Asymptotic analysis of partially and locally dissipated hyperbolic systems

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Introduction

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These systems models physical phenomena with finite speed of propagation or equilibrium laws, such as the compressible Euler equation with damping:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u + \nabla P(\rho) + \dfrac{u}{\varepsilon} = 0. \end{cases} \tag{1}$$

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We are interested in the following questions:

- Limit as $\varepsilon \rightarrow 0$?
- Behavior as $t \rightarrow \infty$?

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$$\mathcal{L}^2 = \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2 + \int_{\mathbb{R}} v\partial_x u,$$

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And since $\mathcal{L}^2 \sim \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2$, we can obtain time-decay estimates.

For the general system, the idea is the same if one assume the (SK) condition:

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$$\forall \xi \in \mathbb{R}^d, \quad \ker L \cap \{\text{eigenvectors of } \sum_j A^j \xi_j\} = \{0\}. \tag{SK}$$

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Such condition is actually equivalent the Kalman rank condition and inspired by hypercoercivity theory, Beauchard and Zuazua defined

$$\mathcal{L}^2 \triangleq \|V\|_{L^2}^2 + \int_{\mathbb{R}^d} \min(\rho, \rho^{-1})\mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \left(LA_\omega^{k-1}\widehat{V} \cdot LA_\omega^k \widehat{V}\right)$$

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Again, we obtain

$$\frac{d}{dt}\mathcal{L} + \kappa \min(1, |\xi|^2)\mathcal{L} \leq 0$$

- With this estimates at hand, one deduces the global existence of small H^s solutions and

$$\|V^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda t}\|V_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)},$$
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- Moreover, this technique also allow to treat situation when the (SK) condition is not satisfied.
- However, these decay estimates do not depict the full story in the low frequencies regime and do not allow to consider the limit as $\varepsilon \to 0$.

"New" observations

- Back to the damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \dfrac{v}{\varepsilon} = 0. \end{cases} \tag{4}$$

A spectral analysis of the matrix

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \dfrac{1}{\varepsilon} \end{pmatrix}$$

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- the threshold between low and high frequencies is at $\dfrac{1}{\varepsilon}$.

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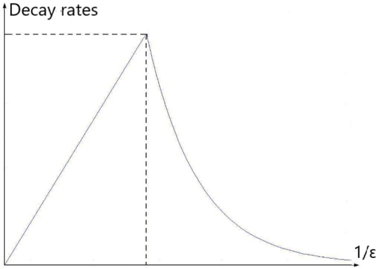

- This is related to the fact that as $\varepsilon \rightarrow 0$, the low frequencies "invade" the whole space of frequency.

Low frequencies in a simple case

Our idea: reproduce exactly what the spectral analysis tells us using:

$$\|f\|^{h}_{\dot{\mathbb{B}}^{s}_{2,1}} \triangleq \sum_{j\geq \frac{1}{\varepsilon}} 2^{js}\|\dot{\Delta}_j f\|_{L^2} \quad \text{and} \quad \|f\|^{\ell}_{\dot{\mathbb{B}}^{s'}_{p,1}} \triangleq \sum_{j\leq \frac{1}{\varepsilon}} 2^{js'}\|\dot{\Delta}_j f\|_{L^p}.$$

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$$\begin{cases} \partial_t u - \varepsilon \partial^2_{xx} u = -\partial_x w \\ \partial_t w + \dfrac{w}{\varepsilon} = -\varepsilon \partial^2_{xx} v. \end{cases}$$

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→ We directly get the behaviour observed in the spectral analysis, not just heat effect.
→ It is possible to study the two equations in a decoupled way as the source terms can be absorbed in the low-frequency regime:

$$\|\partial_x f\|^\ell_{B^s_{p,1}} \leq \|f\|^\ell_{B^s_{p,1}}$$

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To Sum-up

- The hypocoercivity approach does not give the full story of the low-frequency behavior.
- From the low-frequency analysis presented here and the high frequencies computation à la Beauchard et Zuazua, we are able to get a uniform global existence result.
- And from these uniform estimates we can justify, almost directly, the relaxation limit when $\varepsilon \to 0$ in the ill-prepared case.

Relaxation result

Theorem (Danchin, C-B '21- ill-prepared relaxation limit)

Let $d \geq 1$, $p \in [2,4]$ and $\varepsilon > 0$. Let $\bar{\rho}$ be a strictly positive constant and $(\rho - \bar{\rho}, v)$ be the solution obtained with the previous theorem.

