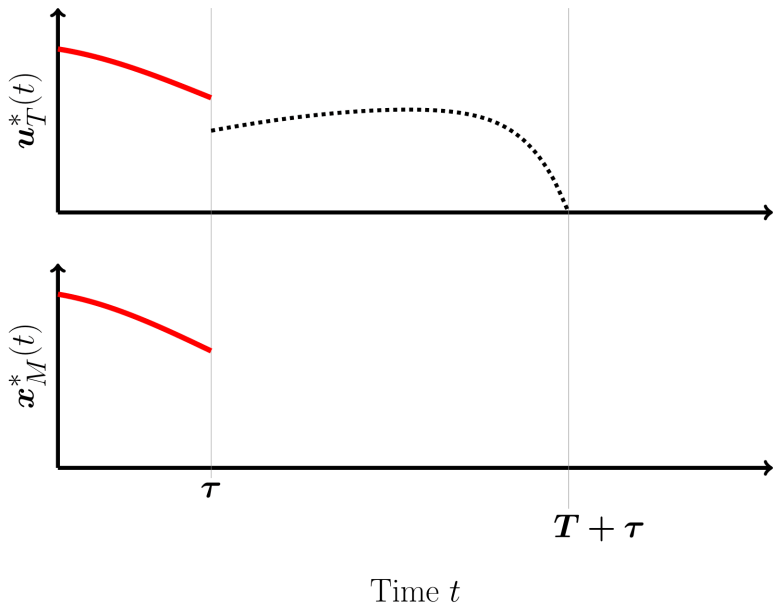


Determine $\boldsymbol{x}_T^*(t, \boldsymbol{x}(0))$ on $[0, \tau]$ by solving

$$\dot{\boldsymbol{x}}_T^*(t, \boldsymbol{x}(0)) = A\boldsymbol{x}_T^*(t, \boldsymbol{x}(0)) + B\boldsymbol{u}_T^*(t, \boldsymbol{x}(0)), \quad \boldsymbol{x}_T^*(0, \boldsymbol{x}(0)) = \boldsymbol{x}(0).$$

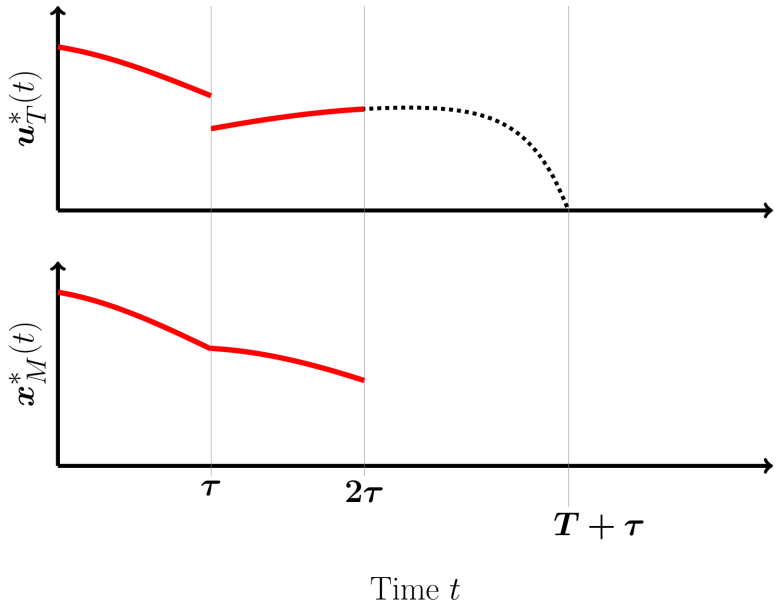
We set $\boldsymbol{x}_M^*(t) = \boldsymbol{x}_T^*(t, \boldsymbol{x}(0))$ on $[0, \tau]$.

Model Predictive Control



Compute $\boldsymbol{u}_T^*(t, \boldsymbol{x}_M^*(\tau))$ on $[\tau, \tau + T]$.

Model Predictive Control in Linear-Quadratic Optimal Control

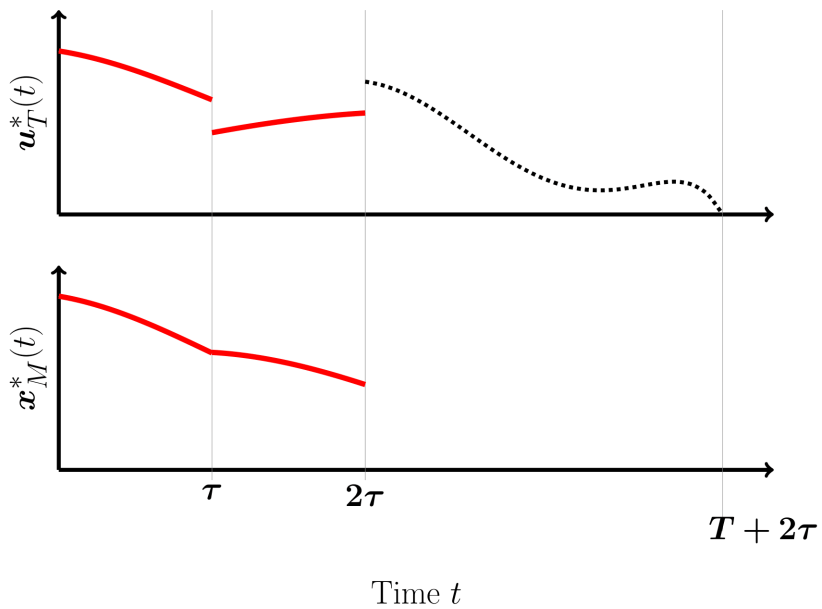


Determine $\boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(\tau))$ on $[\tau, 2\tau]$ by solving

$$\dot{\boldsymbol{x}}_T^*(t, \boldsymbol{x}_M^*(\tau)) = A\boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(\tau)) + B\boldsymbol{u}_T^*(t, \boldsymbol{x}_M^*(\tau)), \quad \boldsymbol{x}_T^*(\tau, \boldsymbol{x}_M^*(\tau)) = \boldsymbol{x}_M^*(\tau).$$

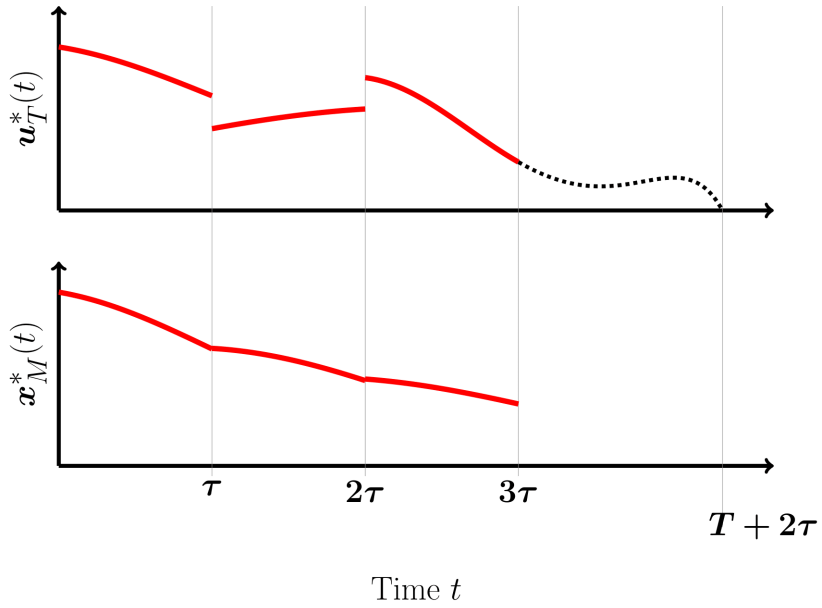
We set $\boldsymbol{x}_M^*(t) = \boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(\tau))$ on $[\tau, 2\tau]$.

Model Predictive Control in Linear-Quadratic Optimal Control



Compute $\boldsymbol{u}_T^*(t, \boldsymbol{x}_M^*(2\tau))$ on $[2\tau, 2\tau + T]$.

