SHARP ESTIMATES FOR THE CHEEGER CONSTANT IN THE PLANAR CASE

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Introduction and Motivations

The Cheeger constant of set $\Omega \subseteq \mathbb{R}^2$ is defined as

$$h(\Omega) = \inf\left\{\frac{P(E)}{|E|} \,:\, E \text{ measurable and } E \subseteq \Omega\right\}, \tag{1}$$

where $P(E)$ is the perimeter of E in the sense of De Giorgi and $|E|$ is the area of E. If Ω has Lipschitz boundary, the infimum in (1) is achieved. The set C_Ω that realizes this minimum is called a *Cheeger set* of Ω. In particular, in the case of planar convex sets, the Cheeger set is unique and we have a characterization for the Cheeger constant.

The Cheeger constant appears in several mathematical contexts:

- Study of plate failure under stress;
- Applications in the context of maximal flow-minimal cut problems. In particular, the problem of computing exact continuous optimal curves and surfaces for image segmentation in $2D$ and $3D$ reconstruction from a stereo image pair has applications in medical image process (see [1]);
- The extension of the Cheeger problem involving anisotropic norms and anisotropic total variation turns out to be useful in the context of image processing (see [2]).

For these reasons, it is useful to have estimates of the Cheeger constant in terms of geometric quantities that can be easily computed.

The Blaschke-Santaló diagrams

A Blaschke–Santaló diagram is a tool that allows to visualize all the possible inequalities between three geometric quantities. More precisely, if we consider three shape functionals (J_1, J_2, J_3), this means that we want to find a system of inequalities describing the set

$$\{(J_1(\Omega), J_2(\Omega)) : J_3(\Omega) = 1,\, \Omega \in \mathcal{K}^2\},$$

where

$$\mathcal{K}^2 := \{\Omega \,|\, \Omega \text{ is an open, bounded and convex set of } \mathbb{R}^2\} \setminus \{\emptyset\}.$$

In [3] our aim is to study the Blaschke-Santaló triplets associated to the Cheeger constant and two between the following geometrical quantities: area, perimeter, inradius, circumradius, minimal width and diameter. Let us see, in particular, what happens for the Blaschke-Santaló diagram associated to the width $(J_1(\cdot))$, the Cheeger constant $(J_2(\cdot))$ and the diameter $(J_3(\cdot))$.

Sharp estimates for $h(\Omega)$ in terms of width and diameter

We use the following notations:

- $d(\Omega)$ is the diameter of Ω, i.e. the maximal distance between two points in $\partial\Omega$;
- $\omega(\Omega)$ is the minimal width of Ω, i.e. the minimal distance between two parallel supporting hyperplanes;

and we define the following sets:

- a symmetrical spherical slice is the set obtained by the intersection between a ball of radius r and a strip of width $\omega < 2r$, that is symmetrical w.r.t. the center of the ball;
- a subequilateral triangle is an isosceles triangle with the two equal angles greater than $\pi/3$.

In [3] we prove the following results:

Lower bound Let $\Omega \in \mathcal{K}^2$. Then, it holds

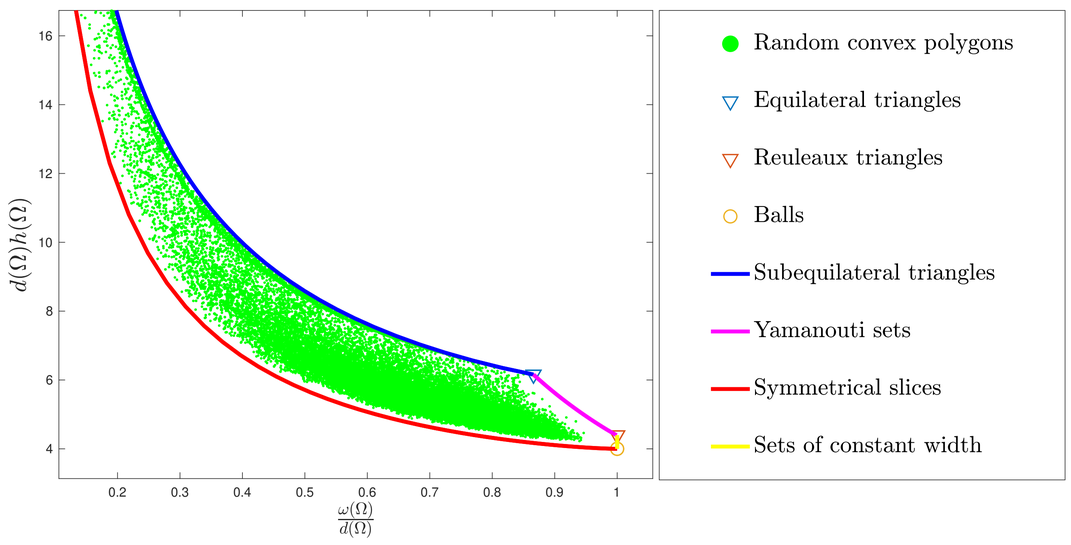
$$h(\Omega) \geq h(K_S),$$

where K_S is the symmetrical spherical slice such that $\omega(\Omega) = \omega(K_S)$ and $d(\Omega) = d(K_S)$. Equality is achieved by the symmetrical spherical slice K_S.

