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Systèmes hyperboliques partiellement dissipatifs et applications à la mécanique des fluides

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Partially dissipative hyperbolic systems and applications to fluid mechanics.

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Résumé

Cette thèse est consacrée à l'étude de la classe des systèmes hyperboliques partiellement dissipatifs satisfaisant la condition de stabilité de Shizuta-Kawashima (souvent appelée "condition (SK)") et à un modèle multi-fluide compressible proche de cette classe mais ne vérifiant pas cette condition, le tout dans un cadre à régularité critique.

Dans sa thèse datant des années 80, Kawashima a découvert un critère systématique (la condition (SK)) assurant l'existence de solutions globales pour des systèmes partiellement dissipatifs et/ou diffusifs. Ce critère a été récemment revisité par Beauchard et Zuazua qui ont fait le lien avec la notion d'observabilité en théorie du contrôle, et ont pu démontrer l'existence globale de solutions dans des situations qui n'étaient pas couvertes par Kawashima. Pour cela, inspirés par la théorie de l'hypo-coercivité de Villani, ils ont construit des fonctionnelles de Lyapunov comprenant des termes d'ordres inférieurs.

Dans la première partie de cette thèse nous établissons l'existence de solutions globales-en-temps pour de petites données initiales dans des espaces de Besov homogènes critiques, puis nous justifions la convergence en temps grand vers des états stationnaires stables avec un taux de convergence algébrique. Pour cela, nous reprenons les arguments de Beauchard et Zuazua directement sur le système localisé en fréquences grâce à la décomposition de Littlewood-Paley et au calcul paradifférentiel, et nous analysons les basses fréquences de manière approfondie. Un point clé de notre analyse est la mise en lumière d'un mode *purement* amorti en basses fréquences qui nous permet d'obtenir des estimations plus précises que celles que l'on obtient avec la théorie de Kawashima. Cela nous permet de conclure que le système est globalement bien posé puis, dans le cas particulier du système d'Euler compressible amorti, de montrer la convergence vers l'équation des milieux poreux lorsque le coefficient d'amortissement tend vers l'infini. Pour ce problème, nous justifions la convergence forte globale-en-espace et dérivons un taux de convergence explicite dans le cadre multi-dimensionnel.

Dans la deuxième partie de cette thèse nous nous intéressons au caractère globalement bien posé et à la relaxation d'un écoulement compressible biphasique proche de la classe étudiée précédemment mais ne vérifiant pas la condition (SK) et n'étant pas symétrique. Plus précisément, nous démontrons l'existence de solutions dans des espaces de Besov homogènes critiques pour un système de Baer-Nunziato amorti et nous justifions la convergence forte de ces solutions, lorsque le paramètre de viscosité tend vers 0, vers un système de Kapila tout en dérivant un taux de convergence explicite pour le processus de relaxation. Le système initial ne satisfaisant pas la condition (SK), nous le reformulons en un couplage entre un sous-système vérifiant la condition (SK) et une équation de transport, de telle sorte que les termes de couplage soient inoffensifs. Et, pour compenser le manque de symétrie, nous considérons une fonctionnelle de Lyapunov à poids non-linéaires.

Partially dissipative hyperbolic systems and applications to fluid mechanics

Abstract

This thesis is devoted to the study of the class of partially dissipative hyperbolic systems satisfying the Shizuta-Kawashima stability condition (often called "(SK) condition") and to a compressible multi-fluid model close to this class but not satisfying this condition, all in a critical regularity framework.

In his thesis in the 80's, Kawashima discovered a systematic criterion (the (SK) condition) ensuring the existence of global solutions for partially dissipative and/or diffusive systems. This criterion was recently revisited by Beauchard and Zuazua who made the link with the notion of observability in control theory, and were able to demonstrate the global existence in situations that were not covered by Kawashima. For this purpose, inspired by Villani's hypocoercivity theory, they constructed Lyapunov functionals including lower-order terms.

In the first part of this thesis we establish the existence of global-in-time solutions for small initial data in critical homogeneous Besov spaces and justify the convergence in large time to stable stationary states with an algebraic convergence rate. To do so, we take up the arguments of Beauchard and Zuazua directly on the frequency-localized system thanks to the Littlewood-Paley decomposition and paradifferential calculus, and we analyze the low frequencies thoroughly. A key point of our analysis is to exhibit a damped mode in the low frequencies regime which allows us to recover more precise estimates than those obtained with Kawashima's theory. It allows us to conclude that the system is globally well-posed and, in the particular case of the damped compressible Euler system, to show the convergence toward the porous media equation when the damping coefficient tends to infinity. For this problem, we justify the strong convergence globally-in-space and derive an explicit convergence rate for the relaxation process in the multi-dimensional setting.

In the second part of this thesis we are interested in the well-posedness and relaxation of a compressible two-phase flow close to the class studied previously but not satisfying the (SK) condition and not being symmetric. More precisely, we prove the global existence of solutions in the framework of critical homogeneous Besov spaces for a damped Baer-Nunziato system and we justify its strong convergence, when the viscosity coefficient tends to 0, to a Kapila model while deriving an explicit convergence rate for the relaxation process. Since the initial system does not satisfy the (SK) condition, we reformulate it into a coupling between a subsystem verifying the (SK) condition and a transport equation, such that the coupling terms are harmless. And to compensate for the lack of symmetry, we consider a Lyapunov functional with nonlinear weights.

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Introduction générale

0.1 Introduction

Cette thèse est consacrée à l'étude théorique des système hyperboliques quasi-linéaires du premier ordre dits *partiellement dissipatifs*. Nous présentons des résultats concernant l'existence, le comportement en temps long de solutions ainsi que l'étude de phénomènes de relaxation pour ces systèmes dans un cadre à régularité critique. Pour cela, notre approche se base sur une localisation en fréquences : la décomposition de Littlewood-Paley et le calcul para-différentiel. De plus, inspirés par l'article récent de Beauchard et Zuazua [4] (eux-mêmes inspirés par la théorie de l'hypo-coercivité), nous performons des estimations d'énergies re-normalisées à partir de fonctionnelle de Lyapunov à poids non-linéaire incluant des termes d'ordre inférieur. Dans un premier temps nous nous intéressons à la classe générale des systèmes hyperboliques partiellement dissipatifs vérifiant la condition de stabilité établie par Shizuta et Kawashima (la condition (SK)) dans les années 80. Puis nous étudions un modèle biphasique issu de la mécanique des fluides compressibles, ne vérifiant pas cette condition de stabilité. Dans les deux cas, les aspects *hyperboliques* et *dissipatifs* seront prédominants.

Les systèmes hyperboliques modélisent des phénomènes physiques régis par des systèmes ayant une vitesse de propagation finie et/ou des lois de conservations. Typiquement, ils apparaissent en dynamique des gaz où la densité massique et l'entropie sont conservées, ou dans des schémas numériques concernant des lois de conservation avec termes de relaxation, par exemple : l'équation d'Euler compressible en dynamique des gaz, l'équation de Maxwell en électromagnétisme, l'équation d'Einstein en relativité générale, l'équation de la magnétohydrodynamique, l'équation des ondes, l'équation de Yang-Mills, l'équation de l'elastodynamique, etc. De plus, leurs applications au monde physique sont multiples, par exemple en mécanique des fluides non visqueux, théorie cinétique, astrophysique, modélisation de trafic routier, circulation dans les vaisseaux sanguins. Dans la plupart des applications au monde physique, il est essentiel de prendre en compte des phénomènes de dissipation et/ou de dispersion qui, par exemple, modélisent des effets d'amortissement dûs à des forces de frottements ou la viscosité d'un fluide. Ces effets naturels ou implémentés par l'homme permettent de "contrôler" la solution et de s'assurer qu'elle converge bien vers un état d'équilibre, c'est pourquoi les lois de conservation couplées à ce type de phénomènes se dénomment des lois d'équilibre.

En pratique, ces effets de dissipation ne sont que partiels et n'agissent pas sur toutes les inconnues d'un système d'équations. En effet, il n'est pas naturel d'inclure un terme d'amortissement (ou un contrôle) dans une équation régissant l'évolution d'une quantité conservée (par exemple la masse). Dans cette thèse, nous verrons dans quelle mesure des effets de dissipation partielle permettent d'assurer l'existence en temps long de solutions et justifierons leur pertinence lorsqu'ils sont vus comme des termes de relaxation.

Les écoulements polyphasiques sont omniprésents dans les applications au monde réel, de

l'ingénierie à la biologie, cf. [40, 84]. Dans le contexte de la mécanique des fluides, le terme polyphasique inclut des écoulements qui sont topologiquement très différents. Comme il est expliqué dans [61], nous distinguons les mélanges avec des écoulements à phases séparées (écoulements en film, écoulements à jet), les écoulements à phases mixtes ou transitoires (poches de gaz dans des liquides) et les écoulements à phases dispersées (flux de bulles, sprays). La compréhension des propriétés qualitatives de ces différents modèles est importante, par exemple, pour construire des schémas numériques plus pertinents qui augmenteraient le pouvoir prédictif de ces modèles.

Ici, c'est le cas des écoulements diphasiques dispersés que nous étudions. Pour pouvoir les décrire, il faut considérer, en plus des variables classiques telles que les densités, les vitesses et les lois d'état, deux variables supplémentaires α_+ et α_-, appelées fractions volumiques, qui mesurent la proportion de l'espace occupé par chaque phase à une position donnée dans l'espace. Il va donc sans dire que pour avoir un système ayant le bon nombre d'équations par rapport au nombre d'inconnues, en plus des équations exprimant la conservation de la masse et de la quantité de mouvement, deux équations supplémentaires sont nécessaires.

Dans [2], Baer et Nunziato ont proposé un modèle où les fractions volumiques vérifient :

$$\partial_t \alpha_\pm + v_I \cdot \nabla \alpha_\pm = \frac{P_\pm - P_\mp}{\mu}, \tag{1}$$

où les fonctions d'état des deux phases sont désignées séparément par $+$ et $-$ et μ peut être vu comme un paramètre de relaxation et, en pratique, est supposé petit. L'inconnue v_I est interprétée comme une *vitesse d'interface* qui dépend des densités, des fractions volumiques et des vitesses. Dans [2], elle coïncide avec la vitesse d'une des phases mais on trouve des hypothèses alternatives dans la littérature, voir par exemple [57] et [90]. Ici, nous supposerons que notre mélange est animé par une vitesse commune pour les deux fluides et comme dans [2] nous supposerons que la vitesse d'interface coïncide avec cette vitesse. De plus, nous incluons un terme d'amortissement dans l'équation de la vitesse pour que ce système rentre dans la classe des systèmes partiellement dissipatifs.

Nous nous intéressons aussi à la justification de phénomènes de relaxation pour les deux systèmes mentionnés ci-dessus. En pratique, le procédé de relaxation est très utile pour étudier numériquement un système, il permet de réduire le nombre de contraintes imposées sur un système et donc d'en simplifier l'étude numérique. Concrètement, ce que nous appelons relaxation ici, sera le fait de remplacer une fermeture algébrique par une équation aux dérivées partielles. Par exemple, le système limite du système bi-fluide comporte l'équation $\alpha_+ + \alpha_- = 1$ qui est relaxée par l'équation (1) dans le système initial (ou système relaxé).

Analyse des systèmes hyperboliques symétriques

L'étude des systèmes hyperboliques symétriques quasilinéaires remonte aux années 50 avec les travaux de Godunov [51, 52] et Lax [67]. Elle est motivée par celle des systèmes de lois de conservation qui apparaissent dans de nombreux modèles liés à la mécanique des fluides. Un système général de lois de conservation à n composantes s'écrit :

$$\frac{\partial V}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j(V)}{\partial x_j} = Q(V), \quad \text{où} \quad V : \begin{cases} \mathbb{R}_+ \times \mathbb{R}^d \to & \mathbb{R}^n \\ (t \ , \ x) \to & V = V(t,x) \end{cases} \tag{2}$$

est l'inconnue du problème à valeurs dans un sous-ensemble ouvert convexe $\mathcal{O}_V$ de $\mathbb{R}^n$ et $Q, F_j : \mathcal{O}_V \to \mathbb{R}^n$ sont des fonctions régulières sur $\mathcal{O}_V$.

Une des caractéristiques principales des systèmes que nous étudions dans ce manuscrit est leur nature hyperbolique, rappelons-en maintenant la définition que l'on peut trouver dans [5] par exemple. Supposons $Q(V) = 0$ pour l'instant. On considère le linéarisé de (2) autour d'un état d'équilibre constant $\bar{V}$:

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{d} \bar{A}^k \frac{\partial V}{\partial x_k} = 0 \tag{3}$$

où pour $k = 1, \ldots, n$, les $\bar{A}^k := A^k(\bar{V})$ sont des matrices réelles $n \times n$ constantes. En appliquant la transformée de Fourier à (3), on obtient :

$$\frac{\partial \widehat{V}}{\partial t} + iA(\xi)\widehat{V} = 0 \quad \text{où} \quad A(\xi) := \sum_{k=1}^{d} \xi_k \bar{A}^k. \tag{4}$$

Pour ce type de système linéaire, la notion d'hyperbolicité se caractérise ainsi.

Définition 0.1.1. *Le système (3) est dit hyperbolique lorsque*

$$\sup_{\xi \in \mathbb{R}} \| \exp(iA(\xi)) \| < +\infty.$$

Cette condition est équivalente au fait que le problème de Cauchy (3) soit bien posé dans l'espace de Sobolev H^s, pour $s \in \mathbb{R}$. Pour le système quasi-linéaire général, la condition que nous emploierons pour assurer l'aspect localement bien posé est légèrement différente. Soit A^k la matrice jacobienne de $F_k = (F_k^1, ..., F_k^n)^t$ définie dans un sous ensemble ouvert $\mathcal{O}_V \subset \mathbb{R}^d$ par :

$$A_k := \begin{pmatrix} \frac{\partial F_1^k}{\partial V_1} & \cdots & \frac{\partial F_k^1}{\partial V_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k^n}{\partial V_1} & \cdots & \frac{\partial F_n^k}{\partial V_n} \end{pmatrix} \quad \text{pour } k = 1, ..., d.$$

Nous pouvons reformuler le système (2) sous la forme non-conservative suivante :

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{d} A_k(V) \frac{\partial V}{\partial x_k} = Q(V). \tag{5}$$

La condition sur le système (5) assurant l'aspect localement bien posé que nous utiliserons est du type suivant.

Définition 0.1.2. *Le système (5) est dit Friedrichs-symétrisable si il existe une fonction régulière $S : V \mapsto S(V)$ définie sur $\mathcal{O}_V$, à valeur dans l'ensemble des matrices symétriques définies positives, telle que pour tout $V \in \mathcal{O}_V$, les matrices $(SA^1)(V), \cdots, (SA^d)(V)$ sont symétriques.*

Dans le cas général, nous considérerons une matrice dépendant de V devant le terme de dérivée temporelle, ce qui modifie légèrement la définition ce dessus, cf. Chapitre 2. Pour plus de détails sur ces notions d'hyperbolicité, le lecteur peut se référer au livre de Benzoni-Gavage et Serre [5] où, en particulier, est démontrée l'hyperbolicité du système d'Euler compressible que nous étudions dans ce manuscrit.

Notons maintenant que ces définitions ne prennent pas en compte le terme $Q(V)$ qui est d'ordre inférieur, cela est du au fait qu'il ne joue pas de rôle dans l'aspect localement bien posé du système. En revanche, comme nous allons le voir par la suite, son action est déterminante pour l'étude du comportement en temps long.

Mentionnons dès à présent que l'étude des systèmes hyperboliques symétriques dans le cas mono-dimensionnel est très différente des méthodes que nous emploierons dans ce manuscrit. Pour plus d'informations sur le sujet, on pourra consulter les références classiques suivantes :

- [66] concernant la théorie de Kruzkov pour les problèmes scalaires (i.e. $n = 1$).
- [1, 18, 19, 50] pour l'existence et l'unicité de solutions a variations bornées (BV) à partir de petites données initiales.
- [41] pour l'existence globale de solutions pour des données arbitraires dans le cas $n = 2$ avec des techniques de compacité par compensation.

Les méthodes développées ci-dessus sont difficilement généralisables et souvent pas adaptées aux dimensions supérieures. Par exemple, Brenner, dans [10, 11], a montré que dans le cadre multidimensionnel les systèmes hyperboliques sont mal posés[1], au sens de Hadamard, dans L^p pour $p \neq 2$, ce qui implique que les espaces BV étant construits sur l'espace des mesures bornées, lui-même proche de l'espace L^1, ne sont pas appropriés pour étudier le cas multi-dimensionnel. Pour plus d'informations sur le sujet se référer à [5].

