Multilevel control

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Motivation

Motivation: Selective Harmonic Modulation (SHM)

This study has been conducted in the context of the research project **CONVADP** (Elkartek program of the Basque Government - participants: **Fundación Deusto**, **Universidad de Mondragón**, **Tecnalia** and **Ingeteam**).

Scope of CONVADP

To develop new technologies to increase the power density in electronic converters for high and low power applications, including energy extraction from eolic turbines or photovoltaic panels, drivers for boats and electrical vehicles.

Employment of a converter in an eolic turbine.
Source: nutechwindparts.com

A widely-employed technique is the **Selective Harmonic Modulation**.

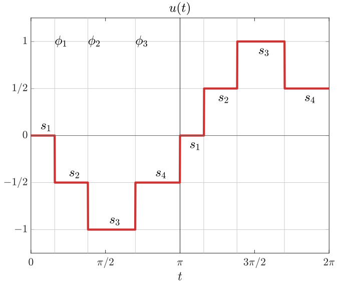
Objective of SHM

To generate a control signal with a desired harmonic spectrum by modulating some specific lower-order Fourier coefficients. This signal is constructed as a step function with a finite number of switches, taking values only in a given finite set.

IMPORTANT FEATURES:

Waveform: the sequence of values that the function takes in its domain.

Switching angles:, the sequence of points where the signal switches from one value to following one.



SHM as an optimal control problem

Mathematical formulation of SHM

$$\mathcal{U} = \{u_1, \ldots, u_L\} \subset \mathbb{R}, \quad L \geq 2$$
$$u_1 = -1,\ u_L = 1 \text{ and } u_\ell < u_{\ell+1}, \qquad \text{for all } \ell \in \{1, \ldots, L\}.$$

GOAL: construct a step function $u(t) : [0, 2\pi) \to \mathcal{U}$ with a finite number of switches, such that some of its lower-order Fourier coefficients take specific values prescribed a priori.

Half-wave symmetry

$$u(t+\pi) = -u(t) \quad \text{for all } t \in [0, \pi).$$

- $u \mapsto u|_{[0,\pi)}$
- $$u(t) = \sum_{\substack{j\in\mathbb{N} \\ j \text{ odd}}} a_j \cos(jt) + \sum_{\substack{j\in\mathbb{N} \\ j \text{ odd}}} b_j \sin(jt) \qquad a_j = \frac{2}{\pi}\int_0^\pi u(\tau)\cos(j\tau)\,d\tau$$
$$b_j = \frac{2}{\pi}\int_0^\pi u(\tau)\sin(j\tau)\,d\tau$$

Mathematical formulation of SHM

We consider **piecewise constant** functions with a **finite number** of switches.

$$u(t) = \sum_{k=0}^{K} s_k \chi_{[\phi_k, \phi_{k+1})}(t), \quad K \in \mathbb{N}$$

Waveform:

$\mathcal{S} = \{s_k\}_{k=0}^{K}$ with $s_k \in \mathcal{U}$ and $s_k \neq s_{k+1}$ for all $k \in \{0, \ldots, K\}$

Switching angles:

$\Phi = \{\phi_k\}_{k=1}^{K}$ such that $0 = \phi_0 < \phi_1 < \ldots < \phi_K < \phi_{K+1} = \pi$

Mathematical formulation of SHM

In practical engineering applications, due to technical limitations, it is preferable to employ signals taking consecutive values in $\mathcal{U}$.

Staircase property

We say that a piecewise constant signal u fulfills the **staircase property** if its waveform $\mathcal{S}$ satisfies

$$(s_k^{min}, s_k^{max}) \cap \mathcal{U} = \varnothing, \qquad \text{for all } k \in \{0, \ldots, K-1\},$$

where $s_k^{min} := \min\{s_k, s_{k+1}\}$ and $s_k^{max} := \max s_k, s_{k+1}$.

