

# On the boundedness of oscillating singular integrals on Lie groups of polynomial growth

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- 1 Table of contents
- 2 Historical Aspects
- 3 Applications to elliptic regularity
- 4 Oscillating singular integrals
- 5 Applications to the wave equation for the fractional Laplacian (pseudo-differential operators)
- 6 Final Remarks

# Table of contents

- Historial Aspects
- Oscillating singular integrals on  $\mathbb{R}^n$
- Oscillating singular integrals on compact Lie groups
- Applications to the wave equation for the fractional Laplacian (joint work with Michael Ruzhansky)
- Final Remarks

- 1 Table of contents
- 2 Historical Aspects**
- 3 Applications to elliptic regularity
- 4 Oscillating singular integrals
- 5 Applications to the wave equation for the fractional Laplacian (pseudo-differential operators)
- 6 Final Remarks

# Fefferman in Bilbao ?



# The beginning...

## СИНГУЛЯРНЫЕ ИНТЕГРАЛЬНЫЕ УРАВНЕНИЯ

С. Г. Михлин

## СОДЕРЖАНИЕ

Выходные . . . . .	30
Глава I. Уравнения с одной независимой переменной . . . . .	32
§ 1. Понятие сингулярного интеграла . . . . .	32
§ 2. Сингулярные интегралы в гильбертовом пространстве . . . . .	36
§ 3. Композиция сингулярных интегралов . . . . .	40
§ 4. Символ сингулярного оператора. Регуляризация . . . . .	43
§ 5. Простейшее однородное уравнение . . . . .	45
§ 6. Теорема эквивалентности . . . . .	48
§ 7. Некоторые теоремы о линейных уравнениях . . . . .	50
§ 8. Теоремы Ф. Петера . . . . .	53
§ 9. Общие решения сингулярного уравнения в случае замкнутого контура . . . . .	54
§ 10. Случай разомкнутого контура . . . . .	57
§ 11. Распространение на гильбертово пространство . . . . .	58
§ 12. Системы сингулярных уравнений . . . . .	65
§ 13. Некоторые приложения . . . . .	67
а) О сходимости одного частного метода . . . . .	67
б) Интегральное уравнение Ф. Трикоми . . . . .	69
в) Дифференциал уртовой волны отогнутой полой . . . . .	72
д) Плоская деформация в вязкоупругой среде . . . . .	77
Глава II. Уравнения со многими независимыми переменными . . . . .	79
§ 14. Многомерные сингулярные интегралы . . . . .	79
§ 15. Некоторые свойства многомерных сингулярных интегралов . . . . .	82
§ 16. Замена переменных в многомерных сингулярных интегралах . . . . .	86
§ 17. Композиция интегралов сингулярных и обыкновенных . . . . .	87
§ 18. Композиция двойных сингулярных интегралов . . . . .	88
§ 19. Композиция двойных сингулярных интегралов. Специальные функции . . . . .	91
§ 20. Композиция многомерных сингулярных интегралов . . . . .	94
§ 21. Раскрытие на гильбертово пространство . . . . .	94
§ 22. Регуляризация и задача эквивалентности . . . . .	100
§ 23. Системы многомерных сингулярных уравнений . . . . .	104
§ 24. Замена в об уравнениях, содержащих интегралы по произвольным многообразиям . . . . .	107
§ 25. Общие теоремы . . . . .	108
§ 26. Задача о ясной производной потенциала простого слоя . . . . .	109
Интересная литература . . . . .	111

## ON THE EXISTENCE OF CERTAIN SINGULAR INTEGRALS.

By

A. P. CALDERÓN and A. ZYGMUND

Dedicated to Professor MARCEL RIESZ, on the occasion of his 63rd birthday

### Introduction.

Let  $f(x)$  and  $K(x)$  be two functions integrable over the interval  $(-\infty, +\infty)$ . It is very well known that their composition

$$\int_{-\infty}^{\infty} f(t)K(x-t)dt$$

exists, as an absolutely convergent integral, for almost every  $x$ . The integral can, however, exist almost everywhere even if  $K$  is not absolutely integrable. The most interesting special case is that of  $K(x) = 1/x$ . Let us set

$$\tilde{f}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt.$$

The function  $\tilde{f}$  is called the conjugate of  $f$  (or the Hilbert transform of  $f$ ). It exists for almost every value of  $x$  in the Principal Value sense:

$$\tilde{f}(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \left( \int_{-\infty}^{x-\epsilon} \frac{f(t)}{x-t} dt + \int_{x+\epsilon}^{\infty} \frac{f(t)}{x-t} dt \right).$$