Systèmes partiellement dissipatifs et aspect globalement bien-posé

Dans le cas multi-dimensionnel, si l'existence locale-en-temps de solutions régulières est bien comprise, il est bien connu que les systèmes classiques de lois de conservation associés à des données régulières admettent des solutions fortes locales-en-temps qui peuvent développer des singularités (ondes de choc) en temps fini même si les données initiales sont de petites perturbations d'une solution constante, voir par exemple les travaux de Majda [72] et de Serre [91]. Néanmoins, dans la plupart des systèmes évolutifs issus de la physique, il est essentiel de prendre en compte des phénomènes de dissipation et/ou de dispersion, qui, parfois, permettent de prévenir l'apparition de singularités. L'étude des systèmes de lois d'équilibres revient alors à comprendre l'interaction entre le terme de dissipation $Q(V)$ et la dynamique du système hyperbolique sous-jacent.

Une condition suffisante assurant l'existence globale-en-temps de solutions étant des petites perturbations autour d'un état constant $\bar{V}$ est l'hypothèse de *dissipation totale* : le terme d'amortissement ou de dissipation agit directement sur chaque composante du système. Dans ce cas, au niveau linéaire, la solution entière tend vers l'état d'équilibre à une vitesse exponentielle, cf. [68]. Mais, en pratique, cette condition est rarement vérifiée, et même si des solutions fortes globales-en-temps existent, la décroissance exponentielle de celles-ci vers l'état d'équilibre est très peu probable.

Une hypothèse plus raisonnable est que la dissipation n'agisse seulement sur certaines composantes du système. Après un changement approprié de coordonnées, il est généralement pos-

[1] En dehors du cas où les matrices correspondant aux termes d'ordre 1 commutent entre elles et sont donc co-diagonalisables

sible de réécrire le terme $Q(V)$ sous la forme suivante :

$$Q(V) = \begin{pmatrix} 0_{\mathbb{R}^{n_1}} \\ q(V) \end{pmatrix} \tag{6}$$

où $q : \mathbb{R}^n \to \mathbb{R}^{n_2}$ et $n_1, n_2 \in \mathbb{N}^*$ tels que $n_1 + n_2 = n$. Sous cette forme, on voit que la dissipation que Q peut engendrer ne peut affecter, a priori, que n_2 composantes, d'où le fait de qualifier ces systèmes de *partiellement dissipatifs*.

Nous allons maintenant voir que cet effet, bien qu'indirect, est parfois suffisant pour assurer l'existence globale pour des données initiales régulières suffisamment petites. Commençons par un exemple bien connu qui nous servira d'application principale tout au long de ce manuscrit : le système d'Euler compressible amorti pour les écoulements isentropiques. Il se formule

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P + \dfrac{\rho u}{\varepsilon} = 0, \end{cases} \tag{7}$$

où $\rho : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ correspond à la densité du fluide et satisfait l'équation de continuité, $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ est la vitesse et ρu vérifie l'équation de bilan de la quantité de mouvement incluant un terme d'amortissement, $\varepsilon > 0$ est le coefficient de friction (ou paramètre de relaxation) et P une loi de pression fonction de la densité. Pour ce système, les travaux de Wang et Yang [99] et de Sideris, Thomases et Wang [94] ont mis en évidence que le mécanisme dissipatif, bien que présent uniquement dans l'équation de la vitesse, peut empêcher la formation de singularités qui se produiraient si $\lambda = 0$ (i.e. $Q \equiv 0$.). En effet, ils prouvent l'existence de solutions globales-en-temps pour de petites données initiales dans des espaces de Sobolev H^s avec $s > \frac{d}{2} + 1$ et étudient l'asymptotique en temps long de ces solutions. Cela est possible grâce à des estimations d'énergie classiques et, pour pallier le manque de dissipation dans l'équation de ρ, ils dérivent des estimations de décroissance en temps sur ρ en réécrivant le terme de pression ∇P en fonction de $\nabla \rho$ puis en isolant ce terme dans l'équation de $\partial_t u$. Notons que des idées similaires avaient été utilisées dans l'étude du système de Navier-Stokes compressible [33] pour récupérer de la décroissance en temps sur la densité. Comme nous allons le voir, leur approche peut être améliorée grâce à une théorie générale sur les système hyperboliques partiellement dissipatif, la théorie de Kawashima.

Dans sa thèse datant des années 80 [64] et avec Shizuta dans [93], Kawashima a établi un critère systématique assurant l'existence de solutions globales-en-temps pour les systèmes hyperboliques-paraboliques quasilinéaires symétriques : *la condition (SK)* (pour Shizuta et Kawashima). En quelques mots, cette condition assure que l'amortissement est assez fort pour empêcher que les solutions provenant de petites perturbation d'états constants explosent (développent des singularités en temps fini). Voyons à quoi ressemble cette condition dans notre cadre. Pour fixer les idées, analysons le système linéaire à n composantes suivant :

$$\partial_t V + \sum_{j=1}^{d} A^j \partial_{x_j} V + LV = 0 \tag{8}$$

où les matrices constantes $A^j \triangleq A^j(\bar{V})$ et L sont symétriques et l'inconnue V vérifient les décompositions suivantes :

$$L = \begin{pmatrix} 0 \\ L_2 \end{pmatrix} \quad \text{et} \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{9}$$

où $V_1 \in \mathbb{R}^{n_1}$ et $V_2 \in \mathbb{R}^{n_2}$ pour $n_1, n_2 \in \mathbb{N}$ tel que $n_1 + n_2 = n$. En raison de l'aspect symétrique des matrices A^j, la méthode d'énergie classique (produit scalaire dans $L^2(\mathbb{R}^d;\mathbb{R}^n)$ de (8) avec V) conduit à

$$\frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2 + (LV|V) = 0.$$

Les hypothèses sur L nous donne directement qu'il existe $\kappa_0 > 0$ tel que

$$(LV|V) \geq \kappa_0 \|V_2\|_{L^2}^2 \quad \text{pour tout } V \in L^2(\mathbb{R}^d;\mathbb{R}^n). \tag{10}$$

Et donc

$$\frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2 + \kappa_0 \|V_2\|_{L^2}^2 \leq 0.$$

Par conséquent, on voit que l'on peut récupérer de l'intégrabilité L^2-en-temps (et donc des propriétés de décroissance vers l'état d'équilibre) pour V_2, la composante de V qui est directement amortie. Mais cette méthode ne permet pas de récupérer assez d'informations sur la composante V_1 de la solution. En pratique, ce manque de coercivité rend l'obtention de résultats d'existence globale-en-temps pour ces systèmes très délicate voire impossible.

C'est à ce moment que la condition (SK) entre en jeu. Nous allons voir que cette condition assure que l'on puisse récupérer assez d'informations sur les composantes non amorties, grâce au couplage entre la partie hyperbolique du système $\partial_t V + \sum_{j=1}^d A^j \partial_{x_j} V$ et l'amortissement, pour compenser le manque de coercivité.

Pour $\xi = \rho\omega$ avec $\omega \in \mathbb{S}^{d-1}$ et $\rho = |\xi|$, on pose $A_\omega \triangleq \sum_{j=1}^d A\omega_j$. La condition (SK) est définie ainsi :

Définition 0.1.3. *Le système (8) satisfait la condition (SK) en $\bar{V}$ (aussi appelée condition de stabilité) si, pour tout $\omega \in \mathbb{S}^{d-1}$, lorsque $\phi \in \mathbb{R}^n$ satisfait $L\phi = 0$ et $\lambda\phi + A_\omega\phi = 0$ pour un $\lambda \in \mathbb{R}$, alors $\phi = 0$.*

Il est clair que cette condition est équivalente à :

$$\forall \omega \in \mathbb{S}^{d-1}, \quad \ker L \cap \{\text{vecteurs propres de } A_\omega\} = \{0\}.$$

Dans l'article [93], Shizuta et Kawashima montrent que si (8)[2] vérifie la condition (SK), alors la solution V de (8) satisfait :

$$\begin{aligned} &\|V^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda t}\|V_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)}, \\ &\|V^\ell(t)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \leq Ct^{-\frac{d}{2}}\|V_0\|_{L^1(\mathbb{R}^d,\mathbb{R}^n)} \end{aligned} \tag{11}$$

où V^h et V^ℓ correspondent, respectivement, aux hautes et basses fréquences de la solution i.e. $\widehat{V^h}(t,\xi) := \widehat{V^h}(t,\xi)\mathbb{1}_{|\xi|\geq 1}$ et $\widehat{V^\ell}(t,\xi) := \widehat{V^\ell}(t,\xi)\mathbb{1}_{|\xi|\leq 1}$. Au niveau linéaire, on observe donc les phénomènes suivants :

- En hautes fréquences, la solution décroît exponentiellement vers l'état d'équilibre.
- En basses fréquences, la solution se comporte comme les solutions de l'équation de la chaleur.

[2] En réalité, ils considèrent un système hyperbolique-parabolique qui diffère de celui que l'on considère dans ce manuscrit. Mais le même raisonnement fonctionne dans notre cadre.

Notons que ce type de comportement peut aussi être *prédit* directement à partir d'une analyse spectrale des valeurs propres d'un système en particulier, comme nous le verrons pour l'équation d'Euler compressible amortie. Pour justifier rigoureusement ces comportements asymptotiques, les auteurs démontrent l'équivalence entre la condition (SK) et l'existence d'une fonction dite *compensatrice* permettant de récupérer des informations sur la composante non amortie. Plus précisément, ils prouvent la proposition suivante.

Proposition 0.1.1. *Les assertions suivantes sont équivalentes :*

1. *Le système (8) satisfait la condition (SK) en $\bar{V} \in \mathbb{R}^n$;*
2. *Il existe une matrice $n \times n$ $K_{\bar{V}}(\omega)$ (appelée fonction compensatrice) avec $\omega \in \mathbb{S}^{d-1}$ satisfaisant les propriétés suivantes :*
   - *(a) $\forall \omega \in \mathbb{S}^{d-1}$, $K_{\bar{V}}(-\omega) = -K_{\bar{V}}(\omega)$;*
   - *(b) $\forall \omega \in \mathbb{S}^{d-1}$, $K_{\bar{V}}(\omega)$ est antisymétrique;*
   - *(c) $\forall \omega \in \mathbb{S}^{d-1}$, $[K_{\bar{V}}(\omega)A_{\bar{V}}(\omega)]' + L$ est définie positive, où $[X]'$ est la partie symétrique de X.*

Par exemple, concernant le système d'Euler compressible avec amortissement, dans [70], les auteurs introduisent la fonction compensatrice

$$K_{\bar{V}}(\xi) = \begin{pmatrix} 0 & \dfrac{1}{P'(\bar{\rho})}\dfrac{\xi^t}{|\xi|} \\ -\xi/|\xi & 0 \end{pmatrix} \quad \text{où } \bar{V} = (\bar{\rho}, 0)$$

dans le but de récupérer le contrôle de $\|\nabla V_1\|_{L^2_T(H^{s-1})}$ qui fait défaut si on ne fait que des estimations d'énergie à cause du manque de coercivité évoqué précédemment. En effet, en appliquant la transformée de Fourier à (8), en prenant le produit scalaire avec $K_{\bar{V}}(\xi)\widehat{V}$ et en utilisant les propriétés de la proposition précédente, il est possible de récupérer les informations nécessaires pour boucler les estimations.

Nous n'entrerons pas dans les détails de leur preuve puisque nous allons développer une méthode qui se base sur une approche différente : celle de Beauchard et Zuazua dans [4]. Mais avant cela, nous allons décrire les nombreux résultats qui ont été obtenus grâce aux avancées de Shizuta et Kawashima.

Il convient tout d'abord d'évoquer le résultat de Yong dans [108], qui a obtenu un résultat d'existence de solutions globales-en-temps de (2) pour des petites données initiales dans des espaces de Sobolev H^s avec $s > \frac{d}{2} + 1$, en supposant les conditions suivantes :

(i) le système (2) satisfait la condition (SK) en $\bar{V}$;

(ii) la fonction Q satisfait $Q(\bar{V}) = 0$;

(iii) le jacobien $D_V q(\bar{V})$ est inversible;

(iv) le système (2) vérifie une condition de *dissipation entropique* :

   (a) il existe une entropie $\eta(V)$ strictement convexe et régulière définie sur un voisinage compact G de $\bar{V}$ telle que $D^2_{VV}\eta(V)D_V F_j(V)$ est symétrique pour tout $V \in G$ et pour tout $j \in \{1, \cdots, d\}$;

(b) il existe une constante positive c_G telle que pour tout $U \in G$,

$$[D_V\eta(V) - D_V\eta(\bar{V})]Q(V) \leq -c_G|Q(V)|^2.$$

L'hypothèse (iv) permet à Yong de reformuler le système de lois d'équilibre (2) sous la forme *normale* suivante :

$$U_t + \sum_{j=1}^{d} A_j(U)U_{x_j} = \mathrm{diag}(0, q_{V_2}(V))U \quad \text{pour } U = \begin{pmatrix} V \\ q(V) \end{pmatrix}$$

où pour tout $1 \leq j \leq d$, les matrices $A_j(U) := V_U^{-1}(U)F_{jV}(V(U))V_U(U)$ sont telles que $A^0A_j(\bar{V})$ est symétrique avec comme symétriseur $A_0 = V_U^*(\bar{U})\eta_{VV}(\bar{V})V_U(\bar{U})$ d'après la condition (iv)-(b). Ensuite, en utilisant la proposition 0.1.1 relative à l'équivalence entre la condition (SK) et l'existence d'une fonction compensatrice (qu'il dénomme *Skew-symmetrizer Theorem*), Yong parvient à obtenir des estimations quantitatives de la solution lui permettant de conclure. Nous reviendrons plus en détails sur l'utilité de cette antisymétrie pour de tels systèmes.

Il est aussi essentiel de mentionner l'article de Hanouzet et Natalini dans [54] qui, la même année, obtiennent un résultat similaire en dimension 1 ($d = 1$) en reformulant le système grâce à des variables dites entropiques et en dérivant des estimations d'énergie grâce à la condition (SK). De plus, ils vérifient l'applicabilité de leur résultat sur plusieurs exemples concrets ne vérifiant pas forcément la condition (SK) comme, entre autres, le modèle de Suliciu, le modèle Kerr-Debye et le modèle Jin-Xin.

Suite aux avancées de Kawashima et Yong, de nombreux résultats concernant l'existence de solutions fortes globales-en-temps pour de petites données initiales ainsi que l'étude du comportement asymptotique en temps long des solutions ont émergé :

- Comme observé par Bianchini, Hanouzet et Natalini dans [8], une analyse approfondie du noyau de Green du système linéarisé autour de $\bar{V}$ couplée à la formule de Duhamel permet d'obtenir un taux de convergence explicite vers l'état d'équilibre dans des espace L^p pour des solutions régulières globales-en-temps, de l'ordre de $\mathcal{O}(t^{-\frac{d}{2}(1-\frac{1}{p})})$ lorsque $t \to +\infty$, pour tout $p \in [\min\{d, 2\}, \infty]$.

- Dans [65], Kawashima et Yong prouvent des estimations un peu plus générales dans les espaces de Sobolev H^s avec $s > \frac{d}{2} + 1$. Pour cela, ils se basent sur des estimations d'énergie à poids (en temps) pour gérer les difficultés provenant des termes quasilinéaires.

- Plus récemment, Kawashima et Xu dans [102] et [105] ont étendu tous les travaux antérieurs sur les systèmes hyperboliques partiellement dissipatifs satisfaisant la condition (SK) et la condition d'entropie de Yong à des espaces de Besov non-homogènes *critiques*. Critique signifiant ici qu'on considère le cadre fonctionnel qui demande la régularité minimale aux données initiales afin d'obtenir des solutions uniques et globales-en-temps. Pour obtenir leurs résultats, ils ont symétrisé le système grâce à l'hypothèse d'entropie de Yong, appliqué un argument de localisation en fréquences reposant sur la décomposition de Littlewood-Paley et ont développé de nouvelles propriétés concernant les espaces de Chemin-Lerner introduits dans [24, 25]. De plus, en se basant sur l'équivalence entre la condition (SK) et l'existence d'une fonction compensatrice ils ont récupéré des propriétés d'intégrabilité L^2-en-temps sur toutes les composantes du système.