Remark

Note that when $\mathcal{U} = \{-1, 1\}$ (**bilevel** problem), this property is satisfied by any piece-wise linear function $u : [0, \pi) \to \mathcal{U}$.

Mathematical formulation of SHM

SHM PROBLEM

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be finite sets of odd numbers of cardinality $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively. For any two given vectors $\mathbf{a}_T \in \mathbb{R}^{N_a}$ and $\mathbf{b}_T \in \mathbb{R}^{N_b}$, we want to construct a function $u : [0, \pi) \to \mathcal{U}$ in staircase form such that the vectors $\mathbf{a} \in \mathbb{R}^{N_a}$ and $\mathbf{b} \in \mathbb{R}^{N_b}$, defined as

$$\mathbf{a} = (a_j)_{j \in \mathcal{E}_a} \quad \text{and} \quad \mathbf{b} = (b_j)_{j \in \mathcal{E}_b}$$

satisfy

$$\mathbf{a} = \mathbf{a}_T \quad \text{and} \quad \mathbf{b} = \mathbf{b}_T.$$

Optimal control for SHM

We propose to formulate the SHM problem as an optimal control one.

The Fourier coefficients of the signal $u(t)$ are identified with the terminal state of a controlled dynamical system of $N_a + N_b$ components defined in the time-interval $[0, \pi)$.

The control of the system is the signal $u(t)$, defined as a function $[0, \pi) \to \mathcal{U}$, which has to steer the state from the origin to the desired values of the prescribed Fourier coefficients.

D. J. Oroya-Villalta, C. Esteve-Yagüe and U.B. - Multilevel Selective Harmonic Modulation via optimal control, 2021.

Step 1: dynamical system for the Fourier coefficients

For all $u \in L^\infty([0,\pi);\mathbb{R})$ we have $a_j = y_a(\pi)$ and $b_j = y_b(\pi)$ with

$$y_a(t) = \frac{2}{\pi}\int_0^t u(\tau)\cos(j\tau)d\tau \in C([0,\pi);\mathbb{R})$$

$$y_b(t) = \frac{2}{\pi}\int_0^t u(\tau)\sin(j\tau)d\tau \in C([0,\pi);\mathbb{R})$$

Fundamental theorem of calculus

The functions $y_a(\cdot)$ and $y_b(\cdot)$ are the unique solutions to the differential equation

$$\begin{cases} \dot{y}_a(t) = \dfrac{2}{\pi}\cos(jt)u(t), \ t \in [0,\pi) \\ y_a(0) = 0 \end{cases} \qquad \begin{cases} \dot{y}_b(t) = \dfrac{2}{\pi}\sin(jt)u(t), \ t \in [0,\pi) \\ y_b(0) = 0 \end{cases}$$

Optimal control for SHM

Step 1: dynamical system for the Fourier coefficients

Hence, for $\mathcal{E}_a$, $\mathcal{E}_b$, $\mathbf{a}_T$, and $\mathbf{b}_T$ given, the SHM problem can be reduced to:

SHM problem - dynamical system formulation

Find a staircase control function u such that the corresponding solution $\mathbf{y} \in C([0,\pi);\mathbb{R}^{N_a+N_b})$ to the dynamical system

$$\begin{cases} \dot{\mathbf{y}}(t) = \frac{2}{\pi}\mathbf{D}(t)u(t), & t \in [0,\pi) \\ \mathbf{y}(0) = 0 \end{cases}$$

satisfies $\mathbf{y}(\pi) = [\mathbf{a}_T;\mathbf{b}_T]^\top$, where

$$\mathbf{D}(t) = \begin{bmatrix} \mathbf{D}^a(t) \\ \mathbf{D}^b(t) \end{bmatrix}, \quad \mathbf{D}^a(t) = \begin{bmatrix} \cos(e_a^1 t) \\ \cos(e_a^2 t) \\ \vdots \\ \cos(e_a^{N_a} t) \end{bmatrix} \in \mathbb{R}^{N_a}, \quad \mathbf{D}^b(t) = \begin{bmatrix} \sin(e_b^1 t) \\ \sin(e_b^2 t) \\ \vdots \\ \sin(e_b^{N_b} t) \end{bmatrix} \in \mathbb{R}^{N_b}$$

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\}$$

Step 2: time reversion

We can reverse the time using the transformation $\mathbf{x}(t) = \mathbf{y}(\pi - t)$. In this way, the SHM problem turns into the following null controllability one.

