From particles to cells: kinetic modeling of living complexity

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MATHEMATICS

Outline

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A brief historical excursus on complex living systems

- **Kant (1790)**, Critique de la raison pure, Traduction Francaise, Press Univ. de France, 1967
  Living systems: Special structures organized and with the ability to chase a purpose.

- **Hartwell - Nobel Laureate 2001, Nature 1999**
  • Biological systems are very different from the physical or chemical systems analyzed by statistical mechanics or hydrodynamics. Statistical mechanics typically deals with systems containing many copies of a few interacting components, whereas cells contain from millions to a few copies of each of thousands of different components, each with very specific interactions.
  • Although living systems obey the laws of physics and chemistry, the notion of function or purpose differentiate biology from other natural sciences. Organisms exist to reproduce, whereas, outside religious belief rocks and stars have no purpose. Selection for function has produced the living cell, with a unique set of properties which distinguish it from inanimate systems of interacting molecules. Cells exist far from thermal equilibrium by harvesting energy from their environment.

Some useful references...

- E. Mayr, What Evolution Is, (Basic Books, New York, 2001).
- A.L. Barabasi, Linked. The New Science of Networks, (Perseus Publishing, Cambridge Massachusetts, 2002.)
- F. Schweitzer, Brownian Agents and Active Particles, (Springer, Berlin, 2003)
- M.A. Nowak, Evolutionary Dynamics, Princeton Univ. Press, (2006).
- N.N. Taleb, The Black Swan: The Impact of the Highly Improbable, 2007.
- N.Bellomo, Modelling Complex Living Systems. A Kinetic Theory and tochastic Game Approach, (Birkhauser-Springer, Boston, 2008).
- K. Sigmund, The Calculus of Selfishness, Princeton Univ. Press, (2011).

Do Living, hence complex, systems exhibit common features?

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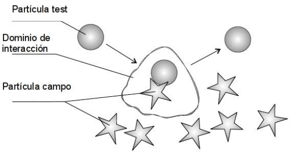
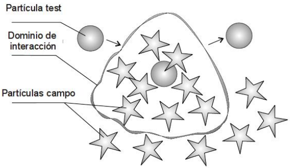
10 selected common features of complex living systems

1. Ability to express a strategy: Living entities are capable to develop specific strategies related to their organization ability depending on the state and entities in their surrounding environment. These can be expressed without the application of any principle imposed by the outer environment.
2. The said ability is not the same for all entities. Indeed, heterogeneous behaviors characterize a great part of living systems. Namely, the characteristics of interacting entities can even differ from an entity to another belonging to the same structure.
3. Large number of components: Complexity in living systems is induced by a large number of variables, which are needed to describe their overall state. Therefore, the number of equations needed for the modeling approach may be too large to be practically treated.

4. Interactions: Interactions are nonlinearly additive and involve immediate neighbors, but in some cases also distant particles, as living systems have the ability to communicate and may possibly choose different observation paths. In some cases, the topological distribution of a fixed number of neighbors can play a prominent role in the development of the strategy and interactions.



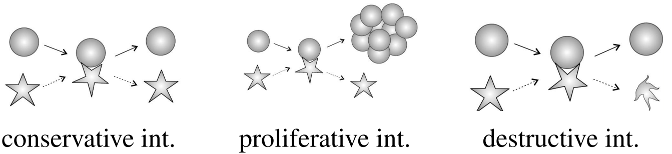
Short range interactions



Long range interactions

5. Stochastic games: Living entities at each interaction play a game with an output that is technically related to their strategy often related to surviving and adaptation ability, namely to a personal search of fitness. The output of the game is not generally deterministic even when a causality principle is identified.
6. Learning ability: Living systems have the ability to learn from past experience. Therefore their strategic ability and the characteristics of interactions among living entities evolve in time.
7. Multiscale aspects: The study of complex living systems always needs a multiscale approach. For instance in biology, the dynamics of a cell at the molecular (genetic) level determines the ability of cells to express specific functions.

8. Darwinian selection and time as a key variable: All living systems are evolutionary. For instance birth processes can generate individuals more fitted to the environment, who in turn generate new individuals again more fitted to the outer environment.



9. Large deviations: The observation of living systems should focus on the emerging behaviors that appear in non-equilibrium conditions. Very similar input conditions reproduce, in several cases, the qualitative behaviors. However, large deviations can be observed corresponding to small changes in the input conditions.
10. Emerging behaviors: Large living systems show collective emerging behaviors that are not directly related to the dynamics of a few interacting entities. Generally, emerging behaviors are reproduced only at a qualitative level.

Emerging behaviors (examples)

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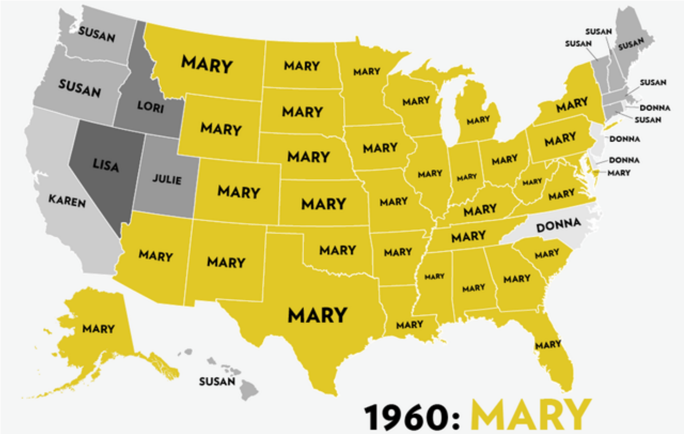
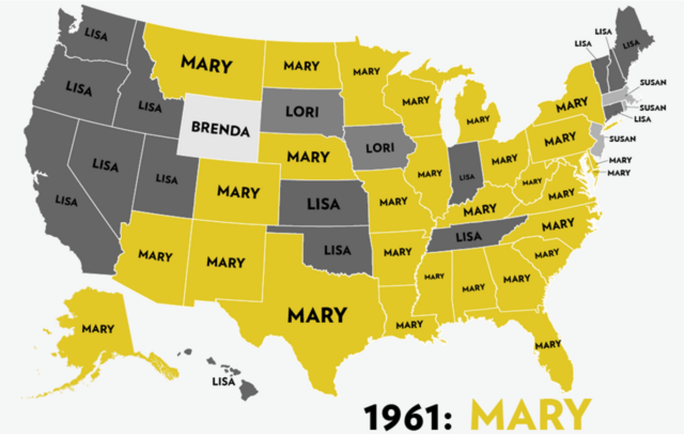


Emerging behaviors (examples)



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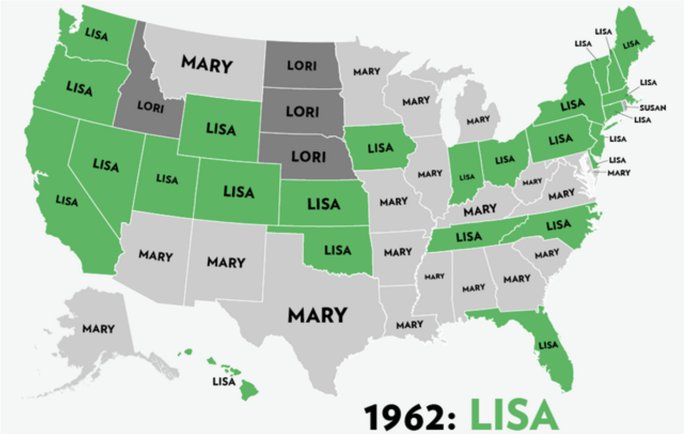

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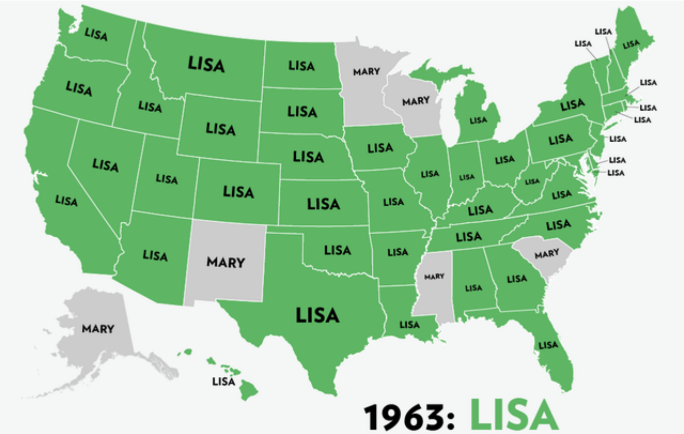

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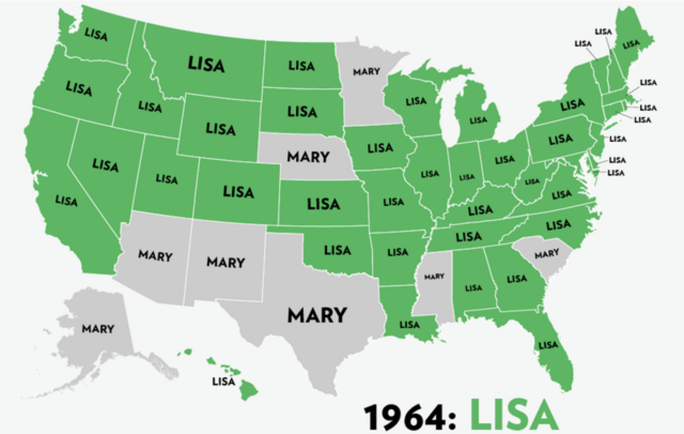

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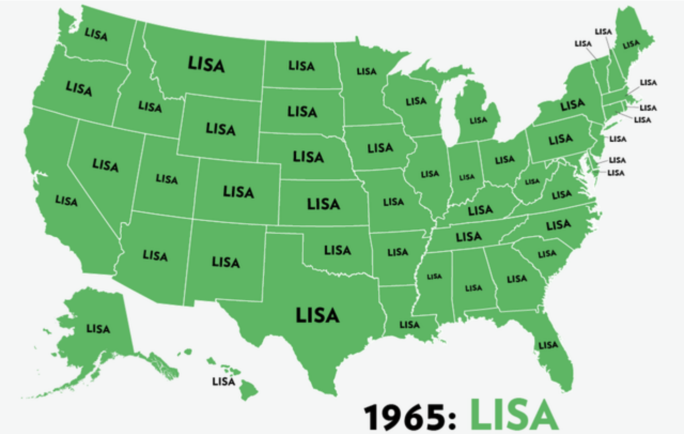

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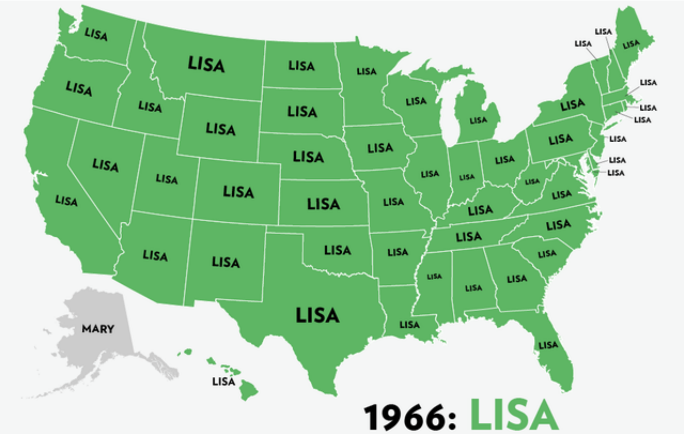

