Fefferman-Stein theory for oscillating singular integrals

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Many many motivations...

- A-priori-estimates for elliptic problems (e.g. Laplace/Poisson type equations) on compact manifolds on $\mathbb{R}^n$ (via estimates for pseudo-differential operators/singular integrals).
  - Ennio De Giorgi, John Nash, Alberto Calderón, Lars Gårding, Lars Hörmander, Charles Fefferman.
- A-priori-estimates for hyperbolic problems (e.g. wave type equations) on compact manifolds or on $\mathbb{R}^n$ (via estimates for Fourier integral operators).
  - Andreas Seeger, Christopher Sogge, Elias Stein, Terence Tao, Michael Ruzhansky.

Elliptic/Hyperbolic/Parabolic problems

- Elliptic problems on a closed manifold M, or on $M = \mathbb{R}^n$.

$$(\mathrm{I}) : P(x, D)u = f, \;\; P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \tag{1}$$

  where the symbol (polynomial associated to the operator $P(x, D)$) $P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha \neq 0$ for $\xi \neq 0$.

- Wave equation for the Laplacian and for (fractional) Laplacians

$$(\mathrm{II}) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta)^\theta u, & u \in \mathscr{D}'([0, T] \times M) \\ u(0, x) = f_0, u_t(0, x) = f_1 \end{cases}, \tag{2}$$

  where $0 < \theta \leq 1$. Here, $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ for $M = \mathbb{R}^n$.

Elliptic/Hyperbolic/Parabolic problems

- Elliptic problems arise generalising the Poisson equation on a closed manifold M, or on $M = \mathbb{R}^n$.

$$(\mathrm{I}) : \Delta u = f, \;\; \Delta(x,\xi) = -|\xi|^2 := -(\xi_1^2 + \cdots + \xi_n^2). \tag{3}$$

  Note that $\Delta(x,\xi) = -|\xi|^2 \neq 0$ for $\xi \neq 0$.

- Wave equation for the Laplacian and for (fractional) Laplacians

$$(\mathrm{II}) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta)^{\theta} u, & u \in \mathscr{D}'([0,T] \times M) \\ u(0,x) = f_0, u_t(0,x) = f_1 \end{cases}, \tag{4}$$

  where $0 \leq \theta \leq 1$.

Let us analyse the elliptic case

- Consider the Poisson equation on a closed manifold $M = G$ with enough symmetries, or on $M = \mathbb{R}^n$. For instance, assume that G is a compact Lie group.
$$(\mathrm{I}) : \Delta u = f, \;\; \Delta(x,\xi) = -|\xi|^2 := -(\xi_1^2 + \cdots + \xi_n^2). \tag{5}$$
Note that $\Delta(x,\xi) = -|\xi|^2 \neq 0$ for $\xi \neq 0$.
- There exists a distribution k on M such that
$$\Delta u(x) = u * k(x) = \int_G u(xy^{-1})k(y)dy.$$
- The Laplacian Δ has a fundamental solution E which means that $k * E = \delta$. The distribution E gives the invertibility of Δ. Indeed,
$$\Delta u * E = u * k * E = u * \delta = u.$$
- We solve the Poisson equation
$$u = \Delta u * E = f * E.$$

Let us estimate the solution in Sobolev spaces modeled on L^p-spaces

$$L^p(G) = \{f : \|f\|_{L^p} := (\int_G |f(x)|^p dx)^{\frac{1}{p}} < \infty\},\ 1 < p < \infty.$$

For $s \in \mathbb{R}$, the Sobolev space $W^{s,p}(G)$ is defined via

$$W^{s,p}(G) = \{g : \|g\|_{W^{s,p}} := \|(1-\Delta)^{\frac{s}{2}} g\|_{L^p} < \infty\},\ s \in \mathbb{R}.$$

For any s, there is a distribution k_s such that

$$(1-\Delta)^{\frac{s}{2}} g(x) = g * k_s(x) = \int_G g(xy^{-1}) k_s(y) dy.$$

For $s = 2$, observe that, for the Poisson equation $\Delta u = f$, with $u = f * E$,

$$\|u\|_{W^{2,p}} = \|(1-\Delta)^{\frac{2}{2}} u\|_{L^p} = \|(1-\Delta)^{\frac{2}{2}} (f * E)\|_{L^p} = \|f * E * k_2\|_{L^p}$$

Let us estimate the solution in L^p-spaces

$$\|u\|_{W^{2,p}} = \|(1-\Delta)^{\frac{2}{2}}(f*E)\|_{L^p} = \|f*E*k_2\|_{L^p}$$

