On the boundedness of oscillating singular integrals on Lie groups of polynomial growth

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Fefferman in Bilbao ?

The beginning...

СИНГУЛЯРНЫЕ ИНТЕГРАЛЬНЫЕ УРАВНЕНИЯ

С. Г. Михлин

СОДЕРЖАНИЕ

ON THE EXISTENCE OF CERTAIN SINGULAR INTEGRALS.

By

A. P. CALDERON and A. ZYGMUND

Dedicated to Professor MARCEL RIESZ, on the occasion of his 65th birthday

Introduction.

Let $f(x)$ and $K(x)$ be two functions integrable over the interval $(-\infty, +\infty)$. It is very well known that their composition

$$\int_{-\infty}^{+\infty} f(t)\, K(x-t)\, dt$$

exists, as an absolutely convergent integral, for almost every x. The integral can, however, exist almost everywhere even if K is not absolutely integrable. The most interesting special case is that of $K(x) = 1/x$. Let us set

$$\tilde{f}(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{x-t}\, dt.$$

The function $\tilde{f}$ is called the conjugate of f (or the Hilbert transform of f). It exists for almost every value of x in the Principal Value sense:

$$\tilde{f}(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right) \frac{f(t)}{x-t}\, dt.$$

Moreover it is known (See [9] or [7], p. 317) to satisfy the M. Riesz inequality

$$\text{(1)} \qquad \left[\int_{-\infty}^{+\infty} |\tilde{f}|^p\, dx\right]^{1/p} \le A_p \left[\int_{-\infty}^{+\infty} |f|^p\, dx\right]^{1/p}, \qquad 1 < p < \infty,$$

where A_p depends on p only. There are substitute result for $p = 1$ and $p = \infty$. The limit $\tilde{f}$ exists almost everywhere also in the case when $f(t)\,dt$ is replaced there by $dF(t)$, where $F(t)$ is any function of bounded variation over the whole interval $(-\infty, +\infty)$. (For all this, see e.g. [7], Chapters VII and XI, where also bibliographical references can be found).

- Usp. Mat. Nauk., and Acta Math.

Hörmander, 1960

ESTIMATES FOR TRANSLATION INVARIANT OPERATORS IN L^p SPACES

BY

LARS HÖRMANDER

Stockholm

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Preface

The theory of bounded translation invariant operators between L^p spaces in several variables has attracted much interest in the literature during the past decade, partly due to its applications in some fields such as the theory of partial differential equations. Through the work of Calderón, Zygmund and others real variable methods have been introduced which have permitted the extension to several variables of results originally based on complex methods in the case of a single variable. Further,

Fundamental theorem for singular integral operators on Lie groups of polynomial growth

- Hörmander, L. Estimates for translation invariant operators in Lp spaces, Acta Math., 104, 93–139, (1960).
- Coifman, R., Weiss, G. Analyse harmonique non-commutative sur certains espaces homogénes. (French). Lecture Notes in Mathematics, Vol. 242, 1971. v+160.

Theorem (Hörmander, Coifman, Weiss, De Guzmán)

Assume that the convolution operator T, defined by,

$$g \mapsto (Tg)(x) := g * K(x) := \int_G f(xy^{-1})K(y)dy : L^2(G) \to L^2(G), \quad (1)$$

is bounded (that is $\|Tg\|_{L^2(G)} \leq C\|g\|_{L^2(G)}$), and its kernel satisfies

$$\sup_{0<R\leq 1} \sup_{|y|\leq R} \int_{|x|\geq 2R} |K(y^{-1}x) - K(x)|dx < \infty. \quad (2)$$

Then $T : L^p(G) \to L^p(G)$ is bounded for all $1 < p < \infty$.

Applications to elliptic regularity ($G = \mathbb{R}^n$ or G is a compact Lie group).

Example

Let k_2 be the right-convolution kernel of the operator $(1-\Delta)$, with Δ being the Laplacian on G. For the Poisson equation $\Delta u = f$, with $u = f * E$, and $K = E * k_2$,

$$\|u\|_{W^{2,p}} = \|f * E * k_2\|_{L^p} = \|f * K\|_{L^p} = \|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

Moreover, for any $s \in \mathbb{R}$,

$$\|u\|_{W^{2+s,p}} = \|f * E * k_2\|_{L^p} = \|f * K\|_{L^p} = \|Tf\|_{L^p} \leq C\|f\|_{W^{s,p}}.$$