- Une autre façon d'appréhender la condition (SK) et permettant d'étendre les résultats existants provient du travail de Beauchard et Zuazua [4]. Dans leur article, les auteurs ont utilisé des techniques de stabilisation provenant de la théorie du contrôle, afin d'obtenir des informations supplémentaires sur le linéarisé de (2). *En effet, ils ont remarqué que la condition (SK) est en fait équivalente à la célèbre condition du rang de Kalman* et, à partir de cette observation, ils ont construit une fonctionnelle de Lyapunov pour le système linéaire (8) en s'inspirant de l'analyse de l'équation des ondes amortie (voir par exemple [55]) et aussi de la théorie de l'hypocoercivité (quantification du retour à l'équilibre) de Villani [97] dans le contexte des systèmes partiellement diffusifs. Grâce à cela, ils ont pu redémontrer les inégalités de décroissance (11) pour le système linéaire (8) d'une manière plus directe que par l'approche de Kawashima.

- Notons que ce lien avait déjà été mis en évidence par Serre dans le chapitre 6 de [92]. Dans ces notes de cours, il montre que la condition (SK) apparaît naturellement lorsque l'on s'intéresse à la question de la convergence vers l'état d'équilibre du système. Plus précisément, en s'appuyant sur des commutateurs itérés (à la Hörmander dans sa théorie de l'hypoellipticité) entre la partie hyperbolique et l'amortissement, Serre justifie, dans le cas mono-dimensionnel, la nécessité de la condition

$$\bigcap_{j=1}^{n-1} \ker(LA^j) = \{0\}$$

  qui est bien équivalente à la condition (SK) et la condition de rang de Kalman.

C'est dans la continuité des résultats mentionnés ci-dessus que nous nous inscrivons dans ce manuscrit. En alliant cette nouvelle connexion avec la théorie de contrôle et les techniques classiques d'obtention de solutions régulières globales en mécanique des fluides, nous nous attaquons à l'étude approfondie des systèmes hyperboliques partiellement dissipatifs posés dans des espaces de Besov homogènes critiques basés sur des espaces L^p avec p potentiellement différent de 2. Ici, *critique* signifie que nous travaillons dans l'espace qui demande le moins de régularité possible sur les conditions initiales et permettant tout de même d'obtenir l'aspect globalement bien posé. Pour bien comprendre cette suite d'améliorations, il est nécessaire d'avoir à l'esprit la suite d'injections suivante pour $s > 1 + d/2$ et $p \geq 2$:

$$H^s \hookrightarrow B^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{p}}_{p,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1} \hookrightarrow \mathcal{C}^1.$$

Ce type de résultats remonte à l'article de Nash [77] concernant l'aspect bien posé localement en temps pour le système de Navier-Stokes compressible avec des données initiales loin du vide. Ensuite, l'existence de solutions régulières globales a été démontrée par Matsumura et Nishida dans [76] pour des données initiales proches d'un équilibre dans $H^3 \times H^3$. Puis, pour saisir les propriétés d'invariance par changement d'échelle du système de Navier-Stokes compressible, Danchin, inspiré par l'étude du système de Navier-Stokes incompressible de Fujita et Kato [44], a écrit une série d'articles e.g. [33, 34] dans des espaces de Besov homogènes critiques.

Méthode de Beauchard-Zuazua

Expliquons maintenant succinctement le raisonnement de Beauchard et Zuazua dans [4] que nous détaillerons dans la deuxième partie de cette thèse.

Le résultat central sur lequel se base leur article est le suivant :

Proposition 0.1.2. *Soient A et L deux matrices $n \times n$ à coefficients réels tel que L soit de la forme (9). Alors, les propriétés suivantes sont équivalentes :*

1. *le couple (A, L) satisfait la condition (SK);*

2. *le couple (A, L) vérifie la condition de rang de Kalman : le rang de $\big(L \;\; AL \;\; \dots \;\; A^{n-1}L\big)$ vaut n;*

3. *pour tout $\varepsilon_0, \cdots, \varepsilon_{n-1} > 0$, la fonction $y \longmapsto \left(\sum_{k=0}^{n-1} \varepsilon_k |LA^k y|^2\right)^{\frac{1}{2}}$ définit une norme sur $\mathbb{C}^n$.*

Nous allons maintenant voir en quoi la norme en jeu dans ces équivalences est pertinente pour l'étude de la stabilité L^2 ainsi que pour obtenir un taux de convergence vers l'état d'équilibre. Pour compenser le manque de coercivité mis en évidence précédemment, les auteurs, inspirés par des outils provenant de la théorie du contrôle ainsi que par le mémoire de Villani [97], introduisent la fonctionnelle de Lyapunov suivante :

$$\mathcal{L} \triangleq \|V\|_{L^2}^2 + \int_{\mathbb{R}^d} \min(|\xi|, |\xi|^{-1}) \mathcal{I}(\xi)\, d\xi \quad \text{avec} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(LA_\omega^{k-1}\widehat{V} \cdot LA_\omega^k \widehat{V}\big)$$

pour $n-1$ paramètres positifs fixés $\varepsilon_1, \cdots \varepsilon_{n-1}$. Le premier terme de cette fonctionnelle Lyapunov correspond à l'énergie et le deuxième à des termes d'ordres inférieurs. L'intérêt de rajouter ces termes par rapport à l'estimation d'énergie classique est qu'après dérivation en temps de la fonctionnelle $\mathcal{L}$, on obtient :

$$\frac{d}{dt}\mathcal{L} + \mathcal{H} \leq 0 \quad \text{avec} \quad \mathcal{H} \triangleq \frac{1}{2} \int_{\mathbb{R}^d} \sum_{k=0}^{n-1} \varepsilon_k \min(1, |\xi|^2) |LA_\omega^k \widehat{V}(\xi)|^2\, d\xi.$$

Il est simple de voir que si $\varepsilon_1, \cdots, \varepsilon_{n-1}$ sont assez petits, alors il existe c et C des constantes strictement positives telles que $c\|V\|_{L^2}^2 \leq \mathcal{L} \leq C\|V\|_{L^2}^2$. La question est donc maintenant de vérifier si $\mathcal{H}$ est similaire à $\|V\|_{L^2}^2$. La réponse dépend des propriétés du support de $\widehat{V}_0$ et des éventuelles annulations de la quantité $\mathcal{N}_{*,\bar{V}}$ définie par

$$\mathcal{N}_{*,\bar{V}} \triangleq \inf\left\{\sum_{k=0}^{n-1} \varepsilon_k |LA_\omega^k x|^2;\; x \in \mathbb{S}^{n-1},\, \omega \in \mathbb{S}^{d-1}\right\}.$$

D'après les équivalences de la Proposition 0.1.2, on déduit alors directement que la condition (SK) autour de $\bar{V}$ est satisfaite par (A_ω, L) pour tout $\omega \in \mathbb{S}^{d-1}$ si et seulement si $\mathcal{N}_{*,\bar{V}} > 0$. Et donc en utilisant le théorème de Fourier-Plancherel et le fait que $\mathcal{L} \sim \|V\|_{L^2}^2$, il existe une constante $\kappa > 0$ telle que

$$\mathcal{H} \geq \kappa \min(1, |\xi|^2)\mathcal{L}.$$

Donc, au niveau linéaire et sous la condition (SK), Beauchard et Zuazua parviennent à récupérer la décomposition obtenue par Kawashima d'une manière simple :

- si le support de $\widehat{V}_0$ a une borne inférieure strictement positive (ce qui correspond au régime hautes fréquences), alors $\mathcal{H} \geq \kappa\|Z\|_{L^2}^2$, ce qui implique une décroissance exponentielle vers l'état d'équilibre,

- si $\widehat{V}_0$ est à support compact (ce qui correspond au régime basses fréquences), alors $\mathcal{H} \geq \kappa\|\nabla V\|_{L^2}^2$, ce qui révèle un comportement parabolique sur toutes les composantes de la solution.

De plus, avec leur approche, il est possible de déduire la forme explicite d'une fonction compensatrice pour le système (8), elle s'écrit :

$$K_{\bar{V}}(\omega) := \sum_{k=1}^{n} \varepsilon_k \left(A_\omega^{*k} L^* L A_\omega^{k-1} - A_\omega^{*k-1} L^* L A_\omega^k \right).$$

Inspiré par cette approche, Danchin dans [35] présente une méthode robuste pouvant être utilisée pour prouver des estimations de régularité ou de décroissance pour des systèmes linéaires ayant une partie antisymétrique et une partie partiellement dissipative (ou diffusive). Il étudie les systèmes de la forme suivante :

$$\partial_t V + A(D)V + L(D)V = 0 \quad \text{où}$$

- $A(D) = (A_{ij}(D))_{1\leq i,j\leq d}$ pour $A_{ij}(D)$ un multiplicateur de Fourier homogène d'ordre α,
- $L(D) = (L_{ij}(D))_{1\leq i,j\leq d}$ pour $L_{ij}(D)$ un multiplicateur de Fourier homogène d'ordre β.

De plus, il suppose que $A(D)$ est antisymétrique[3] :

$$\mathrm{Re}((A(\xi)\eta)\cdot\eta) = 0 \quad \forall(\xi,\eta) \in \mathbb{R}^d \times \mathbb{C}^n,$$

et que $L(D)$ vérifie une propriété d'ellipticité partielle :

$$|\xi|^\beta \mathrm{Re}((L(\xi)\eta)\cdot\eta) \geq \kappa|L(\xi)\eta|^2 \quad \forall(\xi,\eta) \in \mathbb{R}^d \times \mathbb{C}^n$$

pour κ un nombre réel positif. Ensuite, en considérant la même fonctionnelle de Lyapunov que dans [4] et en supposant que la condition (SK) est satisfaite, il montre qu'en basses fréquences i.e. $|\xi| < \kappa^{\frac{1}{\beta-\alpha}}$ (resp. en hautes fréquences i.e. $|\xi| > \kappa^{\frac{1}{\beta-\alpha}}$) les solutions de (0.1) ont les mêmes propriétés de régularité et de décroissance que les solutions de

$$\partial_t U + |D|^\gamma U = 0 \quad \text{où} \quad \gamma = \max(\beta, 2\alpha-\beta) \quad (\text{resp. } \gamma = \min(\beta, 2\alpha-\beta)).$$

Notons que dans le cas $\alpha = 1$ et $\beta = 0$, on retrouve exactement la décomposition de Kawashima. Il montre ensuite comment, avec ces nouvelles considérations, on peut espérer obtenir des résultats d'existence pour des systèmes quasilinéaires partiellement dissipatifs grâce à la décomposition de Littlewood-Paley et la technique du paraproduit introduite par Bony dans [9] (sur lesquels nous revenons en détail dans l'annexe). C'est cette voie que nous explorerons dans la première partie de ce manuscrit.

Au-delà de la condition (SK)

Revenons quelque peu sur la condition (SK). Il est important de noter qu'elle est suffisante mais pas nécessaire pour obtenir l'existence de solutions globales-en-temps. En effet, il existe de multiples systèmes ne satisfaisant pas la condition (SK) mais pour lesquels on peut obtenir

[3] On désigne par $\eta_1 \cdot \eta_2$ le produit Hermitien sur $\mathbb{C}^n$.

l'existence de solutions globales-en-temps. Au cours de la dernière décennie, de nombreuses tentatives ont été faites pour améliorer la condition (SK) et pour traiter des systèmes ne satisfaisant pas cette condition. Nous allons maintenant présenter quelques références sur le sujet.

- Dans [4], à nouveau, en se basant sur leurs améliorations concernant le système linéaire, les auteurs ont prouvé deux résultats importants concernant la version non linéaire de ces systèmes. Tout d'abord, ils ont pu obtenir des estimations quantitatives de la taille du voisinage où il est possible d'obtenir l'existence de solutions régulières globales pour des systèmes non-linéaires satisfaisant la condition (SK). Grâce à cela, ils ont établi un résultat d'existence globale pour le problème de Cauchy associé à une classe de systèmes ne vérifiant pas la condition (SK). Plus précisément, ils ont fait l'hypothèse suivante : il existe un équilibre, pour lequel le système linéarisé sous-jacent satisfait la condition (SK), qui est suffisamment proche de l'état d'équilibre considéré (pour lequel la condition (SK) n'est pas vérifiée).

- Une autre innovation présente dans [4] est l'étude de la stabilité L^2 ainsi que le comportement à l'infini des solutions du système linéaire (8) lorsque la condition (SK) n'est pas vérifiée. Les auteurs ont établi une classification en fonction de la dimension de l'espace d, la dimension du système n et la forme de l'ensemble dégénéré suivant :

$$\mathcal{D} := \{\xi \in \mathbb{R}^d \text{ t.q. } \ker L \cap \{\text{vecteurs propres de } A(\xi)\} \neq \{0\}\} \tag{12}$$

  qui est l'ensemble des valeurs de ξ pour lesquelles la condition (SK) n'est pas vérifiée. Cet ensemble décrit les interactions entre la dynamique libre et l'opérateur d'amortissement. Sous la condition que $\mathcal{N}_{*,\bar{w}} > \mathrm{dist}(\omega, \mathcal{D})^\alpha$ où la valeur de $\alpha \in \mathbb{R}$ dépend de la situation étudiée, ils obtiennent une décomposition plus précise de la solution. Par exemple, dans le cas où la condition (SK) n'est pas satisfaite, que (8) est un système à n composantes tel que $\mathrm{Ker}(L) = \mathrm{Span}(e_1)$ et pour $\mathcal{D}$ un sous-espace vectoriel de codimension $r \in \mathbb{N}^*$, ils ont prouvé la stabilité L^2 de la solution ainsi que la décomposition suivante $V = V_1 + V_2 + V_3 + V_4$ où V_1 et V_2 vérifient les même estimations que précédemment, et

$$\|V_3\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^2)} \leq Ct^{-\frac{r}{2}} \|w_0\|_{L^1(\mathbb{R}^d,\mathbb{R}^2)},$$
$$\|V_4\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^2)} \leq Ct^{-\frac{1}{2}} \|w_0\|_{L^1(\mathbb{R}^d,\mathbb{R}^2)}$$

  où $r = \mathrm{codim}(\mathcal{D})$. Dans cette décomposition, V_1 et V_2 correspondent respectivement aux hautes et basses fréquences "loin de $\mathcal{D}$" et V_3 et V_4 correspondent respectivement aux hautes et basses fréquences "proches de $\mathcal{D}$". Dans ce cas précis, ils ont été capables de conclure en montrant que $\mathcal{N}_{*,\bar{w}} \geq \mathrm{dist}(\omega, \mathcal{D})^2$ et en utilisant une décomposition orthogonale de l'espace.

  Cependant, la classification faite dans [4] n'est pas complète et dépend beaucoup de la structure de $\mathcal{D}$. En général, $\mathcal{D}$ est une variété algébrique, appelée variété de Kalman, qui n'est pas nécessairement une union de sous-espaces vectoriels et donc pour laquelle obtenir une paramétrisation est plus complexe (pour plus d'informations, voir [60, 80, 81]).

- La condition (SK) ne concernant que le linéarisé, elle omet la prise en compte des contributions pouvant provenir des non-linéarités, qui peuvent, dans certains cas, être bénéfiques. En lien avec l'article précédent, dans [7], Bianchini et Natalini ont considéré deux modèles-jouets dans le cas tri-dimensionnel appartenant à la classe des systèmes hyperboliques

partiellement dissipatifs ne vérifiant pas la condition (SK), de telle sorte que certaines directions propres ne subissent aucune dissipation (i.e. certaines variables ne sont pas dissipées du tout et cela persiste en tout point de l'espace des fréquences) et avec des termes sources non-linéaires. Dans ce cadre, ils étudient deux cas :

- soit la non-linéarité inclut des composantes *dissipées* qui comportent assez de dissipation pour la contrôler et il n'y a pas d'apparition de singularités,
- soit la non-linéarité n'inclut aucune forme de dissipation et alors il est nécessaire qu'elle soit *non-résonante* dans les directions où il n'y a pas de dissipation pour empêcher la formation de singularités.

Pour montrer cela, ils combinent l'analyse des fonctions de Green pour les systèmes hyperboliques "faiblement" dissipatifs (faiblement car les termes sources font perdre de la décroissance au sous-système vérifiant la condition (SK)) avec une extension de la notion des formes nulles pour l'équation des ondes par Pusateri et Shatah [85], dans le contexte de la théorie des résonances temps-espaces pour les équations dispersives introduite par Germain, Masmoudi et Shatah. Il convient de noter, pour faire le lien avec les autres articles mentionnés, qu'au moins en dimension 1, l'aspect linéairement dégénéré des valeurs propres est directement relié à la condition nulle, via la formule de décomposition de John (pour plus d'informations cf. [7]).