It is not a trivial fact, but in any case, a wonderful fact, that the kernel

$$K = E * k_2$$

satisfies the Hörmander condition

$$\sup_{0<R\leq 1}\sup_{|y|\leq R}\int_{|x|\geq 2R}|K(y^{-1}x)-K(x)|dx < \infty, \tag{6}$$

and the convolution operator T, defined by,

$$g \mapsto (Tg)(x) := g*K(x) := \int_G f(xy^{-1})K(y)dy : L^2(G) \to L^2(G), \tag{7}$$

is bounded (that is $\|Tg\|_{L^2(G)} \leq C\|g\|_{L^2(G)}$).

Fundamental theorem for singular integral operators

- Hörmander, L. Estimates for translation invariant operators in Lp spaces, Acta Math., 104, 93–139, (1960).
- Coifman, R., De Guzmán, M. Singular integrals and multipliers on homogeneous spaces. Rev. Un. Mat. Argentina, 137–143, (1970) + Coifman, R., Weiss, G. Analyse harmonique non-commutative sur certains espaces homogénes. (French). Lecture Notes in Mathematics, Vol. 242, 1971. v+160.

Theorem (Hörmander, Coifman, Weiss, De Guzmán)

Assume that the convolution operator T, defined by,

$$g \mapsto (Tg)(x) := g * K(x) := \int_G f(xy^{-1})K(y)dy : L^2(G) \to L^2(G), \qquad (8)$$

is bounded (that is $\|Tg\|_{L^2(G)} \leq C\|g\|_{L^2(G)}$), and its kernel satisfies

$$\sup_{0<R\leq 1} \sup_{|y|\leq R} \int_{|x|\geq 2R} |K(y^{-1}x) - K(x)|dx < \infty. \qquad (9)$$

Then $T : L^p(G) \to L^p(G)$ is bounded for all $1 < p < \infty$.

Applying the Fundamental theorem to the Poisson equation...

Theorem (Hörmander, Coifman, Weiss, De Guzmán)

Assume that the convolution operator T, defined by,

$$g \mapsto (Tg)(x) := g * K(x) := \int_G f(xy^{-1})K(y)dy : L^2(G) \to L^2(G), \tag{10}$$

is bounded (that is $\|Tg\|_{L^2(G)} \leq C\|g\|_{L^2(G)}$), and its kernel satisfies

$$\sup_{0<R\leq 1} \sup_{|y|\leq R} \int_{|x|\geq 2R} |K(y^{-1}x) - K(x)|dx < \infty. \tag{11}$$

Then $T : L^p(G) \to L^p(G)$ is bounded for all $1 < p < \infty$.

Example

For the Poisson equation $\Delta u = f$, with $u = f * E$, and $K = E * k_2$,

$$\|u\|_{W^{2,p}} = \|f * E * k_2\|_{L^p} = \|f * K\|_{L^p} = \|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

Conclusion

Example

For the Poisson equation $\Delta u = f$, with $u = f * E$, and $K = E * k_2$,

$$\|u\|_{W^{2,p}} = \|f * E * k_2\|_{L^p} = \|f * K\|_{L^p} = \|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

Moreover, for any $s \in \mathbb{R}$,

$$\|u\|_{W^{2+s,p}} = \|f * E * k_2\|_{L^p} = \|f * K\|_{L^p} = \|Tf\|_{L^p} \leq C\|f\|_{W^{s,p}}.$$

This *elliptic regularity theorem* can be extended to any elliptic differential operator

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$$

of order $m \in \mathbb{N}$ as follows. For the problem $P(x, D)u = f$, one has

$$\|u\|_{W^{m+s,p}} \leq C(\|f\|_{W^{s,p}} + \|u\|_{W^{t,p}}), \; t \in \mathbb{R}.$$

Summarizing

- Harmonic analysis and the study of singular integrals, that are, convolution operators $f \mapsto Tf = f * K$, with kernels satisfying conditions of the type

$$\sup_{0<R\leq 1} \sup_{|y|\leq R} \int_{|x|\geq 2R} |K(y^{-1}x) - K(x)|dx < \infty, \tag{12}$$

  are very good tools for obtaining qualitative properties of elliptic differential problems.