Moreover it is known (See [9] or [7], p. 317) to satisfy the M. Riesz inequality

$$(I) \quad \left\| \int_{-\infty}^{\infty} f(t) \frac{dx}{x-t} \right\|_p \leq A_p \left\| \int_{-\infty}^{\infty} f(t) dt \right\|_p, \quad 1 < p < \infty,$$

where  $A_p$  depends on  $p$  only. There are substitute results for  $p=1$  and  $p=\infty$ . The limit  $\tilde{f}$  exists almost everywhere also in the case when  $f(t)dt$  is replaced there by  $dF(t)$ , where  $F(t)$  is any function of bounded variation over the whole interval  $(-\infty, +\infty)$ . (For all this, see e.g. [7], Chapters VII and XI, where also bibliographical references can be found).

• Usp. Mat. Nauk., and Acta Math.

# Hörmander, 1960

## ESTIMATES FOR TRANSLATION INVARIANT OPERATORS IN $L^p$ SPACES

BY

LARS HÖRMANDER

Stockholm

### Contents

	Page
Preface . . . . .	1
CHAPTER I. General theory.	
1.1. Translation invariant operators as convolutions . . . . .	95
1.2. Basic properties of $M_p^p$ . . . . .	100
1.3. Homomorphisms of $M_p^p$ . . . . .	108
1.4. Analytic operations in $M_p^p$ . . . . .	111
CHAPTER II. Estimates for some special operators.	
2.1. Main theorem . . . . .	113
2.2. Applications . . . . .	118
CHAPTER III. Estimates for some families of operators.	
3.1. Preliminaries . . . . .	125
3.2. $L^2$ estimates . . . . .	128
3.3. Main theorem on mixed $L^2$ estimates . . . . .	130
3.4. Examples of mixed $L^2$ estimates . . . . .	135

### Preface

The theory of bounded translation invariant operators between  $L^p$  spaces in several variables has attracted much interest in the literature during the past decade, partly due to its applications in some fields such as the theory of partial differential equations. Through the work of Calderón, Zygmund and others real variable methods have been introduced which have permitted the extension to several variables of results originally based on complex methods in the case of a single variable. Further,

# Fundamental theorem for singular integral operators on Lie groups of polynomial growth

- Hörmander, L. Estimates for translation invariant operators in  $L^p$  spaces, Acta Math., 104, 93–139, (1960).
- Coifman, R., Weiss, G. Analyse harmonique non-commutative sur certains espaces homogènes. (French). Lecture Notes in Mathematics, Vol. 242, 1971. v+160.

Theorem (Hörmander, Coifman, Weiss, De Guzmán)

Assume that the convolution operator  $T$ , defined by,

$$g \mapsto (Tg)(x) := g * K(x) := \int_G f(xy^{-1})K(y)dy : L^2(G) \rightarrow L^2(G), \quad (1)$$

is bounded (that is  $\|Tg\|_{L^2(G)} \leq C\|g\|_{L^2(G)}$ ), and its kernel satisfies

$$\sup_{0 < R \leq 1} \sup_{|y| \leq R, |x| \geq 2R} \int |K(y^{-1}x) - K(x)|dx < \infty. \quad (2)$$

Then  $T : L^p(G) \rightarrow L^p(G)$  is bounded for all  $1 < p < \infty$ .