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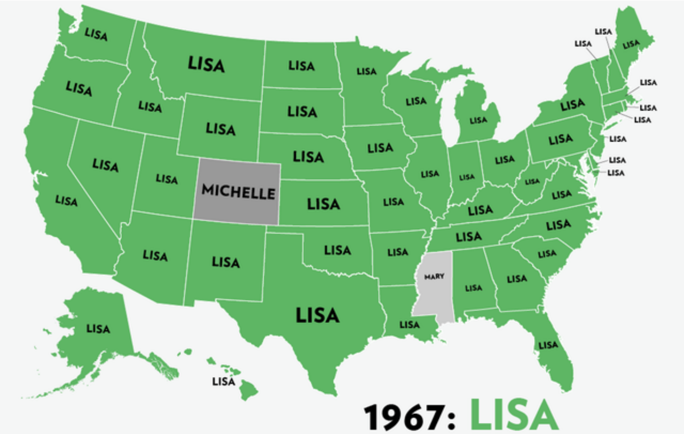

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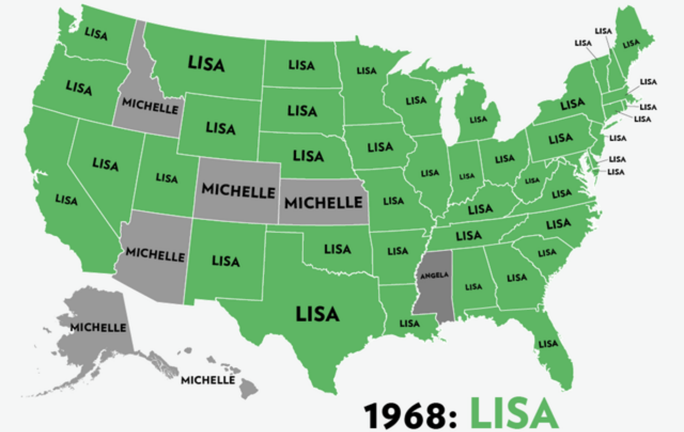

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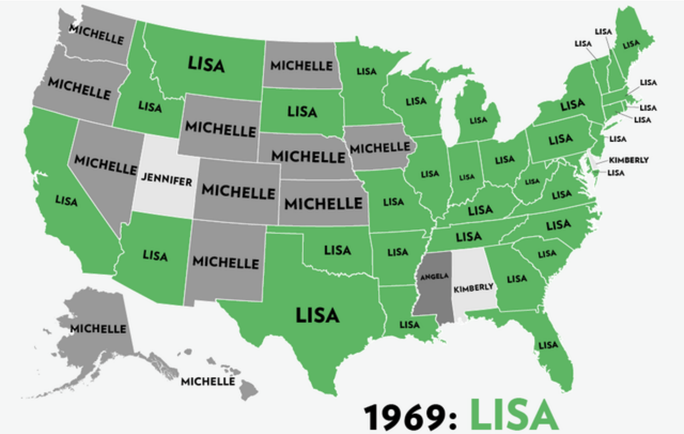

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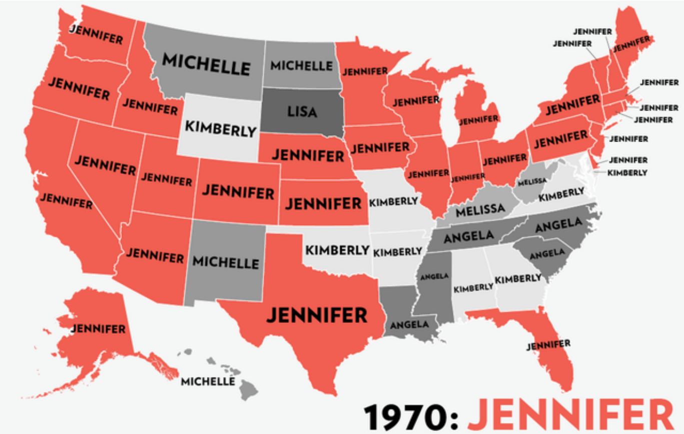

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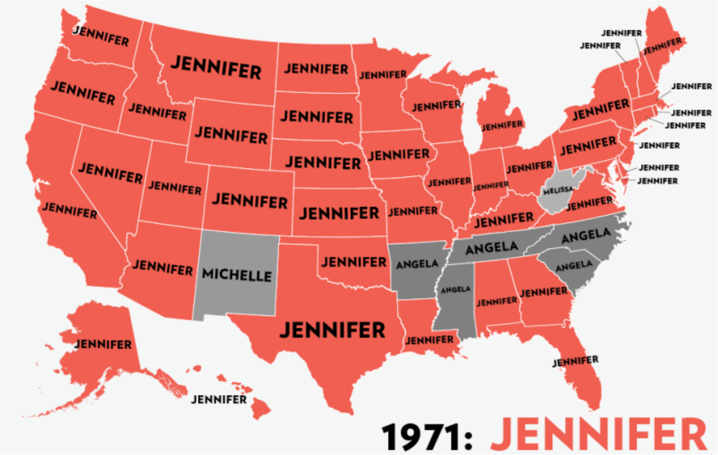

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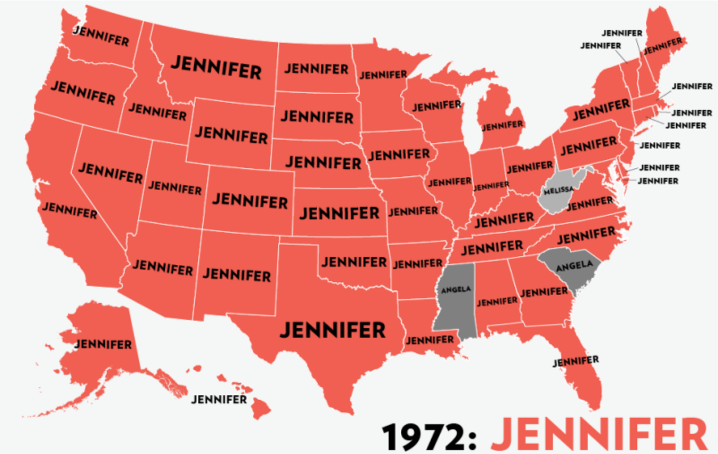

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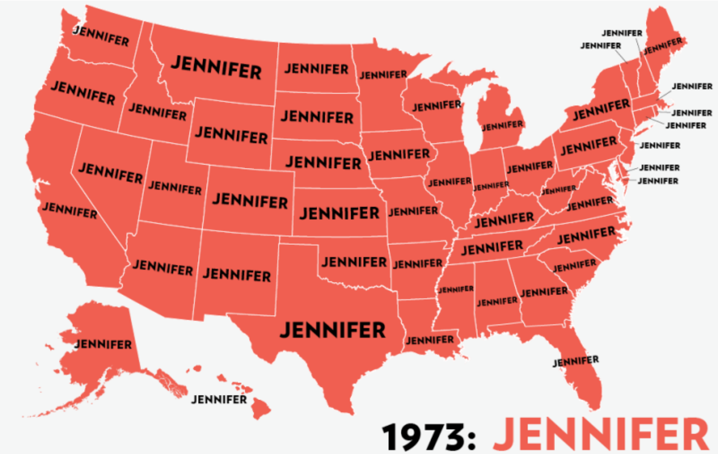

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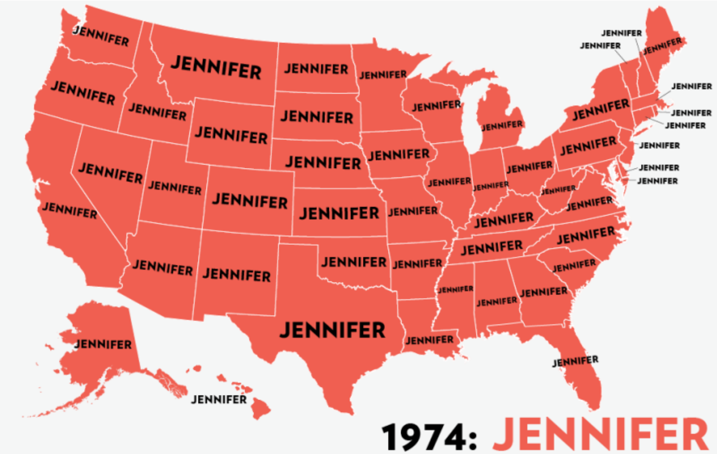

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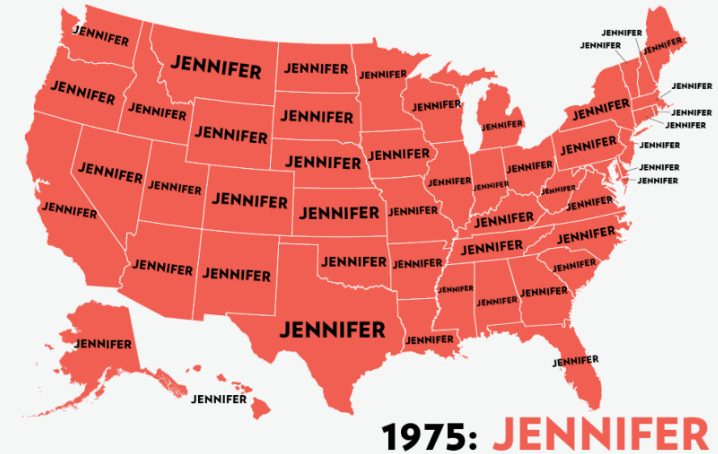

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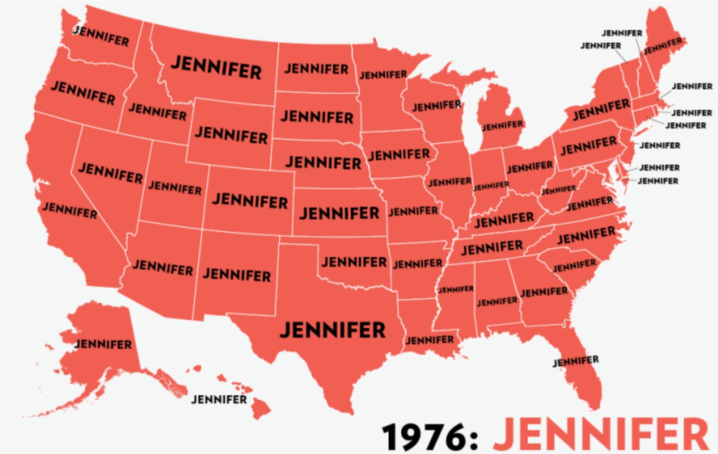