About the wave equation

- In the analysis of the wave equation associated to the Laplacian or to the fractional Laplacian

$$(\mathrm{II}): \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta)^{\theta} u, & u \in \mathscr{D}'([0,T]\times G) \\ u(0,x) = f_0, u_t(0,x) = f_1 \end{cases}, \tag{13}$$

where $0 \leq \theta < 1$, is necessary to study the L^p-boundedness of convolution operators

$$f \mapsto Tf := f * K,$$

where the kernel K satisfies the following oscillating condition

$$\sup_{0<R\leq 1} \sup_{|y|\leq R} \int_{|x|\geq 2R^{1-\theta}} |K(y^{-1}x) - K(x)|dx < \infty. \tag{14}$$

Fefferman and Stein, 1970-1972. Acta Math.

Consider that K satisfies the F.T. condition

$$|\widehat{K}(\xi)| = O((1+|\xi|)^{-\frac{n\theta}{2}}), \quad 0 \leq \theta < 1, \ \widehat{K}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} K(x)dx. \tag{15}$$

Theorem (Fefferman and Stein)

Let T be a convolution operator with a temperate distribution K of compact support and let $0 \leq \theta < 1$. Assume that $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies (23) and the oscillating Hörmander condition

$$\sup_{0\leq R<1} \sup_{|y|<R} \int_{|x|\geq 2R^{1-\theta}} |K(x-y) - K(x)|dx < \infty. \tag{16}$$

Then $T : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ is bounded. Moreover, $T : H^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is bounded where $H^1(\mathbb{R}^n)$ denotes the Hardy space.

- $\|g\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda|\{x : |g(x)| > \lambda\}| < \infty.$
- $\|g\|_{H^1(\mathbb{R}^n)} := \| \sup_{\lambda>0} |e^{-\lambda\Delta} g| \|_{L^1(\mathbb{R}^n)} < \infty.$

By applying Fefferman and Stein theorem to the fractional wave equation...

- Let $s \in \mathbb{R}$. The wave equation for the fractional Laplacian $(-\Delta)^{\theta}$

$$(\text{II}) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta)^{\theta} u, & u \in \mathscr{D}'([0,T] \times \mathbb{R}^n) \\ u(0,x) = f_0, u_t(0,x) = f_1 \end{cases}, \quad (17)$$

where $0 \leq \theta < 1$, satisfies the a-priori-estimates

$$\|u(x,t)\|_{L^{1,\infty}_s} := \|(1-\Delta)^{\frac{s}{2}} u(x,t)\|_{L^{1,\infty}} \leq C_t \left(\|f_0\|_{L^{1,\infty}_{s+\frac{n\theta}{2}}} + \|f_1\|_{L^{1,\infty}_{s+\theta(\frac{n}{2}+1)}} \right), \quad (18)$$

and

$$\|u(x,t)\|_{H^1_s} := \|(1-\Delta)^{\frac{s}{2}} u(x,t)\|_{H^1} \leq C_t \left(\|f_0\|_{H^1_{s+\frac{n\theta}{2}}} + \|f_1\|_{H^1_{s+\theta(\frac{n}{2}+1)}} \right). \quad (19)$$

Remarks

- The proof of these a priori estimates are based on the representation of the solution

$$u(x,t) = e^{it\sqrt{-\Delta}^{\theta}} f_{+} + e^{it\sqrt{-\Delta}^{\theta}} f_{-} \tag{20}$$

where

$$f_{+} := \frac{1}{2}(f_0 - i\sqrt{-\Delta}^{\theta} f_1),\ f_{-} := \frac{1}{2}(f_0 + i\sqrt{-\Delta}^{\theta} f_1).$$

Indeed, one re-writes the solution as follows

$$u(x,t) = (1-\Delta)^{\frac{m}{2}} A_t f_{+} + (1-\Delta)^{\frac{m}{2}} A_t f_{-},\ m = n\theta/2, \tag{21}$$

where the time-dependent kernel K_t of the (pseudo-differential) operator

$$A_t := (1-\Delta)^{-\frac{m}{2}} e^{it\sqrt{-\Delta}^{\theta}},\ 0 \leq \theta < 1. \tag{22}$$

Fefferman and Stein Theorem for compact Lie groups

Let G be a compact Lie group of dimension n. Consider that $K \in L^1_{\text{loc}}(G)$ satisfies the F.T. condition

$$\|\widehat{K}(\xi)\|_{\text{op}} = O((1+|\xi|)^{-\frac{n\theta}{2}}), \quad 0 \leq \theta < 1, \ \widehat{K}(\xi) = \int_G \xi(x)^* K(x)dx. \tag{23}$$

Theorem (C+Ruzhansky, 2022)