Model Predictive Control in Linear-Quadratic Optimal Control

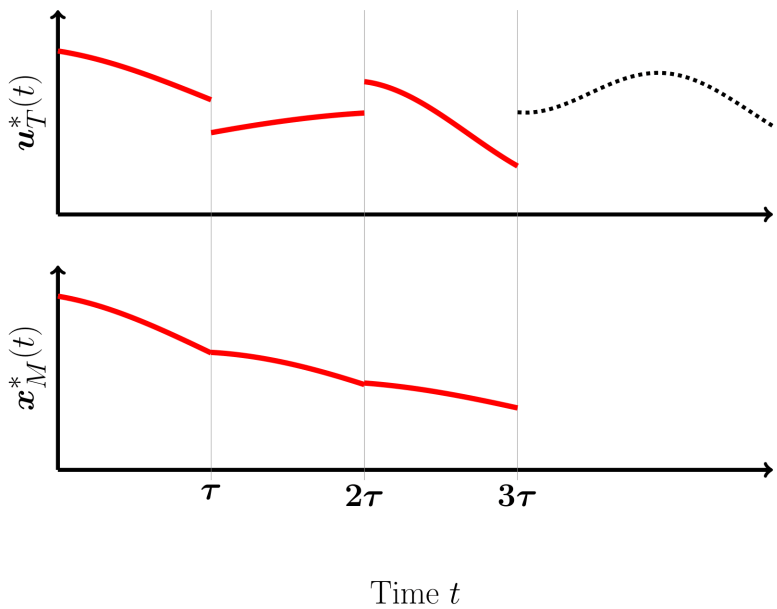


Determine $\boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(2\tau))$ on $[2\tau, 3\tau]$ by solving

$$\dot{\boldsymbol{x}}_T^*(t, \boldsymbol{x}_M^*(2\tau)) = A\boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(2\tau)) + B\boldsymbol{u}_T^*(t, \boldsymbol{x}_M^*(2\tau)), \quad \boldsymbol{x}_T^*(2\tau, \boldsymbol{x}_M^*(2\tau)) = \boldsymbol{x}_M^*(2\tau).$$

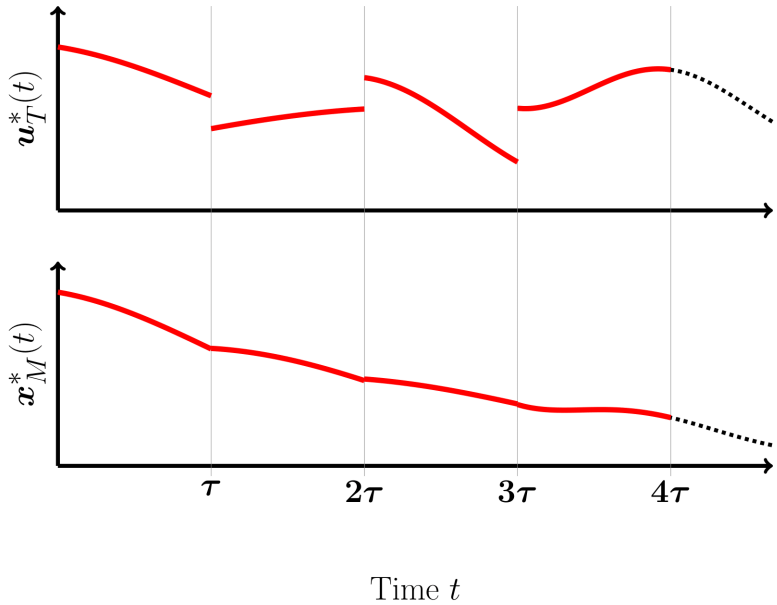
We set $\boldsymbol{x}_M^*(t) = \boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(2\tau))$ on $[2\tau, 3\tau]$.

Model Predictive Control in Linear-Quadratic Optimal Control



Compute $\boldsymbol{u}_T^*(t, \boldsymbol{x}_M^*(3\tau))$ on $[3\tau, 3\tau + T]$.

Model Predictive Control in Linear-Quadratic Optimal Control



Determine $\boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(3\tau))$ on $[3\tau, 4\tau]$ by solving

$$\dot{\boldsymbol{x}}_T^*(t, \boldsymbol{x}_M^*(3\tau)) = A\boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(3\tau)) + B\boldsymbol{u}_T^*(t, \boldsymbol{x}_M^*(3\tau)), \quad \boldsymbol{x}_T^*(3\tau, \boldsymbol{x}_M^*(3\tau)) = \boldsymbol{x}_M^*(3\tau).$$

We set $\boldsymbol{x}_M^*(t) = \boldsymbol{x}_T^*(t, \boldsymbol{x}_M^*(3\tau))$ on $[3\tau, 4\tau]$.

Model Predictive Control in Linear-Quadratic Optimal Control

The MPC dynamics can thus be expressed with a periodic feedback operator:

$$\begin{aligned}\dot{\boldsymbol{x}}_M^*(t) &= \Big(A - BR^{-1}B^T P_T(t \mod \tau)\Big)\boldsymbol{x}_M^*(t) \\ &=: A_{T,\tau}\boldsymbol{x}_M^*(t).\end{aligned}$$

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Stability (Veldmann and Zuazua, 2022)

There exists a constant $\mu_{T,\tau} = \mu_\infty - M_\infty K_0^{RH} e^{-2\mu_\infty(T-\tau)}$, such that

$$|\boldsymbol{x}_M(t)| \leq M_\infty e^{-\mu_{T,\tau} t}|\boldsymbol{x}_0|. \tag{1}$$

For $\mu_{T,\tau} > 0$, the MPC strategy is stabilizing.

- Choosing $T - \tau$ big enough, will result in a stabilizing algorithm.

3 Random Batch Method (RBM)

The RBM in Linear-Quadratic Optimal Control (see Veldman and Zuazua, 2021)

Can we approximate the minimizer $\boldsymbol{u}_T^*(t)$ of

$$\min_{\boldsymbol{u}_T(t)\in L^2(0,T;\mathbb{R}^q)} J_T(\boldsymbol{u}_T(t)) = \int_0^T \left(\boldsymbol{x}_T(t)^T Q \boldsymbol{x}_T(t) + \boldsymbol{u}_T^T(t) R \boldsymbol{u}_T(t)\right) dt,$$
$$\dot{\boldsymbol{x}}_T(t) = A\boldsymbol{x}_T(t) + B\boldsymbol{u}_T(t), \quad \boldsymbol{x}_T(0) = \boldsymbol{x}_0 \ ?$$

<u>Main Idea:</u> We approximate the dynamics by replacing A by - potentially - sparser matrices.

The RBM in Linear-Quadratic Optimal Control (see Veldman and Zuazua, 2021)

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- Decompose the matrix A as

$$A = \sum_{m=1}^{M} A_m.$$

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- Decompose the matrix A as

$$A = \sum_{m=1}^{M} A_m, \quad \text{e.g. } A = A_1 + A_2.$$

- Enumerate the 2^M subsets of $\{1, 2, \ldots, M\}$ as $S_1, S_2, \ldots S_{2^M}$. Assign to each subset S_ω a probability p_ω.

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- Divide $[0,T]$ into K subintervals $[t_{k-1}, t_k)$ of length h. For each $[t_{k-1}, t_k)$, randomly choose an index $\omega_k \in \{1, 2, \ldots, 2^M\}$ according to the probabilities p_{ω_k}. Set $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_K)$.

The RBM in Linear-Quadratic Optimal Control (see Veldman and Zuazua, 2021)

Can we approximate the minimizer $\boldsymbol{u}_T^*(t)$ of

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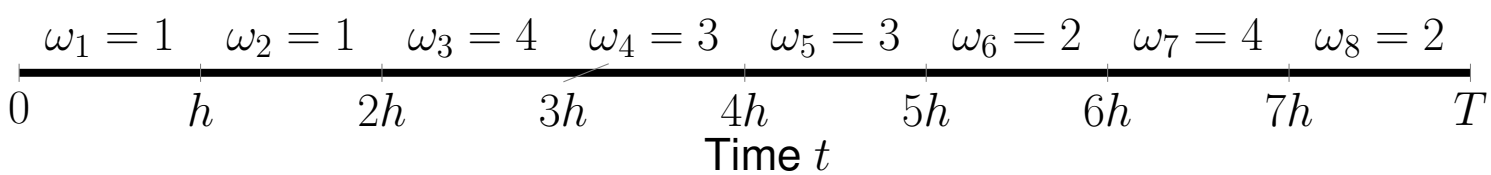

The RBM in Linear-Quadratic Optimal Control (see Veldman and Zuazua, 2021)

- Define the mapping $t \mapsto A_h(\boldsymbol{\omega}, t)$

$$A_h(\boldsymbol{\omega}, t) = \sum_{m \in S_{\omega_k}} \frac{A_m}{\pi_m}, \qquad t \in [t_{k-1}, t_k),$$

where π_m is the probability that m is an element of the selected subset, i.e.

$$\pi_m = \sum_{\omega \in \{\omega | m \in S_\omega\}} p_\omega.$$

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e.g. $A_h(\boldsymbol{\omega}, t) = 2A_1$, for $t \in [0, 2h)$, $A_h(\boldsymbol{\omega}, t) = 2A_1 + 2A_2 = 2A$, for $t \in [2h, 3h)$, ...