- Dans [75], Mascia et Natalini ont considéré une classe de systèmes hyperboliques ne vérifiant pas la condition (SK) et ont prouvé des résultats d'existence de solutions régulières globales-en-temps pour de petites données initiales dans des espaces de Sobolev H^s avec $s > \frac{d}{2} + 1$. Leur analyse nécessite une condition de dégénérescence linéaire dans les directions non-dissipées pour empêcher la formation de singularités. Un exemple trivial vérifiant cette condition est un système de deux équations découplées, la première étant linéaire et l'autre totalement dissipée.

- En lien direct avec le précédent résultat, dans [86], Qu et Wang démontrent un résultat d'existence globale pour des systèmes hyperboliques quasilinéaires pour lesquels il existe une unique famille propre ne satisfaisant pas la condition (SK). En d'autres termes, pour $n \geq 3$, ils supposent que $n-1$ vecteurs propres à droite satisfont la condition (SK) mais que la première famille de vecteurs à droite ne la satisfait pas. Cela peut se reformuler en conditions de structures sur la première famille propre : soit λ_1 la première valeur propre de $A(\xi)$ et r_1 (resp. l_1) un vecteur propre à droite (resp. à gauche) associé à λ_1 tels que pour tout $\omega \in \mathbb{S}^{d-1}$:

  - La première famille de vecteurs propres (l_1, r_1) vérifie une condition d'isotropie :
  $$r_1(V,\omega) = r_1(V) \quad \text{et} \quad l_1(V,\omega) = l_1(V).$$
  - $Q(\bar{V})$ est dégénéré le long des premières trajectoires des caractéristiques :
  $$\nabla_V Q(\bar{V}) r_1(\omega) = 0.$$
  - Pour toutes les autres familles de vecteurs propres à droites r_j pour $j = 2, ..., n$,
  $$\nabla_V Q(\bar{V}) r_j(\omega) \neq 0.$$

  Cette condition et la précédente impliquent que la condition (SK) est vérifiée par $n-1$ vecteurs propres mais pas par r_1.

– La première famille propre est linéairement dégénérée :

$$\nabla_V \lambda_1(V,\omega) \cdot r_1(\omega) = 0.$$

Bien que la méthode qu'ils développent permette l'obtention de solutions globales-en-temps, elle ne permet pas de récupérer des informations de décroissance en temps sur toutes les composantes de la solution. En d'autres termes, du fait de la condition de dégénérescence de la première famille propre, la condition (SK) n'est pas nécessaire pour le système entier mais seulement pour un sous-système. Une des applications de leur résultat est le système d'Euler complet amorti pour lequel l'entropie ne satisfait qu'une équation de transport et n'est donc que transportée par le flot (il est donc naturel de ne pas pouvoir récupérer de décroissance pour celle-ci). Les hypothèses structurelles qu'ils font sont donc là pour garantir qu'il n'est pas nécessaire d'avoir des informations de décroissance sur la composante non dissipée pour obtenir des estimations a priori permettant de conclure.

Nous verrons que le système multi-fluide étudié dans le chapitre 4 de ce manuscrit est proche de la classe d'équations considérée par Qu et Wang. En effet, une condition de dégénérescence linéaire dans une direction particulière est vérifiée par notre système et elle est essentielle pour conclure. Cependant la complexité de la pression et les équations sur les fractions volumiques font que le système que l'on considère ne rentre pas dans le cadre de leur résultat.

Phénomènes de relaxation

Dans ce manuscrit nous étudions deux phénomènes de relaxation assez différents. Les deux s'inscrivent dans l'étude du comportement asymptotique des solutions des systèmes hyperboliques partiellement dissipatifs :

$$\frac{\partial V}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j(V)}{\partial x_j} = \frac{Q(V)}{\varepsilon} \tag{13}$$

lorsque le paramètre ε tend vers 0. L'étude des problèmes de relaxation associés aux systèmes de lois de conservation remonte aux travaux de Chen et al [26]. En utilisant la méthode de Chapman-Enskog, Chen a déduit une approximation plus précise du système limite :

$$\partial_t V + \sum_{j=1}^{d} \partial_j F_j(V) - \varepsilon \sum_{i,j=1}^{d} \partial_j (B_{ij}(V)\partial_j V) = \frac{L(V)}{\varepsilon}. \tag{14}$$

Pour ce dernier système, Giovangigli, Yong et Matuszewski dans [46–49] ont étudié un problème de limite de relaxation dans le cadre de la dynamique des gaz parfaits hors de l'équilibre thermodynamique. En supposant l'existence d'une entropie compatible avec les opérateurs de diffusion et de dissipation, ils ont prouvé l'existence locale de solutions pour le problème de Cauchy ainsi que des estimations d'erreurs entre les solutions de (14) à ε fixé et les solutions du système limite. Ils ont été ensuite en mesure d'appliquer leurs résultats à un système bi-fluide à deux températures pour lequel le processus de relaxation consiste à faire s'accorder ces températures. Il est important de noter que leurs résultats n'étant que locaux en temps, la condition (SK) n'est pas pertinente dans leur analyse (le terme dissipatif étant responsable du comportement en temps grand du système limite). Néanmoins, pour obtenir des estimations uniformes, ils s'appuient sur la méthode de Kawashima.

Discutons maintenant de quelques résultats concernant l'étude du système (13).

- Dans [59, 74], Hsiao, et Marcati et Rubino étudient la relaxation du système (13) vers un système parabolique pour des solutions classiques ou des solutions faibles entropiques grâce à des techniques de compacité par compensation.
- Dans [4], les auteurs stipulent que si la condition (SK) est satisfaite ainsi qu'une condition structurelle supplémentaire sur les matrices[4], alors il est possible d'obtenir des estimations uniformes par rapport au paramètre de relaxation $\varepsilon > 0$.
- Dans le même contexte, dans [82], Peng et Wasiolek ont justifié la limite de relaxation parabolique en s'appuyant sur la méthode de [4] pour obtenir des estimations uniformes.

De nombreux résultats concernant (13) sont des généralisations de résultats concernant le système d'Euler compressible amorti. Pour ce système, on peut différencier trois catégories de résultats :

(i) Résultats obtenus grâce aux fonctions de courant :

- Dans [62], Junca et Rascle ont justifié la relaxation du système d'Euler compressible amorti mono-dimensionnel (loin du vide) vers l'équation des milieux poreux pour des solutions BV arbitrairement grandes globales-en-temps et ont fourni un taux de convergence explicite du processus de relaxation. Leur approche est basée sur l'utilisation de fonctions de courant qui sont liées aux coordonnées lagrangiennes de masse.
- Très récemment, Peng, Li et Zhao dans [69] ont généralisé l'approche précédente à des classes de systèmes hyperboliques partiellement dissipatifs vérifiant la condition (SK) dans le cadre des espaces de Sobolev avec $d = 1$ et ont obtenu un taux de convergence explicite pour le processus de relaxation.
- Par la suite, en utilisant des techniques similaires, Liang et Shuai dans [109] ont obtenu un taux de convergence explicite pour le processus de relaxation dans le cadre périodique multi-dimensionnel (dans le tore $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$). La limitation au cadre périodique venant du fait qu'ils doivent utiliser l'inégalité de Poincaré pour contrôler les termes de dérivées partielles mixtes de la fonction de courant qui n'apparaissent pas dans le cas mono-dimensionnel.

(ii) Résultats concernant les solutions faibles du système d'Euler, potentiellement en présence du vide :

- Marcati et Milani dans [73], ont prouvé la convergence de solutions faibles du système d'Euler vers des solutions faibles de l'équation des milieux poreux dans le cas mono-dimensionnel et ont justifié que la vitesse du système limite vérifie la loi de Darcy.
- Dans [43, 71], les auteurs ont étudié le processus de relaxation pour le problème à frontière libre du système d'Euler mono-dimensionnel compressible amorti en présence du vide. Sous une condition de régularité Hölderienne pour la vitesse du son proche de la frontière et en présence du vide, ils démontrent la convergence de solutions régulières vers les solutions auto-similaires Barenblatt du système des milieux poreux et obtiennent un taux de convergence explicite. Leur preuve s'appuie sur la considération de fonctionnelles à poids en temps et en espace permettant d'appréhender le comportement de la solution près du vide ainsi que des estimations elliptiques pour justifier le comportement en temps grand.

(iii) Résultats de relaxation pour des solutions régulières :

[4] L'hypothèse est que la matrice de convection ne dépende que de la composante amortie.

- Dans [29, 70], Coulombel, Goudon et Lin justifient la convergence forte locale-en-espace du système d'Euler vers l'équation des milieux poreux dans des espaces de Sobolev avec indice de régularité $s > \frac{d}{2} + 1$. Un ingrédient principal, après l'obtention d'estimations a priori uniformes *à la Kawashima*, est l'utilisation d'un changement de variable diffusif dans les inconnues via l'introduction d'une variable en temps lent que l'on retrouve aussi dans [62, 75].

- Dans [104, 107], Xu, Wang et Kawashima justifient la limite de relaxation dans des espaces de Besov critiques non-homogènes grâce à une procédure de localisation en fréquences, les résultats fondateurs de Kawashima et le changement de variable introduit dans [75].

Un des buts principaux de ce manuscrit est de compléter cette dernière section en justifiant la convergence forte globale-en-espace (et en temps) des solutions régulières d'Euler tout en exhibant un taux de convergence explicite pour le processus de relaxation. De plus, la méthode que nous développons est adaptable, sous certaines conditions raisonnables, à la justification de la limite de relaxation du système général (13) mais aussi pour des systèmes ne vérifiant pas la condition (SK).

Voyons maintenant comment déduire la limite de relaxation du système d'Euler compressible amorti lorsque ε tend vers 0. En reprenant les notations dans [29, 70], on introduit la variable *en temps lent* $\tau = \varepsilon t$ et on effectue le changement de variable diffusif suivant :

$$(\tilde{\rho}_\varepsilon, \tilde{J}_\varepsilon)(\tau, x) = (\rho, \frac{\rho u}{\varepsilon})(t, x). \tag{15}$$

Le couple $(\tilde{\rho}_\varepsilon, \tilde{J}_\varepsilon)$ vérifie alors le système suivant :

$$\begin{cases} \partial_\tau \tilde{\rho}_\varepsilon + \nabla \cdot \widetilde{J}_\varepsilon = 0, \\ \varepsilon^2 \left(\partial_\tau \widetilde{J}_\varepsilon + \operatorname{div} \left(\dfrac{\widetilde{J}_\varepsilon \otimes \widetilde{J}_\varepsilon}{\widetilde{\rho}_\varepsilon} \right) \right) + \nabla P(\widetilde{\rho}_\varepsilon) + \widetilde{J}_\varepsilon = 0. \end{cases} \tag{16}$$

En utilisant les bornes uniformes obtenues pour $\tilde{\rho}_\varepsilon$ et $\tilde{J}_\varepsilon$ lors de la preuve du caractère bien posé, on peut montrer qu'au sens des distribution, on a

$$-\tilde{J}_\varepsilon - \nabla P(\widetilde{\rho}_\varepsilon) \overset{*}{\rightharpoonup} 0 \quad \text{et donc} \quad \partial_\tau \tilde{\rho}_\varepsilon - \Delta P(\tilde{\rho}_\varepsilon) \overset{*}{\rightharpoonup} 0.$$

Grâce à ces propriétés de convergence faible-$*$ et des arguments de compacité du type lemme d'Aubin-Lions, on peut récupérer de la convergence forte locale-en-espace. Et donc, lorsque ε tend vers 0, $\widetilde{\rho}_\varepsilon$ converge fortement (localement-en-espace) vers $\mathcal{N}$, la solution de l'équation des milieux poreux :

$$\begin{cases} \partial_\tau \mathcal{N} - \Delta P(\mathcal{N}) = 0, \\ \mathcal{N}_{\tau=0} = \rho_0 \end{cases} \tag{17}$$

où la vitesse du fluide v vérifie une loi de Darcy :

$$v = \frac{\nabla P(\mathcal{N})}{\mathcal{N}}.$$

0.2 Présentation des principaux résultats obtenus

Nos premiers résultats concernent l'étude d'un modèle-jouet rentrant dans le cadre des systèmes hyperboliques partiellement dissipatifs. Nous souhaitons obtenir le plus d'informations possible sur ce système simple dans le but de pouvoir aborder des systèmes plus généraux par la suite.

0.2.1 Chapitre 1 : Etude d'un modèle-jouet

Dans le premier chapitre de cette thèse, on se place dans le cadre mono-dimensionnel ($d = 1$) et on s'intéresse au modèle-jouet suivant :

$$(TM_\lambda) \qquad \begin{cases} \partial_t u + v\partial_x u + \partial_x v = 0 & \text{dans} \quad \mathbb{R}^+ \times \mathbb{R}, \\ \partial_t v + v\partial_x v + \partial_x u + \lambda v = 0 & \text{dans} \quad \mathbb{R}^+ \times \mathbb{R}, \\ (u,v)|_{t=0} = (u_0, v_0). & \text{sur} \quad \mathbb{R}. \end{cases}$$

Ce modèle peut s'interpréter comme une simplification du système d'Euler compressible avec amortissement (7) ayant pour loi de pression $P(\rho) = \frac{1}{2}\rho^2$. En effet, si on considère l'inconnue u comme étant l'écart par rapport à la densité de référence normalisée à 1 (i.e. $\bar{\rho} = 1$) alors, la première équation est une simplification de la conservation de la masse, et la deuxième équation correspond à l'évolution de la vitesse v avec un terme de friction de magnitude $\lambda > 0$ (qui peut également s'interpréter comme un paramètre de relaxation). En particulier ce système est quasi-linéaire, symétrique et satisfait la condition (SK).

Afin de disposer d'une méthode robuste et adaptable à des systèmes plus complexes, nous ne calculons pas explicitement la solution du système linéarisé (TM_λ) autour de $(0,0)$, mais utilisons plutôt des méthodes d'énergie avancées (différentes de celles de S. Kawashima dans sa thèse [64]) et un changement approprié d'inconnues. Plus précisément, nous introduirons une vitesse effective (mode amorti) jouant le même rôle que le flux effectif visqueux dans les travaux de Hoff [58] et, plus récemment, de Haspot [56] dédiés au système de Navier-Stokes compressible.

Dans le but d'obtenir un résultat dans le cadre des espaces de Besov homogènes, il est crucial d'étudier en profondeur le comportement des basses fréquences qui, dans les précédents travaux sur le sujet, étaient mélangées au reste des fréquences. Comme suggéré par l'analyse du comportement asymptotique des solutions de Kawashima et plus tard redémontré par Beauchard et Zuazua, les basses fréquences des solutions se comportent comme les solutions de l'équation de la chaleur. Bien que dominant pour l'ensemble de la solution, il s'avère que ce comportement n'est pas assez précis pour appréhender les basses fréquences. En effet, une analyse spectrale approfondie nous montre que :

- en basses fréquences, la matrice du système correspondant à la fréquence ξ a deux valeurs propres réelles qui tendent respectivement vers 1 et ξ^2 pour ξ proche de 0;
- en hautes fréquences, deux valeurs propres complexes conjuguées coexistent, elles sont asymptotiquement égales à $\dfrac{1}{2}(\xi^2 \pm i\xi)$.

Cela nous dit qu'en basses fréquences une valeur propre a un comportement parabolique mais que l'autre est purement amortie. De plus, cette analyse révèle l'existence d'un mode amorti qui permet de diagonaliser et de découpler le système dans le régime des basses fréquences. C'est sur cette dernière propriété que nous allons nous appuyer pour gérer les basses fréquences.