SHM via null controllability

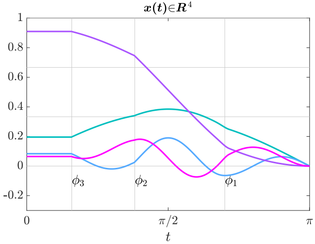
Let $\mathcal{U}$, $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\mathbf{a}_T$ and $\mathbf{b}_T$ be given. We look for a staircase function

$$u : [0, \pi) \to [-1, 1]$$

such that the solution to the initial-value problem

$$\begin{cases} \dot{\mathbf{x}}(t) = \dfrac{2}{\pi}\mathbf{C}(t)u(t), & t \in [0, \pi) \\ \mathbf{x}(0) = [\mathbf{a}_T, \mathbf{b}_T]^\top =: \mathbf{x}_0 \end{cases}$$

with $\mathbf{C} = -\mathbf{D}$ satisfies $\mathbf{x}(\pi) = 0$.



Evolution in the time horizon $[0, \pi)$ of the dynamics $\mathbf{x}$ with $\mathcal{E}_a = \mathcal{E}_b = \{1, 3\}$.

Step 3: optimal control problem for SHM

$\mathcal{A}_{ad} := \left\{ u : [0, \pi) \to [-1, 1] \text{ measurable satisfying the staircase property} \right\}$

Optimal control problem for SHM

Let $\mathcal{U}$, $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\mathbf{a}_T$ and $\mathbf{b}_T$ be given. We look for an admissible control $u \in \mathcal{A}_{ad}$ solution to the following optimal control problem:

$$\min_{u \in \mathcal{A}_{ad}} \frac{1}{2} \|\mathbf{x}(\pi)\|^2. \qquad (OCP1)$$

Remark

The cost functional (OCP1) is quadratic and the existence of at least one minimizer is ensured for any target $[\mathbf{a}_T, \mathbf{b}_T]^\top$.

Such a minimizer solves the SHM problem if and only if the minimum of (OCP1) is zero. Otherwise, the target $[\mathbf{a}_T, \mathbf{b}_T]^\top$ is unreachable.

Due to the limitations on the size of controls ($u \in \mathcal{U}$) and the time horizon ($T = \pi$), not every target $[\mathbf{a}_T, \mathbf{b}_T]^\top \in \mathbb{R}^{N_a + N_b}$ is reachable.

Step 4: penalized optimal control problem for SHM

The optimal control problem (OCP1) is defined on the non-convex set $\mathcal{A}_{ad}$ to take into account the staircase constraints on u.

In order to have a convex optimal control problem, we add a penalization term for the control to the cost functional, and remove the staircase constraint on the control.

Penalized OCP for SHM

Fix $\varepsilon > 0$ and a convex function $\mathcal{L} \in C([-1,1];\mathbb{R})$. Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\mathbf{a}_T$ and $\mathbf{b}_T$ be given. We look for a control

$$u \in \mathcal{A} := \left\{ u : [0,\pi) \to [-1,1] \text{ measurable} \right\}$$

solution to the following optimal control problem:

$$\min_{u \in \mathcal{A}} \left(\frac{1}{2} \|\mathbf{x}(\pi)\|^2 + \varepsilon \int_0^\pi \mathcal{L}(u(t))dt \right). \qquad (OCP2)$$

The staircase property for u can be ensured by a suitable choice of the penalization term $\mathcal{L}$.