Let the positive function $\mathcal{N}_0$ such that $\mathcal{N}_0 - \bar{\rho}$ is small enough in $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$, and let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$ be the unique solution associated to the Cauchy problem:

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \end{cases}$$

If we assume that

$$\|\widetilde{\rho}_0^{\varepsilon} - \mathcal{N}_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^\infty(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^{\varepsilon})}{\widetilde{\rho}^{\varepsilon}} + \widetilde{v}^{\varepsilon} \right\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

Localized damping

Damping active outside of a ball

We consider the one-dimensional linear hyperbolic system

$$\begin{cases} \partial_t U + A\partial_x U = -BU\mathbf{1}_\omega, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ U(0,x) = U_0(x), & x \in \mathbb{R}, \end{cases}$$

where $U = (u_1, u_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and

$$\omega := \mathbb{R} \setminus B_R(0) = \{x \in \mathbb{R} : \|x\| \geq R\} \quad \text{for a fixed } R > 0.$$

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We assume :

- $B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ with $D > 0$
- The matrix A is a *strictly hyperbolic matrix*, i.e. A has n real distinct eigenvalues

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In other words: **we are in the same situation as before but the damping is only effective in ω (the complementary of a ball).**

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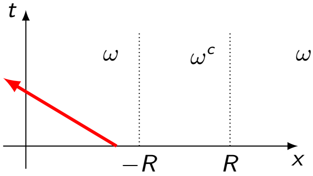
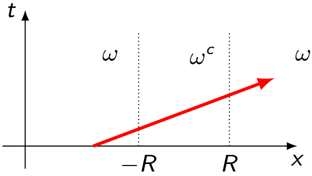
$\rightarrow$ This motivates us to develop a method involving only the consideration of the characteristics curves and a semigroup-wise decomposition.

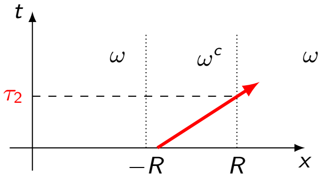
Propagation of characteristics and their location with respect to the region $\omega = \mathbb{R} \setminus B_R$ where the damping is active.



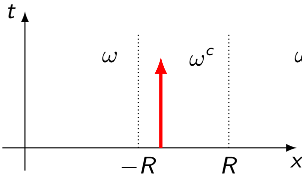
(a) **Case 1:** The initial support is in the damped region and the characteristics are going away from the un-damped region.



(b) **Case 2:** The initial support is in the damped region and the characteristics cross the un-damped region



(c) **Case 3:** The initial support is in the un-damped region



(d) **Case 4:** There is one zero eigenvalue. $\rightarrow$ Standing wave

Reformulation of the system

As A is symmetric with n real distinct eigenvalues, there exists a matrix $P \in O(n, \mathbb{R})$ such that

$$P^{-1}AP = \Lambda \quad \text{where} \quad \Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n).$$

Setting $V = P^{-1}U$, the system can be reformulated into

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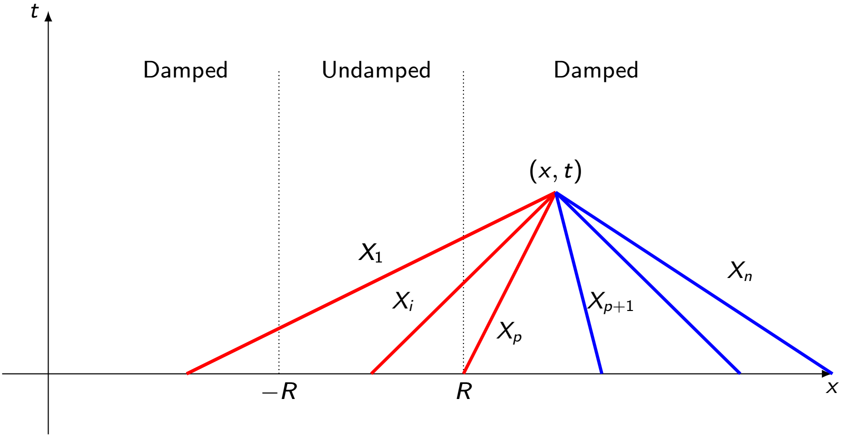
For all $1 \leq i \leq n$, the characteristic lines X_i of each equations passing through the point $(x_0, t_0) \in \mathbb{R} \times [0, T]$ are given by

$$X_i\,(t, x_0, t_0) := \lambda_i(t - t_0) + x_0, \quad t \in [0, T].$$

Figure: Characteristics passing through a point $(x, t) \in \mathbb{R} \times \mathbb{R}_+$.