Let T be a convolution operator with a temperate distribution K of compact support and let $0 \leq \theta < 1$. Assume that $K \in L^1_{\text{loc}}(G \setminus \{0\})$ satisfies (23) and the oscillating Hörmander condition

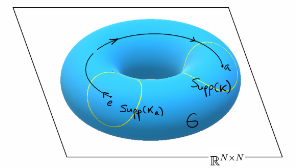
$$\sup_{0 \leq R < 1} \sup_{|y|<R} \int_{|x| \geq 2R^{1-\theta}} |K(x-y) - K(x)| dx < \infty. \tag{24}$$

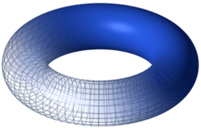
Then $T : L^1(G) \to L^{1,\infty}(G)$ is bounded. Moreover, $T : H^1(G) \to L^1(G)$ is bounded where $H^1(G)$ denotes the Hardy space on G.

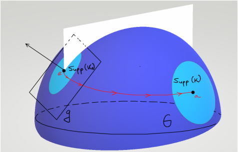
- $\|g\|_{L^{1,\infty}(G)} := \sup_{\lambda>0} \lambda |\{x : |g(x)| > \lambda\}| < \infty.$
- $\|g\|_{H^1(G)} := \| \sup_{\lambda>0} |e^{-\lambda\Delta} g| \|_{L^1(G)} < \infty.$

About compact Lie groups (our setting)

- Lie groups = manifolds with symmetries.
- compact Lie groups = are diffeomorphic to closed subgroups of $\mathrm{U}(N) = \{M \in \mathbb{C}^{N\times N} : M^* = M^{-1}\}$ for N large enough.







- Examples : the torus $\mathbb{T}^n \cong (\mathbb{R}/\mathbb{Z})^n$, Linear Lie groups (groups of matrices), $\mathrm{SU}(n)$, $\mathrm{SO}(n)$, etc. In particular, $\mathrm{SU}(2) \cong \mathbb{S}^3$.
- If M is a closed, connected and simply connected, then $M \cong \mathbb{S}^3$ (the Poincaré conjecture proved by Perelman). Our approach (with Turunen and Wirth, and with Cardona) induces global pseudo-differential theories on M.

By applying Fefferman and Stein theorem on compact Lie groups to the fractional wave equation...

- Let $s \in \mathbb{R}$. The wave equation for the fractional Laplacian $(-\Delta)^{\theta}$

$$(\mathrm{II}) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta_G)^{\theta} u, & u \in \mathscr{D}'([0,T] \times G) \\ u(0,x) = f_0,\, u_t(0,x) = f_1 \end{cases}, \tag{25}$$

where $0 \leq \theta < 1$, satisfies the a-priori-estimates

$$\|u(x,t)\|_{L^{1,\infty}_s} := \|(1-\Delta_G)^{\frac{s}{2}} u(x,t)\|_{L^{1,\infty}} \leq C_t \left(\|f_0\|_{L^{1,\infty}_{s+\frac{n\theta}{2}}} + \|f_1\|_{L^{1,\infty}_{s+\theta(\frac{n}{2}+1)}} \right), \tag{26}$$

and

$$\|u(x,t)\|_{H^1_s} := \|(1-\Delta_G)^{\frac{s}{2}} u(x,t)\|_{H^1} \leq C_t \left(\|f_0\|_{H^1_{s+\frac{n\theta}{2}}} + \|f_1\|_{H^1_{s+\theta(\frac{n}{2}+1)}} \right). \tag{27}$$

Remark

- Cardona, D. Ruzhansky, M. Oscillating singular integral operators on compact Lie groups revisited. submitted. arXiv :2202.10531.
- Delgado, J. Ruzhansky, M. Lp bounds for pseudo-differential operators on compact Lie groups, J. Inst. Math. Jussieu, 18, no. 3, 531-559, 2019.
- Cardona, D., Delgado, J., Ruzhansky, M. Lp-bounds for pseudo-differential operators on graded Lie groups. J. Geom. Anal. Vol. 31, 11603-11647, (2021). arXiv :1911.03397

The case $\theta = 1$ (the hyperbolic case)

$$(\text{II}) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u, & u \in \mathscr{D}'([0,T] \times \mathbb{R}^n) \\ u(0,x) = f_0, u_t(0,x) = f_1 \end{cases}. \tag{28}$$

The solution is given by

$$u(x,t) = e^{it\sqrt{-\Delta}} f_+ + e^{it\sqrt{-\Delta}} f_- \tag{29}$$

where

$$f_+ := \frac{1}{2}(f_0 - i\sqrt{-\Delta}^{\theta} f_1),\ f_- := \frac{1}{2}(f_0 + i\sqrt{-\Delta}^{\theta} f_1).$$

Indeed, one re-writes the solution as follows

$$u(x,t) = (1-\Delta)^{\frac{m}{2}} A_t f_+ + (1-\Delta)^{\frac{m}{2}} A_t f_-,\ m = (n-1)/2, \tag{30}$$

where the time-dependent (Lagrangian distribution) K_t of the (Fourier integral) operator

$$A_t := (1-\Delta)^{-\frac{m}{2}} e^{it\sqrt{-\Delta}}, \tag{31}$$

satisfies...