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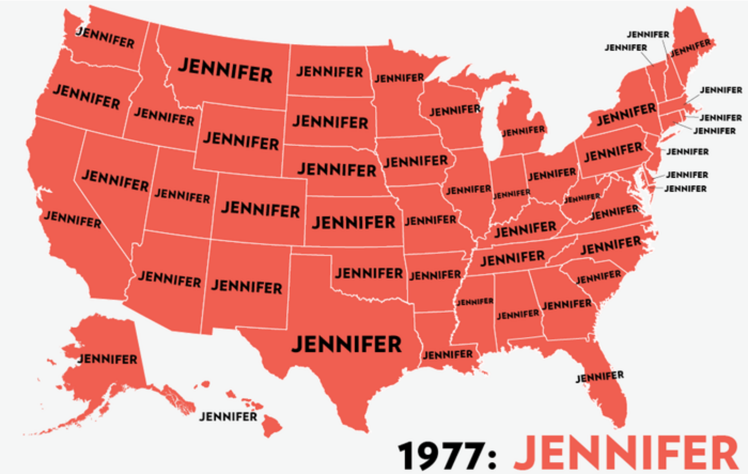

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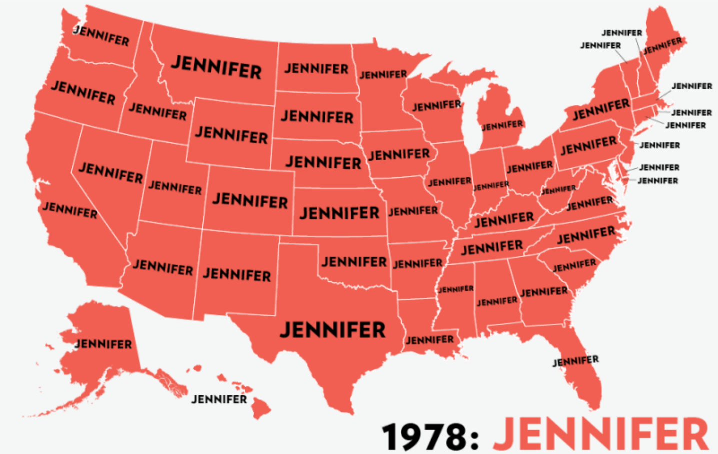

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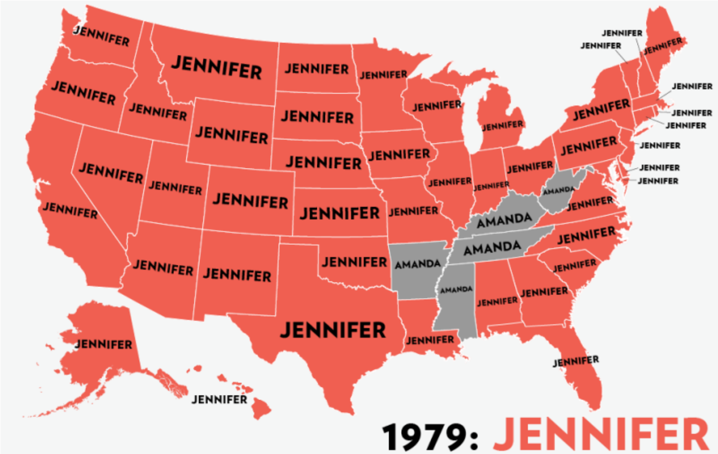
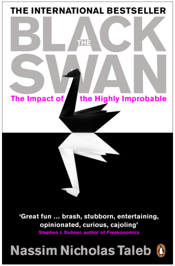


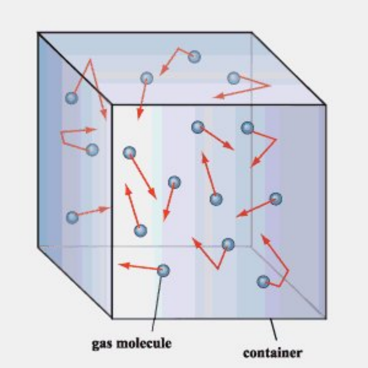
What is the black swan?



"A Black Swan is a highly improbable event with three principal characteristics: It is unpredictable; it carries a massive impact; and, after the fact, we concoct an explanation that makes it appear less random, and more predictable, than it was."
Since it is very difficult to predict directly the onset of a black swan, it is useful looking for the presence of early signals.

Outline

Kinetic theory of active particles



Hallmarks of the kinetic theory of active particles

- The overall system is subdivided into *functional subsystems* constituted by entities, called *active particles*, whose individual state is called *activity*;
- The state of each functional subsystem is defined by a suitable, time dependent, distribution function over the activity variable;
- Interactions are modeled by games, more precisely stochastic games, where the state of the interacting particles and the output of the interactions are known in probability;
- Interactions are delocalized and nonlinearly additive;
- The evolution of the distribution function is obtained by a balance of particles within elementary volumes of the space of the microscopic states, where the dynamics of inflow and outflow of particles is related to interactions at the microscopic scale.

Some applications

- There are plenty of references for applications of this framework to model complex living systems
  - Crowd dynamics and evacuation problems [Bellomo, Bellouquid, DK (2013)], [Gibelli, DK, Liao (2025), [Agnelli, Armas, DK (2025)]
  - Crowd dynamics and disease contagion [Agnelli, Buffa, DK, Torres (2023)], [Agnelli, Armas, DK (2025)]
  - Epidemics propagation [DK et al (2020)], [Burini, DK (2024, 2026)]
  - Onset and evolution of criminality [Bellomo, Colasuonno, DK, Soler (2015)]
  - Vehicular traffic dynamics [Fermo and Tosin (2013)]
  - Immune competition, tumour growth, antibiotic resistance [Bellouquid, De Angelis, DK (2013)], [DK, Trucco (2020), [Burini, DK, Serrano (2026)].
  - Migration phenomena [DK (2013)]

KTAP: Formalities

Each particle has a microscopic state **w**:

$$\mathbf{w} = (\mathbf{x}, \mathbf{v}, u) \in D_{\mathbf{w}} = D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u$$

where

- $\mathbf{x} \in D_{\mathbf{x}}$ is the geometric microscopic state (for example, the spatial position)
- $\mathbf{v} \in D_{\mathbf{v}}$ is the mechanical microscopic state (for example, the velocity)
- $u \in D_u$ is the activity

KTAP: Formalities

The description of the global state of the system is given by a function:

$$f : [0, T] \times D_{\mathbf{w}} \to [0, \infty)$$

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$f(t,\mathbf{w})d\mathbf{w}$ denotes the number of active particles whose state at time t lies in $[\mathbf{w}, \mathbf{w}+d\mathbf{w}]$. Therefore

$$n_\Lambda(t) = \int_{\Lambda \times D_{\mathbf{v}} \times D_u} f(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{x}\,d\mathbf{v}\,du$$

denotes the number of particles in $\Lambda \subset D_{\mathbf{x}}$ at time t.

From the point of view of measure theory, the existence of f can be guaranteed through the Radon–Nikodym Theorem. Indeed, if we look for a completely additive function $\phi(A)(t)$ that allows us to "count" the number of active particles at time t in each subset $A \subseteq D_{\mathbf{w}}$, such theorem ensures the existence of a non-negative integrable function f such that $\phi(A)(t) = \int_A f(t,\mathbf{w})d\mathbf{w}$.

KTAP: Formalities

Knowing f allows one to compute macroscopic quantities:

- population size at time t at position $\mathbf{x}$

$$n[f](t,\mathbf{x}) = \int_{D_{\mathbf{v}} \times D_u} f(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{v}\,du$$

- total population size at time t

$$N[f](t) = \int_{D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u} f(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{x}\,d\mathbf{v}\,du$$

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$N[f](t)$ is not necessarily constant since there may be births and/or deaths, migrations, etc.

KTAP: Formalities

If the number of active particles is constant then f can be normalized and can be thought of as a probability density.

Modeling microscopic interactions

Microscopic interactions depend on two quantities:

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- Encounter rate:

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is the frequency of encounters between a particle with state $\mathbf{w}_*$ and one with state $\mathbf{w}^*$.

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$$\eta(\mathbf{w}_*, \mathbf{w}^*)$$

  is the frequency of encounters between a particle with state $\mathbf{w}_*$ and one with state $\mathbf{w}^*$.

- Transition probability density:

$$\varphi(\mathbf{w}_*, \mathbf{w}^*; \mathbf{w}) : D_{\mathbf{w}} \times D_{\mathbf{w}} \times D_{\mathbf{w}} \to \mathbb{R}_+$$

  is the probability that a particle with state $\mathbf{w}_*$, when interacting with another with state $\mathbf{w}^*$, moves to a state $\mathbf{w}$.

Modeling microscopic interactions

φ must satisfy

$$\int_{D_{\mathbf{w}}} \varphi(\mathbf{w}_*, \mathbf{w}^*; \mathbf{w}) d\mathbf{w} = 1, \quad \forall \mathbf{w}_*, \mathbf{w}^*$$

Modeling microscopic interactions

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- Incoming flux C^+

$$C^+[f](t,\mathbf{w}) = \int_{D_\mathbf{w}\times D_\mathbf{w}} \eta(\mathbf{w}_*,\mathbf{w}^*)\varphi(\mathbf{w}_*,\mathbf{w}^*;\mathbf{w})f(t,\mathbf{w}_*)f(t,\mathbf{w}^*)d\mathbf{w}_*\,d\mathbf{w}^*$$

Modeling microscopic interactions

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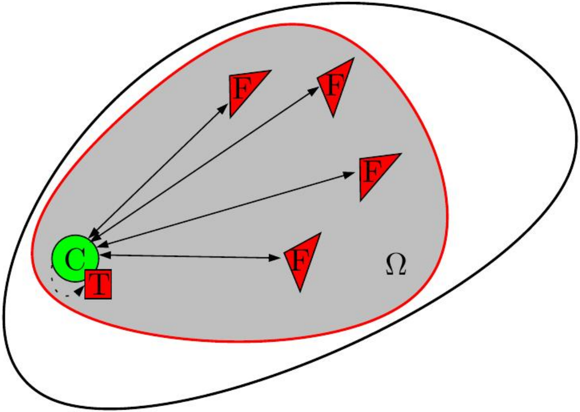
- Outgoing flux C^-

$$C^-[f](t,\mathbf{w}) = f(t,\mathbf{w})\int_{D_\mathbf{w}} \eta(\mathbf{w},\mathbf{w}^*)f(t,\mathbf{w}^*)d\mathbf{w}^*$$

Modeling interactions



Active particles interact with other particles in their action domain

Modeling microscopic interactions

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- $\mu(\mathbf{w}_*, \mathbf{w}^*; \mathbf{w})$ is the proliferation rate in state $\mathbf{w}$ due to the interaction of a particle with state $\mathbf{w}_*$ with another with state $\mathbf{w}^*$

Modeling microscopic interactions

When we have non-conservative interactions (proliferative and/or destructive) we define:

- $\mu(\mathbf{w}_*, \mathbf{w}^*; \mathbf{w})$ is the proliferation rate in state $\mathbf{w}$ due to the interaction of a particle with state $\mathbf{w}_*$ with another with state $\mathbf{w}^*$
- $\nu(\mathbf{w}, \mathbf{w}^*)$ is the destruction rate in state $\mathbf{w}$ due to the interaction of a particle with state $\mathbf{w}^*$