- 1 Table of contents
- 2 Historical Aspects
- 3 Applications to elliptic regularity**
- 4 Oscillating singular integrals
- 5 Applications to the wave equation for the fractional Laplacian (pseudo-differential operators)
- 6 Final Remarks

# Applications to elliptic regularity ( $G = \mathbb{R}^n$ or $G$ is a compact Lie group).

## Example

Let  $k_2$  be the right-convolution kernel of the operator  $(1 - \Delta)$ , with  $\Delta$  being the Laplacian on  $G$ . For the Poisson equation  $\Delta u = f$ , with  $u = f * E$ , and  $K = E * k_2$ ,

$$\|u\|_{W^{2,p}} = \|f * E * k_2\|_{L^p} = \|f * K\|_{L^p} = \|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

Moreover, for any  $s \in \mathbb{R}$ ,

$$\|u\|_{W^{2+s,p}} = \|f * E * k_2\|_{L^p} = \|f * K\|_{L^p} = \|Tf\|_{L^p} \leq C\|f\|_{W^{s,p}}.$$

This *elliptic regularity theorem* can be extended to any elliptic differential operator

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \quad P(x, D)u = f,$$

$$\|u\|_{W^{m+s,p}} \leq C(\|f\|_{W^{s,p}} + \|u\|_{W^{t,p}}), \quad t \in \mathbb{R}.$$

# Summarizing

- Harmonic analysis and the study of singular integrals, that are, convolution operators  $f \mapsto Tf = f * K$ , with kernels satisfying conditions of the type

$$\sup_{0 < R \leq 1} \sup_{|y| \leq R} \int_{|x| \geq 2R} |K(y^{-1}x) - K(x)| dx < \infty, \quad (3)$$

are very good tools for obtaining qualitative properties of elliptic differential problems.

- 1 Table of contents
- 2 Historical Aspects
- 3 Applications to elliptic regularity
- 4 Oscillating singular integrals**
- 5 Applications to the wave equation for the fractional Laplacian (pseudo-differential operators)
- 6 Final Remarks

# About the wave equation

- In the analysis of the wave equation associated to the Laplacian or to the fractional Laplacian

$$(II) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta)^\theta u, & u \in \mathcal{D}'([0, T] \times G) \\ u(0, x) = f_0, u_t(0, x) = f_1 \end{cases}, \quad (4)$$

where  $0 \leq \theta < 1$ , is necessary to study the  $L^p$ -boundedness of convolution operators

$$f \mapsto Tf := f * K,$$

where the kernel  $K$  satisfies the following **oscillating condition**

$$\sup_{0 < R \leq 1} \sup_{|y| \leq R} \int_{|x| \geq 2R^{1-\theta}} |K(y^{-1}x) - K(x)| dx < \infty. \quad (5)$$

# Fefferman 1970; Fefferman and Stein, 1972. Acta Math.

## INEQUALITIES FOR STRONGLY SINGULAR CONVOLUTION OPERATORS

BY

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### Contents

I. Introduction . . . . .	9
II. Air on the $g$ -function . . . . .	14
III. Weakly strongly singular integrals . . . . .	21
IV. Results on the operators $T_\lambda$ . . . . .	28
References . . . . .	35

### I. Introduction

Suppose that  $f$  is an  $L^p$  function on the torus  $T^n = S^1 \times \dots \times S^1$ . Must the partial sums of the multiple Fourier series of  $f$  converge to  $f$  in the  $L^p$  norm? For the one-dimensional case,  $T = S^1$ , an affirmative answer has been known for many years. More specifically, suppose that  $f \in L^p(S^1)$  has the Fourier expansion  $f \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$ , and set  $f_m(\theta) = \sum_{|k| \leq m} a_k e^{ik\theta}$ . Then  $f_m$  converges to  $f$  in  $L^p(S^1)$ , as  $m \rightarrow \infty$ —provided  $1 < p < +\infty$  (see [14]).

A whole slew of  $n$ -dimensional analogues of this theorem suggest themselves. Here are two natural conjectures.