This *elliptic regularity theorem* can be extended to any elliptic differential operator

$$P(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha,\ P(x,D)u = f,$$

$$\|u\|_{W^{m+s,p}} \leq C(\|f\|_{W^{s,p}} + \|u\|_{W^{t,p}}),\ \ t \in \mathbb{R}.$$

Summarizing

- Harmonic analysis and the study of singular integrals, that are, convolution operators $f \mapsto Tf = f * K$, with kernels satisfying conditions of the type

$$\sup_{0<R\leq 1} \sup_{|y|\leq R} \int_{|x|\geq 2R} |K(y^{-1}x) - K(x)| dx < \infty, \tag{3}$$

  are very good tools for obtaining qualitative properties of elliptic differential problems.

About the wave equation

- In the analysis of the wave equation associated to the Laplacian or to the fractional Laplacian

$$(\mathrm{II}) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta)^{\theta} u, & u \in \mathscr{D}'([0,T] \times G) \\ u(0,x) = f_0, u_t(0,x) = f_1 \end{cases}, \tag{4}$$

where $0 \leq \theta < 1$, is necessary to study the L^p-boundedness of convolution operators

$$f \mapsto Tf := f * K,$$

where the kernel K satisfies the following oscillating condition

$$\sup_{0<R\leq 1} \sup_{|y|\leq R} \int_{|x|\geq 2R^{1-\theta}} |K(y^{-1}x) - K(x)| dx < \infty. \tag{5}$$

Fefferman 1970 ; Fefferman and Stein, 1972. Acta Math.

INEQUALITIES FOR STRONGLY SINGULAR CONVOLUTION OPERATORS

BY

CHARLES FEFFERMAN

Princeton University, Princeton, N. J., U.S.A. (1)

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I. Introduction

Suppose that f is an L^p function on the torus $T^n = S^1 \times S^1 \times \ldots \times S^1$. Must the partial sums of the multiple Fourier series of f converge to f in the L^p norm? For the one-dimensional case, $T = S^1$, an affirmative answer has been known for many years. More specifically, suppose that $f \in L^p(S^1)$ has the Fourier expansion $f \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, and set $f_m(\theta) = \sum_{k=-m}^{m} a_k e^{ik\theta}$. Then f_m converges to f in $L^p(S^1)$, as $m \to \infty$ — provided $1 < p < +\infty$ (see [14]).

A whole slew of n-dimensional analogues of this theorem suggest themselves. Here are two natural conjectures.

(I) Let $f \in L^p(T^n)$ have the multiple Fourier expansion

$$f(\theta_1 \ldots \theta_n) = \sum_{k_1 \ldots k_n = -\infty}^{\infty} a_{k_1 \ldots k_n} e^{i(k_1\theta_1 + \ldots + k_n\theta_n)}.$$

For each positive integer m, set

$$f_m(\theta_1 \ldots \theta_n) = \sum_{|k_1| \leq m, |k_2| \leq m, \ldots, |k_n| \leq m} a_{k_1 \ldots k_n} e^{i(k_1\theta_1 + \ldots + k_n\theta_n)}.$$

Then $f_m \to f$ in $L^p(T^n)$, as $m \to \infty$.

(1) This work was supported by the National Science Foundation.

H^p SPACES OF SEVERAL VARIABLES

BY

C. FEFFERMAN and E. M. STEIN

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Introduction

The classical theory of H^p spaces could be considered as a chapter of complex function theory—although a fundamental one, with many intimate connections to Fourier analysis. (1) From our present-day perspective we can see that its heavy dependence on such special tools as Blaschke products, conformal mappings, etc. was not an insurmountable obstacle barring its extension in several directions. Thus the more recent n-dimensional theory (begun in [24], but with many roots in earlier work) succeeded in some measure

(1) See Zygmund [28], Chapter III in particular.

10 – 722902 *Acta mathematica* 129. Imprimé le 2 Octobre 1972

Fefferman 1970 ; Fefferman and Stein, 1972. Acta Math.

Consider that K satisfies the F.T. condition

$$|\widehat{K}(\xi)| = O((1+|\xi|)^{-\frac{n\theta}{2}}), \quad 0 \leq \theta < 1, \ \widehat{K}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} K(x)dx. \tag{6}$$

Theorem (Fefferman and Stein)

Let T be a convolution operator with a temperate distribution K of compact support and let $0 \leq \theta < 1$. Assume that $K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies (14) and the oscillating Hörmander condition

$$\sup_{0\leq R<1} \sup_{|y|<R} \int_{|x|\geq 2R^{1-\theta}} |K(x-y) - K(x)|dx < \infty. \tag{7}$$

Then $T : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ is bounded. Moreover, $T : H^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is bounded where $H^1(\mathbb{R}^n)$ denotes the Hardy space.