Présentons maintenant quelques calculs sur le linéarisé du modèle-jouet pour mieux comprendre la méthode que nous voulons mettre en place. Supposons que l'on ait une solution régulière (u, v) du système suivant :

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + v = 0. \end{cases} \tag{18}$$

La première chose que nous devons faire est de localiser en fréquences le système grâce à l'opérateur $\dot{\Delta}_j$ provenant de la théorie de Littlewood-Paley. En appliquant $\dot{\Delta}_j$ à (18) et en notant $f_j := \dot{\Delta}_j f$, on obtient

$$\begin{cases} \partial_t u_j + \partial_x v_j = 0, \\ \partial_t v_j + \partial_x u_j + v_j = 0. \end{cases} \tag{19}$$

En considérant le mode amorti $w = v + \partial_x u$ comme une inconnue à la place de v, le système se reformule ainsi :

$$\begin{cases} \partial_t u_j - \partial_{xx}^2 u_j = -\partial_x w_j, \\ \partial_t w_j + w_j = -\partial_{xx}^2 v_j. \end{cases}$$

Sous cette forme, on peut voir directement le comportement parabolique dans l'équation de u et l'aspect amorti dans l'équation de w. Sachant que cette analyse se restreint au régime basses fréquences, les termes sources ne poseront aucun problème à être absorbés étant donné qu'en Fourier les dérivées sont équivalentes à multiplier par 2^j et qu'en basses fréquences j est suffisamment négatif. On peut donc étudier le système d'équations de manière découplée et récupérer les comportements attendus d'une manière directe en basses fréquences. De plus, cette diagonalisation du système permet d'effectuer des estimations au-delà du cadre énergétique, c'est-à-dire dans des espaces L^p avec $p \geq 2$. En pratique, cela est possible pour un système lorsque ses valeurs propres sont réelles, ce qui est bien le cas ici mais seulement dans le régime basses fréquences. En hautes fréquences, le système ayant des valeurs propres imaginaires, nous devons nous restreindre au cadre d'énergie L^2. Voyons maintenant comment procéder dans ce régime.

Même si les hautes fréquences sont bien comprises dans tous les articles concernant les solutions fortes de ces systèmes, nous allons quelque peu simplifier la méthode d'obtention des estimations a priori et éviter d'introduire le concept de fonction compensatrice. Inspiré par les articles de Beauchard et Zuazua [4] et de Danchin [33, 35], nous considérons une fonctionnelle de Lyapunov permettant de retrouver les propriétés voulues. Pour le système linéarisé du modèle-jouet (19), elle s'écrit

$$\mathcal{L}_j^2 \triangleq \|u_j\|_{L^2}^2 + \|v_j\|_{L^2}^2 + \|\partial_x u_j\|_{L^2}^2 + \|\partial_x v_j\|_{L^2}^2 + \int_{\mathbb{R}} v_j \, \partial_x u_j$$

où les quatre premiers termes correspondent à l'énergie basse et hautes fréquences et le dernier terme à un terme d'ordre inférieur permettant de récupérer de l'intégrabilité en temps sur u. En dérivant en temps cette fonctionnelle, on arrive à

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + \|(v_j, \partial_x v_j)\|_{L^2}^2 + \frac{1}{2}\|\partial_x u_j\|_{L^2}^2 - \frac{1}{2}\|\partial_x v_j\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} v_j \, \partial_x u_j \leq 0.$$

Sachant qu'il existe c et C deux constantes strictement positives telles que

$$C^{-1}\|(u_j, v_j, \partial_x u_j, \partial_x v_j)\|_{L^2}^2 \leq \mathcal{L}_j^2 \leq C\|(u_j, v_j, \partial_x u_j, \partial_x v_j)\|_{L^2}^2$$

$$\text{et} \quad \|(v_j, \partial_x v_j)\|_{L^2}^2 + \frac{1}{2}\left(\|\partial_x u_j\|_{L^2}^2 - \|\partial_x v_j\|_{L^2}^2 + \int_{\mathbb{R}} v_j \, \partial_x u_j \, dx\right) \geq c \min(1, 2^{2j})\mathcal{L}_j^2,$$

on arrive aisément à

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + c\min(1, 2^{2j})\mathcal{L}_j^2 \leq 0.$$

Donc, après intégration en temps, on retrouve le comportement purement amorti des hautes fréquences et par la même occasion le comportement parabolique des basses fréquences. Nous verrons que pour le système quasi-linéaire, la gestion des termes supplémentaires ne pose pas de problème. Une des difficultés principales est l'adaptation d'outils d'analyse fonctionnelle tels que les estimations de commutateurs et les lois de produits à notre cadre fonctionnel.

Les analyses ci-dessus révèlent que dans le but d'obtenir les estimations de dissipation optimales il est préférable de séparer la solution en une partie basses fréquences et une partie hautes fréquences. Cela est possible grâce à la décomposition de Littlewood-Paley qui permet de localiser en fréquences.

Cela nous amène au premier théorème du premier chapitre, pour lequel les notations sont définies en annexe.

Théorème 0.2.1. *Soit $2 \leq p \leq 4$ et $\lambda > 0$. Il existe $k = k(p) \in \mathbb{Z}$ et $c_0 = c_0(p) > 0$ tels que pour le seuil entre les hautes et basses fréquences $J_\lambda \triangleq \lfloor \log_2 \lambda \rfloor + k$, si on suppose que $u_0^{\ell,J_\lambda}, v_0^{\ell,J_\lambda} \in \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ et $u_0^{h,J_\lambda}, v_0^{h,J_\lambda} \in \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$ avec*

$$\|(u_0, v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \lambda^{-1}\|(u_0, v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \leq c_0,$$

alors le système (TM_λ) admet une unique solution globale (u, v) dans l'espace $E_p^{J_\lambda}$ défini par

$$\begin{aligned}
&u^{\ell,J_\lambda} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^1(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}), \quad u^{h,J_\lambda} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}) \cap L^1(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}),\\
&v^{\ell,J_\lambda} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^1(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}), \quad v^{h,J_\lambda} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}) \cap L^1(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})\\
&\lambda v + \partial_x u \in L^1(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \quad \text{et} \quad v \in L^2(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}).
\end{aligned}$$

De plus, nous avons l'estimation a priori suivante :

$$X_{p,\lambda}(t) \lesssim \|(u_0, v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \lambda^{-1}\|(u_0, v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \quad \text{pour tout } t \geq 0,$$

où

$$\begin{aligned}
X_{p,\lambda}(t) \triangleq \|(u,v)\|^{\ell,J_\lambda}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \lambda^{-1}\|(u,v)\|^{h,J_\lambda}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \lambda^{-1}\|u\|^{\ell,J_\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})} + \|(u,v)\|^{h,J_\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}\\
+ \|\lambda v + \partial_x u\|_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \lambda^{\frac{1}{2}}\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}.
\end{aligned}$$

- Le mode amorti $w = \lambda v + \partial_x u$ satisfait, comme pressenti, une meilleure intégrabilité temporelle en basses fréquences que le reste de la solution. C'est une des clés pour boucler les estimations globalement en temps et pour obtenir des résultats concernant la limite de relaxation.
- Comparé aux articles de Kawashima et Xu [102, 105] où les auteurs obtiennent des résultats dans des espaces de Besov non-homogènes critiques construits sur L^2 pour une classe de systèmes contenant (TM_λ), l'homogénéité de nos normes nous permet d'obtenir la dépendance exacte des normes par rapport au paramètre d'amortissement λ.

- Le seuil J_λ entre les basses et hautes fréquences dépend du coefficient de friction et doit être choisi assez petit via la constante k pour pouvoir absorber les termes avec "trop de dérivées" en basses fréquences.

Nous montrons ensuite un résultat concernant les estimations de décroissance vers l'état stationnaire pour le système d'Euler compressible avec amortissement. Ce résultat améliore les précédents résultats concernant le comportement asymptotique des solutions dans le cas monodimensionnel, notamment [8, 106]. De plus, en adaptant les idées de Guo et Wang dans [53] et de Xu et Xin dans [101] pour le système de Navier-Stokes compressible, nous écartons l'hypothèse de petitesse supplémentaire sur les basses fréquences des données initiales qui est habituellement requise pour obtenir des estimations de décroissance. Voici le deuxième résultat que nous obtenons pour (TM_λ) :

Théorème 0.2.2. *Supposons que les hypothèses du Théorème 0.2.1 sont satisfaites pour $p = 2$. Alors, il existe une fonctionnelle de Lyapunov associée à la solution (u, v) construite dans ce théorème, qui est équivalente à $\|(u,v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$.*

Si de plus on suppose que $(u_0, v_0) \in \dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ pour $\sigma_1 \in \left]-\frac{1}{2}, \frac{1}{2}\right]$ alors il existe une constante $C > 0$ dépendant seulement de σ_1 telle que

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\,\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0.$$

De plus il existe une constante κ_0 dépendant seulement de σ_1, λ et de la norme de la donnée initiale telle que, si

$$\langle t\rangle \triangleq 1 + \kappa_0 t, \quad \alpha_2 \triangleq \sigma_1 + \frac{1}{2} \quad \text{et} \quad C_{0,\lambda} \triangleq \lambda^{1+\alpha_2}\|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}},$$

alors u et v vérifient les estimations de décroissance suivantes :

$$\begin{aligned}
\lambda^{\frac{3}{2}-\sigma}\,\|\langle\lambda t\rangle^{\alpha}(u,v)(t)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq CC_{0,\lambda}, \quad \sigma \in [-\sigma_1, 1/2], \quad \alpha \triangleq \frac{\sigma+\sigma_1}{2},\\
\|\langle\lambda t\rangle^{\alpha_2}(u,v)(t)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,\infty}} &\leq CC_{0,\lambda},\\
\lambda^{\sigma_1+\frac{3}{2}}\,\|\langle\lambda t\rangle^{\alpha_1}v\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} &\leq CC_{0,\lambda}, \qquad \alpha_1 \triangleq \frac{1}{2}\Big(\frac{1}{2}+\sigma_1\Big).
\end{aligned}$$

- Le fait que v subisse directement la dissipation, contrairement à u, explique pourquoi le taux de décroissance des basses fréquences de v est meilleur que celui de u.

- Grâce à l'injection $L^1 \hookrightarrow \dot{\mathbb{B}}^{-\frac{1}{2}}_{2,\infty}$, on voit que l'hypothèse ci-dessus avec $\sigma_1 = 1/2$ englobe la condition classique présente dans l'article de Matsumura et Nishida [76] où les données initiales sont dans L^1 afin d'obtenir un taux de décroissance similaire à celui de l'équation de la chaleur.

- En choisissant les bons exposants, on peut retomber sur les conditions utilisées dans [8] mais avec des meilleurs taux de décroissance vers l'état d'équilibre.

Le reste de cette section est consacré à l'adaptation de ces deux résultats au système d'Euler

compressible amorti puis à des systèmes plus généraux de la forme suivante :

$$\begin{cases} \partial_t u + \alpha \partial_x v + V^1 \partial_x u + W^1 \partial_x v = 0, \\ \partial_t v + \beta \partial_x u + V^2 \partial_x u + W^2 \partial_x v + \lambda v + \kappa \lambda v^q = 0 \end{cases}$$

où κ est un paramètre réel, $q \geq 2$, un entier, $V^1 = V^1(v)$ et $V^2 = V^2(v)$ sont des fonctions régulières s'annulant en 0, $W^1 = W^1(u, v)$ et $W^2 = W^2(u, v)$ des fonctions régulières s'annulant en $(0, 0)$, et α, β, λ sont des constantes strictement positives. Les conditions particulières sur les nouvelles fonctions entrant en jeu viennent du fait que v possède de meilleures propriétés en terme de régularité basses fréquences que u.

0.2.2 Chapitre 2 : Les systèmes hyperboliques partiellement dissipatifs satisfaisant la condition (SK) dans le cas multi-dimensionnel

Dans le deuxième chapitre de cette thèse nous généralisons la méthode développée dans la première partie aux systèmes hyperboliques quasi-linéaires partiellement dissipatifs dans l'espace entier $\mathbb{R}^d$ avec $d \geq 2$. Nous établissons l'existence de solutions fortes globales et d'estimations de décroissance dans des espaces de Besov homogènes critiques lorsque le système considéré est symétrisable et satisfait la condition (SK). En particulier, nos résultats s'appliquent au système d'Euler compressible avec amortissement dans l'équation de vitesse.

Plus précisément on s'intéresse aux systèmes dans $\mathbb{R}^d$ à n-composantes suivants :

$$A^0(V)\frac{\partial V}{\partial t} + \sum_{j=1}^{d} A^j(V)\frac{\partial V}{\partial x_j} = H(V). \tag{20}$$

Afin d'assurer l'hyperbolicité et donc l'aspect bien posé localement en temps de notre système, nous supposons que ces systèmes sont symétrisables au sens de Friedrichs :

Il existe une fonction régulière $S : V \mapsto S(V)$ définie sur $\mathcal{O}_V$ à valeurs dans l'ensemble des matrices symétriques définies positives telle que pour tout $V \in \mathcal{O}_V$, les matrices $(SA^0)(V), \cdots, (SA^d)(V)$ sont symétriques et $SA^0(V)$ est définie positive.

Grâce à une décomposition de notre solution en une partie directement amortie et une partie indirectement amortie, il est possible de reformuler le système (20) sous la forme suivante :

$$\begin{cases} \widetilde{A}^0_{1,1}(V)\partial_t Z_1 + \sum_{j=1}^{d} \left(\widetilde{A}^j_{1,1}(V)\partial_j Z_1 + \widetilde{A}^j_{1,2}(V)\partial_j Z_2 \right) = 0, \\ \widetilde{A}^0_{2,2}(V)\partial_t Z_2 + \sum_{j=1}^{d} \left(\widetilde{A}^j_{2,1}(V)\partial_j Z_1 + \widetilde{A}^j_{2,2}(V)\partial_j Z_2 \right) + L_2 Z_2 = Q(Z) \end{cases} \tag{21}$$

où $Z = V - \bar{V}$ et les différentes matrices sont décomposées selon $Z = \left(\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right)$.

Ce découpage en deux systèmes d'équations donne à ce système une structure très similaire au système étudié dans le premier chapitre de cette thèse. Cela nous permet d'employer des arguments similaires et d'obtenir des résultats pour ces systèmes généraux.

Comme pour le modèle-jouet, un ingrédient essentiel dans notre analyse est l'introduction

d'un mode amorti W correspondant à la partie de la solution subissant l'amortissement maximal en basses fréquences. Dans ce contexte, on le définit ainsi :

$$W \triangleq -L_2{}^{-1}\widetilde{A}^0_{2,2}(V)\partial_t Z_2 = Z_2 + \sum_{j=1}^{d} L_2{}^{-1}\big(\widetilde{A}^j_{2,1}(V)\partial_j Z_1 + \widetilde{A}^j_{2,2}(V)\partial_j Z_2\big) - L_2{}^{-1}Q(Z)\cdot$$

Ce mode amorti est bien sûr analogue à l'inconnue w définie précédemment et satisfait l'équation suivante :

$$\widetilde{A}^0_{2,2}(V)\partial_t W + L_2 W = \widetilde{A}^0_{2,2}(V)L_2{}^{-1}\sum_{j=1}^{d}\partial_t\big(\widetilde{A}^j_{2,1}(V)\partial_j Z_1 + \widetilde{A}^j_{2,2}(V)\partial_j Z_2\big) - \widetilde{A}^0_{2,2}(V)L_2{}^{-1}\partial_t Q(Z)\cdot$$

On peut voir que le membre de gauche assure la dissipation maximale sur W et que les membres de droite ne contiennent que des termes au moins quadratiques, ou linéaires mais avec "assez" de dérivées pour qu'ils soient négligeables dans le régime basses fréquences. De plus, pour des petites solutions i.e. si Z est petit dans une certaine norme, alors W est comparable à Z_2 et il est donc possible de récupérer des meilleures informations sur Z_2 que pour l'ensemble de la solution Z.

Concernant les hautes fréquences, en s'inspirant des arguments développés par Beauchard et Zuazua dans [4], on considère la fonctionnelle suivante :

$$\mathcal{L}_j \triangleq \|Z_j\|^2_{L^2_{\widetilde{A}_0(V)}} + 2^{-j}\mathcal{I}_j \quad \text{si} \quad j \geq 0,$$

avec

$$\mathcal{I}_j \triangleq \int_{\mathbb{R}^d}\sum_{k=1}^{n-1}\varepsilon_k \Im\Big((LA^{k-1}_\omega\widehat{Z_j})\cdot(LA^k_\omega\widehat{Z_j})\Big)\,d\xi,$$

où les $\varepsilon_1, \cdots, \varepsilon_{n-1} > 0$ seront choisis assez petits pour que $\mathcal{L} \sim \|Z\|^2_{L^2}$. La première partie de cette fonctionnelle correspond à l'énergie et permet de récupérer les informations de base sur la solution. La deuxième partie correspond à des termes d'ordres inférieurs permettant, après avoir dérivé en temps $\mathcal{I}_q$, de compenser le manque de coercivité et de récupérer des informations de dissipation sur les composantes non amorties. Considérer cette fonctionnelle de Lyapunov permet d'éviter d'utiliser les arguments du type fonction compensatrice et nous donne aisément accès à des estimations de décroissance.