Theorem

Let $\mathcal{U}$ and $\mathbf{x}_0$ be given. For any $\alpha > 0$ and $\beta \in \mathbb{R}$, set the function

$$\mathcal{P}(u) = \alpha(u-\beta)^2.$$

Consider (OCP2) with

$$\mathcal{L}(u) = \begin{cases} \lambda_\ell(u) & \text{if } u \in [u_\ell, u_{\ell+1}) \\ \mathcal{P}(1) & \text{if } u = u_L \end{cases} \quad \text{for all } \ell \in \{1,\dots,L-1\},$$

$$\lambda_\ell(u) := \frac{(u-u_\ell)\mathcal{P}(u_{\ell+1}) + (u_{\ell+1}-u)\mathcal{P}(u_\ell)}{u_{\ell+1}-u_\ell}.$$

Assume in addition that $\mathcal{L}$ has a unique minimum in $[-1,1]$. Then, (OCP2) admits a unique minimizer u_ε which has the multilevel and staircase structure. Moreover, u_ε is continuous with respect to $\mathbf{x}_0$ in the strong topology of $L^1(0,\pi)$. Finally, the associated optimal trajectory $\mathbf{x}_\varepsilon$ satisfies $\|\mathbf{x}_\varepsilon(\pi)\|^2_{\mathbb{R}^N} \leq 4\pi\varepsilon\|\mathcal{L}\|_\infty$.

Proof (sketch)

Existence and uniqueness of the minimizer: they can be obtained via a standard argument since the functional is convex with respect to the control, the admissible controls in $\mathcal{A}$ are uniformly bounded and the dynamical constraints are linear.

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Existence and uniqueness of the minimizer: they can be obtained via a standard argument since the functional is convex with respect to the control, the admissible controls in $\mathcal{A}$ are uniformly bounded and the dynamical constraints are linear.

Continuity of solutions: the argument uses the fact that the optimal solutions are uniformly bounded in $BV(0, \pi) \hookrightarrow L^1(0, \pi)$ with compact embedding. More details can be found in

D. J. Oroya-Villalta, C. Esteves-Yagüe and U. B., Multilevel Selective Harmonic Modulation via optimal control, 2021.

Proof (sketch)

Multilevel structure and staircase property: introduce the Hamiltonian

$$\mathcal{H}(t, \mathbf{p}, u) = \varepsilon \mathcal{L}(u) - \mu(t)u(t), \quad \mu(t) := \frac{2}{\pi}(\mathbf{p}(t) \cdot \mathbf{D}(t))$$

and derive the optimality conditions via Pontryagin's Maximum Principle.

1. The adjoint system reads as

$$\begin{cases} \dot{\mathbf{p}}^*(t) = -\nabla_x \mathcal{H}(u(t), \mathbf{p}^*(t), t) = 0, \ t \in [0, \pi) \\ \mathbf{p}^*(\pi) = \mathbf{x}^*(\pi) \end{cases} \quad \rightarrow \quad \mathbf{p}^*(t) = \mathbf{x}^*(\pi).$$

2. **Optimality condition:**

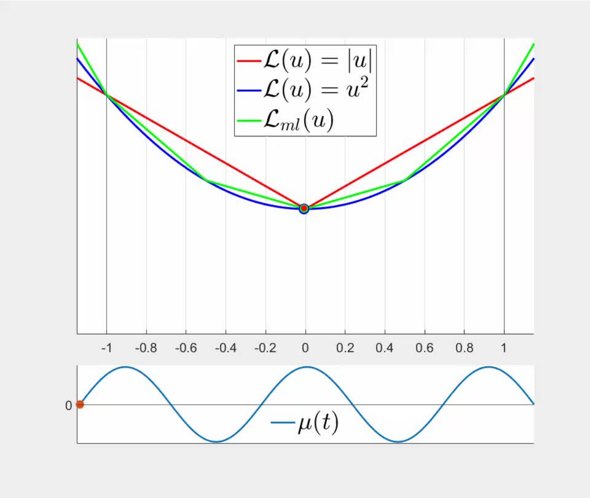
$$u^*(t) \in \operatorname*{argmin}_{|u| \leq 1} \Big[\varepsilon \mathcal{L}(u) - \mu^*(t) u \Big]$$

$$\mu^*(t) := \frac{2}{\pi}(\mathbf{x}^*(\pi) \cdot \mathbf{D}(t)) = \sum_{j \in \mathcal{E}_a} a_j^*(\pi) \cos(jt) + \sum_{j \in \mathcal{E}_b} b_j^*(\pi) \sin(jt).$$