- Compute the minimizer $\boldsymbol{u}_R^*(\boldsymbol{\omega}, t)$ of the 'simpler' LQR problem

$$\min_{\boldsymbol{u}_R \in L^2(0,T;\mathbb{R}^q)} J_h(\boldsymbol{\omega}, \boldsymbol{u}) = \int_0^T \left(\boldsymbol{x}_R(\boldsymbol{\omega}, t)^T Q \boldsymbol{x}_R(\boldsymbol{\omega}, t) + \boldsymbol{u}_R(t)^T R \boldsymbol{u}_R(t)\right) dt,$$
$$\dot{\boldsymbol{x}}_R(\boldsymbol{\omega}, t) = A_h(\boldsymbol{\omega}, t)\boldsymbol{x}_R(\boldsymbol{\omega}, t) + B\boldsymbol{u}_R(t), \quad \boldsymbol{x}_R(\boldsymbol{\omega}, 0) = \boldsymbol{x}_0.$$

The RBM in Linear-Quadratic Optimal Control

The RBM dynamics can be expressed with a randomized feedback form:

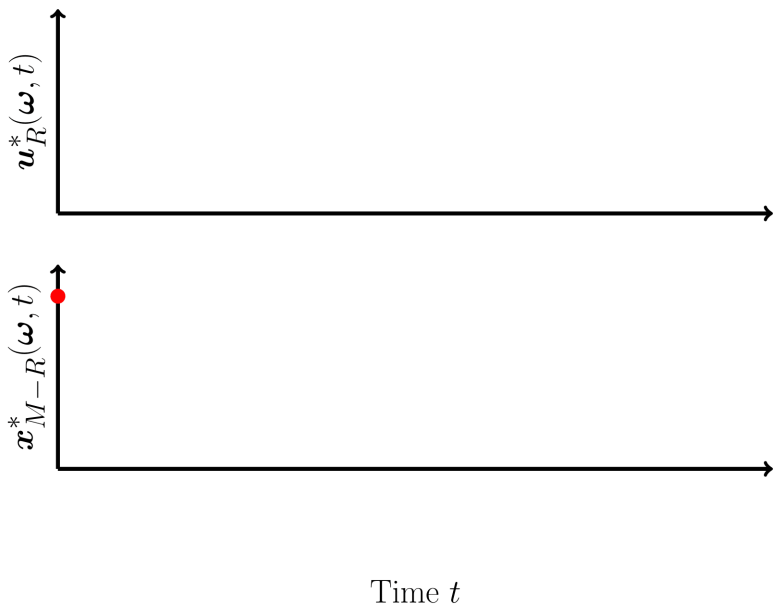
$$\begin{aligned}\dot{\boldsymbol{x}}_R^*(\boldsymbol{\omega},t) &= A_h(\boldsymbol{\omega},t)\boldsymbol{x}_R^*(\boldsymbol{\omega},t) + B\boldsymbol{u}_R^*(\boldsymbol{\omega},t)\\ &= A_h(\boldsymbol{\omega},t)\boldsymbol{x}_R^*(\boldsymbol{\omega},t) - BR^{-1}B^T P_R(\boldsymbol{\omega},t)\boldsymbol{x}_R^*(\boldsymbol{\omega},t)\\ &= (A_h(\boldsymbol{\omega},t) - BR^{-1}B^T P_R(\boldsymbol{\omega},t))\boldsymbol{x}_R^*(\boldsymbol{\omega},t),\end{aligned}$$

where $P_R(\boldsymbol{\omega},t)$ solves the Randomized Riccati Differential Equation

$$\begin{aligned}-\dot{P}_R(\boldsymbol{\omega},t) &= A_h(\boldsymbol{\omega},t)^T P_R(\boldsymbol{\omega},t) + P_R(\boldsymbol{\omega},t)A_h(\boldsymbol{\omega},t)\\ &\quad - P_R(\boldsymbol{\omega},t)BR^{-1}B^T P_T(\boldsymbol{\omega},t) + Q,\\ P_R(\boldsymbol{\omega},T) &= 0.\end{aligned}$$

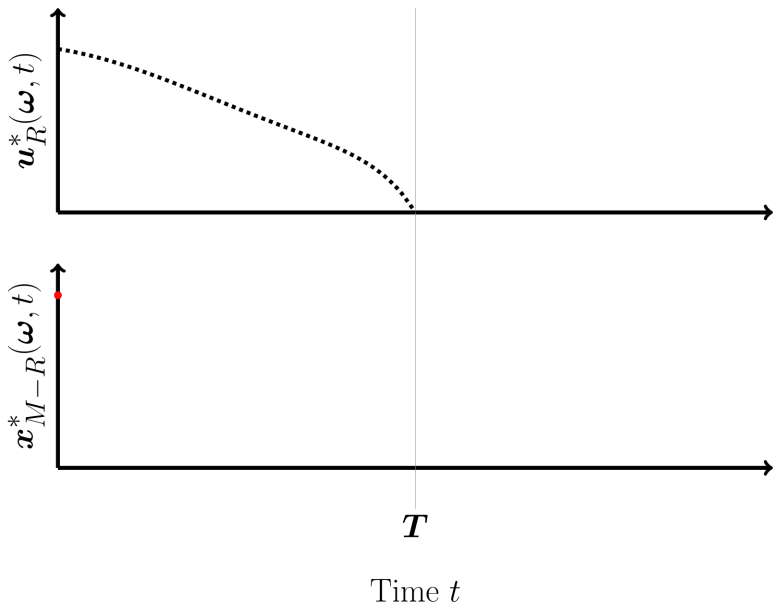
4 Model Predictive Control with Random Batch Method

RBM-MPC



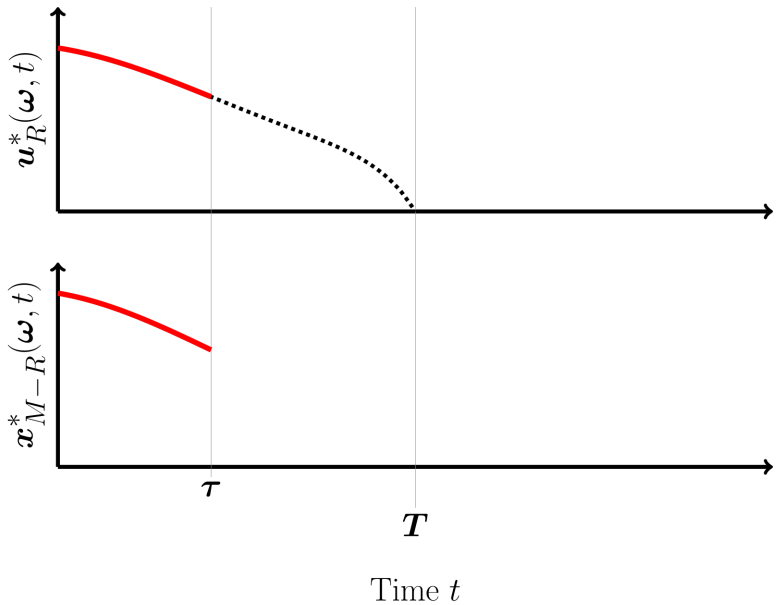
<u>Main Idea:</u> Approximate $\boldsymbol{x}_\infty^*$ recursively by finite-time horizon solutions $\boldsymbol{x}_T^*(\boldsymbol{\omega}, t)$, which are based on randomized predictions.

RBM-MPC



Compute $\boldsymbol{u}_R^*(\boldsymbol{\omega}, \boldsymbol{x}(0))$ on $[0, T]$, $\Omega_{-1} = \emptyset$.

RBM-MPC

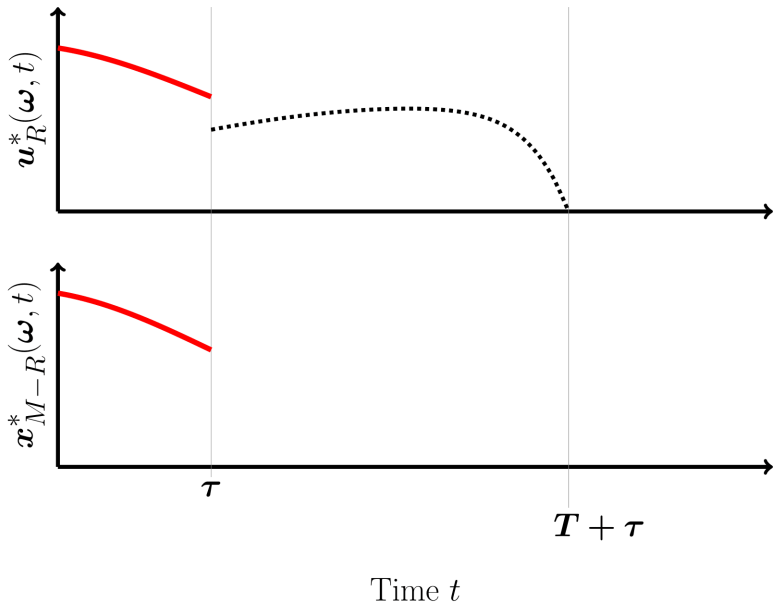


Determine $\boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}(0))$ on $[0, \tau]$ by solving

$$\begin{aligned}\dot{\boldsymbol{x}}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}(0)) &= A\boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}(0)) + B\boldsymbol{u}_R^*(\boldsymbol{\omega}, t, \boldsymbol{x}(0)),\\ \boldsymbol{x}_T^*(\boldsymbol{\omega}, \tau, \boldsymbol{x}(0)) &= \boldsymbol{x}(0).\end{aligned}$$