Modeling microscopic interactions

In this way one can compute macroscopic quantities due to proliferative and destructive events

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- Proliferation

$$P[f](t,\mathbf{w}) = \int_{D_{\mathbf{w}} \times D_{\mathbf{w}}} \eta(\mathbf{w}_*, \mathbf{w}^*)\mu(\mathbf{w}_*, \mathbf{w}^*; \mathbf{w}) f(t, \mathbf{w}_*) f(t, \mathbf{w}^*) d\mathbf{w}_* d\mathbf{w}^*$$

Modeling microscopic interactions

In this way one can compute macroscopic quantities due to proliferative and destructive events

- Proliferation

$$P[f](t,\mathbf{w}) = \int_{D_{\mathbf{w}}\times D_{\mathbf{w}}} \eta(\mathbf{w}_*,\mathbf{w}^*)\mu(\mathbf{w}_*,\mathbf{w}^*;\mathbf{w})f(t,\mathbf{w}_*)f(t,\mathbf{w}^*)d\mathbf{w}_*d\mathbf{w}^*$$

- Destruction

$$D[f](t,\mathbf{w}) = f(t,\mathbf{w})\int_{D_{\mathbf{w}}} \eta(\mathbf{w},\mathbf{w}^*)\nu(\mathbf{w},\mathbf{w}^*)f(t,\mathbf{w}^*)d\mathbf{w}^*$$

Mathematical structures

Variation rate of the number of active particles	=	Inlet flux rate caused by conservative interactions	−	Outlet flux rate caused by conservative interactio
	+	Net flux of active particles caused by proliferative/ destructive interactions	+	Net flux from/to an external agent

Mathematical structures

The previous expression would be formulated as follows:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = C^{+}[f,f] - C^{-}[f,f] + P[f,f] - D[f,f] + E$$

transport term:
net balance of particles in the volume element
of the microscopic state space
due to transport

interaction term:
net balance of particles in the volume element
of the microscopic state space
due to interactions

Mathematical structures

The previous expression would be formulated as follows:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = C^{+}[f,f] - C^{-}[f,f] + P[f,f] - D[f,f] + E$$

$$\begin{aligned}
= & \int_{D_{\mathbf{w}} \times D_{\mathbf{w}}} \eta(\mathbf{w}_*, \mathbf{w}^*) \varphi(\mathbf{w}_*, \mathbf{w}^*; \mathbf{w}) f(t, \mathbf{w}_*) f(t, \mathbf{w}^*) d\mathbf{w}_* \, d\mathbf{w}^* \\
- & f(t, \mathbf{w}) \int_{D_{\mathbf{w}}} \eta(\mathbf{w}, \mathbf{w}^*) f(t, \mathbf{w}^*) d\mathbf{w}^* \\
+ & \int_{D_{\mathbf{w}} \times D_{\mathbf{w}}} \eta(\mathbf{w}_*, \mathbf{w}^*) \mu(\mathbf{w}_*, \mathbf{w}^*; \mathbf{w}) f(t, \mathbf{w}_*) f(t, \mathbf{w}^*) d\mathbf{w}_* d\mathbf{w}^* \\
- & f(t, \mathbf{w}) \int_{D_{\mathbf{w}}} \eta(\mathbf{w}, \mathbf{w}^*) \nu(\mathbf{w}, \mathbf{w}^*) f(t, \mathbf{w}^*) d\mathbf{w}^* + E.
\end{aligned}$$

Outline

Particular cases

For each application one can particularize the general structure.

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For each application one can particularize the general structure. Examples:

- Models that do not depend on geometric/mechanical variables (social dynamics, opinion formation)
- Models with 1D geometric/mechanical variable (vehicular traffic)
- Models with 2D geometric/mechanical variable (crowd dynamics)
- Models with 3D geometric/mechanical variable (swarm dynamics)
- Proliferation/destruction phenomena (biological systems)
- Models where variables can be discretized (the equations become discrete)

Example 1: Jamarat Bridge

A. Johansson, D. Helbing, H. Al-Abideen, S. Al-Bosta, From crowd dynamics to crowd safety: A video-based analysis, Advances in Complex Systems 11 (2008)

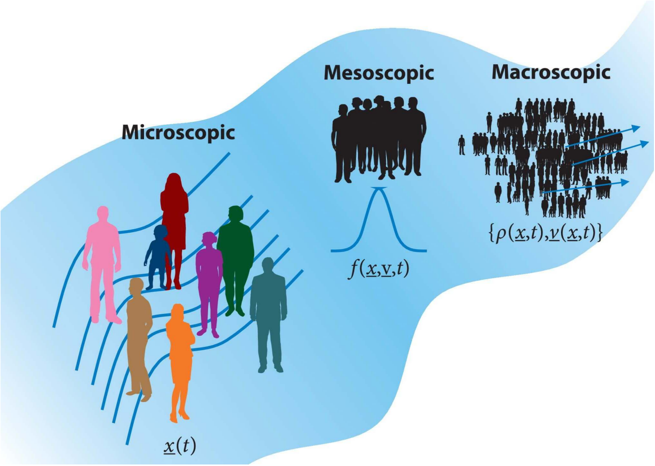
- **Where:** Mina/Makkah, Saudi Arabia
- **When:** January 12, 2006 (last day of the Hajj, muslim pilgrimage)
- **Panic triggering:** stop-and-go waves, crowd turbulence;
- **Who:** 346 deaths, 289 injuries
- **Ped density:** global density 8 ped/m^2, local density 10 ped/m^2
- **Remedial measure:** surveillance cameras, reshaping of the Jamarahs and footbridge

Example 2: Brooklyn Bridge

The New York Times. May 31, 1883

- **When:** May 24, 1883 (six days after its opening);
- **Panic triggering:** vibrations, a rumor that the Bridge was going to collapse;
- **Who:** 12 deaths
- **Ped density:** n.a., 150.300 people crossing the bridge in one day
- **Remedial measure:** a parade of 21 elephants, Jumbo included, organised by the Barnum circus to squelch doubts about the bridge's stability.

The choice of the scale for crowd dynamics

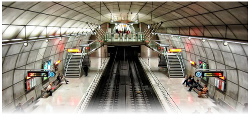


[N. Bellomo, A. Bellouquid, D.K., *From the Microscale to Collective Crowd Dynamics*, SIAM MMS (2013)]

- **What**: Evacuation of pedestrians from a venue with exits.
- **Why**:

eVACUATE

A holistic, scenario-independent,
situation-awareness and guidance system
For sustaining the Active Evacuation Route for large crowds



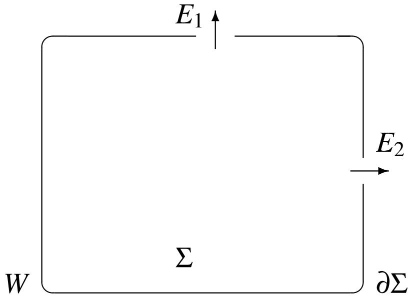
[N. Bellomo, A. Bellouquid, D.K., *From the Microscale to Collective Crowd Dynamics*, SIAM MMS (2013)]

- **What**: Evacuation of pedestrians from a venue with exits.
- **How**: Developing a kinetic approach in order to include
  - interactions with boundary

  - flow through doors

Representation of the system

- Pedestrians move in a bounded domain $\Sigma \subset \mathbb{R}^2$ assumed to be convex (this can be relaxed).
- Boundary includes an exit zone $E \subset \partial\Sigma$, E that could be a disjoint union of subsets.
- $W = \partial\Sigma \setminus E$ is the wall.



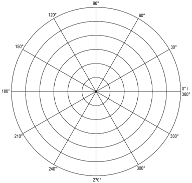
Description of the system

- Pedestrians are viewed as **active particles**
- Microscopic state (continuous-discrete hybrid features):
  - Position: $\mathbf{x} = (x, y)$
  - Velocity : $\mathbf{v} = (v, \theta)$
    - changes in velocity direction θ: probabilistic
    - changes in the velocity modulus v: deterministic

Modeling velocity

- Direction θ takes values in the discrete set:

$$I_\theta = \left\{ \theta_i = \frac{i-1}{n} 2\pi \,:\, i = 1, \ldots, n \right\}$$



- Changes of direction θ are modeled with a probabilistic approach.
- Speed v is modeled as a deterministic variable that depends on macro effects (congestion level).

Generalized distribution function

- The generalized distribution function

$$f_i(t,\mathbf{x}) = f(t,\mathbf{x},\theta_i)$$

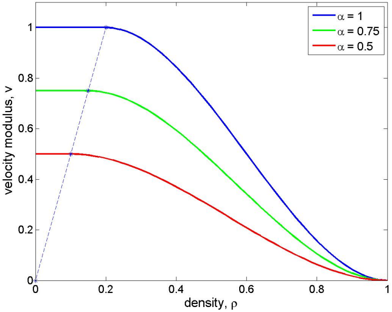
is such that $f_i(t,\mathbf{x})d\mathbf{x}$ = number of pedestrians that, at time t, are in the infinitesimal rectangle $[x, x+dx] \times [y, y+dy]$ and are moving with direction θ_i.

- Local density: $\rho(t,\mathbf{x}) = \sum_{i=1}^{n} f_i(t,\mathbf{x})$

Velocity modulus

The velocity modulus is assumed to depend on

1. level of congestion (perceived)
2. quality of environment, described by a parameter $\alpha \in [0,1]$



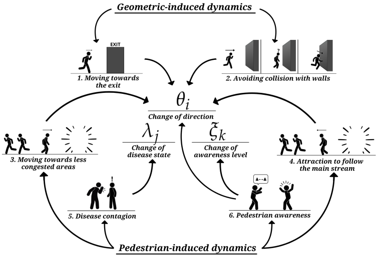
In the free flow zone ($\rho \leq \rho_c(\alpha) = \alpha/5$) pedestrians move with the maximal speed $v_m(\alpha) = \alpha$ allowed by the environment. In the slowdown zone ($\rho > \rho_c(\alpha)$) pedestrians have a velocity modulus which is heuristically modeled by the 3rd order polynomial joining the points $(\rho_c(\alpha), v_m(\alpha))$ and $(1,0)$ and having horizontal tangent in such points.

Interaction dynamics

We take into account the following effects:

1. Geometric effects
2. Congestion effects

   * ε close to 0 → search for less congested areas.
   * ε close to 1 → pure tendency to follow the stream.



Interaction dynamics

Interactions involve:

- *Test particle* $(\mathbf{x}, \theta_i)$: representative of the system at $\mathbf{x}$
- *Field particle* $(\mathbf{x}, \theta_k)$: interacts with candidate (and test) particles
- *Candidate particle* $(\mathbf{x}, \theta_h)$: may reach in probability the test state after a binary interaction with a field particle or with the environment

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We consider that transitions only occur to adjacent directions: the particle h will change its direction choosing among the three allowed outputs: θ_{h-1}, θ_h and θ_{h+1}

Geometric effects: tendency to move towards the exit

In an evacuation, individuals try to reach the exit through the shortest path.

- Given a candidate particle at $\mathbf{x}$, we define $d_E(\mathbf{x}) = \min_{\mathbf{y}\in E} \|\mathbf{x}-\mathbf{y}\|$ and the unit vector $\vec{\mathbf{v}}(\mathbf{x})$, which points from $\mathbf{x}$ to the exit.
- The tendency towards the exit is proportional to $1 - d_E(\mathbf{x})$ (it increases as the particle approaches the door).