With our choice of $\mathcal{L}(u)$, the above argmin is a singleton for a.e. $t \in [0, \pi)$, except for a finite number of times (the switching angles).

Staircase property:

$$\mathcal{H}(u) = \varepsilon\mathcal{L}(u) - \mu(t)u$$



Adjoint formulation

Adjoint formulation of the SHM problem

Applying the **Fenchel-Rockafellar theory**, we can build the following dual problem

$$\mathbf{p}_{\varepsilon,\pi} = \underset{\mathbf{p}_\pi \in \mathbb{R}^N}{\operatorname{argmin}} \, \mathcal{J}_\varepsilon(\mathbf{p}_\pi)$$

$$\mathcal{J}_\varepsilon(\mathbf{p}_\pi) = \int_0^\pi \mathcal{L}^\star\left(\mathbf{C}^\top(t)\mathbf{p}_\pi\right) dt + \frac{\varepsilon}{2} \|\mathbf{p}_\pi\|^2_{\mathbb{R}^N} + \langle \mathbf{x}_0, \mathbf{p}_\pi \rangle,$$

where

$$C(\mathbb{R}) \ni \mathcal{L}^\star(v) = \sup_{u \in \mathbb{R}} \Big(uv - \mathcal{L}(u) \Big)$$

is the **convex conjugate** of $\mathcal{L}$ and is still a piece-wise linear function.

Adjoint formulation of the SHM problem

Theorem

For any $\varepsilon > 0$, there exists a unique minimizer $\mathbf{p}_{\varepsilon,\pi} \in \mathbb{R}^N$ of the functional $\mathcal{J}_\varepsilon$. Moreover, this minimizer is related with the minimizer u_ε of (OCP2) through the formulas

$$u_\varepsilon(t) \in \partial\mathcal{L}^\star\left(\mathbf{C}^\top(t)\mathbf{p}_{\varepsilon,\pi}\right), \qquad \text{for a.e. } t \in [0,\pi)$$

and

$$\mathbf{x}_\varepsilon(\pi) = -\varepsilon\mathbf{p}_{\varepsilon,\pi}.$$

U. B. and E. Zuazua, Selective Harmonic Modulation by duality, 2021.

Numerical experiments

Numerical experiments

Control set

Test 1: $\mathcal{U} = \{-1, 0, 1\}$

Test 2: $\mathcal{U} = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$

Common parameters

$\mathcal{E}_a = \mathcal{E}_b = \{1, 5, 7, 11, 13\}$

$\mathbf{a}_T = \mathbf{b}_T = (m, 0, 0, 0, 0)$
$m \in [-0.8, 0.8]$

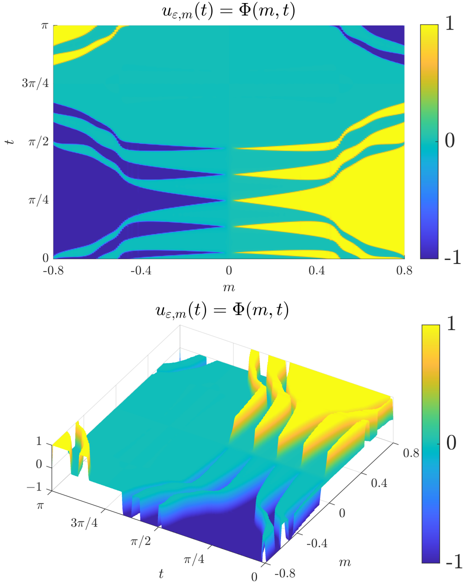
$\varepsilon = 10^{-6}$

For all the experiments, we plot the function

$$\begin{array}{rccc} \Phi: & [-0.8, 0.8] \times [0, \pi] & \longrightarrow & \mathcal{U} \\ & (m, t) & \longmapsto & u_m^*(t), \end{array}$$

where for each $m \in [-0.8, 0.8]$, $u_m^*(\cdot)$ represents the solution to the SHM problem with the desired target frequencies.