Seeger, Sogge, Stein, 1991 (Annals of Math.) + Terence Tao, 2004 (J. Aust. Math. Soc.)

Theorem (Seeger, Sogge, Stein + Tao)

Let us consider for any $t > 0$, the Fourier integral operator

$$A_t := (1-\Delta)^{-\frac{m}{2}} e^{it\sqrt{-\Delta}}. \tag{32}$$

Then, $A_t : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ is bounded. Moreover, $A_t : H^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is bounded where $H^1(\mathbb{R}^n)$ denotes the Hardy space.

- $\|g\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda|\{x : |g(x)| > \lambda\}| < \infty.$
- $\|g\|_{H^1(\mathbb{R}^n)} := \| \sup_{\lambda>0} |e^{-\lambda\Delta} g| \|_{L^1(\mathbb{R}^n)} < \infty.$

By applying Seeger, Sogge and Stein+ Terry Tao's theorem ..

- Let $s \in \mathbb{R}$. The wave equation

$$(\mathrm{II}) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u, & u \in \mathscr{D}'([0,T] \times \mathbb{R}^n) \\ u(0,x) = f_0, u_t(0,x) = f_1 \end{cases}, \quad (33)$$

satisfies the a-priori-estimates

$$\|u(x,t)\|_{L^{1,\infty}_s} := \|(1-\Delta)^{\frac{s}{2}} u(x,t)\|_{L^{1,\infty}} \leq C_t \left(\|f_0\|_{L^{1,\infty}_{s+\frac{n-1}{2}}} + \|f_1\|_{L^{1,\infty}_{s+\frac{n+1}{2}}} \right), \quad (34)$$

and

$$\|u(x,t)\|_{H^1_s} := \|(1-\Delta)^{\frac{s}{2}} u(x,t)\|_{H^1} \leq C_t \left(\|f_0\|_{H^1_{s+\frac{n-1}{2}}} + \|f_1\|_{H^1_{s+\frac{n+1}{2}}} \right). \quad (35)$$

Sharpness of these estimates for general classes of Fourier integral operators

- Ruzhansky, M. On the sharpness of Seeger-Sogge-Stein orders. Hokkaido Math. J., 28, no. 2, 357–362, 1999.
- Cardona, D., Ruzhansky, M. Sharpness of Seeger-Sogge-Stein orders for the weak (1,1) boundedness of Fourier integral operators., to appear in, Archiv der Mathematik, 2022.

Final remarks

- Harmonic analysis (the sudy of the Fourier transform on Euclidean and non-Euclidean structures) is a powerful, malleable tool that can be shaped and used differently by various analysts (and non-analysts) to analize elliptic, subelliptic, hyperbolic and parabolic problems (using a-priori-estimates, Carleman estimates, etc).
-
  - Cardona, D. Ruzhansky, M. Oscillating singular integral operators on compact Lie groups revisited. submitted.
  - Cardona, D. Ruzhansky, M. Boundedness of oscillating singular integrals on Lie groups of polynomial growth, submitted.
  - Cardona, D. Ruzhansky, M. [v1 : Weak (1,1) continuity and Lp-theory for oscillating singular integral operators],[v2 : Oscillating singular integral operators on graded Lie groups revisited], submitted.
  - Cardona, D., Ruzhansky, M. Sharpness of Seeger-Sogge-Stein orders for the weak (1,1) boundedness of Fourier integral operators., to appear in Archiv der Mathematik arXiv :2104.09695
  - Cardona, D., Delgado, J., Ruzhansky, M. Lp-bounds for pseudo-differential operators on graded Lie groups. J. Geom. Anal. Vol. 31, 11603-11647, (2021).

Thank you for your attention !

- https ://sites.google.com/site/duvancardonas/home
- Thank you ! Gracias ! eskerrik asko ! milesker ! mila esker ! esker mila ! esker aunitz !