Upper bound Let $\Omega \in \mathcal{K}^2$. Then, it holds

$$h(\Omega) \leq h(T_I), \qquad \text{if } \omega(\Omega) \leq \frac{\sqrt{3}}{2} d(\Omega),$$

where T_I is the subequilateral triangle such that $\omega(\Omega) = \omega(T_I)$ and $d(\Omega) = d(T_I)$. Equality is achieved by the subequilateral triangle T_I.



Blaschke-Santaló diagram for the triplet (ω, h, d)

Generating random convex polygons

We want to provide a numerical approximation of the Blaschke-Santaló diagram for the triplet (ω, h, d). To do so, a natural idea is to generate a large number of convex polygons and for each of them to compute the involved functionals. The main difficulty is to design an efficient and fast algorithm that allows to obtain an uniform distribution of the generated random convex polygons.

- one easy way to generate random convex polygons is by rejection sampling. We generate a random set of points in a square; if they form a convex polygon, we return it, otherwise we try again.The probability of a set of n points uniformly generated inside a given square to be in convex position is equal to $p_n = \left(\frac{\binom{2n-2}{n-1}}{n!}\right)^2$. Thus, the random variable X_n corresponding to the expected number of iterations needed to obtain a convex distribution follows a geometric law of parameter p_n, which means that its expectation is given by $\mathbb{E}(X_n) = \frac{1}{p_n} = \left(\frac{n!}{\binom{2n-2}{n-1}}\right)^2$. For example, if $N = 20$, the expected number of iterations is approximately equal to 2.10^9, and since one iteration is performed in an average of 0.7 seconds, the algorithm will need about 50 years to provide one convex polygon with 20 sides.
- Another natural approach is to generate random points and take their convex hull. This method is quite fast, as one can compute the convex hull of N points in a $\mathcal{O}(N\log(N))$ time, but it is not quite relevant since it yields to a biased distribution.

In order to avoid the issues stated above, we use an algorithm based on the work of P. Valtr [4], where the author computes the probability of a set of n points uniformly generated inside a given square to be in convex position, with a fast and non biased method. We generate 10^5 random convex polygons of unit area and number of sides between 3 and 30 for which we compute the involved functionals, obtaining clouds of dots that provide approximations of the diagrams.

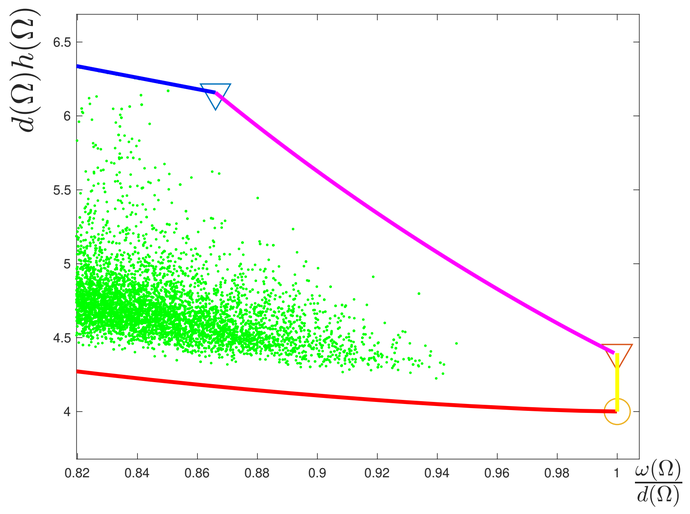
A new conjecture

We conjecture that, if $\sqrt{3}/2 d(\Omega) \leq \omega(\Omega) \leq d(\Omega)$, then for all $\in \mathcal{K}^2$

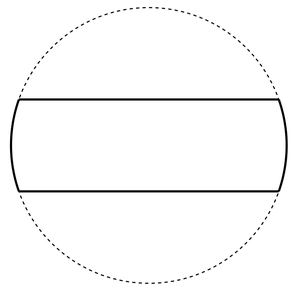
$$h(\Omega) \leq h(Y)$$

where Y is the Yamanouti set with $\omega(Y) = \omega(\Omega)$ and $d(Y) = d(\Omega)$. A *Yamanouti set* Y is a set obtained by an equilateral triangle by constructing on each side an arc of circle centered in the opposite vertex and with radius less or equal than the side itself. The Yamanouti set is the convex hull of the set obtained in this way.

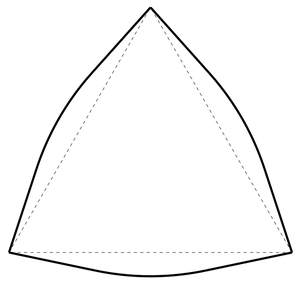
Zoom in: conjectured inequality



Symmetrical spherical slice



Yamanouti set



Open Problems

- Prove the conjectured inequalities given by numerical simulations.
- Study the Blaschke-Santalò diagrams for the Cheeger constant in dimension $n > 2$.
- Study the Blaschke-Santalò diagrams for the Cheeger constant in the anisotropic case.

Selected publications

[1] B. Appleton, H. Talbot (2006) **Globally minimal surfaces by continuous maximal flows.** IEEE, Transactions on Pattern Analysis and Machine Intelligence.

[2] V. Caselles, G. Facciolo, E. Meinhardt (2009). **Anisotropic Cheeger Sets and Applications.** SIAM Journal on Imaging Sciences .

[3] I. Ftouhi, A.L. Masiello, G.P (2022). **Sharp estimates for the Cheeger constant in the planar case.** arXiv:2206.13158.

[4] P. Valtr. (1995). **Probability that n random points are in convex position.** Discrete Comput. Geom.

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