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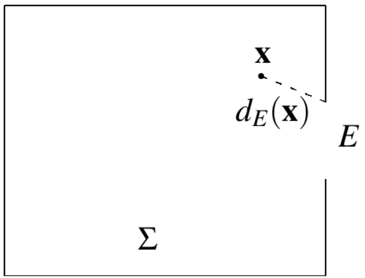
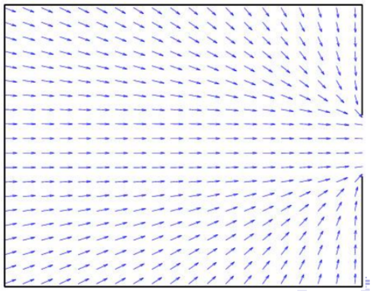
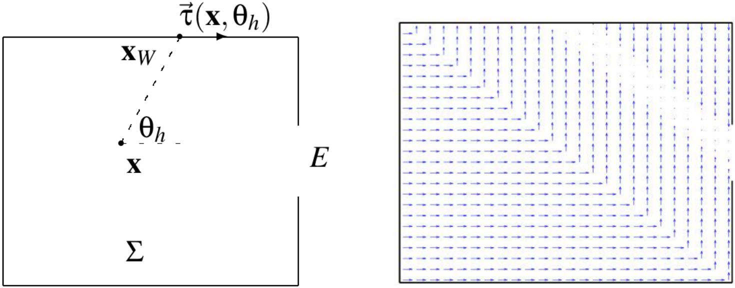




Geometric effects: tendency to avoid walls.

- A particle at $\mathbf{x}$ moving with direction θ_h would collide with the wall at $\mathbf{x}_W$. We take the unit tangent vector $\vec{\tau}(\mathbf{x}, \theta_h)$ at the point $\mathbf{x}_W$ which would help the pedestrian approach the exit.
- The tendency to avoid walls is proportional to $1 - d_W(\mathbf{x}, \theta_h)$, where $d_W(\mathbf{x}, \theta_h)$ is the distance from $\mathbf{x}$ to the wall W in direction θ_h.

Geometric effects: tendency to avoid walls.



Chosen geometric direction

- $$\vec{\omega}_G(\mathbf{x},\theta_h) = (1 - d_E(\mathbf{x}))\,\vec{\nu}_E(\mathbf{x}) + (1 - d_W(\mathbf{x},\theta_h))\,\vec{\tau}_W(\mathbf{x},\theta_h)$$

  the direction θ_G is the **optimal geometric direction** that a pedestrian at $\mathbf{x}$ moving with direction θ_h should take to reach the exit and avoid walls.

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  the direction θ_G is the **optimal geometric direction** that a pedestrian at $\mathbf{x}$ moving with direction θ_h should take to reach the exit and avoid walls.

- Geometric transition probability or "Game Table":

$$\boxed{\mathcal{A}_h(i) = \beta_h(\alpha)\delta_{s,i} + (1-\beta_h(\alpha))\delta_{h,i}, \quad i = h-1, h, h+1}$$

$$s := \arg\min_{j\in\{h-1,h+1\}} \{d(\theta_G,\theta_j)\},$$

$$\beta_h(\alpha) = \begin{cases} \alpha & \text{if} \quad d(\theta_h,\theta_G) \geq \Delta\theta \\ \alpha\frac{d(\theta_h,\theta_G)}{\Delta\theta} & \text{if} \quad d(\theta_h,\theta_G) < \Delta\theta \end{cases}$$

Congestion effects: Void vs Stream

1. Tendency to move towards less congested areas
   - A candidate particle h chooses to move towards an area with lower density

$$\vec{\gamma}(\mathbf{x},\theta_{\boldsymbol{h}}) = (\cos\theta_l, \sin\theta_l), \qquad \text{where } l := \underset{j\in\{h-1,h,h+1\}}{\arg\min} \{\partial_j \rho(t,\mathbf{x})\}.$$

   - Proportional to $\rho(t,\mathbf{x})$.

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   - Proportional to $\rho(t,\mathbf{x})$.

2. Tendency to follow the stream.
   - A candidate particle h interacts with a field particle k and may decide to "follow it" $\vec{\sigma}_{\boldsymbol{k}} = (\cos\theta_k, \sin\theta_k)$
   - Proportional to $\rho(t,\mathbf{x})$.

Chosen direction based on interactions

- $$\vec{\omega}_P(\mathbf{x},\theta_h,\theta_k) = \varepsilon\,\vec{\sigma}_k + (1-\varepsilon)\vec{\gamma}(\mathbf{x},\theta_h)$$

  the direction θ_P is the preferred direction based on interactions.

  * ε close to 0 $\rightarrow$ search for less congested areas.

  * ε close to 1 $\rightarrow$ pure tendency to follow the stream.

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  the direction θ_P is the preferred direction based on interactions.

  * ε close to 0 $\rightarrow$ search for less congested areas.

  * ε close to 1 $\rightarrow$ pure tendency to follow the stream.

- "Congestion" transition probability:

$$\boxed{\mathcal{B}_{hk}(i)[\rho] = \beta_{hk}(\alpha)\rho\delta_{r,i} + (1-\beta_{hk}(\alpha)\rho)\delta_{h,i}, \quad i = h-1, h, h+1.}$$

$$r := \operatorname*{arg\,min}_{j\in\{h-1,h+1\}} \{d(\theta_P,\theta_j)\},$$

$$\beta_{hk}(\alpha) = \begin{cases} \alpha & \text{if} \quad d(\theta_h,\theta_P) \geq \Delta\theta \\ \alpha\frac{d(\theta_h,\theta_P)}{\Delta\theta} & \text{if} \quad d(\theta_h,\theta_P) < \Delta\theta \end{cases}$$

Interaction terms (RHS).

$$\mathcal{J}_i[f](t,\mathbf{x}) = \mathcal{J}_i^G[f](t,\mathbf{x}) + \mathcal{J}_i^P[f](t,\mathbf{x})$$

$$\mathcal{J}_i^G[f](t,\mathbf{x}) = \mu[\rho(t,\mathbf{x})]\left(\sum_{h=1}^{n} \mathcal{A}_h(i) f_h(t,\mathbf{x}) - f_i(t,\mathbf{x})\right)$$

$$\mathcal{J}_i^P[f](t,\mathbf{x}) = \eta[\rho(t,\mathbf{x})]\left(\sum_{h,k=1}^{n} \mathcal{B}_{hk}(i)[\rho] f_h(t,\mathbf{x}) f_k(t,\mathbf{x}) - f_i(t,\mathbf{x})\rho(t,\mathbf{x})\right)$$

$\mathcal{J}_i^G$: difference between the gain and the loss of particles moving in direction θ_i due to geometric effects

$\mathcal{J}_i^P$: difference between the gain and the loss of particles moving in direction θ_i due to interactions among particles

Interaction rates

- *interaction rate* $\mu_h[\rho]$: frequency of encounters between candidate particle with direction θ_h and the environment. We assume $\mu_h[\rho] \propto (1-\rho)$.
- *interaction rate* $\eta_{hk}[\rho]$: frequency of encounters between particles h and k. We assume $\eta_{hk}[\rho] \propto \rho$.

Mathematical model

$$
\begin{aligned}
\partial_t f_i(t,\mathbf{x}) + \mathrm{div}_{\mathbf{x}}(\mathbf{v}_i[\rho](t,\mathbf{x}) f_i(t,\mathbf{x})) &= \mathcal{J}_i^G[f](t,\mathbf{x}) + \mathcal{J}_i^P[f](t,\mathbf{x}) \\
&= \mu[\rho(t,\mathbf{x})] \left(\sum_{h=1}^{n} \mathcal{A}_h(i) f_h(t,\mathbf{x}) - f_i(t,\mathbf{x}) \right) \\
&\quad + \eta[\rho(t,\mathbf{x})] \left(\sum_{h,k=1}^{n} \mathcal{B}_{hk}(i)[\rho] f_h(t,\mathbf{x}) f_k(t,\mathbf{x}) - f_i(t,\mathbf{x}) \rho(t,\mathbf{x}) \right)
\end{aligned}
$$

Case-studies

We select some specific cases in order to analyze the **influence on evacuation time** of

1. the **size of the exit**
2. the **initial distribution**
3. the **parameter ε**

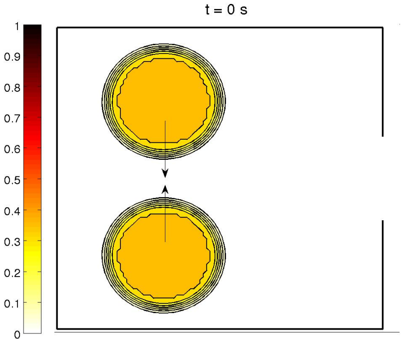
Case-studies: parameters

In all simulations we consider:

- Σ square domain of side $L = 10\,m$
- 8 different velocity directions in $I_\theta = \left\{\frac{i-1}{8}2\pi : i = 1,\ldots,8\right\}$
- Environment quality $\alpha = 1$
- Velocity modulus referred to $V_M = 2\,m/s$
- Density referred to $\rho_M = 7\,ped/m^2$
- Reference time $T_M = 13\,s$
- Interaction rates $\mu = 1 - \rho$ and $\eta = \rho$

The influence of exit size

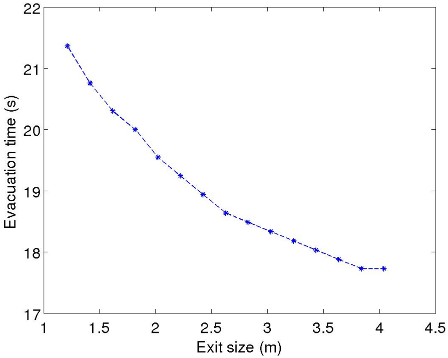
- Approximately 45 pedestrians are distributed into two circular clusters with opposite directions θ_3 and θ_7.
- Different sizes of the exit zone chosen in the range from $1{,}25\,m$ to $4\,m$.



The influence of exit size

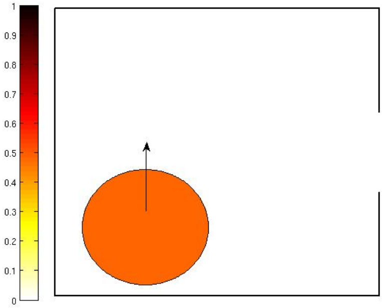
The influence of exit size

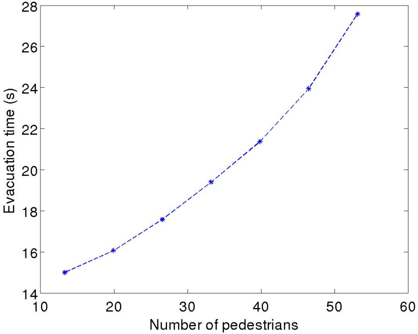
- Evacuation time increases with decreasing size of the exit.
- For sufficiently large sizes, evacuation time does not change significantly.