Voici le premier résultat que nous obtenons concernant (20), réécrit avec l'inconnue Z.

Théorème 0.2.3. *Soit $d \geq 2$ et $\bar{V}$ un état d'équilibre constant tel que $H(\bar{V}) = 0$. On suppose que (21) vérifie la condition (SK). Alors, il existe une constante positive α telle que pour tout $Z_0 \in \dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ satisfaisant*

$$\mathcal{Z}_0 \triangleq \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq \alpha,$$

le système (21) associé à la donnée initiale Z_0 admet une unique solution globale Z dans l'espace E défini par

$$Z \in \mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \quad Z^h \in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \quad Z^\ell_1 \in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}) \quad \textit{et} \quad W \in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}).$$

De plus, il existe une fonctionnelle de Lyapunov équivalence à $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$, *et une constante* C *dépendant seulement des matrices* A^j *et* H, *telles que*

$$\mathcal{Z}(t) \leq C\mathcal{Z}_0 \quad \textit{pour tout } t \geq 0$$

où

$$\mathcal{Z}(t) \triangleq \|Z\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})} + \|Z\|^{h}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|W\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|Z_2\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}.$$

- Les remarques concernant le théorème d'existence pour le modèle-jouet s'appliquent à ce théorème.
- Ce résultat ne peut pas s'étendre au cas $d = 1$ car l'indice de régularité du régime basses fréquences devient négatif et donc certains termes non-linéaires ne peuvent pas être bornés convenablement.

En suivant une approche similaire au premier chapitre, on obtient un résultat concernant les estimations de décroissance en temps vers l'état d'équilibre constant $\bar{V}$. Ce résultat généralise tous les précédents résultats de convergence vers l'état d'équilibre obtenus par le passé.

Théorème 0.2.4. *Sous les hypothèses du Théorème 0.2.3 et si, de plus,* $Z_0 \in \dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ *pour* $\sigma_1 \in \left]-\frac{d}{2}, \frac{d}{2}\right]$ *alors, il existe une constante* C *dépendant seulement de* σ_1 *telle que*

$$\|Z(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\,\|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0.$$

De plus, si $\sigma_1 > 1 - d/2$,

$$\langle t\rangle \triangleq \sqrt{1+t^2}, \quad \alpha_1 \triangleq \frac{\sigma_1 + \frac{d}{2} - 1}{2} \quad \textit{et} \quad C_0 \triangleq \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}},$$

alors la solution Z *vérifie les estimations de décroissance suivantes :*

$$\begin{aligned}
\sup_{t\geq 0}\left\|\langle t\rangle^{\frac{\sigma+\sigma_1}{2}} Z(t)\right\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq CC_0 \quad \textit{si} \quad -\sigma_1 < \sigma \leq d/2 - 1,\\
\sup_{t\geq 0}\left\|\langle t\rangle^{\frac{\sigma+\sigma_1}{2}+\frac{1}{2}} Z_2(t)\right\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq CC_0 \quad \textit{si} \quad -\sigma_1 < \sigma \leq d/2 - 2,\\
\sup_{t\geq 0}\|\langle t\rangle^{\alpha_1} Z_2(t)\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq CC_0 \quad \textit{si} \quad \min(d/2-2, -\sigma_1) < \sigma \leq d/2 - 1\\
\textit{et} \quad \sup_{t\geq 0}\left\|\langle t\rangle^{2\alpha_1} Z(t)\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} &\leq CC_0.
\end{aligned}$$

A nouveau, en invoquant l'injection $L^1 \hookrightarrow \dot{\mathbb{B}}^{-\frac{d}{2}}_{2,\infty}$ le résultat ci-dessus englobe la condition d'intégrabilité classique $Z_0 \in L^1$ nécessaire pour obtenir ce type d'estimations.

La suite de ce chapitre est consacrée à l'étude du système (20) sous des hypothèses de structure additionnelles permettant de montrer des résultats dans la même veine que les deux mentionnés ci-dessus mais avec des hypothèses moins exigeantes sur les basses fréquences des

données initiales. En outre, si on ajoute un paramètre de relaxation sur la partie dissipative, ce nouveau cadre fonctionnel permet aussi d'obtenir une dépendance par rapport au paramètre d'amortissement précise et exploitable pour étudier la limite de relaxation comme nous le verrons dans le Chapitre 3.

0.2.3 Chapitre 3 : le cas multi-dimensionnel dans le cadre L^p et relaxation du système d'Euler

Dans ce chapitre nous considérons les systèmes à n composantes suivants :

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{d} A^k(V) \frac{\partial V}{\partial x_k} = \frac{LV}{\varepsilon}$$

où ε est une petite constante strictement positive, les fonctions A^k sont régulières et linéaires à valeurs dans l'ensemble des matrices symétriques. A nouveau, nous faisons une hypothèse de dissipation partielle : le terme LV n'agit que sur une partie de la solution, ce qui peut s'écrire :

$$LV = \begin{pmatrix} 0 \\ L_2 V \end{pmatrix}$$

où $0 \in \mathbb{R}^{n_1}$, $L_2 V \in \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$ and $n_1 + n_2 = n$. Le système que nous étudions ici est une simplification du modèle étudié dans le chapitre précédent. Néanmoins, en corollaire de notre théorème principal, nous obtenons un résultat concernant la limite de relaxation du système d'Euler compressible amorti vers l'équation des milieux poreux, ce qui est le but principal de ce chapitre.

A nouveau on considère un état d'équilibre constant $\bar{V}$ tel que $L\bar{V} = 0$. En définissant $Z \triangleq V - \bar{V}$, on a

$$\partial_t Z + \sum_{k=1}^{d} A^k(V) \partial_k Z + \frac{LZ}{\varepsilon} = 0 \cdot \tag{22}$$

Tout d'abord, comme dans le premier chapitre, nous nous concentrons sur le cas $\varepsilon = 1$ qui n'est pas restrictif grâce au changement de variable :

$$Z(t,x) \triangleq \check{Z}\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\Big)$$

qui nous permet de récupérer la dépendance par rapport à ε grâce à l'homogénéité de nos normes.

Ensuite, pour gérer les basses fréquences, nous suivons l'approche du premier chapitre et considérons le mode amorti W introduit dans le chapitre 2 comme une nouvelle inconnue. Cela nous permet de diagonaliser le système en une équation purement amortie et une équation parabolique qui ne sont couplés que par des termes au moins quadratiques, ou des termes linéaires d'ordre supérieur (en termes de dérivées) et qui seront donc négligeables si le seuil J_0 entre les basses et hautes fréquences est assez petit. En effet, en utilisant la définition de W dans l'équation de Z_1, on obtient

$$\partial_t Z_1 + \sum_{k=1}^{d} \bar{A}^k_{1,1} \partial_k Z_1 - \sum_{k=1}^{d} \sum_{\ell=1}^{d} \bar{A}^k_{1,2} L_2^{-1} \bar{A}^\ell_{2,1} \partial_k \partial_\ell Z_1 = f_1 + f_2 + f_3 + f_4 + f_5.$$

Afin de pouvoir estimer cette équation de manière découplée, nous avons besoin de l'éllipticité

de l'opérateur

$$\mathcal{A} \triangleq \bar{A}^k_{1,2} L_2^{-1} \bar{A}^\ell_{2,1} \partial_k \partial_\ell.$$

et aussi de la nullité des $\bar{A}^k_{1,1}$. Dans ce cas particulier, il s'avère que la condition (SK) est en fait équivalente à l'éllipticité de cet opérateur. Nous avons le lemme suivant :

Lemme 0.2.1. *Supposons que* $\forall\ k \in \{1, \cdots, d\},\ \bar{A}^k_{1,1} = 0$. *Alors les assertions suivantes sont équivalentes :*

- *Le système* (0.2) *satisfait la condition (SK) en* $\bar{V}$*;*
- *l'opérateur* $\mathcal{A}$ *est fortement elliptique.*

De plus, si une de ces assertions est vraie et si $Supp(\mathcal{F}Z_1) \subset \{\xi \in \mathbb{R}^d : R_1\lambda \leq |\xi| \leq R_2\lambda\}$ *pour* $0 < R_1 < R_2$ *alors, pour tout* $p \in [2, \infty[$, *il existe* $c = c(p, d, R_1, R_2) > 0$ *tel que*

$$\int_{\mathbb{R}^d} \langle \sum_{k=1}^d \sum_{\ell=1}^d \bar{A}^k_{1,2} L_2^{-1} \bar{A}^\ell_{2,1} \partial_k \partial_\ell Z_1, |Z_1|^{p-2} Z_1 \rangle \geq c\lambda^2 \|Z_1\|^p_{L^p}.$$

Une fois ce lien établi, nous pouvons aisément étudier les basses fréquences en étudiant l'équation de Z_1 et de W de manière découplée, puis, à partir de cela nous récupérons les propriétés sur Z_2.

Concernant les hautes fréquences, nous employons l'approche de Beauchard et Zuazua développée dans le chapitre 2. La difficulté réside dans la mise en place de nouvelles estimations pour les commutateurs, lois de produits et lemmes de composition pour pouvoir estimer la partie basses fréquences des non-linéarités qui n'appartient pas à un espace basé sur L^2.

Cela nous amène au théorème suivant.

Théorème 0.2.5. *Soient* $d \geq 2$, $\varepsilon > 0$ *et* $p \in [2, \min(4, \frac{2d}{d-2})]$, *on suppose que la condition (SK) est satisfaite en* $\bar{V}$ *et que pour tout* $k \in \{1, \cdots, d\}$, $\bar{A}^k_{1,1} = 0$, $Z \mapsto A^k_{1,1}(\bar{V} + Z)$ *est linéaire par rapport à* Z_2 *et* $Z \mapsto A^k(\bar{V} + Z)$ *est linéaire par rapport à* Z. *Il existe un* $k_p \in \mathbb{Z}$ *et* $c_0 = c_0(p) > 0$ *tel que pour* $J_\varepsilon \triangleq \lfloor -\log_2 \varepsilon \rfloor + k_p$, *si on suppose* $Z_0^{\ell,J_\varepsilon} \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ *et* $Z_0^{h,J_\varepsilon} \in \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ *avec*

$$\|Z_0\|^{\ell,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

alors le système (22) *admet une unique solution globale* Z *dans l'espace* $E_p^{J_\varepsilon}$ *défini par*

$$Z_1^{\ell,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}), \quad Z^{h,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}) \cap L^1(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}),$$
$$Z_2^{\ell,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}), \quad W_\varepsilon \in L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \quad et \quad Z_2 \in L^2(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}).$$

De plus, nous avons les bornes suivantes :

$$X_{p,\varepsilon}(t) \lesssim \|Z_0\|^{\ell,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \quad pour\ tout\ t \geq 0,$$

où

$$X_{p,\varepsilon}(t) \triangleq \|Z\|^{\ell,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon \|Z\|^{h,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \varepsilon \|Z_1\|^{\ell,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})} + \|Z\|^{h,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}$$
$$+ \|W_\varepsilon\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon^{-\frac{1}{2}} \|Z_2\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}$$

et

$$W_\varepsilon \triangleq \frac{Z_2}{\varepsilon} + \sum_{k=1}^{d} L_2^{-1}\big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2\big)\cdot$$

Ce résultat nous permet, presque directement, d'obtenir des bornes uniformes pour la solution du système d'Euler compressible amorti et d'étudier la limite de relaxation lorsque $\varepsilon \to 0$. Le système (7) n'étant pas symétrique, nous ne pouvons pas lui appliquer le Théorème 0.2.1 directement. Pour cela, nous allons employer la symmétrisation dûe à Makino, Ukai et Kawashima dans [95].

Pour une loi de pression satisfaisant :

$$P(\rho) = A\rho^\gamma \quad \text{for } \gamma > 1 \quad \text{and} \quad A > 0. \tag{23}$$

on considère l'inconnue

$$c = \frac{2}{\gamma - 1}\sqrt{\frac{\partial P}{\partial \rho}} = \frac{(4\gamma A)^{\frac{1}{2}}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}},$$

appelée vitesse du son. En définissant $\tilde{\gamma} = \dfrac{\gamma - 1}{2}$ et $\widetilde{c} = c - \bar{c}$ où $\bar{c}$ est une constante positive, on peut réécrire le système (7) sous la forme :

$$\begin{cases} \partial_t \widetilde{c} + v \cdot \nabla \widetilde{c} + \tilde{\gamma}(\widetilde{c} + \bar{c})\operatorname{div} v = 0, \\ \partial_t v + v \cdot \nabla v + \tilde{\gamma}(\widetilde{c} + \bar{c})\nabla \widetilde{c} + \dfrac{1}{\varepsilon} v = 0. \end{cases} \tag{24}$$

Ce système est symétrique, vérifie la condition (SK) et les hypothèses de structure mentionnées ci-dessus. Donc, le Théorème 0.2.1 peut lui être appliqué avec

$$\check{W} = v \cdot \nabla v + \frac{v}{\varepsilon} + \tilde{\gamma}(\widetilde{c} + \bar{c})\nabla \widetilde{c} = v \cdot \nabla v + \frac{v}{\varepsilon} + \tilde{\gamma} c \nabla c$$

pour mode amorti. En utilisant un lemme de composition, on obtient une solution $(\rho - \bar{\rho}, v)$ du système initial (7) vérifiant des propriétés réduites mais suffisantes pour nous permettre d'étudier la limite de relaxation.

En effet, grâce aux bornes uniformes provenant de ce théorème, il est alors possible d'étudier le comportement asymptotique de la solution de (7) lorsque $\varepsilon \to 0$, qui, en réalité, n'est pas si intuitif. En effet, à première vue, lorsque le coefficient d'amortissement devient de plus en plus grand on s'attend à ce que le phénomène de dissipation soit de plus en plus dominant. Cependant, il se produit ce qu'on appelle l'effet d'*overdamping* : le taux de décroissance n'augmente plus lorsque le coefficient d'amortissement augmente mais se comporte plutôt comme $\min(\varepsilon, 1/\varepsilon)$. Cela peut se voir concrètement avec une analyse spectrale du système d'Euler amorti ou dans [110] pour l'oscillateur harmonique amorti. Dans l'espace de Fourier, le système linéarisé d'Euler

avec paramètre de relaxation égal à ε a pour matrice

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \dfrac{1}{\varepsilon} \end{pmatrix}.$$

Il est facile de voir que :

- le discriminant du polynôme caractéristique étant égal à $\dfrac{1}{\varepsilon^2} - 4\xi^2$, la coupure entre les hautes et basses fréquences se situe au niveau de $\dfrac{1}{2\varepsilon}$.
- en basses fréquences (i.e. $|\xi| \ll \varepsilon^{-1}$), cette matrice a deux valeurs propres réelles qui tendent respectivement vers $\dfrac{1}{\varepsilon}$ et $\varepsilon\xi^2$ pour ξ proche de 0;
- en hautes fréquences (i.e. $|\xi| \gg \varepsilon^{-1}$), deux valeurs propres complexes conjuguées coexistent, dont les parties réelles sont asymptotiquement égales à $\dfrac{1}{2\varepsilon}$.

Dans les approches précédentes, toutes les fréquences étaient "mélangées" et donc l'aspect dissipatif en basses fréquences n'était pas pleinement exploité car le comportement global qui ressort d'un mixage des fréquences est toujours le "moins bon". Dans l'approche que nous proposons, nous exploitons les bornes obtenues sur le mode amorti (qui est directement lié à la valeur propre asymptotiquement égale à $1/\varepsilon$ en basses fréquences) et la dépendance du seuil entre les hautes et basses fréquences par rapport à ε. En effet, si pour $\varepsilon < 1$, on effectue les mêmes calculs que précédemment mais avec le seuil entre les hautes et basses fréquences J_0 ne dépendant pas de ε, on se retrouve avec un coefficient ε devant toutes les normes L^1-en-temps de la solution, ce qui bloque totalement l'étude de la relaxation. Plus précisément, notre seuil $J_\varepsilon = \lfloor -\log_2 \varepsilon \rfloor + k_p$, correspond à l'endroit où les termes d'ordres 0 et les termes d'ordre 1 ont le même poids (paramètre inclus). Lorsque $\varepsilon \to 0$, heuristiquement, cela implique que les basses fréquences recouvrent tout l'espace de fréquences. Ceci peut être, d'une certain manière, justifié en voyant que le comportement du système limite est purement parabolique tout comme le système initial dans le régime de basses fréquences.