(I) Let  $f \in L^p(T^n)$  have the multiple Fourier expansion

$$f(\theta_1, \dots, \theta_n) = \sum_{k_1, \dots, k_n = -\infty}^{\infty} a_{k_1, \dots, k_n} e^{i(k_1\theta_1 + \dots + k_n\theta_n)}.$$

For each positive integer  $m$ , set

$$f_m(\theta_1, \dots, \theta_n) = \sum_{|k_1| \leq m, |k_2| \leq m, \dots, |k_n| \leq m} a_{k_1, \dots, k_n} e^{i(k_1\theta_1 + \dots + k_n\theta_n)}.$$

Then  $f_m \rightarrow f$  in  $L^p(T^n)$ , as  $m \rightarrow \infty$ .

(\*) This work was supported by the National Science Foundation.

## $H^p$ SPACES OF SEVERAL VARIABLES

BY

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### Table of contents

I. Introduction . . . . .	137
II. Duality of $H^1$ and BMO . . . . .	141
1. Functions of bounded mean oscillation: preliminaries . . . . .	141
2. Duality of $H^1$ and BMO . . . . .	144
3. Some applications to $H^1$ . . . . .	149
III. Applications to $L^p$ boundedness . . . . .	153
4. The function $f^\#$ . . . . .	153
5. Intermediate spaces . . . . .	156
6. $L^p$ boundedness of certain convolution operators . . . . .	158
IV. Characterization of $H^p$ in terms of boundary properties of harmonic functions . . . . .	161
7. Area integral and non-tangential max. function . . . . .	161
8. Characterizations of $H^p$ . . . . .	167
9. Lemmas for harmonic functions . . . . .	172
10. Passage to "arbitrary" approximate identities . . . . .	177
V. Real-variable theory of $H^p$ . . . . .	183
11. Equivalence of several definitions . . . . .	183
12. Applications . . . . .	188

### Introduction

The classical theory of  $H^p$  spaces could be considered as a chapter of complex function theory—although a fundamental one, with many intimate connections to Fourier analysis.<sup>(\*)</sup> From our present-day perspective we can see that its heavy dependence on such special tools as Blaschke products, conformal mappings, etc. was not an insurmountable obstacle barring its extension in several directions. Thus the more recent  $n$ -dimensional theory (begin in [24], but with many roots in earlier work) succeeded in some measure

(\*) See Zygmund [28], Chapter III in particular.

# Fefferman 1970 ; Fefferman and Stein, 1972. Acta Math.

Consider that  $K$  satisfies the F.T. condition

$$|\widehat{K}(\xi)| = O((1 + |\xi|)^{-\frac{n\theta}{2}}), \quad 0 \leq \theta < 1, \quad \widehat{K}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} K(x) dx. \quad (6)$$

## Theorem (Fefferman and Stein)

Let  $T$  be a convolution operator with a temperate distribution  $K$  of compact support and let  $0 \leq \theta < 1$ . Assume that  $K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$  satisfies (14) and the oscillating Hörmander condition

$$\sup_{0 \leq R < 1} \sup_{|y| < R} \int_{|x| \geq 2R^{1-\theta}} |K(x-y) - K(x)| dx < \infty. \quad (7)$$

Then  $T : L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)$  is bounded. Moreover,  $T : H^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$  is bounded where  $H^1(\mathbb{R}^n)$  denotes the Hardy space.

- $\|g\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{\lambda > 0} \lambda |\{x : |g(x)| > \lambda\}| < \infty.$
- $\|g\|_{H^1(\mathbb{R}^n)} := \|\sup_{\lambda > 0} |e^{-\lambda \Delta} g|\|_{L^1(\mathbb{R}^n)} < \infty.$

- 1 Table of contents
- 2 Historical Aspects
- 3 Applications to elliptic regularity
- 4 Oscillating singular integrals
- 5 Applications to the wave equation for the fractional Laplacian (pseudo-differential operators)**
- 6 Final Remarks



By applying Fefferman and Stein theorem to the fractional wave equation...