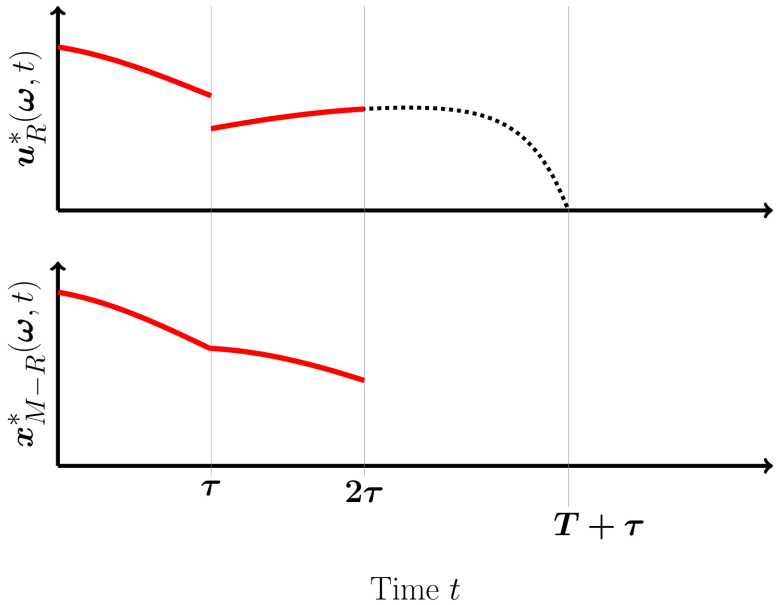
We set $\boldsymbol{x}_{R-M}^*(\boldsymbol{\omega}, t) = \boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}(0))$ on $[0, \tau]$, $\Omega_0 = \boldsymbol{\omega}$.

RBM-MPC



Compute $\boldsymbol{u}_R^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_0, \tau))$ on $[\tau, \tau + T]$.

RBM-MPC

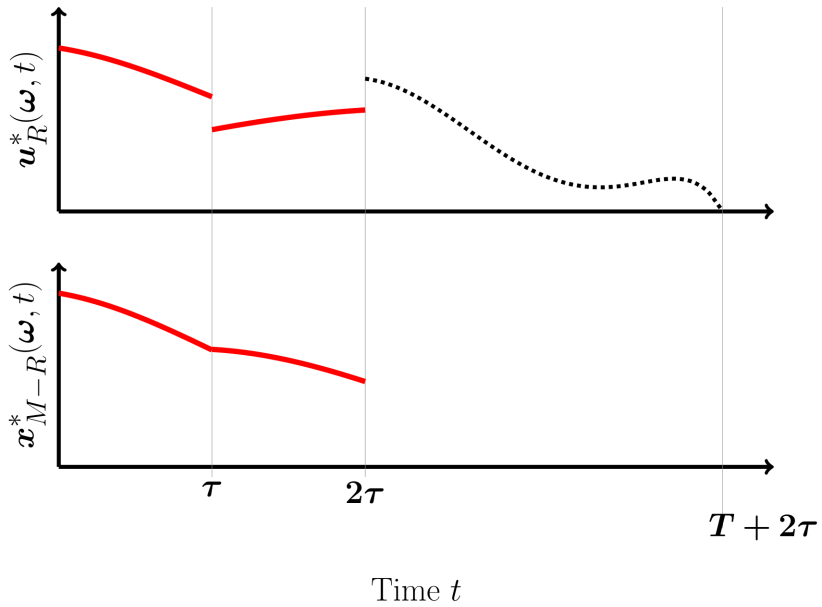


Determine $\boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_0, \tau))$ on $[\tau, 2\tau]$ by solving

$$\dot{\boldsymbol{x}}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_0, \tau)) = A\boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_0, \tau)) + B\boldsymbol{u}_R^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_0, \tau)),$$
$$\boldsymbol{x}_T^*(\boldsymbol{\omega}, \tau, \boldsymbol{x}_{R-M}^*(\Omega_0, \tau)) = \boldsymbol{x}_{R-M}^*(\Omega_0, \tau).$$

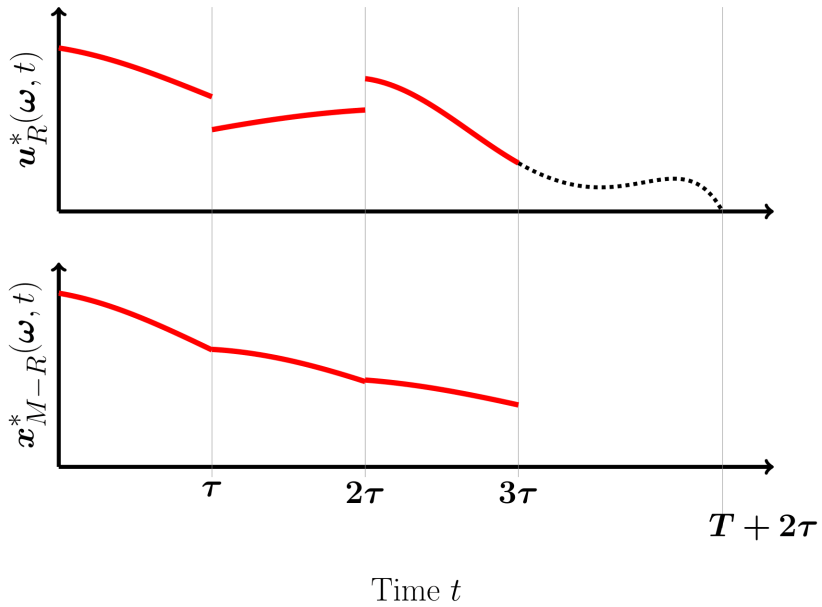
We set $\boldsymbol{x}_{R-M}^*(\boldsymbol{\omega}, \Omega_0, t) = \boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_0, \tau))$ on $[\tau, 2\tau]$, $\Omega_1 = \{\Omega_0, \boldsymbol{\omega}\}$.

RBM-MPC



Compute $\boldsymbol{u}_R^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_1, 2\tau))$ on $[2\tau, 2\tau + T]$.

RBM-MPC

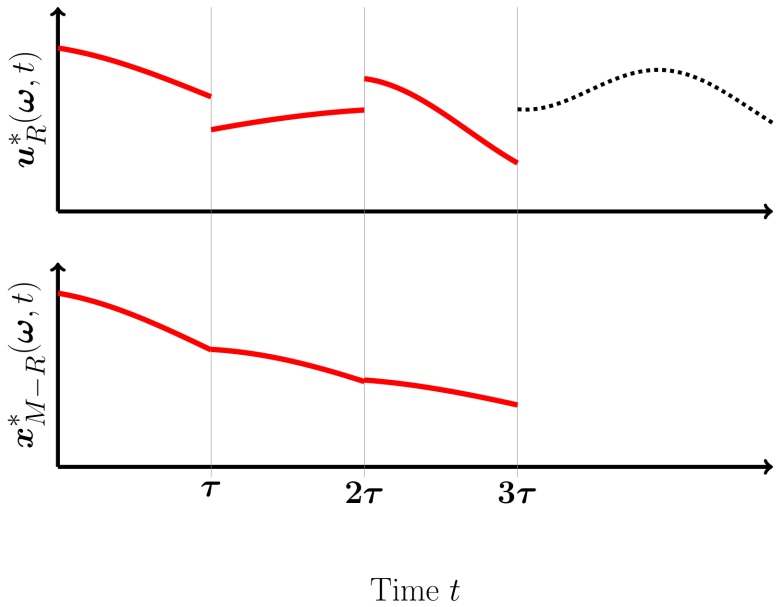


Determine $\boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_M^*(\Omega_1, 2\tau))$ on $[2\tau, 3\tau]$ by solving

$$\begin{aligned} \dot{\boldsymbol{x}}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_1, 2\tau)) &= A\boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_1, 2\tau)) + B\boldsymbol{u}_R^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_1, 2\tau)), \\ \boldsymbol{x}_T^*(\boldsymbol{\omega}, 2\tau, \boldsymbol{x}_{R-M}^*(\Omega_1, 2\tau)) &= \boldsymbol{x}_{R-M}^*(\Omega_1, 2\tau). \end{aligned}$$