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These considerations led us to the following Theorem.

Main Theorem

Theorem (De Nitti-Zuazua-CB '22)

Assume that the matrix A is symmetric, strictly hyperbolic and does not admit the eigenvalue 0 and that the couple (A, B) satisfies the (SK) condition.
Let $U_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
Then, there exists a constant $C > 0$ and a finite time $\bar{\tau} > 0$ such that for $t \geq \bar{\tau}$, the solution satisfies

$$\|U^h(\cdot, t)\|_{L^2(\mathbb{R})} \leq Ce^{-\gamma(t-\bar{\tau})}\|U_0\|_{L^2(\mathbb{R})},$$
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The decay estimates are delayed by the time each characteristic spend in the undamped region

Idea of proof

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1. We define S_d the dissipative semigroup associated to the equation without localization. This semigroup is active when all the characteristics are outside the undamped region. Recall that we have

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Then, for every $(x,t) \in \mathbb{R}^2$, we can always find suitable times t_1, t_2 such that each components of the solution can be rewritten:

$$v_i(x,t) = S_{d,i}(t)S_{c,i}(t_1)S_{d,i}(t_2)v_{i,0}(x). \tag{6}$$

where $S_{d,i}$ and $S_{c,i}$ are the semigroup associated to each components.

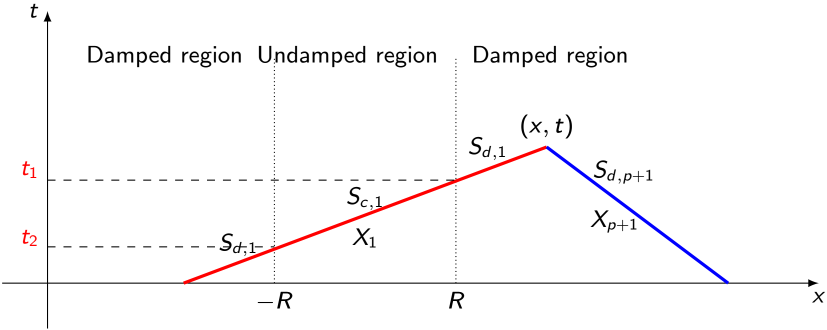
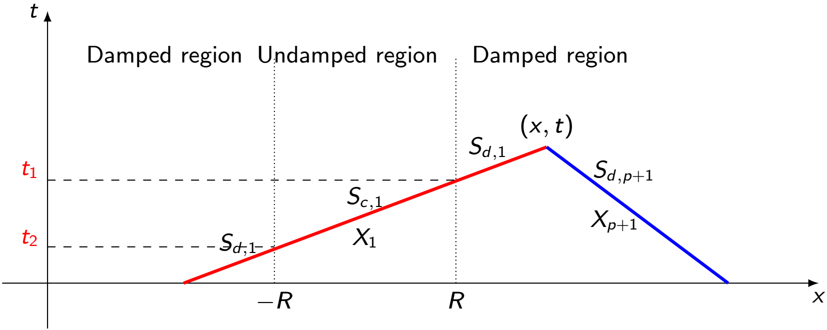


Figure: Illustration on the semigroups and the quantities t_1 and t_2.



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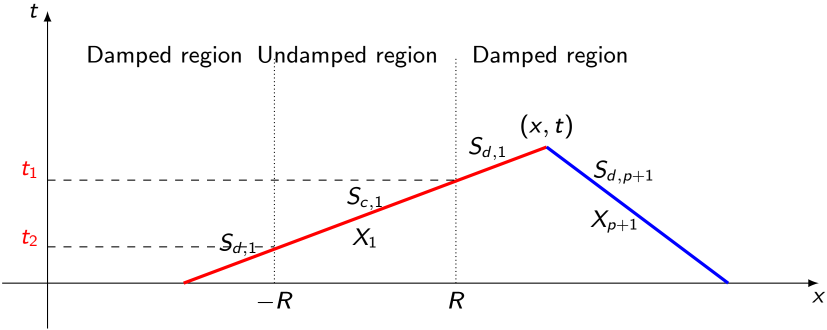

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But the difference $t_1 - t_2$ is always uniformly bounded!