- Let  $s \in \mathbb{R}$ . The wave equation for the fractional Laplacian  $(-\Delta)^\theta$

$$(II) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta)^\theta u, & u \in \mathcal{D}'([0, T] \times \mathbb{R}^n) \\ u(0, x) = f_0, u_t(0, x) = f_1 \end{cases}, \quad (8)$$

where  $0 \leq \theta < 1$ , satisfies the a-priori-estimates

$$\|u(x, t)\|_{L_s^{1, \infty}} := \|(1 - \Delta)^{\frac{s}{2}} u(x, t)\|_{L^{1, \infty}} \leq C_t \left( \|f_0\|_{L_{s+\frac{n\theta}{2}}^{1, \infty}} + \|f_1\|_{L_{s+\theta(\frac{n}{2}+1)}^{1, \infty}} \right), \quad (9)$$

and

$$\|u(x, t)\|_{H_s^1} := \|(1 - \Delta)^{\frac{s}{2}} u(x, t)\|_{H^1} \leq C_t \left( \|f_0\|_{H_{s+\frac{n\theta}{2}}^1} + \|f_1\|_{H_{s+\theta(\frac{n}{2}+1)}^1} \right). \quad (10)$$

# Remarks

- The proof of these a priori estimates are based on the representation of the solution

$$u(x, t) = e^{it\sqrt{-\Delta}^\theta} f_+ + e^{it\sqrt{-\Delta}^\theta} f_- \quad (11)$$

where

$$f_+ := \frac{1}{2}(f_0 - i\sqrt{-\Delta}^\theta f_1), \quad f_- := \frac{1}{2}(f_0 + i\sqrt{-\Delta}^\theta f_1).$$

Indeed, one re-writes the solution as follows

$$u(x, t) = (1 - \Delta)^{\frac{m}{2}} A_t f_+ + (1 - \Delta)^{\frac{m}{2}} A_t f_-, \quad m = n\theta/2, \quad (12)$$

where the time-dependent kernel  $K_t$  of the (pseudo-differential) operator

$$A_t := (1 - \Delta)^{-\frac{m}{2}} e^{it\sqrt{-\Delta}^\theta}, \quad 0 \leq \theta < 1. \quad (13)$$

# Fefferman and Stein Theorem for compact Lie groups

Let  $G$  be a compact Lie group of dimension  $n$ . Consider that  $K \in L^1_{\text{loc}}(G)$  satisfies the F.T. condition

$$\|\widehat{K}(\xi)\|_{\text{op}} = O((1 + |\xi|)^{-\frac{n\theta}{2}}), \quad 0 \leq \theta < 1, \quad \widehat{K}(\xi) = \int_G \xi(x)^* K(x) dx. \quad (14)$$

## Theorem (C+Ruzhansky, 2022)

Let  $T$  be a convolution operator with a temperate distribution  $K$  of compact support and let  $0 \leq \theta < 1$ . Assume that  $K \in L^1_{\text{loc}}(G \setminus \{0\})$  satisfies (14) and the oscillating Hörmander condition

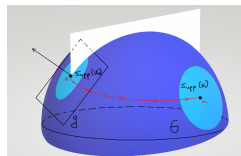
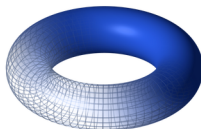
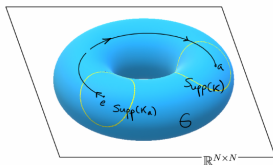
$$\sup_{0 \leq R < 1} \sup_{|y| < R} \int_{|x| \geq 2R^{1-\theta}} |K(x-y) - K(x)| dx < \infty. \quad (15)$$

Then  $T : L^1(G) \rightarrow L^{1,\infty}(G)$  is bounded. Moreover,  $T : H^1(G) \rightarrow L^1(G)$  is bounded where  $H^1(G)$  denotes the Hardy space on  $G$ .

- $\|g\|_{L^{1,\infty}(G)} := \sup_{\lambda > 0} \lambda |\{x : |g(x)| > \lambda\}| < \infty.$
- $\|g\|_{H^1(G)} := \left\| \sup_{\lambda > 0} |e^{-\lambda \Delta} g| \right\|_{L^1(G)} < \infty.$

# About compact Lie groups (our setting)

- Lie groups = **manifolds with symmetries**.
- compact Lie groups = are diffeomorphic to closed subgroups of  $U(N) = \{M \in \mathbb{C}^{N \times N} : M^* = M^{-1}\}$  for  $N$  large enough.



