Controllability on some PDEs with dynamic boundary conditions

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Control results of PDEs with dynamic boundary conditions

Introduction

In this talk, for simplicity, we restrict our attention to the case of **Cubic Ginzburg-Landau equation with dynamic boundary conditions** [2]:

$$\begin{cases} \partial_t u - a(1+\alpha i)\Delta u + f(u) = \mathbb{1}_\omega h & \text{in } \Omega \times (0,T), \\ \partial_t u_\Gamma + a(1+\alpha i)\partial_\nu u - b(1+\alpha i)\Delta_\Gamma u_\Gamma + f(u_\Gamma) = 0 & \text{on } \Gamma \times (0,T), \\ u\big|_\Gamma = u_\Gamma & \text{on } \Gamma \times (0,T), \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma, \end{cases}$$

where

- $i := \sqrt{-1}$ is the imaginary unit.
- ∂_ν denotes the normal derivative operator,
- Δ_Γ is the Laplace-Beltrami operator acting on Γ,
- $f(w) = c(1+\gamma i)|w|^2 w$,
- $a, b > 0$, $c \neq 0$, $\alpha, \gamma \in \mathbb{R}$.

[2] Carreño, N., Mercado, A., & Morales, R. (2025). Local null controllability of a cubic Ginzburg-Landau equation with dynamic boundary conditions. (to appear in Journal of Evolution Equations) arXiv:2301.03429

The objective is to establish the **local null controllability** in X of this system: $\exists\, \delta > 0$ such that, for every initial state $(u_0, u_{\Gamma,0}) \in X$ which fulfills

$$\|(u_0, u_{\Gamma,0})\|_X \leq \delta,$$

we can find a control $h \in L^2(\omega \times (0,T))$ such that the solution (u, u_Γ) fulfills

$$u(\cdot, T) = 0 \text{ in } \Omega, \quad u_\Gamma(\cdot, T) = 0 \text{ on } \Gamma.$$

Remarks

- The control h acts only on a portion of the domain Ω. Therefore, the second equation needs to be controlled by the side condition $u\big|_{\Gamma} = u_{\Gamma}$ on $\Gamma \times (0, T)$.
- We proved a **local null controllability result** [3] in the case of $d = 2$ or $d = 3$ in the space $\mathbb{H}^1$, where

$$\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) \,:\, y\big|_{\Gamma} = y_\Gamma\}.$$

- To obtain this result, we first deduce a null controllability result for the associated adjoint linear system by duality **(observability inequality)**. Then, by an **inverse mapping theorem**, we deduce the local null controllability of the cubic GL with dynamic boundary conditions.

[3] Carreño, N., Mercado, A., & Morales, R. (2025). Local null controllability of a cubic Ginzburg-Landau equation with dynamic boundary conditions. (to appear in Journal of Evolution Equations) arXiv:2301.03429

- The main ingredient to prove the **observability inequality** is a new **Carleman estimate**.
- For $\lambda, m > 1$ and $\omega' \Subset \omega \Subset \Omega$, we introduce the **weight functions**

$$\varphi(x,t) := (t(T-t))^{-1} \left(e^{2\lambda m \|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty) + \eta^0(x)} \right),$$
$$\xi(x,t) := (t(T-t))^{-1} e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}$$

for $(x,t) \in \overline{\Omega} \times (0,T)$, where $\eta^0 \in C^2(\overline{\Omega})$ satisfies

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \Gamma, \quad |\nabla \eta^0| > 0 \text{ in } \overline{\Omega \setminus \omega'}.$$

A Carleman estimate

Theorem (Carreño, Mercado, M., 2025)

Let $\omega \Subset \Omega$. Set $\omega' \Subset \omega$ and η^0 as before. Then, there exist constants $C, \lambda_0, s_0 > 0$ such that for all $\lambda \geq \lambda_0$ and $s \geq s_0$:

$$\begin{aligned}
&\int_0^T \int_\Omega e^{-2s\varphi}(s^3\lambda^4\xi^3|v|^2 + s\lambda^2\xi|\nabla v|^2 + s^{-1}\xi^{-1}(|\partial_t v|^2 + |\Delta v|^2)) \\
&+ \int_0^T \int_\Gamma e^{-2s\varphi}(s^3\lambda^3\xi^3|v_\Gamma|^2 + s\lambda\xi\left(|\nabla_\Gamma v_\Gamma|^2 + |\partial_\nu v|^2\right)) \\
&+ \int_0^T \int_\Gamma e^{-2s\varphi} s^{-1}\xi^{-1}(|\partial_t v_\Gamma|^2 + |\Delta_\Gamma v_\Gamma|^2) \\
\leq & C s^3\lambda^4 \int_0^T \int_\omega e^{-2s\varphi}\xi^3|v|^2 + C\int_0^T \int_\Omega |\partial_t v + a(1-\alpha i)\Delta v|^2 \\
&+ C\int_0^T \int_\Gamma e^{-2s\varphi}|\partial_t v_\Gamma - a(1-\alpha i)\partial_\nu v + b(1-\alpha i)\Delta_\Gamma v_\Gamma|^2
\end{aligned}$$

for all $(v, v_\Gamma) \in H^1(0,T;\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}^2)$.