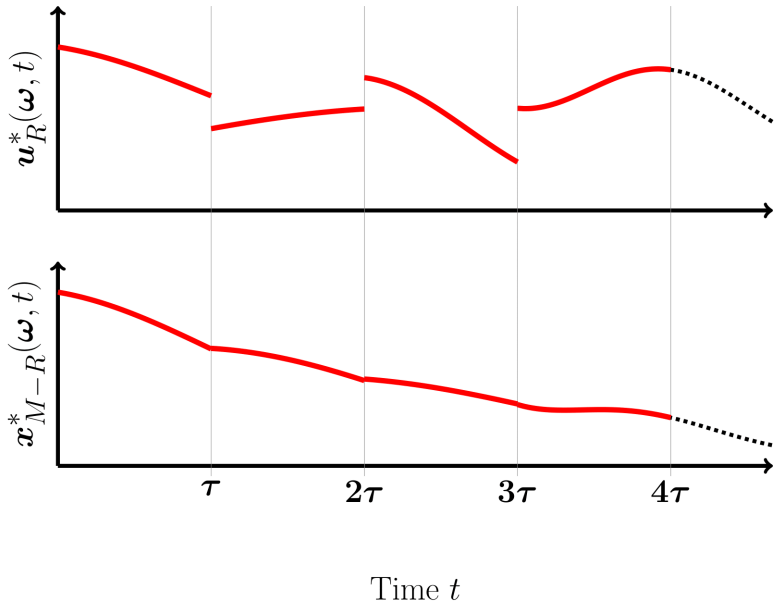
We set $\boldsymbol{x}_{R-M}^*(\boldsymbol{\omega}, \Omega_1, t) = \boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_1, 2\tau))$ on $[2\tau, 3\tau]$, $\Omega_2 = \{\Omega_1, \boldsymbol{\omega}\}$.

RBM-MPC



Compute $\boldsymbol{u}_R^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_2, 3\tau))$ on $[3\tau, 3\tau + T]$.

RBM-MPC



Determine $\boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_2, 3\tau))$ on $[3\tau, 4\tau]$ by solving

$$\begin{aligned} \dot{\boldsymbol{x}}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_2, 3\tau)) &= A\boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_2, 3\tau)) + B\boldsymbol{u}_R^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_2, 3\tau)), \\ \boldsymbol{x}_T^*(\boldsymbol{\omega}, 3\tau, \boldsymbol{x}_{R-M}^*(\Omega_2, 3\tau)) &= \boldsymbol{x}_{R-M}^*(\Omega_2, 3\tau). \end{aligned}$$

We set $\boldsymbol{x}_{R-M}^*(\boldsymbol{\omega}, \Omega_2, t) = \boldsymbol{x}_T^*(\boldsymbol{\omega}, t, \boldsymbol{x}_{R-M}^*(\Omega_2, 3\tau))$ on $[3\tau, 4\tau]$, $\Omega_3 = \{\Omega_2, \boldsymbol{\omega}\}$.

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

where

$$\begin{aligned}\dot{\boldsymbol{x}}^*_\infty(t) &= A\boldsymbol{x}^*_\infty(t) + B\boldsymbol{u}^*_\infty(t)\\ &= A\boldsymbol{x}^*_\infty(t) - BR^{-1}B^T P_\infty \boldsymbol{x}^*_\infty(t)\end{aligned}$$

and

$$\begin{aligned}\dot{\boldsymbol{x}}^*_{R-M}(\boldsymbol{\omega}, t) &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t) + B\boldsymbol{u}^*_R(t)\\ &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t) - BR^{-1}B^T P_R(\boldsymbol{\omega}, t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega}, t).\end{aligned}$$

Here, $\boldsymbol{x}^*_R(\boldsymbol{\omega}, t)$ solves for $t \in (\tau_i, \tau_{i+1}]$, $i \geq 0$:

$$\begin{aligned}\dot{\boldsymbol{x}}^*_R(\boldsymbol{\omega}, t) &= A_h(\boldsymbol{\omega}, t)\boldsymbol{x}^*_R(\boldsymbol{\omega}, t) + B\boldsymbol{u}^*_R(\boldsymbol{\omega}, t)\\ \boldsymbol{x}^*_R(\boldsymbol{\omega}, \tau_i) &= \boldsymbol{x}^*_{R-M}(\tau_i).\end{aligned}$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Using the formula for the dynamics yields

$$\dot{\boldsymbol{e}}(\boldsymbol{\omega}, t) = A(\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t)) + B(\boldsymbol{u}^*_\infty(t) - \boldsymbol{u}^*_R(\boldsymbol{\omega}, t))$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Using the formula for the dynamics yields

$$\begin{aligned}\dot{\boldsymbol{e}}(\boldsymbol{\omega}, t) &= A(\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t)) + B(\boldsymbol{u}^*_\infty(t) - \boldsymbol{u}^*_R(\boldsymbol{\omega}, t)) \\ &= A\boldsymbol{e}(\boldsymbol{\omega}, t) - BR^{-1}B^T(P_\infty \boldsymbol{x}^*_\infty(t) - P_R(\boldsymbol{\omega}, t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega}, t)),\end{aligned}$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Using the formula for the dynamics yields

$$\begin{aligned}
\dot{\boldsymbol{e}}(\boldsymbol{\omega}, t) &= A(\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t)) + B(\boldsymbol{u}^*_\infty(t) - \boldsymbol{u}^*_R(\boldsymbol{\omega}, t)) \\
&= A\boldsymbol{e}(\boldsymbol{\omega}, t) - BR^{-1}B^T(P_\infty \boldsymbol{x}^*_\infty(t) - P_R(\boldsymbol{\omega}, t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega}, t)) \\
&= A_\infty \boldsymbol{e}(\boldsymbol{\omega}, t) - BR^{-1}B^T(P_\infty \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t) - P_R(\boldsymbol{\omega}, t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega}, t))
\end{aligned}$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Using the formula for the dynamics yields

$$\begin{aligned}
\dot{\boldsymbol{e}}(\boldsymbol{\omega}, t) &= A(\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t)) + B(\boldsymbol{u}^*_\infty(t) - \boldsymbol{u}^*_R(\boldsymbol{\omega}, t)) \\
&= A\boldsymbol{e}(\boldsymbol{\omega}, t) - BR^{-1}B^T(P_\infty \boldsymbol{x}^*_\infty(t) - P_R(\boldsymbol{\omega}, t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega}, t)) \\
&= A_\infty \boldsymbol{e}(\boldsymbol{\omega}, t) - BR^{-1}B^T(P_\infty \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t) - P_R(\boldsymbol{\omega}, t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega}, t)) \\
&= A_\infty \boldsymbol{e}(\boldsymbol{\omega}, t) - BR^{-1}B^T(P_\infty - P_R(\boldsymbol{\omega}, t \mod \tau))\boldsymbol{x}^*_R(\boldsymbol{\omega}, t) \\
&\quad + P_\infty(\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t) - \boldsymbol{x}^*_R(\boldsymbol{\omega}, t)).
\end{aligned}$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Applying the variation of constants formula, taking norms and the expected value, yields

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq \int_0^t ||e^{A_\infty(t-s)}||\,||BR^{-1}B^T||\Big(\mathbb{E}(||P_\infty - P_R(s \mod \tau)||\,|\boldsymbol{x}^*_R(s)|) + ||P_\infty||\,\mathbb{E}(|\boldsymbol{x}^*_{R-M}(s) - \boldsymbol{x}^*_R(s)|)\Big)\,ds.$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Applying the variation of constants formula, taking norms and the expected value, yields

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq \int_0^t ||e^{A_\infty(t-s)}||\, ||BR^{-1}B^T|| \Big(\mathbb{E}(||P_\infty - P_R(s \mod \tau)||\, |\boldsymbol{x}^*_R(s)|) + ||P_\infty||\, \mathbb{E}(|\boldsymbol{x}^*_{R-M}(s) - \boldsymbol{x}^*_R(s)|) \Big)\, ds.$$

We have $||e^{A_\infty t}|| \leq M_\infty e^{-\mu_\infty t}$.