The influence of the initial distribution

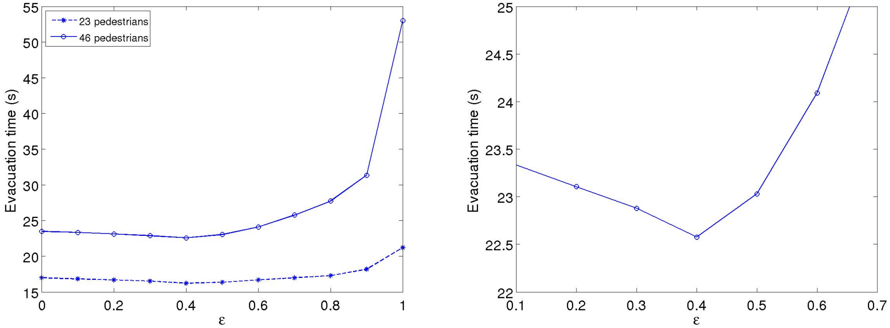
- Fixed area occupied by a crowd moving with direction θ_3.
- Different initial constant densities ρ, varying in the interval $[0.2, 0.8]$ (number of people ranging from 15 to 55).
- Door size $2\,m$.





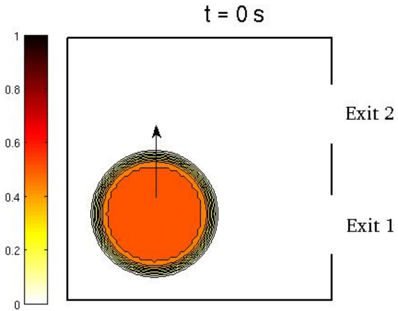
The role of the parameter ε

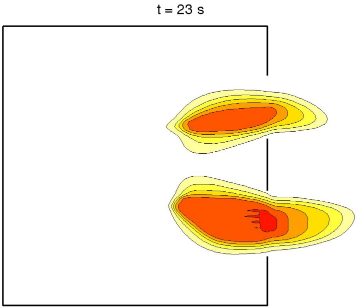
- Two different initial conditions (same area, different density).
- The optimal value of ε is approximately 0,4.
- $\varepsilon \approx 0 \rightarrow$ longer path to reach the exit, while $\varepsilon \approx 1 \rightarrow$ higher levels of congestion and reduction of the velocity modulus.



Room with two exits

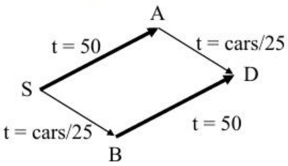
- Ability of the model to let pedestrians decide which is the most convenient way to the exit, by taking into account not only the proximity to each door, but also the less crowded path.
- Two doors of size $2{,}2\,m$ and approximately 55 pedestrians.





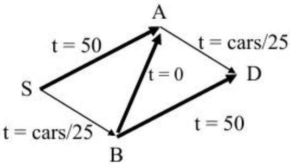
Room with two exits

And what about a paradox?



- **Road Networks**
- Suppose that 1,000 drivers wish to travel from S (start) to D (destination)
  - Two possible paths:
    - $S \to A \to D$ and $S \to B \to D$
  - The roads $S \to A$ and $B \to D$ are very long and very wide
    - $t = 50$ minutes for each, no matter how many drivers
  - The roads $S \to B$ and $A \to D$ are very short and very narrow
    - Time for each = (number of cars)/25
  - Nash equilibrium:
    - 500 cars go through A, 500 cars through B
    - Everyone's time is 50 + 500/25 = 70 minutes
    - If a single driver changes to the other route
      - There now are 501 cars on that route, so his/her time goes up

Braess paradox



- Suppose we add a new road from B to A
- It's so wide and so short that it takes 0 minutes
- New Nash equilibrium:
  - All 1000 cars go S→B→A→D
  - Time is 1000/25 + 1000/25 = 80 minutes
- To see that this is an equilibrium:
  - If driver goes *S→A→D*, his/her cost is 50 + 40 = 90 minutes
  - If driver goes *S→B→D*, his/her cost is 40 + 50 = 90 minutes
  - Both are dominated by S→B→A→D
- To see that it's the ***only*** Nash equilibrium:
  - For every traffic pattern, compute the times a driver would get on all three routes
  - In every case, S→B→A→D dominates *S→A→D* and *S→B→D*
- Carelessly adding capacity can actually be hurtful!

Braess paradox in the real world: Cheonggyecheon river

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Environment

Heart and soul of the city

The demolition of a vast motorway through the centre of South Korea's capital and the restoration of a river and park in its place proves that mega-cities can be changed for the better. John Vidal reports

John Vidal

Wednesday 1 November 2006 15.14 GMT

One year ago this month, several million people headed to a park in the centre of Seoul, the capital of South Korea and seventh largest city in the world. They didn't go for a rock festival, a football match or a political gathering, but mostly to just marvel at the surroundings, to get some fresh air and to paddle in the river that runs through it.

Preliminaries

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- We consider a model the evolution of wealth in a territory (e.g., a country) where the government implements wealth distribution policies.
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- What about the population size?

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  - favoring the rich
  - favoring the poor
  - standardize social classes
  - etc.

The mathematical model

- Let us consider a large population of individuals homogeneously distributed in a certain territory.
- Individuals are regarded as *active particles* and they are characterized by a discrete scalar variable $u \in I_u = \{u_1 = -1, \dots, u_n = 1\} \subset [-1, 1]$, called the *activity*, which represents the social state of the individuals:

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    - $u_1 = -1$ and $u_n = 1$ are, respectively, the lowest and the highest values of the social state.
    - In general, the study of the model seems to be more interesting when the number n of social classes is odd, since in this case a middle class $u_{\frac{n+1}{2}} = 0$ can be identified.
    - The use of a discrete variable is technically convenient since social levels are identified, in practice, only within intervals corresponding to discrete states.


Pirámide social argentina 2013
(Ingresos familiares)
Clases:
Top (ABC1)
Media alta (C2)
Media típica (C3)
Media baja (D1)
Baja (D2/E)
Piso x nivel mensual
%
Promedio mensual
$ 33.200
$ 12.000
$ 6.700
$ 3.100
7
17
30
32
15
$ 66.500
$ 20.800
$ 10.200
$ 4.900
$ 2.200

The mathematical model

- The overall state of the system is described by the generalized distribution functions

$$f_i : [0, T] \to \mathbb{R}_{\geq 0}, \quad i = 1, \ldots, n, \tag{1}$$

such that $f_i(t)$ denotes the probability to find an active particles that, at time t, has social state u_i. Denote by f the vector $[f_1(t), \ldots, f_n(t)]^t \in \mathbb{R}^n$.

- the total size $N(t)$ of the whole system is given by the sum of all its components:

$$N(t) = \sum_{i=1}^{n} f_i(t). \tag{2}$$

- If the total population is conserved, then $N(t) = N(0)$ for all t and f can be normalized with respect to such a number.

The mathematical model

- Higher order moments make possible to calculate another macroscopic variables. For instance, the total wealth $W(t)$ is given by the first order moment

$$W(t) = \sum_{i=1}^{n} u_i f_i(t). \tag{3}$$

- The number of poor and wealthy active particles at time t are defined by

$$N^-(t) = \sum_{i=1}^{\frac{n-1}{2}} f_i(t), \qquad N^+(t) = \sum_{i=\frac{n+3}{2}}^{n} f_i(t), \tag{4}$$

respectively. In this way, the middle class $u_{\frac{n+1}{2}} = 0$ is regarded as economically neutral.

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respectively. In this way, the middle class $u_{\frac{n+1}{2}} = 0$ is regarded as economically neutral.

  - $0 \leq N^{\pm} \leq 1$
  - $N^{-} + N^{+} \leq 1$

The mathematical model

- The *social gap* in the population is :

$$S(t) = N^{-}(t) - N^{+}(t). \tag{5}$$

Notice that $S(t) \in [-1, 1]$, negative values of S correspond to a wealthy society while positive values of S correspond to a poorer one.

- The variance of the wealth distribution will be also useful in our discussion. It is defined as the sencond order moment with respect to u minus the total wealth, namely

$$\sigma^2(t) = \sum_{i=1}^{n} u_i^2 f_i(t) - W(t)^2. \tag{6}$$

Mathematical structures

$$\boxed{\begin{array}{l}\text{Variation rate of}\\ \text{the number of}\\ \text{active particles}\end{array}} = \boxed{\begin{array}{l}\text{Inlet flux rate}\\ \text{caused by}\\ \text{conservative interactions}\end{array}} - \boxed{\begin{array}{l}\text{Outlet flux rate}\\ \text{caused by}\\ \text{conservative interactions}\end{array}}$$

$$+ \boxed{\begin{array}{l}\text{Net flux of}\\ \text{active particles}\\ \text{caused by proliferative/}\\ \text{destructive interactions}\end{array}} + \boxed{\begin{array}{l}\text{Net flux from/to an}\\ \text{external agent}\end{array}}$$

Mathematical structures

Variation rate of the number of active particles	=	Inlet flux rate caused by conservative interactions	−	Outlet flux rate caused by conservative interactions
	+	Net flux of active particles caused by proliferative/ destructive interactions	+	Net flux from/to an external agent

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = G[f] - L[f] + P[f] - D[f] + E$$

Mathematical structures

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$$+ \boxed{\begin{array}{l}\text{Net flux of}\\ \text{active particles}\\ \text{caused by proliferative/}\\ \text{destructive interactions}\end{array}} + \boxed{\begin{array}{l}\text{Net flux from/to an}\\ \text{external agent}\end{array}}$$

$$\frac{df_i}{dt} = G_i[f] - L_i[f]$$

Interaction terms

The mathematical framework to describe microscopic interactions between two subsystems can be described by means of two different functions:

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The mathematical framework to describe microscopic interactions between two subsystems can be described by means of two different functions:

- The *encounter rate* η_{hk}, that describes the interaction rate between a candidate (or test) h–particle and a field k–particle.

Interaction terms

The mathematical framework to describe microscopic interactions between two subsystems can be described by means of two different functions:

- The *encounter rate* η_{hk}, that describes the interaction rate between a candidate (or test) h–particle and a field k–particle.
- The *transition probability density* $\mathcal{B}_{hk}^i$, that describes the probability density that a candidate h–particle falls into the state u_i after an interaction with a field k–particle. This function satisfies

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Then, , for $i = 1,\ldots,n$:

$$\frac{d}{dt} f_i(t) = \sum_{h=1}^{n} \sum_{k=1}^{n} \eta_{hk} \mathcal{B}^i_{hk} f_h(t) f_k(t) - f_i(t) \sum_{k=1}^{n} \eta_{ik} f_k(t). \tag{8}$$

Interaction terms

A possible approach for the modeling of the interaction rate assumes that the closer the states of the two particles are, the higher is the frequency in which they interact:

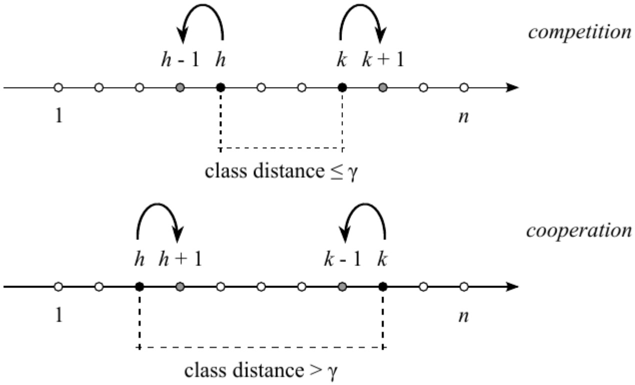
$$\eta_{hk} = \eta_0 \left(1 - \varepsilon \left|u_h - u_k\right|\right), \tag{9}$$

where $0 \leq \varepsilon \leq 1/2$.