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- This means that if one of the conservative semigroups $S_{c,i}$ is active on a time-interval then the whole solution does not experience any decay on this time-interval;
- Of course, it is possible that some components decay "on their own". But in general the whole solution does not decay.
- Still, when one semigroups $S_{d,i}$ is active, the L^p norms of the solutions stay bounded thanks to the positive semidefiniteness of B.

Recalling that $\forall i \in [1, p]$

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$$\mathcal{I}(x,t) = \bigcup_{i=1}^{p} [t_{1,i}(x,t), t_{2,i}(x,t)] \tag{7}$$

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And computations of the following type:

$$\begin{aligned}
\|v_1(.,t)\|_{L^2(\mathbb{R})} &= \|S_{d,1}(t)S_{c,1}(t_1)S_{d,1}(t_2)v_{1,0}\|_{L^2(\mathbb{R})} \\
&\leq e^{-c(t-t_1)}\|S_{c,1}(t_1)S_{d,1}(t_2)v_{1,0}\|_{L^2} \\
&\leq e^{-c(t-t_1)}\|S_{d,1}(t_2)v_{1,0}\|_{L^2} \\
&\leq e^{-c(t-t_1)}e^{-c(t_2-0)}\|v_{1,0}\|_{L^2} \\
&\leq e^{-c(t-(t_1-t_2))}\|v_{1,0}\|_{L^2}
\end{aligned}$$

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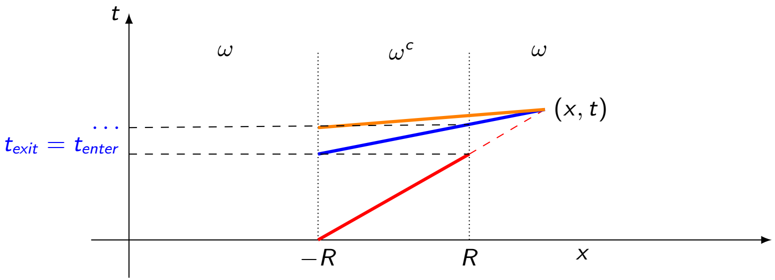
- Indeed the characteristics may overlap in the undamped region for short time and therefore "reduce the delay".
- The result from our theorem is optimal for times $t \leq \bar{\tau}$ if

$$\frac{|\lambda_i|}{|\lambda_{i+1}|} = \frac{|\lambda_{i+1}|}{|\lambda_{i+2}|} \quad \forall i \in [1, p-2] \quad \text{or} \quad \forall i \in [p+1, n-2]), \qquad (9)$$



- What happens when this proportionality condition is not satisfied is nontrivial and depend on the length of the finite union of finite intervals $|\mathcal{I}|$.

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- We are able to provide a precise result in the case of three negative eigenvalues.

Asymptotic for 3 components with negative eigenvalues

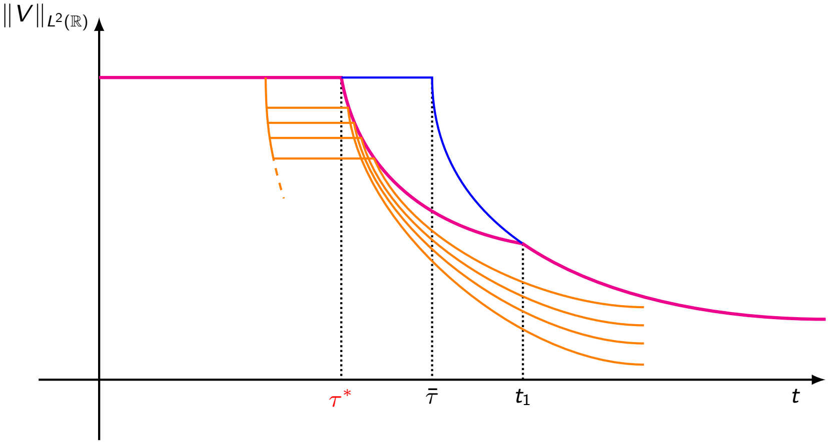

Figure: The magenta curve is the exact upper bound of the energy.