Voyons maintenant comment obtenir précisément notre résultat principal. Comme dans [29, 62, 70, 73], nous introduisons la variable de temps lent $\tau = \varepsilon t$ et on définit le changement de variable diffusif suivant :

$$(\widetilde{c}^\varepsilon, \tilde{v}^\varepsilon)(\tau, x) = (c, \frac{v}{\varepsilon})(t, x).$$

Le couple $(\widetilde{c}^\varepsilon - \bar{c}, \tilde{v}^\varepsilon)$ vérifie

$$\begin{cases} \partial_t \tilde{c}^\varepsilon + \operatorname{div}(\widetilde{c}^\varepsilon \tilde{v}^\varepsilon) = 0, \\ \varepsilon^2 (\partial_t \tilde{v}^\varepsilon + \tilde{v}^\varepsilon \cdot \nabla \tilde{v}^\varepsilon) + \tilde{\gamma} \widetilde{c}^\varepsilon \nabla \widetilde{c}^\varepsilon + \tilde{v}^\varepsilon = 0. \end{cases} \tag{25}$$

A partir des bornes uniformes provenant du théorème d'existence et le fait que

$$\tilde{\gamma} \widetilde{c}^\varepsilon \nabla \widetilde{c}^\varepsilon = \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon},$$

il est possible de récupérer les informations suivantes sur l'inconnue initiale $\widetilde{\rho}^\varepsilon$.

Lemme 0.2.2. *Soit* $(c-\bar{c}, v)$ *la solution du système* (24) *obtenue avec le Théorème 0.2.1. Alors, pour tout* $t \geq 0$, $\widetilde{\rho}^\varepsilon$ *satisfait l'estimation suivante :*

$$\|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \frac{1}{\varepsilon}\left\|\widetilde{v}^\varepsilon + \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon}\right\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq c_0.$$

Avec ce lemme en main, nous pouvons estimer la différence des solutions de l'équation limite et de l'équation initiale :

$$\partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0$$

et

$$\partial_t \tilde{\rho}^\varepsilon + \operatorname{div}(\widetilde{\rho}^\varepsilon \tilde{v}^\varepsilon) = 0.$$

En utilisant le mode amorti $\tilde{W}^\varepsilon = \widetilde{v}^\varepsilon + \dfrac{\nabla P(\widetilde{\rho}^\varepsilon)}{\rho}$, l'équation de $\widetilde{\rho}^\varepsilon$ peut se réécrire:

$$\partial_t \widetilde{\rho}^\varepsilon - \Delta P(\widetilde{\rho}^\varepsilon) = S^\varepsilon \quad \text{avec} \quad S^\varepsilon = -\operatorname{div}(\widetilde{\rho}^\varepsilon \widetilde{W}^\varepsilon).$$

Le lemme 0.2.2 nous dit alors que

$$\|\widetilde{W}^\varepsilon\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} = \mathcal{O}(\varepsilon) \quad \text{et} \quad \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} = \mathcal{O}(1),$$

ce qui implique, avec des lois de produits et estimations de composition, que S^ε est un $\mathcal{O}(\varepsilon)$ dans $L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})$.

Maintenant que les équations sont sous une forme similaire, nous pouvons estimer leur différence. Pour cela, on définit $\delta D^\varepsilon \triangleq \widetilde{\rho}^\varepsilon - \mathcal{N}$ qui vérifie

$$\partial_t \delta D^\varepsilon - \Delta(P(\widetilde{\rho}^\varepsilon) - P(\mathcal{N})) = S^\varepsilon.$$

Avec la condition

$$\|\widetilde{\rho}^\varepsilon_0 - \mathcal{N}_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} \leq \varepsilon$$

et en utilisant le fait qu'il existe une fonction régulière H_1 s'annulant en $\bar{\rho} = \bar{\mathcal{N}}$ telle que

$$P(\widetilde{\rho}^\varepsilon) - P(\bar{\rho}) = P'(\bar{\rho})\,(\widetilde{\rho}^\varepsilon - \bar{\rho}) + H_1(\widetilde{\rho}^\varepsilon)\,(\widetilde{\rho}^\varepsilon - \bar{\rho}),$$

on arrive à

$$\begin{aligned}\|\delta D^\varepsilon\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\delta D^\varepsilon\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \lesssim{}& \|\delta D^\varepsilon_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} + \|S^\varepsilon\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} \\ &+ \|\delta D^\varepsilon\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \|(\widetilde{\rho}^\varepsilon - \bar{\rho}, \mathcal{N} - \bar{\mathcal{N}})\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \\ &+ \|\delta D^\varepsilon\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \|(\widetilde{\rho}^\varepsilon - \bar{\rho}, \mathcal{N} - \bar{\mathcal{N}})\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}.\end{aligned}$$

A partir de là, comme les quantités du membre de droite peuvent être absorbées par les membres de gauche grâce au Lemme 0.2.2 et un résultat d'existence pour l'équation des milieux poreux (Proposition 3.5.1), on déduit que

$$\|\delta D^\varepsilon\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\delta D^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \lesssim \varepsilon.$$

Cela nous permet d'obtenir le taux de relaxation souhaité et nous conduit au résultat suivant.

Théorème 0.2.6. *Soient $d \geq 2$, $\varepsilon > 0$ et $p \in [2, \min(4, \frac{2d}{d-2})]$. Soient $\bar{\rho}$ une constante strictement positive et $(\rho - \bar{\rho}, v)$ la solution du problème de Cauchy (7) avec $(\rho_0 - \bar{\rho})^\ell, v_0^\ell \in \dot{B}^{\frac{d}{p}}_{p,1}$ et $(\rho_0 - \bar{\rho})^h, v_0^h \in \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ obtenue via le Théorème 0.2.1. Soient la fonction positive $\mathcal{N}_0$ telle que $\mathcal{N}_0 - \bar{\rho}$ est assez petit dans $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$, et $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{p}+2}_{p,1})$ l'unique solution globale correspondante du problème de Cauchy suivant :*

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \end{cases}$$

obtenue avec la Proposition 3.5.1. Si on suppose que

$$\|\widetilde{\rho}_0^\varepsilon - \mathcal{N}_0\|_{\mathbb{B}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

alors, lorsque $\varepsilon \to 0$, on a

$$\widetilde{\rho}^\varepsilon - \mathcal{N} \longrightarrow 0 \quad \textit{fortement dans} \quad L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}),$$

et

$$\widetilde{v}^\varepsilon + \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} \longrightarrow 0 \quad \textit{fortement dans} \quad L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}).$$

De plus, nous avons les estimations quantitatives suivantes :

$$\|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \widetilde{v}^\varepsilon \right\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

A notre connaissance, ceci est le premier résultat prouvant la convergence forte globale en espace et en temps de solutions fortes régulières du système d'Euler compressible vers l'équation des milieux poreux dans le cas multi-dimensionnel et dans l'espace entier. De plus, nous sommes en mesure d'exhiber un taux de convergence explicite pour le procédé de relaxation. Mentionnons que le fait de travailler dans des espaces basés sur des espaces de Lebesgue L^p avec $2 \leq p \leq \min(4, \frac{2d}{d-2})$ pour étudier ce phénomène de relaxation n'est pas dans le but d'améliorer un résultat déjà existant dans le cadre L^2 mais simplement car notre méthode nous le permet.

0.2.4 Chapitre 4 : Caractère bien-posé et limite de relaxation d'un système multi-fluide

Dans le dernier chapitre de ce manuscrit, nous étudions un mélange de deux fluides compressibles remplissant l'espace entier $\mathbb{R}^d$ avec $d \geq 2$. Dans [17], en s'appuyant sur les précédents travaux [12–14], les auteurs ont montré qu'il était possible d'obtenir un système de Baer-Nunziato visqueux en suivant une procédure d'homogénéisation. L'hypothèse de base est que si on zoome dans le mélange, alors, à l'échelle mésoscopique, les deux phases sont séparées. En supposant que chaque phase vérifie l'équation de Navier-Stokes compressible dans son propre domaine, les auteurs ont été capables d'écrire un système fermé pour le mélange. Lorsque l'on revient à l'échelle macroscopique, la densité du mélange de fluides est supposée osciller entre les deux densités de référence. La propagation de ces oscillations est quantifiée via des mesures de Young et on peut montrer qu'elles sont des combinations convexes de mesures de Dirac au temps initial, et que la structure est préservée par la suite. Il est important de noter que les équations obtenues pour

les fractions volumiques sont de la forme (1) et que le paramètre de relaxation est proportionnel à la viscosité des deux phases.

Plus précisément, nous étudions la version multi-dimensionnelle du système obtenu par Bresch, Burtea et Lagoutière dans [17] qui se formule de la façon suivante :

$$\begin{cases} \partial_t \alpha_\pm + u \cdot \nabla \alpha_\pm = \pm \dfrac{\alpha_+ \alpha_-}{2\mu + \lambda} \left(P_+ \left(\rho_+ \right) - P_- \left(\rho_- \right) \right), \\ \partial_t \left(\alpha_\pm \rho_\pm \right) + \operatorname{div} \left(\alpha_\pm \rho_\pm u \right) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{A}_{\mu,\lambda} u + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = \alpha_+ P_+ \left(\rho_+ \right) + \alpha_- P_- \left(\rho_- \right). \end{cases} \tag{BN}$$

Une hypothèse essentielle que nous faisons est que le flux du mélange est animé par un seul vecteur vitesse. Notons que cette hypothèse est loin de simplifier les calculs et qu'on la retrouve dans de nombreuses applications. De plus, nous ajoutons un terme de friction dans l'équation de la vitesse avec un paramètre $\eta \geq 1$. Ce terme modélise une force de traînée qui est cruciale pour démontrer l'aspect globalement bien posé de notre système et nous permet d'employer les techniques développées dans les chapitres précédentes.

Notre but principal est de justifier la limite de relaxation (visqueuse) lorsque les données initiales $(\alpha_{\pm 0}, \rho_{\pm 0}, u_0)$ sont proches d'un état d'équilibre constant $(\bar{\alpha}_\pm, \bar{\rho}_\pm, 0)$. Formellement, lorsque $\nu := 2\mu + \lambda$ tend vers 0, le système (BN) converge vers le système de Kapila suivant :

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ \partial_t \left(\alpha_\pm \rho_\pm \right) + \operatorname{div} \left(\alpha_\pm \rho_\pm u \right) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = P_+ \left(\rho_+ \right) = P_- \left(\rho_- \right). \end{cases} \tag{K}$$

Dans la mesure où nous nous attendons à ce que les pressions à la limite soient égales et afin d'éviter l'apparition de couches limites près de l'instant initial, nous supposerons que les pressions sont à l'équilibre à l'infini :

$$\bar{P} \triangleq P_+ \left(\bar{\rho}_+ \right) = P_- \left(\bar{\rho}_- \right). \tag{26}$$

Concernant l'aspect globalement bien posé de (BN), il convient de remarquer que ce système ne rentre pas dans le cadre des théories générales exposées dans les chapitres précédents ou dans les travaux antérieurs [4, 46–49].

- Une première raison à cela est que le système (BN) n'est pas un système de lois de conservation car les équations sur les fractions volumiques $\alpha_\pm$ ne peuvent pas se réécrire sous forme conservative.

- Ensuite, l'entropie naturellement associée au système étant linéaire par rapport aux fractions volumiques, elle n'est pas définie positive.

- Enfin, le système quasi-linéaire associé au système ayant une valeur propre nulle, il ne satisfait pas la condition (SK).

Cependant, il s'avère que la situation n'est pas si mauvaise car le sous-espace propre associé à la valeur propre 0 est de dimension 1 et donc, la partie non-dégénérée (i.e. associée au valeurs propres non nulles) satisfait la condition (SK). Cette analyse nous suggère d'isoler le mode non amorti et de réécrire le reste du système sous la forme d'un système satisfaisant la condition (SK).

Le mode non amorti sera alors vu comme un paramètre et n'apparaîtra que dans des termes non linéaires en facteur de composantes amorties vérifiant de meilleures propriétés d'intégrabilité en temps.

Plus précisément, nous reformulons le système sous la forme suivante :

$$\begin{cases} \partial_t y + u \cdot \nabla y = 0, \\ \partial_t w + u \cdot \nabla w + \big(\bar{H}_1 + H_1(w,r,y)\big) \operatorname{div} u + \big(\bar{H}_2 + H_2(w,r,y)\big) \dfrac{w}{\nu} = 0, \\ \partial_t r + u \cdot \nabla r + \big(\bar{H}_3 + H_3(w,r,y)\big) \operatorname{div} u = \big(\bar{H}_4 + H_4(w,r,y)\big) \dfrac{w^2}{\nu}, \\ \partial_t u + u \cdot \nabla u - \dfrac{1}{\rho} \mathcal{A}_{\mu,\lambda} u + \eta u + \dfrac{1}{\rho} \nabla r + (\gamma_+ - \gamma_-) \dfrac{1}{\rho} \nabla w = 0, \end{cases} \tag{27}$$

où y est la variable non amortie, w est proportionnel à $P_+ - P_-$ et r peut s'interpréter comme une pression effective (i.e. la partie de la pression qui ne disparaît pas après passage à la limite).

Évidemment, il ne sera pas possible de récupérer des propriétés d'intégrabilité L^1-en-temps sur l'inconnue y car elle ne satisfait qu'une équation de transport. Cela implique indirectement qu'il est impossible, avec notre méthode, de récupérer un contrôle L^1-en-temps sur ρ dans un espace $B^s_{2,1}$ avec $s \in \mathbb{R}$. Ceci est une difficulté majeure dans l'étude de ce système. Pour remédier à cela, il est nécessaire de traiter l'équation de y indépendamment des autres et le système (w, r, u) comme un système partiellement dissipatif satisfaisant la condition (SK) grâce à la méthode développée dans les précédents chapitres. Cependant, le système vérifié par (w, r, u) n'est ni symétrique ni symétrisable et il comporte un terme d'ordre deux. Il ne rentre donc pas directement dans le cadre des chapitres précédents. Pour compenser l'absence de symétrie, il sera nécessaire d'inclure des poids non-linéaires dans la fonctionnelle de Lyapunov hautes fréquences afin de pouvoir récupérer des estimations a priori. Et le terme d'ordre deux ne jouant pas de rôle majeur dans les questions de régularité, son effet sera la plupart du temps négligé. Mentionnons aussi que la forme spécifique des non-linéarités, notamment w^2/ν dans l'équation de r, jouent un rôle majeur pour pouvoir conclure.

Grâce à cela, nous sommes capables d'obtenir un théorème d'existence pour (27) et avec l'utilisation d'un lemme de composition, nous pouvons revenir aux variables initiales et obtenir le résultat suivant.[5]

Théorème 0.2.7. *Soit $d \geq 2$. On suppose que : $\mu \geq 0, \lambda + \mu \geq 0, \nu \leq 1$ et $\eta \geq 1$. De plus on suppose que l'hypothèse (26) est satisfaite et que $\bar{\alpha}_\pm \in (0,1)$ et $\bar{\rho}_\pm > 0$. Il existe une constante $c_1 > 0$ indépendante des coefficients de viscosité μ et λ telle que pour toute donnée initiale $(\alpha_{+0}, \alpha_{-0}, \rho_{+0}, \rho_{-0}, u_0)$ satisfaisant*

$$\|(\alpha_{\pm 0} - \bar{\alpha}_\pm, \rho_{\pm 0} - \bar{\rho}_\pm, u_0)\|_{B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}} \leq c_1,$$

alors le système (BN) admet une unique solution globale $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$ telle que

$$\begin{cases} (\alpha_\pm - \bar{\alpha}_\pm, \rho_\pm - \bar{\rho}_\pm, u) \in \mathcal{C}_b(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}), \\ \dfrac{P_+(\rho_+) - P_-(\rho_-)}{2\mu + \lambda} \in L^1(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}}) \quad \text{et} \quad u \in L^1(\mathbb{R}_+; B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}). \end{cases}$$

De plus, les estimations suivantes sont satisfaites uniformément par rapport aux paramètres de

[5] Se référer au début du Chapitre 4 pour les nouvelles notations.

relaxation μ et λ:

$$\|(\alpha_\pm - \bar{\alpha}_\pm, \rho_\pm - \bar{\rho}_\pm, u)\|_{L^\infty(\mathbb{R}_+;B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})} + \|u\|_{L^1(\mathbb{R}_+;B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1})} \\ \frac{1}{2\mu+\lambda}\|P_+(\rho_+) - P_-(\rho_-)\|_{L^1(\mathbb{R}_+;B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}})} \leq Cc_1,$$

où $C > 0$ est une constante universelle.