- $\|g\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda|\{x : |g(x)| > \lambda\}| < \infty.$
- $\|g\|_{H^1(\mathbb{R}^n)} := \|\sup_{\lambda>0} |e^{-\lambda\Delta} g|\|\|_{L^1(\mathbb{R}^n)} < \infty.$

By applying Fefferman and Stein theorem to the fractional wave equation...

- Let $s \in \mathbb{R}$. The wave equation for the fractional Laplacian $(-\Delta)^{\theta}$

$$(\mathrm{II}) : \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta)^{\theta} u, & u \in \mathscr{D}'([0,T] \times \mathbb{R}^n) \\ u(0,x) = f_0, u_t(0,x) = f_1 \end{cases}, \tag{8}$$

where $0 \leq \theta < 1$, satisfies the a-priori-estimates

$$\|u(x,t)\|_{L^{1,\infty}_s} := \|(1-\Delta)^{\frac{s}{2}} u(x,t)\|_{L^{1,\infty}} \leq C_t \left(\|f_0\|_{L^{1,\infty}_{s+\frac{n\theta}{2}}} + \|f_1\|_{L^{1,\infty}_{s+\theta(\frac{n}{2}+1)}} \right), \tag{9}$$

and

$$\|u(x,t)\|_{H^1_s} := \|(1-\Delta)^{\frac{s}{2}} u(x,t)\|_{H^1} \leq C_t \left(\|f_0\|_{H^1_{s+\frac{n\theta}{2}}} + \|f_1\|_{H^1_{s+\theta(\frac{n}{2}+1)}} \right). \tag{10}$$

Remarks

- The proof of these a priori estimates are based on the representation of the solution

$$u(x,t) = e^{it\sqrt{-\Delta}^{\theta}} f_{+} + e^{it\sqrt{-\Delta}^{\theta}} f_{-} \tag{11}$$

where

$$f_{+} := \frac{1}{2}(f_0 - i\sqrt{-\Delta}^{\theta} f_1),\ f_{-} := \frac{1}{2}(f_0 + i\sqrt{-\Delta}^{\theta} f_1).$$

Indeed, one re-writes the solution as follows

$$u(x,t) = (1-\Delta)^{\frac{m}{2}} A_t f_{+} + (1-\Delta)^{\frac{m}{2}} A_t f_{-},\ m = n\theta/2, \tag{12}$$

where the time-dependent kernel K_t of the (pseudo-differential) operator

$$A_t := (1-\Delta)^{-\frac{m}{2}} e^{it\sqrt{-\Delta}^{\theta}},\ 0 \leq \theta < 1. \tag{13}$$

Fefferman and Stein Theorem for compact Lie groups

Let G be a compact Lie group of dimension n. Consider that $K \in L^1_{\text{loc}}(G)$ satisfies the F.T. condition

$$\|\widehat{K}(\xi)\|_{\text{op}} = O((1+|\xi|)^{-\frac{n\theta}{2}}), \quad 0 \leq \theta < 1, \ \widehat{K}(\xi) = \int_G \xi(x)^* K(x) dx. \qquad (14)$$

Theorem (C+Ruzhansky, 2022)

Let T be a convolution operator with a temperate distribution K of compact support and let $0 \leq \theta < 1$. Assume that $K \in L^1_{\text{loc}}(G \setminus \{0\})$ satisfies (14) and the oscillating Hörmander condition

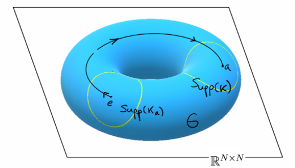
$$\sup_{0 \leq R < 1} \sup_{|y| < R} \int_{|x| \geq 2R^{1-\theta}} |K(x-y) - K(x)| dx < \infty. \qquad (15)$$

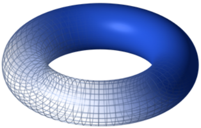
Then $T : L^1(G) \to L^{1,\infty}(G)$ is bounded. Moreover, $T : H^1(G) \to L^1(G)$ is bounded where $H^1(G)$ denotes the Hardy space on G.