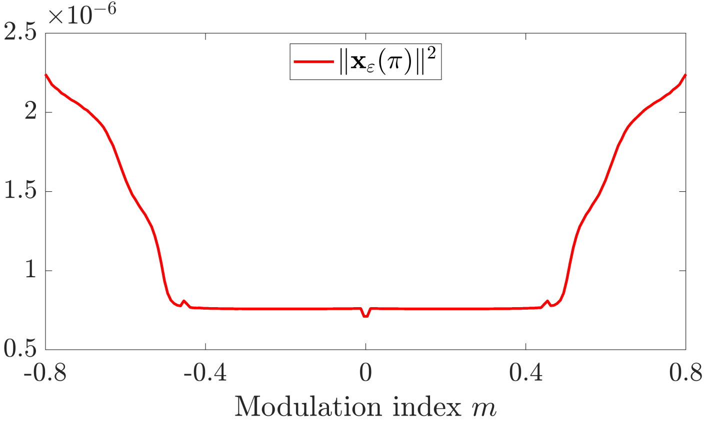
Test case 1



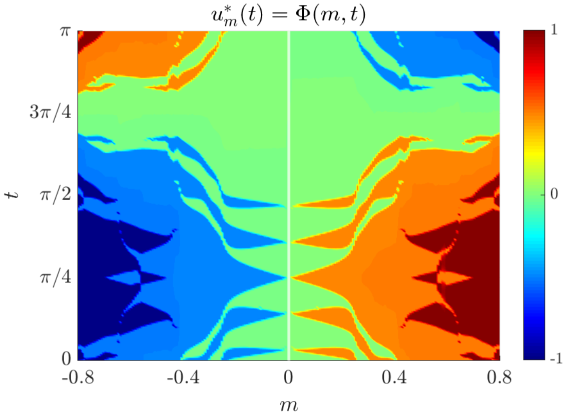
$u_{\varepsilon,m}(t) = \Phi(m,t)$
π
$3\pi/4$
$\pi/2$
$\pi/4$
0
t
-0.8
-0.4
0
0.4
0.8
m
1
0
-1
$u_{\varepsilon,m}(t) = \Phi(m,t)$
1
0
−1
π
$3\pi/4$
$\pi/2$
$\pi/4$
0
t
0.8
0.4
0
-0.4
-0.8
m
1
0
-1

Test case 1 - error

$\|\mathbf{x}_\varepsilon(\pi)\|^2_{\mathbb{R}^{10}}$ for all values of the modulation index $m \in [-0.8, 0.8]$.



Test case 2



Generalization

Multilevel control for ODE systems

The concept of multilevel control can be generalized to linear finite-dimensional controlled systems

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x}(t) + Bu(t), \quad t \in (0, T) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

satisfying the Kalman rank condition.

U. B. and E. Zuazua, Multilevel control by duality, 2021.

Multilevel control for ODE systems

Conservative or dissipative dynamics: we can construct a multilevel and staircase control solving

$$\mathbf{p}^*_{T,ml} = \underset{\mathbf{p}_T\in\mathbb{R}^N}{\operatorname{argmin}} \mathcal{J}_{ml}(\mathbf{p}_T) \qquad \begin{cases} -\mathbf{p}'(t) = A^\top \mathbf{p}(t), & t \in (0,T) \\ \mathbf{p}(T) = \mathbf{p}_T \end{cases}$$

$$\mathcal{J}_{ml}(\mathbf{p}_T) = \int_0^T \mathcal{L}(B^\top \mathbf{p}(t))\,dt + \langle \mathbf{x}_0, \mathbf{p}(0)\rangle_{\mathbb{R}^N},$$

provided that the time horizon T **is large enough**.