Interaction terms

Regarding to the transition probability densities, we take here the **cooperative–competitive paradigm**. It states that there exists a social threshold $\gamma \in \{1, \dots, n-1\}$ such that if the distance between the interacting social classes $|h-k|$ is lower or equal that γ then individuals undergo a competitive interaction: the richer increases his social state while the poorer decreases it. On the other hand, if $|h-k| > \gamma$, then the opposite (altruistic) effect is observed. In all the cases, it is assumed that transitions are wealth preserving and they occur with probability $\alpha \in [0, 1]$.

Interaction terms



Interaction terms

$$
\begin{array}{ll}
h=k & \begin{cases} \mathcal{B}^h_{hh}=1, \\ \mathcal{B}^i_{hh}=0, \quad \forall i\neq h, \end{cases} \\
h\neq k & \left\{ \begin{array}{ll}
|k-h|\leq\gamma & \left\{ \begin{array}{ll}
h=1,n & \begin{cases} \mathcal{B}^h_{hk}=1, \\ \mathcal{B}^i_{hk}=0, \quad \forall i\neq h, \end{cases} \\
h\neq 1,n & \left\{ \begin{array}{ll}
h<k & \left\{ \begin{array}{ll}
k\neq n & \begin{cases} \mathcal{B}^{h-1}_{hk}=\alpha, \\ \mathcal{B}^h_{hk}=1-\alpha, \\ \mathcal{B}^i_{hk}=0, \quad \forall i\neq h,h-1, \end{cases} \\
k=n & \begin{cases} \mathcal{B}^h_{hn}=1, \\ \mathcal{B}^i_{hn}=0, \quad \forall i\neq h, \end{cases}
\end{array} \right. \\
h>k & \left\{ \begin{array}{ll}
k\neq 1 & \begin{cases} \mathcal{B}^h_{hk}=1-\alpha, \\ \mathcal{B}^{h+1}_{hk}=\alpha, \\ \mathcal{B}^i_{hk}=0, \quad \forall i\neq h,h+1, \end{cases} \\
k=1 & \begin{cases} \mathcal{B}^h_{h1}=1, \\ \mathcal{B}^i_{h1}=0, \quad \forall i\neq h, \end{cases}
\end{array} \right.
\end{array} \right.
\end{array} \right. \\
|k-h|>\gamma & \left\{ \begin{array}{ll}
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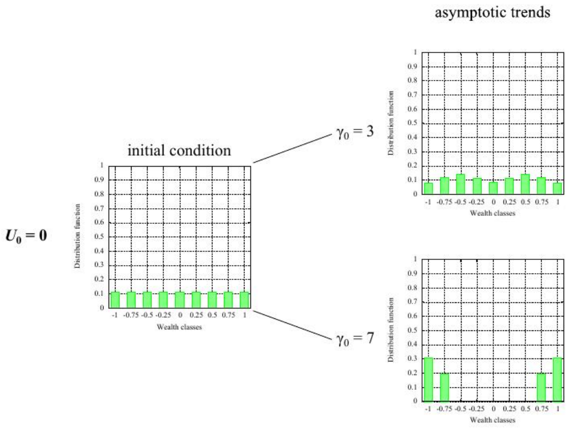
\end{array} \right.
\end{array} \right.
\end{array}
\tag{10}
$$

Some results for constant γ and α



From Bellomo, Herrero, Tosin (2013)

Formulation of the optimal control problem

$$\begin{aligned} \frac{df}{dt}(t) &= J\left(f\left(t\right),a\left(t\right)\right), \quad t \in [0,T], \\ f\left(0\right) &= f_0. \end{aligned} \tag{11}$$

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It provides the *dynamics of the system*, $f : [0, T] \to \mathbb{R}^n$ is called the *state* of the system and $a : [0, T] \to A \subset \mathbb{R}^m$ is the *control function* taking values in the *admissible region* A. In our case $m = 2$, $a(t) = (\alpha(t), \gamma(t))$ is assumed to be a measurable function and $A = [0, 1] \times \{1, \ldots, n-1\}$.

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We recall that a pair of functions $(f(\cdot), a(\cdot)) : [0,T] \to \mathbb{R}^n \times \mathbb{R}^m$ is an admissible pair of the problem if it solves the system given by equation (11).

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"I have just risen to power ($t = 0$) and I would like to get, at time T, a society with a given social profile. Which policies concerning the distance between social classes and the permeability of the society for wealth exchanges shall I implement in order to obtain my desired social profile?"
We introduce the following *objective function*:

$$P(a(\cdot)) = \int_0^T \varphi(t, f(t), a(t))\, dt + \psi(f(T)), \tag{12}$$

where φ can be interpreted as a running cost and $\psi(f(T))$ as a terminal payoff.

We are now able to pose the optimal control problem:

$$\begin{aligned}
\underset{a(\cdot)}{\text{minimize}} \quad & P\left(a\left(\cdot\right)\right) \\
\text{subject to} \quad & \frac{df}{dt}(t) = J\left(f\left(t\right), a\left(t\right)\right), \quad t \in \left[0, T\right], \\
& f\left(0\right) = f_0, \\
& a\left(t\right) \in A, \quad \forall t \in \left[0, T\right].
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Notice that (13) is a nonlinear constrained optimization problem. The problem is box and ODE–constrained, since on the one hand a is restricted to a *box* in $\mathbb{R}^2$, while on the other the state f must satisfy the system of ordinary differential equations given by the social dynamics model (11). In addition, the admissible set $A = [0,1] \times \{1, \ldots, n-1\}$ let the first variable (α) to be continuous, but the second one (γ) is an integer.

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$$
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$$

The problem cannot be tackled with a traditional nonlinear optimization method, since small changes in the variable γ do not change the solution of the system of ODEs (11), as can be noted by observing eq. (10). That is why a typical descent or gradient–based method will fail, as it will move in the direction of α only, fixing γ.

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\end{aligned}
\tag{13}
$$

Notice that $f \in \{x \in \mathbb{R}^n : x_i \geq 0, \sum x_i = 1\}$ due to the construction of the model.

Algorithm: Solving the optimal control problem.

Step 1: Subdivide the interval $[0,T]$ into M subintervals of the same length, and give an initial piecewise constant guess for the control $a^{(0)}(t) = a^{(0)}(t_k)$, with $t \in [t_k, t_{k+1}]$, $k = 0, 1, \ldots, M-1$.

Step 2: In the i-th step ($i = 0, 1, \ldots$), given the initial conditions $f(0) = f_0$ and the control $a^{(i)}$, solve the initial value problem (11) to get a solution $f_k^{(i)} = f^{(i)}(t_k)$, $k = 0, \ldots, M-1$.

Step 3: Evaluate functional (12) to obtain

$$P\left(a^{(i)}(\cdot)\right) = \int_0^T \varphi\left(t, f^{(i)}(t), a^{(i)}(t)\right) dt + \psi\left(f^{(i)}(t_M)\right).$$

Step 4: Use a convenient method to calculate $a^{(i+1)}(t_k)$, $k = 0, 1, \ldots, M-1$, such that $P\left(a^{(i+1)}(\cdot)\right) \leq P\left(a^{(i)}(\cdot)\right)$.

Step 5: Return to step 2 and repeat the procedure until the stopping criteria is satisfied.

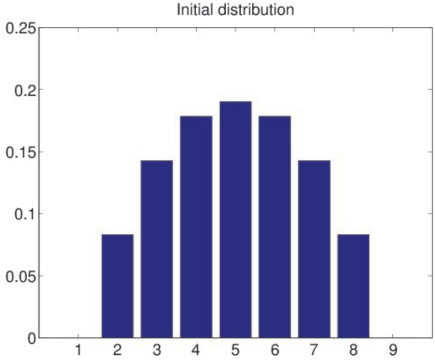
Numerical results

We propose some specific case studies characterized by:

- *The initial state of the society:* At $t = 0$, different initial profiles for wealth distribution are considered, such as poor and rich societies. A random initial distribution is also proposed, since we will see that it is useful to extract some interesting information. In addition, the society is also initially characterized by its values of γ and α, that give an idea of how fair and permeable interactions are.
- *The objective of the controller:* The ruler becomes the responsible of the population's administration at $t = 0$ and must decide what kind of society he/she glimpses at some future time. Accordingly, he/she will choose the functions φ and ψ.

Case study 1: Minimizing the variance and the social gap in a fair society

Let us consider a society with an initial distribution that is well–centered around the middle class with a social gap equal to zero.

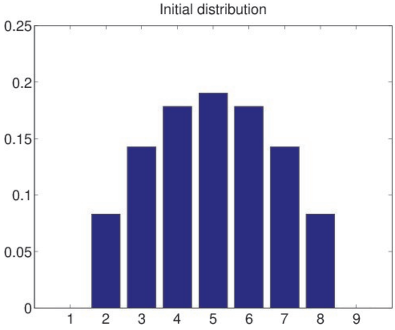


Case study 1: Minimizing the variance and the social gap in a fair society

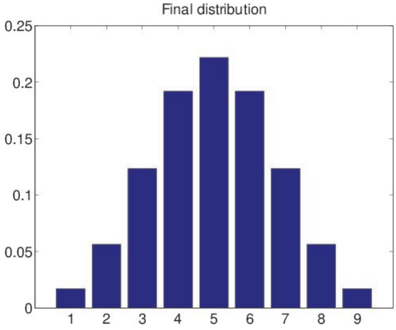
We run our optimization problem taking $\varphi(t,f(t),a(t)) = S(t,f(t),a(t)) + \sigma^2(t,f(t),a(t))$ and $\psi \equiv 0$ in the objective function (12).

Namely, the ruler rises to power and would like to minimize, even more, the variance of wealth distribution and the social gap.