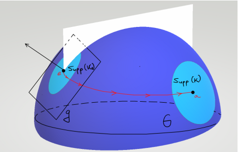
- $\|g\|_{L^{1,\infty}(G)} := \sup_{\lambda>0} \lambda |\{x : |g(x)| > \lambda\}| < \infty$.
- $\|g\|_{H^1(G)} := \| \sup_{\lambda>0} |e^{-\lambda\Delta} g| \|_{L^1(G)} < \infty$.

About compact Lie groups (our setting)

- Lie groups = manifolds with symmetries.
- compact Lie groups = are diffeomorphic to closed subgroups of $\mathrm{U}(N) = \{M \in \mathbb{C}^{N\times N} : M^* = M^{-1}\}$ for N large enough.







- Examples : the torus $\mathbb{T}^n \cong (\mathbb{R}/\mathbb{Z})^n$, Linear Lie groups (groups of matrices), $\mathrm{SU}(n)$, $\mathrm{SO}(n)$, etc. In particular, $\mathrm{SU}(2) \cong \mathbb{S}^3$.
- If M is a closed, connected and simply connected, then $M \cong \mathbb{S}^3$ (the Poincaré conjecture proved by Perelman). Our approach (with Turunen and Wirth, and with Cardona) induces global pseudo-differential theories on M.

By applying Fefferman and Stein theorem on compact Lie groups to the fractional wave equation...

- Let $s \in \mathbb{R}$. The wave equation for the fractional Laplacian $(-\Delta)^\theta$

$$(\text{II}): \begin{cases} \frac{\partial^2 u}{\partial t^2} = -(-\Delta_G)^\theta u, & u \in \mathscr{D}'([0,T] \times G) \\ u(0,x) = f_0, \, u_t(0,x) = f_1 \end{cases}, \qquad (16)$$

where $0 \leq \theta < 1$, satisfies the a-priori-estimates

$$\|u(x,t)\|_{L^{1,\infty}_s} := \|(1-\Delta_G)^{\frac{s}{2}} u(x,t)\|_{L^{1,\infty}} \leq C_t \left(\|f_0\|_{L^{1,\infty}_{s+\frac{n\theta}{2}}} + \|f_1\|_{L^{1,\infty}_{s+\theta(\frac{n}{2}+1)}} \right), \qquad (17)$$

and

$$\|u(x,t)\|_{H^1_s} := \|(1-\Delta_G)^{\frac{s}{2}} u(x,t)\|_{H^1} \leq C_t \left(\|f_0\|_{H^1_{s+\frac{n\theta}{2}}} + \|f_1\|_{H^1_{s+\theta(\frac{n}{2}+1)}} \right). \qquad (18)$$

Remark

- Cardona, D. Ruzhansky, M. Oscillating singular integral operators on compact Lie groups revisited. submitted. arXiv :2202.10531.
- Delgado, J. Ruzhansky, M. Lp bounds for pseudo-differential operators on compact Lie groups, J. Inst. Math. Jussieu, 18, no. 3, 531-559, 2019.
- Cardona, D., Delgado, J., Ruzhansky, M. Lp-bounds for pseudo-differential operators on graded Lie groups. J. Geom. Anal. Vol. 31, 11603-11647, (2021). arXiv :1911.03397

Final remarks

- Harmonic analysis (the sudy of the Fourier transform on Euclidean and non-Euclidean structures) is a powerful, malleable tool that can be shaped and used differently by various analysts (and non-analysts) to analize elliptic, subelliptic, hyperbolic and parabolic problems (using a-priori-estimates, Carleman estimates, etc).
-

  - Cardona, D. Ruzhansky, M. Oscillating singular integral operators on compact Lie groups revisited. submitted.
  - Cardona, D. Ruzhansky, M. Boundedness of oscillating singular integrals on Lie groups of polynomial growth, submitted.
  - Cardona, D. Ruzhansky, M. [v1 : Weak (1,1) continuity and Lp-theory for oscillating singular integral operators],[v2 : Oscillating singular integral operators on graded Lie groups revisited], submitted.
  - Cardona, D., Ruzhansky, M. Sharpness of Seeger-Sogge-Stein orders for the weak (1,1) boundedness of Fourier integral operators., to appear in Archiv der Mathematik arXiv :2104.09695
  - Cardona, D., Delgado, J., Ruzhansky, M. Lp-bounds for pseudo-differential operators on graded Lie groups. J. Geom. Anal. Vol. 31, 11603-11647, (2021).

Thank you for your attention !

- https ://sites.google.com/site/duvancardonas/home

- Thank you ! Gracias !