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Applying the variation of constants formula, taking norms and the expected value, yields

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq \int_0^t ||e^{A_\infty(t-s)}||\,||BR^{-1}B^T||\Big(\mathbb{E}(||P_\infty - P_R(s \mod \tau)||\,|\boldsymbol{x}^*_R(s)|) + ||P_\infty||\,\mathbb{E}(|\boldsymbol{x}^*_{R-M}(s) - \boldsymbol{x}^*_R(s)|)\Big)\,ds.$$

For $s \in [\tau_i, \tau_{i+1}]$, we have

$$|\boldsymbol{x}^*_R(\boldsymbol{\omega}, s)| \leq |\boldsymbol{x}^*_{R-M}(\tau_i)| + \sqrt{\tau}|B\boldsymbol{u}^*_R|_{L^2(\tau_i,\tau_{i+1})} \leq |\boldsymbol{x}^*_{R-M}(\tau_i)|(1 + \sqrt{\tau T}C).$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Applying the variation of constants formula, taking norms and the expected value, yields

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq \int_0^t ||e^{A_\infty(t-s)}||\,||BR^{-1}B^T|| \Big(\mathbb{E}(||P_\infty - P_R(s \mod \tau)||\,|\boldsymbol{x}^*_R(s)|) + ||P_\infty||\,\mathbb{E}(|\boldsymbol{x}^*_{R-M}(s) - \boldsymbol{x}^*_R(s)|) \Big)\, ds.$$

For $s \in [\tau_i, \tau_{i+1}]$, we have

$$|\boldsymbol{x}^*_R(\boldsymbol{\omega}, s)| \leq |\boldsymbol{x}^*_{R-M}(\tau_i)| + \sqrt{\tau}|B\boldsymbol{u}^*_R|_{L^2(\tau_i,\tau_{i+1})} \leq |\boldsymbol{x}^*_{R-M}(\tau_i)|(1 + \sqrt{\tau T}C)$$

and

$$\begin{aligned} &\mathbb{E}(||P_\infty - P_R(s \mod \tau)||\,|\boldsymbol{x}^*_R(s)|) \\ &\quad \leq \big(||P_\infty - P_T(s \mod \tau)|| + \mathbb{E}(||P_T(s \mod \tau) - P_R(s \mod \tau)||)\big)\mathbb{E}(|\boldsymbol{x}^*_R(s)|) \end{aligned}$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Applying the variation of constants formula, taking norms and the expected value, yields

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq \int_0^t ||e^{A_\infty(t-s)}||\,||BR^{-1}B^T||\Big(\mathbb{E}(||P_\infty - P_R(s \mod \tau)||\,|\boldsymbol{x}^*_R(s)|) + ||P_\infty||\,\mathbb{E}(|\boldsymbol{x}^*_{R-M}(s) - \boldsymbol{x}^*_R(s)|)\Big)\,ds.$$

For $s \in [\tau_i, \tau_{i+1}]$, we have

$$|\boldsymbol{x}^*_R(\boldsymbol{\omega}, s)| \leq |\boldsymbol{x}^*_{R-M}(\tau_i)| + \sqrt{\tau}|B\boldsymbol{u}^*_R|_{L^2(\tau_i,\tau_{i+1})} \leq |\boldsymbol{x}^*_{R-M}(\tau_i)|(1 + \sqrt{\tau T}C)$$

and

$$\mathbb{E}(||P_\infty - P_R(s \mod \tau)||\,|\boldsymbol{x}^*_R(s)|) \leq \Big(\underbrace{||P_\infty - P_T(s \mod \tau)||}_{\leq K_0 e^{-2\mu_\infty(T-\tau)}} + \underbrace{\mathbb{E}(||P_T(s \mod \tau) - P_R(s \mod \tau)||)}_{C_{[Q,R,B,T,A]}(\sqrt{\mathsf{Var}[A]h}+\mathsf{Var}[A]h)}\Big)\mathbb{E}(|\boldsymbol{x}^*_R(s)|)$$

Error Estimate

For $t > 0$, we want to determine the expected error

$$\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|),$$

Applying the variation of constants formula, taking norms and the expected value, yields

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq \int_0^t ||e^{A_\infty(t-s)}||\,||BR^{-1}B^T||\Big(\mathbb{E}(||P_\infty - P_R(s \mod \tau)||\,|\boldsymbol{x}^*_R(s)|) + ||P_\infty||\,\mathbb{E}(|\boldsymbol{x}^*_{R-M}(s) - \boldsymbol{x}^*_R(s)|)\Big)\,ds.$$

For $s \in [\tau_i, \tau_{i+1}]$, we have

$$\mathbb{E}(|\boldsymbol{x}^*_R(s) - \boldsymbol{x}^*_{R-M}(s)|) \leq \sqrt{||A||\tau^2 + 2\tau}\sqrt{h\,\mathsf{Var}[A]}\Big(1 + \sqrt{\tau T}C\Big)\mathbb{E}(|\boldsymbol{x}^*_R(\tau_i)|).$$

Error Estimate

This reduces to

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq C_{[T,\tau,h]} \int_0^t e^{-\mu_\infty(t-s)} \mathbb{E}(|\boldsymbol{x}^*_{R-M}(\tau_s)|)\, ds,$$

- $\tau_s = s - (s \mod \tau)$
- $\lim_{T-\tau\to\infty,\, h\to 0} C_{[T,\tau,h]} \to 0$

Exponential Stability

Do the RBM-MPC dynamics,

$$
\begin{aligned}
\dot{\boldsymbol{x}}^*_{R-M}(\boldsymbol{\omega}, t) &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t) + B\boldsymbol{u}^*_R(\boldsymbol{\omega}, t) \\
&= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, t) - BR^{-1}B^T P_R(\boldsymbol{\omega}, t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega}, t) \\
\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega}, 0) &= \boldsymbol{x}_0,
\end{aligned}
$$

result in a stable trajectory?

Exponential Stability

Do the RBM-MPC dynamics,

$$\begin{aligned}\dot{\boldsymbol{x}}^*_{R-M}(\boldsymbol{\omega},t) &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},t) + B\boldsymbol{u}^*_R(\boldsymbol{\omega},t)\\ &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},t) - BR^{-1}B^TP_R(\boldsymbol{\omega},t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega},t)\\ \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},0) &= \boldsymbol{x}_0,\end{aligned}$$

result in a stable trajectory?

Stability of the RBM-MPC Trajectory

There exists a constant

$$\mu_{R-M} := \mu_{T,\tau} - S_{[T,\tau,h]} = \mu_\infty - M_\infty K_0^{RH} e^{-2\mu_\infty(T-\tau)} - S_{[T,\tau,h]}, \tag{2}$$

such that

$$\mathbb{E}(|\boldsymbol{x}^*_{R-M}(t)|) \leq M_\infty |\boldsymbol{x}_0| e^{-t\,\mu_{R-M}}. \tag{3}$$

For $\mu_{R-M} > 0$, the RBM-MPC trajectory is exponentially stable.

Exponential Stability

Do the RBM-MPC dynamics,

$$\begin{aligned}\dot{\boldsymbol{x}}^*_{R-M}(\boldsymbol{\omega},t) &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},t) + B\boldsymbol{u}^*_R(\boldsymbol{\omega},t)\\ &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},t) - BR^{-1}B^TP_R(\boldsymbol{\omega},t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega},t)\\ \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},0) &= \boldsymbol{x}_0,\end{aligned}$$

result in a stable trajectory?

Stability of the RBM-MPC Trajectory

There exists a constant

$$\mu_{R-M} := \mu_{T,\tau} - S_{[T,\tau,h]} = \mu_\infty - M_\infty K_0^{RH} e^{-2\mu_\infty(T-\tau)} - S_{[T,\tau,h]}, \tag{4}$$

such that

$$\mathbb{E}(|\boldsymbol{x}^*_{R-M}(t)|) \leq M_\infty |\boldsymbol{x}_0| e^{-t\,\mu_{R-M}}. \tag{5}$$

For $\mu_{R-M} > 0$, the RBM-MPC trajectory is stable.

Exponential Stability

Do the RBM-MPC dynamics,

$$\begin{aligned}\dot{\boldsymbol{x}}^*_{R-M}(\boldsymbol{\omega},t) &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},t) + B\boldsymbol{u}^*_R(\boldsymbol{\omega},t)\\ &= A\boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},t) - BR^{-1}B^TP_R(\boldsymbol{\omega},t \mod \tau)\boldsymbol{x}^*_R(\boldsymbol{\omega},t)\\ \boldsymbol{x}^*_{R-M}(\boldsymbol{\omega},0) &= \boldsymbol{x}_0,\end{aligned}$$

result in a stable trajectory?

Stability of the RBM-MPC Trajectory

There exists a constant

$$\mu_{R-M} := \mu_{T,\tau} - S_{[T,\tau,h]} = \mu_\infty - M_\infty K_0^{RH} e^{-2\mu_\infty(T-\tau)} - S_{[T,\tau,h]}, \tag{6}$$

such that

$$\mathbb{E}(|\boldsymbol{x}^*_{R-M}(t)|) \leq M_\infty |\boldsymbol{x}_0| e^{-t\,\mu_{R-M}}. \tag{7}$$

For $\mu_{R-M} > 0$, the RBM-MPC trajectory is stable.

- $\lim_{h\to 0} S_{[T,\tau,h]} = O(\sqrt{h})$

Convergence to x^*_∞

For $\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|)$ we found

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq C_{[T,\tau,h]} \int_0^t e^{-\mu_\infty(t-s)} \mathbb{E}(|\boldsymbol{x}^*_{R-M}(\tau_s)|)\, ds,$$

with $\tau_s = s - (s \mod \tau)$.

$$\mathbb{E}(|\boldsymbol{x}^*_{R-M}(\tau_s)|) \leq M_\infty |\boldsymbol{x}_0| e^{-(s-(s \mod \tau))\,\mu_{R-M}}.$$

Convergence to x^*_∞

For $\mathbb{E}(|\boldsymbol{e}(t)|) := \mathbb{E}(|\boldsymbol{x}^*_\infty(t) - \boldsymbol{x}^*_{R-M}(t)|)$ we found

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq C_{[T,\tau,h]} \int_0^t e^{-\mu_\infty(t-s)} \mathbb{E}(|\boldsymbol{x}^*_{R-M}(\tau_s)|)\, ds,$$

with $\tau_s = s - (s \mod \tau)$.