- Examples : the torus  $\mathbb{T}^n \cong (\mathbb{R}/\mathbb{Z})^n$ , Linear Lie groups (groups of matrices),  $SU(n)$ ,  $SO(n)$ , etc. In particular,  $SU(2) \cong \mathbb{S}^3$ .
- If  $M$  is a closed, connected and simply connected, then  $M \cong \mathbb{S}^3$  (**the Poincaré conjecture proved by Perelman**). Our approach (with Turunen and Wirth, and with Cardona) induces global pseudo-differential theories on  $M$ .

By applying Fefferman and Stein theorem on compact Lie groups to the fractional wave equation...

- Let  $s \in \mathbb{R}$ . The wave equation for the fractional Laplacian  $(-\Delta)^\theta$

$$(II) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta_G)^\theta u, & u \in \mathcal{D}'([0, T] \times G) \\ u(0, x) = f_0, u_t(0, x) = f_1 \end{cases}, \quad (16)$$

where  $0 \leq \theta < 1$ , satisfies the a-priori-estimates

$$\|u(x, t)\|_{L_s^{1, \infty}} := \|(1 - \Delta_G)^{\frac{s}{2}} u(x, t)\|_{L^{1, \infty}} \leq C_t \left( \|f_0\|_{L_{s+\frac{n\theta}{2}}^{1, \infty}} + \|f_1\|_{L_{s+\theta(\frac{n}{2}+1)}^{1, \infty}} \right), \quad (17)$$

and

$$\|u(x, t)\|_{H_s^1} := \|(1 - \Delta_G)^{\frac{s}{2}} u(x, t)\|_{H^1} \leq C_t \left( \|f_0\|_{H_{s+\frac{n\theta}{2}}^1} + \|f_1\|_{H_{s+\theta(\frac{n}{2}+1)}^1} \right). \quad (18)$$

# Remark

- Cardona, D. Ruzhansky, M. Oscillating singular integral operators on compact Lie groups revisited. submitted. arXiv :2202.10531.
- Delgado, J. Ruzhansky, M.  $L_p$  bounds for pseudo-differential operators on compact Lie groups, J. Inst. Math. Jussieu, 18, no. 3, 531-559, 2019.
- Cardona, D., Delgado, J., Ruzhansky, M.  $L_p$ -bounds for pseudo-differential operators on graded Lie groups. J. Geom. Anal. Vol. 31, 11603-11647, (2021). arXiv :1911.03397

- 1 Table of contents
- 2 Historical Aspects
- 3 Applications to elliptic regularity
- 4 Oscillating singular integrals
- 5 Applications to the wave equation for the fractional Laplacian (pseudo-differential operators)
- 6 Final Remarks**

# Final remarks

- Harmonic analysis (the study of the Fourier transform on Euclidean and non-Euclidean structures) is a powerful, malleable tool that can be shaped and used differently by various analysts (and non-analysts) to analyze elliptic, subelliptic, hyperbolic and parabolic problems (using a-priori-estimates, Carleman estimates, etc).
- - ▶ Cardona, D. Ruzhansky, M. Oscillating singular integral operators on compact Lie groups revisited. submitted.
  - ▶ Cardona, D. Ruzhansky, M. Boundedness of oscillating singular integrals on Lie groups of polynomial growth, submitted.
  - ▶ Cardona, D. Ruzhansky, M. [v1 : Weak (1,1) continuity and  $L_p$ -theory for oscillating singular integral operators],[v2 : Oscillating singular integral operators on graded Lie groups revisited], submitted.
  - ▶ Cardona, D., Ruzhansky, M. Sharpness of Seeger-Sogge-Stein orders for the weak (1,1) boundedness of Fourier integral operators., to appear in Archiv der Mathematik arXiv :2104.09695
  - ▶ Cardona, D., Delgado, J., Ruzhansky, M.  $L_p$ -bounds for pseudo-differential operators on graded Lie groups. J. Geom. Anal. Vol. 31, 11603-11647, (2021).



# Thank you for your attention !

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- Thank you ! Gracias !