Case study 1. Minimizing the variance and the social gap in a fair society



(a)



(b)

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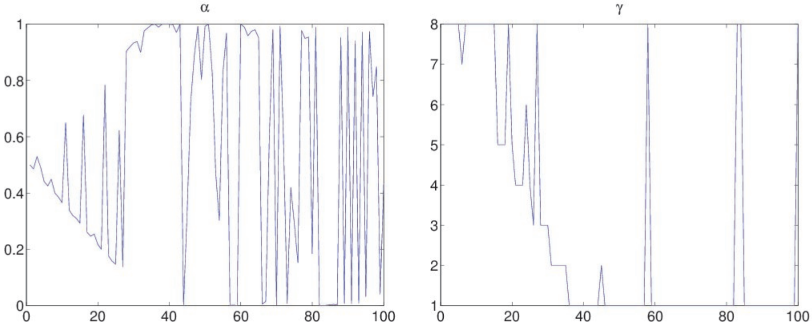

(c) (d)

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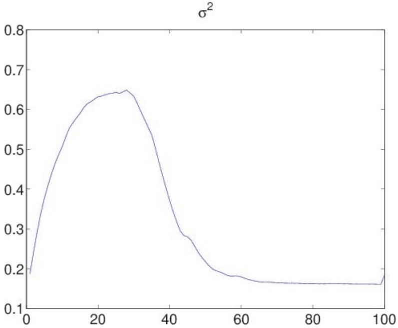

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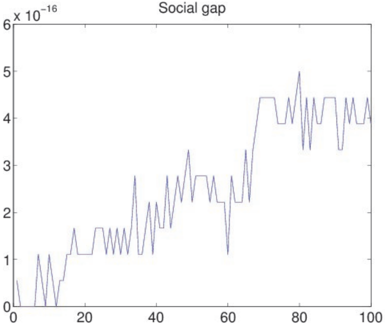

(f)

Case study 2: Minimizing the variance in a poor society

We consider a poor society with a very unfair wealth distribution. The ruler may take decisions in order to minimize the variance of the distribution, taking $\varphi(t, f(t), a(t)) = \sigma^2(t, f(t), a(t))$ and $\psi \equiv 0$ in the objective function

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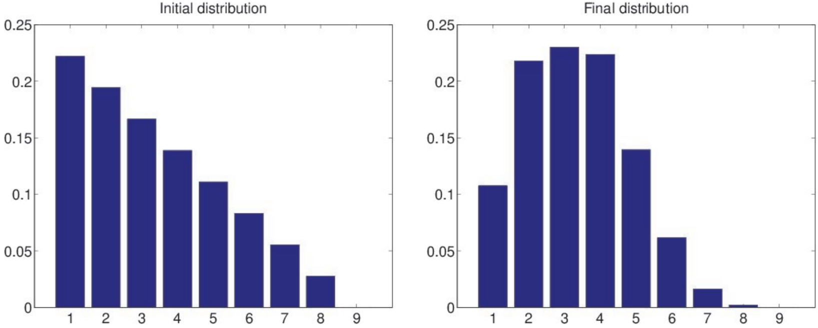

(a) (b)

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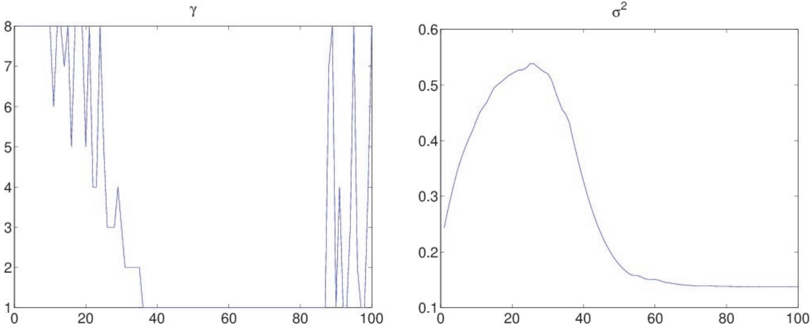
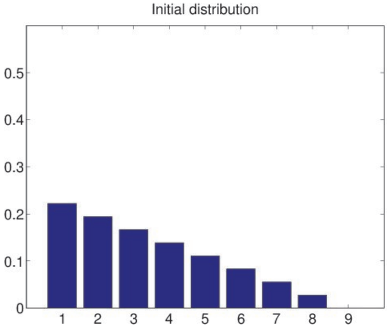


(c) (d)

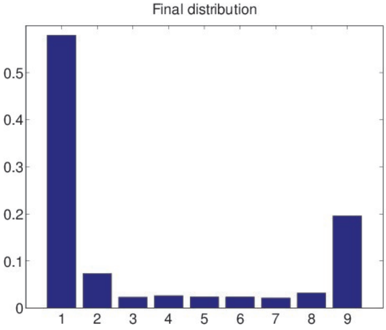
Case study 3: Maximizing the wealthy class and minimizing the terminal variance in a poor society

This case is proposed as a "counterexample" of what should not be done in order to get a fair society.
The initial wealth distribution is the same as in case 2, but the selected functional is obtained by choosing
$\varphi(t, f(t), a(t)) = -N^+(t, f(t), a(t))$ and $\psi(f) = \sigma^2$, that is the ruler would like to maximize the number of rich people over the whole time interval and to minimize the variance of the distribution only at final time.

Case study 3: Maximizing the wealthy class and minimizing the terminal variance in a poor society



(a)



(b)

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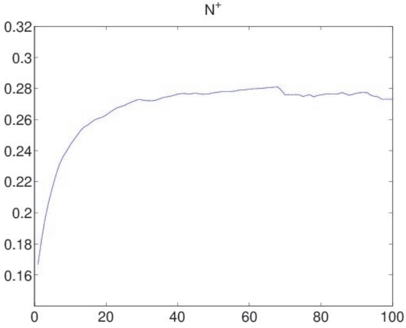
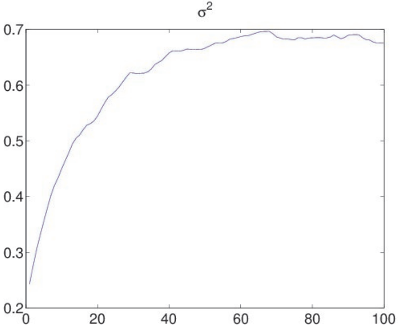


(c)

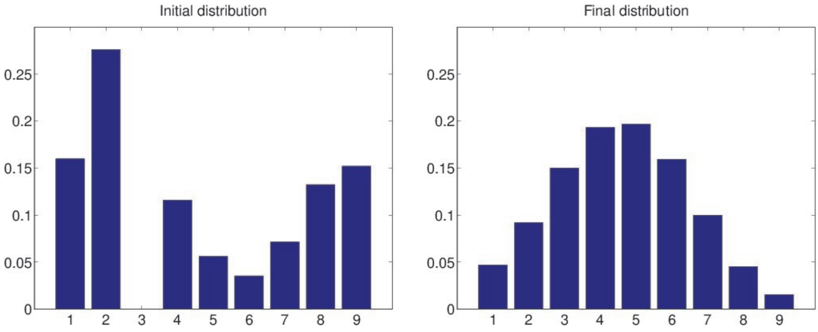


(d)

Case study 4: Regularizing a disordered society

Let us finally consider a toy example in which the society has an initial random profile generated with a normalized uniform distribution.
Several experiments have been performed, with different initial random distributions and objective functionals and we found that using $\varphi(t,f(t),a(t)) = \sigma^2(t,f(t),a(t)) - N^+(t,f(t),a(t))$ and $\psi \equiv 0$ the ruler is able to obtain an “ideal” profile

Case study 4: Regularizing a disordered society



(a) (b)

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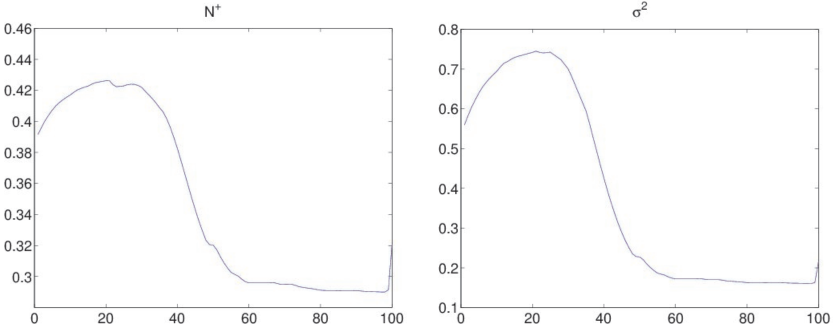
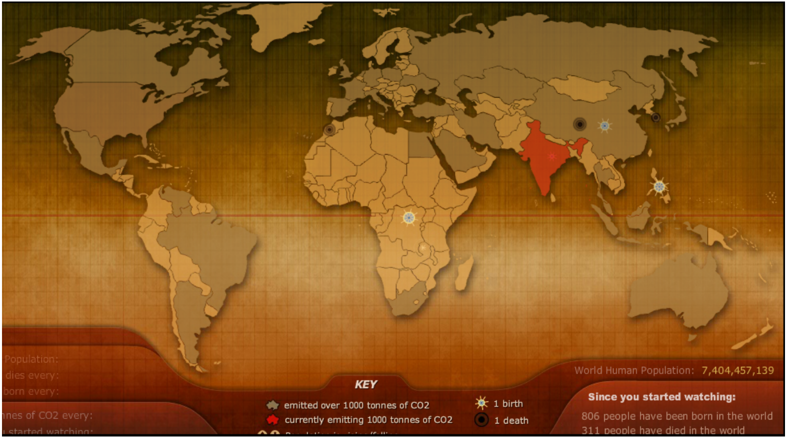


(c) (d)

The system conserves total population...



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- However, the assumption of the short–time interval is not such appropriate. In fact, according to the website *Breathingeart.net*, for instance in China one baby is born every 1.9 seconds and 1 person dies every 3.4 seconds or in Italy a different behavior is observed, with a born every 1.1 minute a death every 50.1 seconds.

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- This suggests that the dynamics of proliferation and destruction should be taken into account in the modeling approach.

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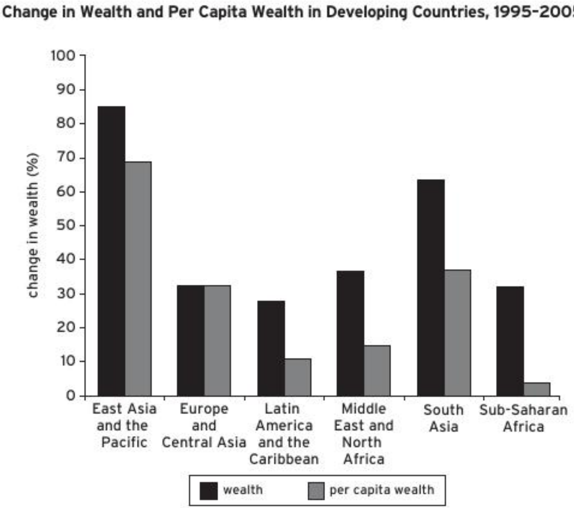
- From this definition it becomes apparent that wealth can be created or destroyed in a society. Indeed, measuring changes in wealth permits to measure the sustainability of development
- For instance, according to the World Bank, between 1995 and 2005, global wealth increased in per capita terms by 17 percent in constant 2005 U.S. dollars. It is also documented that the fastest grow was observed in the lower–middle–income countries while the majority of the world's wealth (about 82 %) is hold by the high–income countries in the Organisation for Economic Cooperation and Development (OECD).

The system conserves total wealth...



Outline

Looking ahead... interdiscipline is needed!

- Empirical data
- Empirical data on microscopic behaviours
- Social dynamics (diffusion of types of behaviours)
- Multiscale problems
- Computational problems


감사합니다
Natick
Grazie
Danke
Ευχαριστίες
Dalu
Obrigado
Thank You
Köszönöm
Tack
Спасибо
Dank
Gracias
谢谢
Merci
Seé
ありがとう