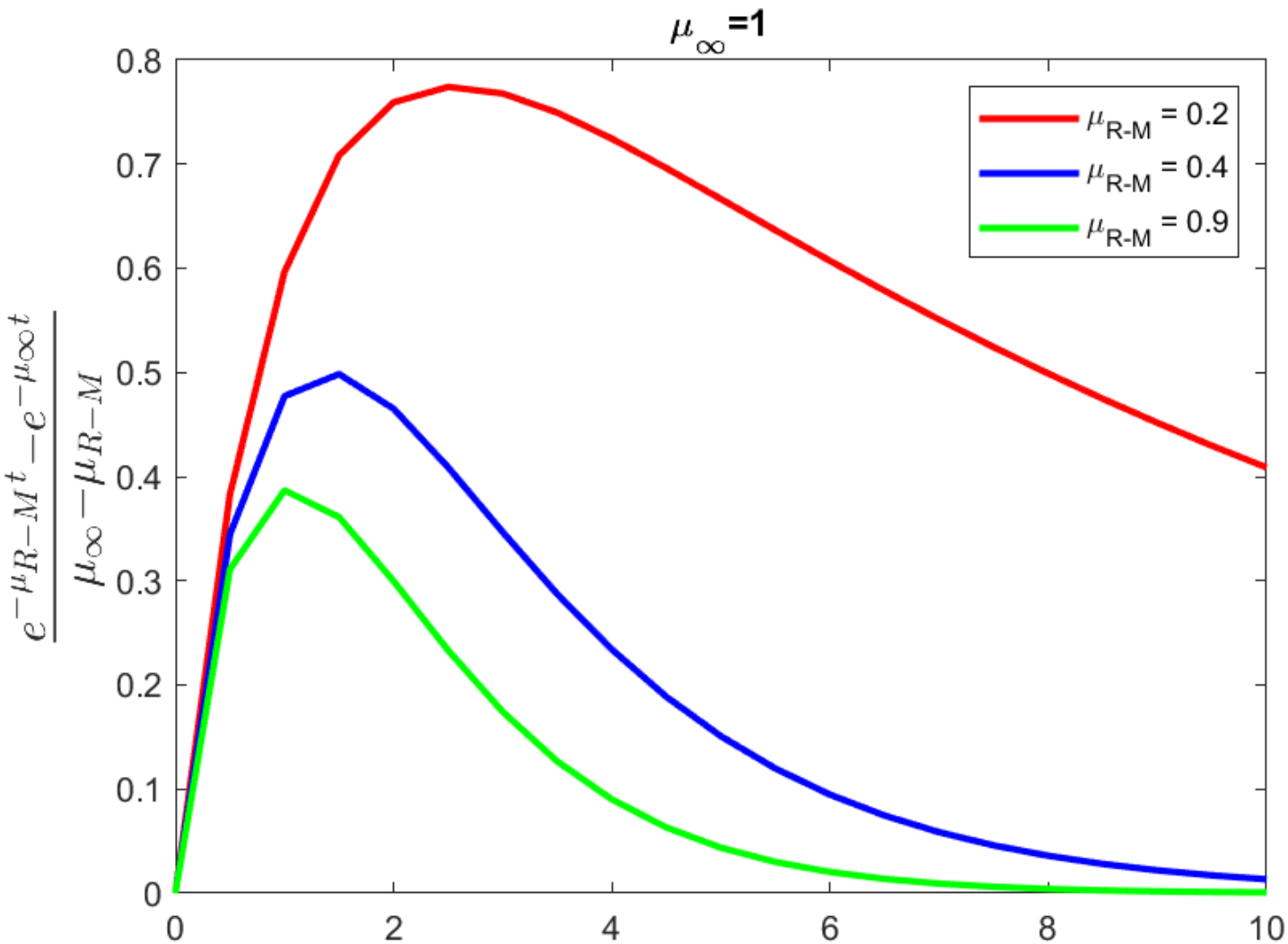
$$\mathbb{E}(|\boldsymbol{x}^*_{R-M}(\tau_s)|) \leq M_\infty |\boldsymbol{x}_0| e^{-(s-(s \mod \tau))\,\mu_{R-M}}.$$

Convergence of the RBM-MPC Trajectory

There exists a $C_{[T,\tau,h]} > 0$ such that

$$\mathbb{E}(|\boldsymbol{e}(t)|) \leq C_{[T,\tau,h]} M_\infty |\boldsymbol{x}_0| e^{\mu_{R-M}\,\tau} \frac{e^{-\mu_{R-M} t} - e^{-\mu_\infty t}}{\mu_\infty - \mu_{R-M}}. \tag{8}$$

Convergence to x^*_∞



5 Numerical Illustration

The 1-D Wave Equation in the LQ Optimal Control Setting

Wave Equation with a control term $u(t) \in L^2([0,T];\mathbb{R})$

$$\frac{\partial^2 s(x,t)}{\partial t^2} = c^2 \frac{\partial^2 s(x,t)}{\partial x^2} + \delta(x-L)u(t),$$

- $\Omega = [0, L]$
- $0 \leq t \leq T$
- $s : \Omega \times [0,T] \to \mathbb{R}$: vertical deflection.

The 1-D Wave Equation in the LQ Optimal Control Setting

Wave Equation with a control term $u(t) \in L^2([0,T];\mathbb{R})$

$$\frac{\partial^2 s(x,t)}{\partial t^2} = c^2 \frac{\partial^2 s(x,t)}{\partial x^2} + \delta(x-L)u(t),$$

- $\Omega = [0, L]$
- $0 \leq t \leq T$
- $s : \Omega \times [0,T] \to \mathbb{R}$: vertical deflection.

The Spatial Discretization yields the linear standard form:

$$\dot{\boldsymbol{x}}(t) = \underbrace{\begin{pmatrix} 0_{N_x,N_x} & I_{N_x,N_x} \\ c^2 \Delta_D & 0_{N_x,N_x} \end{pmatrix}}_{A} \boldsymbol{x}(t) + \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{B} u(t), \quad \text{with } \boldsymbol{x}(t) = \begin{pmatrix} s_1(t) \\ \dots \\ s_{N_x}(t) \\ \dot{s}_1(t) \\ \dots \\ \dot{s}_{N_x}(t) \end{pmatrix}$$

The 1-D Wave Equation in the LQ Optimal Control Setting

We set $N_x = 11$ and decompose the matrix A as

$$A = \begin{pmatrix} 0_{N_x,N_x} & I_{N_x,N_x} \\ c^2 \Delta_D & 0_{N_x,N_x} \end{pmatrix} = \underbrace{\begin{pmatrix} 0_{N_x,N_x} & \frac{1}{2} I_{N_x,N_x} \\ c^2 \Delta_{D,1} & 0_{N_x,N_x} \end{pmatrix}}_{A_1} + \underbrace{\begin{pmatrix} 0_{N_x,N_x} & \frac{1}{2} I_{N_x,N_x} \\ c^2 \Delta_{D,2} & 0_{N_x,N_x} \end{pmatrix}}_{A_2}, \tag{9}$$