Some related references

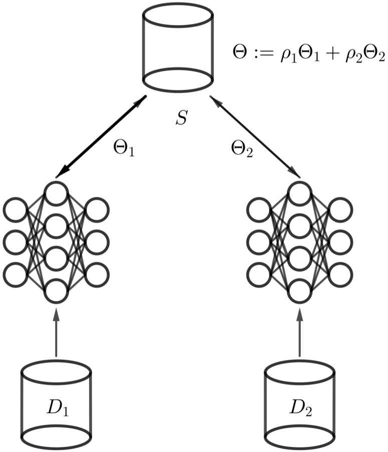
- Mercado, A & Morales, R. (2023) Exact controllability for a Schrödinger equation with dynamic boundary conditions. SIAM Journal on Control and Optimization, 61(6), 3501-3525.
- Lecaros, R., Morales R., Pérez, A. & Zamorano, S. (2023) Discrete Carleman estimates and application to controllability for a fully-discrete parabolic operator with dynamic boundary conditions. Journal of Differential Equations, 365, 832-881.
- Chorfi, S.-E., Hasanov, A., & Morales, R. (2025) Identification of source terms in the Schrödinger equation with dynamic boundary conditions from final data. (To appear in Zeitschrift für angewandte Mathematik und Physik.)
- Santos, M. C., Carreño, N, & Morales, R. (2024). An insensitizing control problem involving tangential gradient terms for a reaction-diffusion equation with dynamic boundary conditions. arXiv:2407:09882.

Research Proposal (Control and ML)

Federated Learning (FL)

- Allows multiple devices(clients) to **collaboratively train a model** without centralizing data with a central server.
- it is particularly useful in scenarios where **data privacy, security and decentralization** are important.
  - Healthcare.
  - Finance.
  - Mobile Devices (Gboard in Google).
  - Smart homes.
- It consists in each client **trains a local model** using its own data, and only the **model updates** are shared.

Federated Learning



Formulation

Let $K \geq 1$ be the number of clients. Associated to each client $k \in [K] = \{1, \dots, K\}$, we have

- A dataset $\{(x_i^k, y_i^k)\}_{i \in N^k} \subset \mathbb{R}^d \times \mathbb{R}^d$, with $d \geq 1$ and $N^k \geq 1$,
- A NODE with $p \geq 1$ neurons of the form:

$$\begin{cases} \dot{x}^k(t) = W(t)\sigma(A(t)x^k(t) + b(t)) \quad t \in (0, T), \\ x_i^k(0) = x_i^k \quad \forall i \in [N^k], \end{cases}$$

where

- $\sigma : \mathbb{R}^p \to \mathbb{R}^p$ is the activation function,
- $\Theta := (W(t), A(t), b(t))$ is the control.

Controllability notions

Exact controllability

We look for $\Theta := (W(t), A(t), b(t)) \in \mathcal{U}$ such that the following condition holds:

$$\phi_T^k(x_i^k, W, A, b) = y_i^k$$

for each $k \in [K]$ and $i \in [N^k]$.

This definition makes sense, but in practice it could be **difficult to verify** the exact controllability notion.

For each $k \in [K]$, we define the following **cost functional**

$$J^k(\Theta) := \frac{1}{2} \sum_{i=1}^{N^k} |\phi_T^k(x_i^k, W, A, b) - y_i^k|^2, \quad \Theta := (W, A, b) \in \mathcal{U}.$$

Then, the FL problem can be formulated as a **Multi-Objective Optimal Control** problem of the form:

$$\min_{\Theta \in \mathcal{U}} (J^1(\Theta), \ldots, J^K(\Theta)). \tag{1}$$

The element $\Theta^\star \in \mathcal{U}$ which satisfies

$$J^k(\Theta^\star) \leq J^k(\Theta) \quad \forall \Theta \in \mathcal{U}, \quad \forall k \in [K],$$

with at least one of these inequalities being strict, it is called **Pareto optimal solution**.

Goals

1. Determine the existence of solutions for the problem (1).[4]
2. Characterize Pareto optimal solutions (Pareto fronts).
3. Establish an interpretation for Θ in the context of FL.
4. Implement numerical methods to find Θ.

[4] Liu, K., Wang, Z., & Zuazua, E. (2024). A Potential Game Perspective in Federated Learning. arXiv preprint arXiv:2411.11793.

Thank you!