CONTROL

$$u^*_{ml} \in \partial\Big(\mathcal{L}(B^\top \mathbf{p}^*_{ml})\Big)$$

Multilevel control for ODE systems

General dynamics: for general dynamics that satisfy the Kalman rank condition but are neither purely conservative nor purely dissipative, we can construct a multilevel and staircase control **for any** $T > 0$ solving

$$\mathbf{p}^*_{T,ml} = \underset{\mathbf{p}_T \in \mathbb{R}^N}{\operatorname{argmin}} \mathcal{J}_{ml}(\mathbf{p}_T)$$

$$\mathcal{J}_{ml}(\mathbf{p}_T) = \frac{1}{2}\left(\int_0^T \mathcal{L}(B^\top \mathbf{p}(t))\,dt\right)^2 + \langle \mathbf{x}_0, \mathbf{p}(0)\rangle_{\mathbb{R}^N}.$$

Control

$$u^*_{ml} \in \Lambda_{T,ml}\partial\Big(\mathcal{L}(B^\top \mathbf{p}^*_{ml})\Big) \quad \textit{with} \quad \Lambda_{T,ml} := \int_0^T \mathcal{L}(B^\top \mathbf{p}^*_{ml}(t))\,dt$$

Numerical experiments

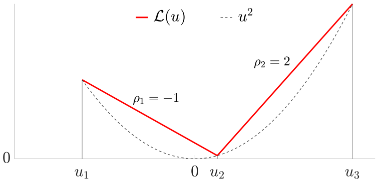
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \qquad T = 4$$

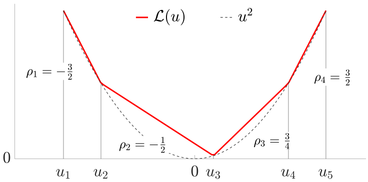
Control set

$$\mathcal{R} = \left\{ -1, 2 \right\}$$

Control set

$$\mathcal{R} = \left\{ -\frac{3}{2}, -\frac{1}{2}, \frac{3}{4}, \frac{3}{2} \right\}$$





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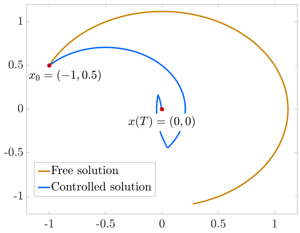

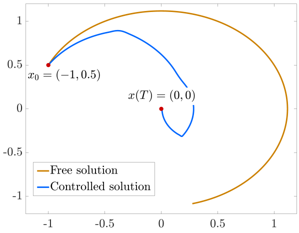

Numerical experiments

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Control set

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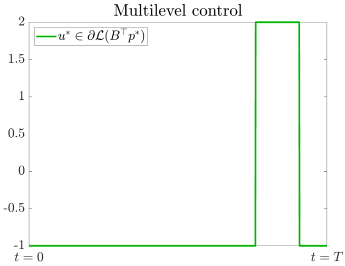

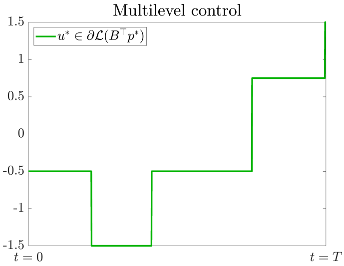

Open problems

Open problems

Minimal number of switching angles

In practical applications, to optimize the converters' performance, it is required to maintain the number of switches in the SHM signal the lowest possible.

Characterization of the solvable set

It would be interesting to have a full characterization of the solvable set for the SHM problem, thus determining the entire range of Fourier coefficients which can be reached by means of our approach.

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