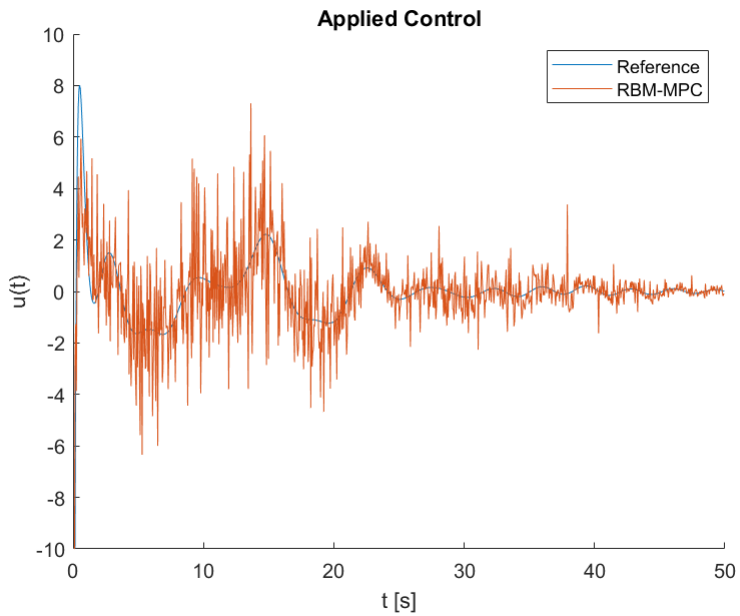
with

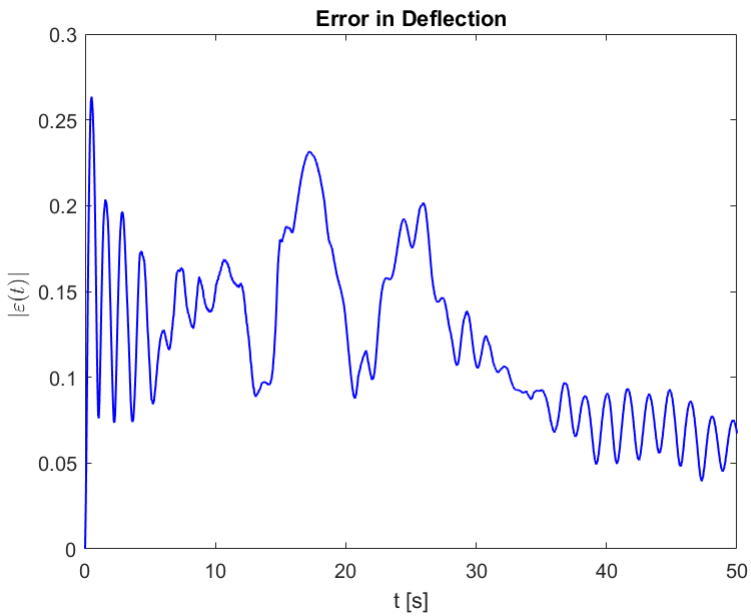
$$\Delta_{D,1} = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 2 & & & & & & & \\ 1 & -2 & 1 & & & & & & \\ & & \ddots & & & & & & \\ & & 1 & -2 & 1 & & & & \\ & & & 1 & -1 & 0 & & & \\ & & & & 0 & 0 & 0 & & \\ & & & & & & \ddots & & \\ & & & & & 0 & 0 & 0 & \\ & & & & & & & 0 & 0 \end{pmatrix} \qquad \Delta_{D,2} = \frac{1}{\Delta x^2} \begin{pmatrix} 0 & 0 & & & & & & & \\ 0 & 0 & 0 & & & & & & \\ & & \ddots & & & & & & \\ & & 0 & 0 & 0 & & & & \\ & & & 0 & -1 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 2 & -2 \end{pmatrix}.$$

To the subsets $\{A_1\}$ and $\{A_2\}$, we assign each a possibility of $\frac{1}{2}$ and to the subsets $\{A_1, A_2\}$ and $\emptyset$ the probability 0.

Simulation

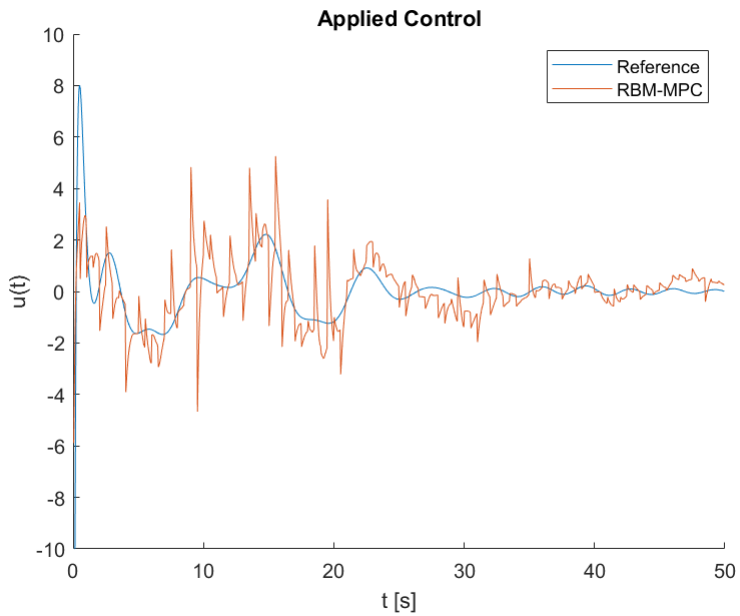
Applied Control and Error of the Trajectory w.r.t x_∞^*

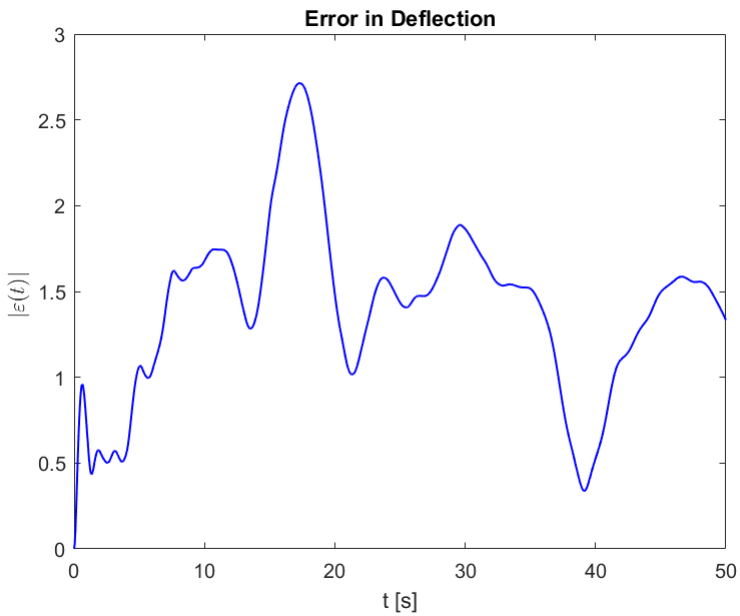




Here: $h = \tau = \Delta t = 0.050$ sec, $T - \tau = 9.950$ sec.

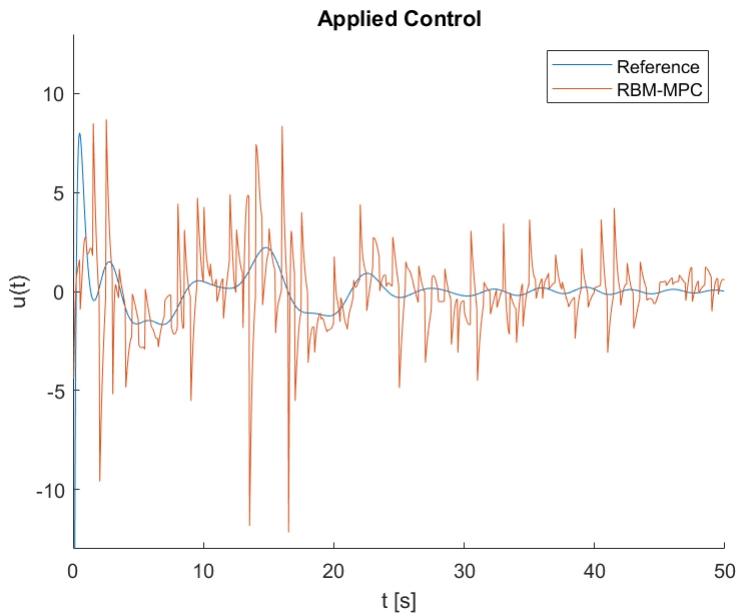
Applied Control and Error of the Trajectory w.r.t x_∞^*

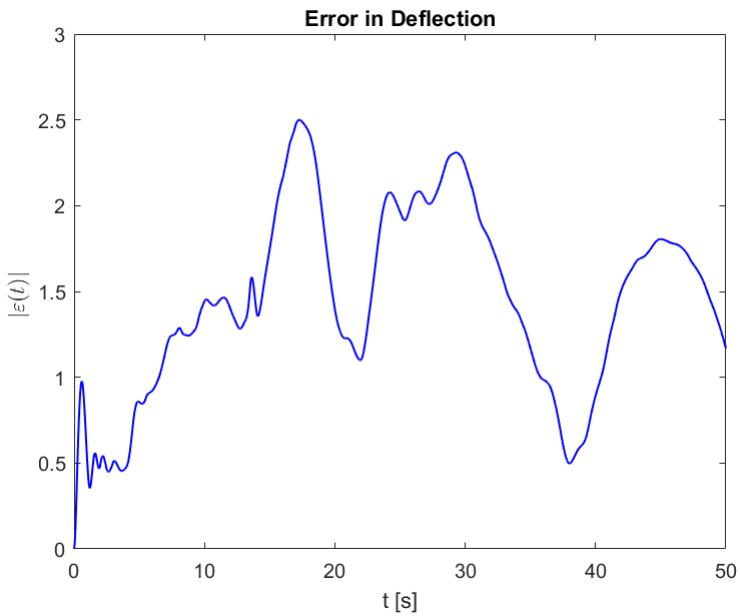




Here: $h = \Delta t = 0.050$ sec, $T - \tau = 9.50$ sec.

Applied Control and Error of the Trajectory w.r.t x_∞^*





Here: $h = 5\Delta t = 0.250$ sec, $T - \tau = 9.50$ sec.

6 Perspectives

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- Numerical Convergence Rates
- Possible Extension:
  - Non-linear Dynamics
  - Constraints on Control or State
  - Decomposition of the matrix B

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Thank you for your attention!