Concernant l'étude de la limite de relaxation du système (BN) vers le système (K), nous tirons profit des estimations uniformes du théorème d'existence pour obtenir un taux de convergence explicite du processus de relaxation en estimant la différence des solutions des deux systèmes. Pour cela, en utilisant le fait que

$$\frac{\alpha_+^\nu\alpha_-^\nu}{\nu}\left(P_+^\nu - P_-^\nu\right) = \frac{-\alpha_+^\nu\alpha_-^\nu}{\gamma_+\alpha_-^\nu P_+^\nu + \gamma_-\alpha_+^\nu P_-^\nu}\Big(\partial_t(P_+^\nu - P_-^\nu) + u^\nu\cdot\nabla(P_+^\nu - P_-^\nu) + (\gamma_+P_+^\nu - \gamma_-P_-^\nu)\operatorname{div}u^\nu\Big),$$

puis en insérant cette égalité dans le système satisfait par $(\alpha_\pm, P_\pm)$, on peut écrire l'équation vérifiée par la différence des solutions. Mais alors, le terme $\partial_t(P_+^\nu - P_-^\nu)$ qui apparaît en terme source est problématique car il ne vérifie pas de propriété de décroissance en temps. Nous reformulons donc notre système en introduisant des variables permettant de compenser ce terme et on se retrouve avec le système suivant :

$$\begin{cases} \partial_t\delta Y_+ + u^\nu\cdot\nabla\delta Y_+ = \delta S_1, \\ \partial_t\delta Q_+ + u^\nu\cdot\nabla\delta Q_+ + \big(\bar{\Gamma}_3 + (\Gamma_3 - \bar{\Gamma}_3)\big)\operatorname{div}\delta u = \delta S_2, \\ \partial_t\delta u + u^\nu\cdot\nabla\delta u + \delta u + \big(\frac{1}{\bar{\rho}} + (\frac{1}{\rho^\nu} - \frac{1}{\bar{\rho}})\big)\nabla\delta Q_+ = \delta S_3, \\ (\delta Y_+, \delta Q_+, \delta u)|_{t=0} = (0,0,0). \end{cases}$$

Ce système étant sous la forme d'un système hyperbolique partiellement dissipatif, nos techniques peuvent s'appliquer pour en estimer les solutions. De plus, en utilisant des lois de produits, on peut montrer que les termes sources sont des $O(\sqrt{\nu})$ dans l'espace $L^\infty(B^{\frac{d}{2}-\frac{3}{2}}\cap B^{\frac{d}{2}-\frac{1}{2}})$.

Voici le résultat que l'on obtient concernant la convergence des solutions du système (BN) vers les solutions du système (K).

Théorème 0.2.8. *Soit $d \geq 3$ et les mêmes hypothèses sur les coefficients que dans le Théorème 0.2.7. Soit $(\alpha_+^\nu, \alpha_-^\nu, \rho_+^\nu, \rho_+^\nu, u^\nu)$ (resp. $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$) la solution du problème de Cauchy (BN), associé à la valeur initiale $(\alpha_{+0}^\nu, \alpha_{-0}^\nu, \rho_{+0}^\nu, \rho_{-0}^\nu, u_0^\nu)$, provenant du Théorème 0.2.7 (resp. (K)-$(\alpha_{+0}, \alpha_{-0}, \rho_{+0}, \rho_{-0}, u_0)$ provenant du Théorème 4.2.2) telles que*

$$\|(\frac{\alpha_{+0}^\nu\rho_{+0}^\nu}{\alpha_{+0}^\nu\rho_{+0}^\nu + \alpha_{0-}^\nu\rho_{-0}^\nu} - \frac{\alpha_{+0}\rho_{+0}}{\alpha_{+0}\rho_{+0} + \alpha_{-0}\rho_{-0}}, P_{\pm0}^\nu - P_{\pm0}, u_0^\nu - u_0)\|_{B^{\frac{d}{2}-\frac{3}{2}}\cap B^{\frac{d}{2}-\frac{1}{2}}} \\ + \|P_{+0}^\nu - \frac{\gamma_+\alpha_-^\nu P_+^\nu}{\gamma_+\alpha_-^\nu P_+^\nu + \gamma_-\alpha_+^\nu P_-^\nu}(P_{+0}^\nu - P_{-0}^\nu) - P_{+0}\|_{B^{\frac{d}{2}-\frac{3}{2}}\cap B^{\frac{d}{2}-\frac{1}{2}}} \leq C\sqrt{\nu}$$

Alors il existe une constante $C > 0$ indépendante de ν telle que $(\alpha_+^\nu, \alpha_-^\nu, \rho_+^\nu, \rho_+^\nu, u^\nu)$ converge

vers $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$ *au sens suivant :*

$$\|(\alpha_\pm^\nu - \alpha_\pm, \rho_\pm^\nu - \rho_\pm, \rho_-^\nu - \rho_-, u^\nu - u)\|_{L^\infty(B^{\frac{d}{2}-\frac{1}{2}})} + \|\rho_\pm^\nu - \rho_\pm\|_{L^2(B^{\frac{d}{2}-\frac{1}{2}})} + \|u^\nu - u\|_{L^1(B^{\frac{d}{2}-\frac{1}{2}})} \leq C\sqrt{\nu}.$$

Ce théorème est, à notre connaissance, un des premiers résultats justifiant la convergence forte d'un système bi-fluide avec deux pressions vers un système de type Kapila. De plus nous sommes en mesure de quantifier cette convergence avec un taux explicite.

Chapitre 1

Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications

This chapter is based on the paper [30].

1.1 Introduction

In this chapter we develop a method for investigating global strong solutions of partially dissipative hyperbolic systems in the critical regularity setting. Our focus is on the one-dimensional setting (the multi-dimensional case being considered in the next chapter) and, for expository purpose, the first part of the chapter is devoted to implementing our method on a simple 'toy model' that may be seen as a simplification of the one-dimensional compressible Euler system with damping, and pressure law $P(\rho) = \frac{1}{2}\rho^2$, namely

$$(TM_\lambda) \qquad \begin{cases} \partial_t u + v\partial_x u + \partial_x v = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}, \\ \partial_t v + v\partial_x v + \partial_x u + \lambda v = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}, \\ (u,v)|_{t=0} = (u_0, v_0). & \text{on} \quad \mathbb{R} \end{cases}$$

where the unknown u may be seen as the discrepancy to the reference density normalized to 1, (then, the first equation is a simplification of the mass balance), while the unknown v stands for the velocity of the fluid, and the second equation corresponds to the evolution of velocity with a friction term of magnitude $\lambda > 0$ (which could also be interpreted as a relaxation parameter). More elaborated systems (including the compressible Euler system with general increasing pressure law) are considered at the end of the chapter.

The method we develop here offers us these two main features:

- Compared to the recent efforts on the matter, our use of *hybrid* Besov spaces with different regularity exponent in low and high frequency allows to consider more general data and to track the exact dependency on the dissipation parameter for the solution.
- Our approach enables us to go beyond the L^2 framework in the treatment of the low frequencies of the solution, which is totally new, to the best of our knowledge.

1.1.1 Background and motivations

The study of the global existence issue for so-called partially dissipative hyperbolic systems of balance laws goes back to the seminal work of Kawashima [64]. Recall that a general n-component systems of balance laws in $\mathbb{R}^d$ reads:

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j(w)}{\partial x_j} = Q(w). \tag{1.1}$$

Here the unknown $w = w(t,x)$ with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$ is valued in an open convex subset $\mathcal{O}_w$ of $\mathbb{R}^n$ and $Q, F_j : \mathcal{O}_w \to \mathbb{R}^n$ are given n-vector valued smooth functions on $\mathcal{O}_w$.

It is well known that classical systems of conservation laws (that is with $Q(w) = 0$) supplemented with smooth data admit local-in-time strong solutions that may develop singularities (shock waves) in finite time even if the initial data are small perturbations of a constant solution (see for instance the works by Majda in [72] and Serre in [91]). A sufficient condition for global existence for small perturbations of a constant solution $\bar{w}$ of (1.1) is the *total dissipation hypothesis*, namely the damping (or dissipation) term $Q(w)$ acts directly on each component of the system, making the whole solution to tend to $\bar{w}$ exponentially fast. However, in most evolutionary systems coming from physics, that condition is not verified, and even though global-in-time strong solutions do exist, exponential decay is very unlikely. A more reasonable assumption is that dissipation acts only on some components of the system. After suitable change of coordinates, we may write:

$$Q(w) = \begin{pmatrix} 0 \\ q(w) \end{pmatrix} \tag{1.2}$$

where $0 \in \mathbb{R}^{n_1}, q(w) \in \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$ and $n_1 + n_2 = n$. This so-called *partial dissipation hypothesis* arises in many applications such as gas dynamics or numerical simulation of conservation laws by relaxation scheme. A well known example is the damped compressible Euler system for isentropic flows that we will be investigated at the end of the chapter. For this system, the works by Wang and Yang [99] and Sideris, Thomases and Wang [94] pointed out that the dissipative mechanism, albeit only present in the velocity equation, can prevent the formation of singularities that would occur if $Q \equiv 0$.

Looking for conditions on the systems of the form (1.1)-(1.2) guaranteeing global existence of strong solutions for small perturbations of a constant solution $\bar{w}$ goes back to the paper of Shizuta and Kawashima [93], the thesis of Kawashima [64] and, more recently, to the paper of Yong [108]. Their researches reveal the importance of a rather explicit linear stability criterion, that is nowadays called the (SK) (for *Shizuta-Kawashima*) condition and of the existence of an entropy that provides a suitable symmetrisation of the system. Roughly speaking, (SK) condition ensures that the partial damping provided by (1.2) acts on all the components of the solution, although indirectly, so that all the solutions of (1.1) emanating from small perturbations of $\bar{w}$ eventually tend to $\bar{w}$, while the paper by Yong provides tools to get quantitative estimates on the solutions when $Q(\bar{w}) = 0$. As observed by Bianchini, Hanouzet and Natalini [8], in many situations, a careful analysis of the Green kernel of the linearized system about $\bar{w}$ allows to get explicit (and optimal) algebraic rates of convergence in L^p of smooth global solutions to $\bar{w}$. Let us finally mention that a more general approach has been proposed by Beauchard and Zuazua in [4], that allows to handle partially dissipative systems that need not satisfy the (SK) condition.

Recently, Kawashima and Xu in [102] and [105] extended all the prior works on partially dissipative hyperbolic systems satisfying the (SK) and entropy conditions (including the com-

pressible Euler system with a damping term) to 'critical' non-homogeneous Besov spaces of L^2 type. To obtain their results, they symmetrized the system thanks to the entropy hypothesis, applied a frequency localization argument relying on the Littlewood-Paley decomposition and used new properties concerning the Chemin-Lerner's spaces introduced in [24, 25]. They took advantage of the equivalence between the condition (SK) and the existence of a compensating function so as to exhibit global-in-time L^2 integrability properties of all the components of the system.

In this chapter, we focus on the particular situation where the space dimension is $d = 1$ and the number of components is $n = 2$. Our goal is to propose a method and a functional framework with *different* regularities for low and high frequencies. For the high frequencies, we do not really have the choice as it is known that the optimal regularity for local well-posedness in the context of general quasilinear hyperbolic systems, is given by the 'critical' Besov space $\mathbb{B}^{\frac{3}{2}}_{2,1}$. The novelty here is that we propose to look at the low frequencies of the solution in another space, not necessarily related to L^2. The advantage is not only that we will be able to consider a larger class of initial data that may be less decaying at infinity, but also that one can easily keep track of the dependency of the solution with respect to the dissipation coefficient, and thus have some information on the large dissipation asymptotics. Various considerations lead us to think that a suitable space for low frequencies is the homogeneous Besov space $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ (with, possibly, $p > 2$) that corresponds to the critical embedding in L^∞.

In order to have a robust method that can be adapted to more involved systems, we shall not compute explicitly the solution of the linearized system (TM_λ) about $(0, 0)$, but rather use modified energy arguments (different from those of S. Kawashima in his thesis [64]) and suitable change of unknowns. More specifically, we will introduce a 'modified' velocity that plays the same role as the 'viscous effective flux' in the works of Hoff [58] and, more recently, of Haspot [56] dedicated to the compressible Navier-Stokes equations.

Our approach enables us to obtain more accurate estimates and a weaker smallness condition than in prior works (in particular [4, 54, 64, 106, 108]). We will see that it is enough to assume that the low frequencies of the data have Besov regularity for some Lebesgue index that may be greater than 2. Also, we will improve the decay that was obtained for the compressible Euler system with damping in [106] and, adapting an idea from Xu and Xin in [101] for the compressible Navier-Stokes system will enable us to discard the additional smallness assumption on the low frequencies that is usually required to obtain the decay estimates.

1.2 Main results

Before stating our mains results, we recall that all the notation used below are defined in the Appendix A.2.

We state our main global existence result for (TM_λ).

Theorem 1.2.1. *Let $2 \le p \le 4$ and $\lambda > 0$. There exist $k = k(p) \in \mathbb{Z}$ and $c_0 = c_0(p) > 0$ such that for $J_\lambda \triangleq \lfloor \log_2 \lambda \rfloor + k$, if we assume that $u_0^{\ell,J_\lambda}, v_0^{\ell,J_\lambda} \in \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ and $u_0^{h,J_\lambda}, v_0^{h,J_\lambda} \in \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$ with*

$$\|(u_0, v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \lambda^{-1} \|(u_0, v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \le c_0,$$

then System (TM_λ) *admits a unique global solution* (u,v) *in the space* $E_p^{J_\lambda}$ *defined by*

$$\begin{aligned}
&u^{\ell,J_\lambda} \in \mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}), \quad u^{h,J_\lambda} \in \mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}) \cap L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}),\\
&v^{\ell,J_\lambda} \in \mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}), \quad v^{h,J_\lambda} \in \mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}) \cap L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})\\
&\lambda v + \partial_x u \in L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \quad \text{and} \quad v \in L^2(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}).
\end{aligned}$$

Moreover we have the following a priori estimate:

$$X_{p,\lambda}(t) \lesssim \|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \lambda^{-1}\|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \quad \text{for all } t \geq 0,$$

where

$$\begin{aligned}
X_{p,\lambda}(t) \triangleq \|(u,v)\|^{\ell,J_\lambda}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \lambda^{-1}\|(u,v)\|^{h,J_\lambda}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \lambda^{-1}\|u\|^{\ell,J_\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})} + \|(u,v)\|^{h,J_\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}\\
+ \|v\|^{\ell,J_\lambda}_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1})} + \|\lambda v + \partial_x u\|_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \lambda^{\frac{1}{2}}\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}.
\end{aligned}$$

Remark 1.2.1. *Somehow, the function* $\lambda v + \partial_x u$ *may be seen as a damped mode of the system, which explains its better time integrability. This is actually the key to close the estimates globally in time, and this enables us to prove similar results for more general systems (see Sections 1.5 and 1.6).*

Remark 1.2.2. *Kawashima and Xu in [102] obtained a result in critical nonhomogeneous Besov spaces built on* L^2 *for a class of system containing* (TM_λ)*. In their functional setting however, it seems difficult to track the exact dependency of the smallness condition and of the estimates with respect to the damping parameter* λ*.*

Remark 1.2.3. *In the* L^2 *case, the method we here propose is robust enough to be adapted to higher dimension and to systems with more components, as we will see in the next chapters.*

Remark 1.2.4. *In Section 1.5 a statement similar to the above one is obtained for the isentropic compressible Euler system with a damping term in the velocity equation. To our knowledge, it is the first result (partially) in the* L^p *setting for this system. Obtaining a similar result in higher dimension is the purpose of Chapter 3.*

The above theorem gives us for free some insight on the diffusive relaxation limit of (TM_λ) in the case of fixed initial data[1].

Corollary 1.2.1. *Under the hypotheses of Theorem 1.2.1, we have* $u \to u_0$ *and* $v \to 0$ *when* λ *goes to infinity. More precisely,*

$$\|v\|_{L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \leq Cc_0\lambda^{-1/2} \quad \text{and} \quad \|u(t)-u_0\|_{\dot{\mathbb{B}}^0_{p,1}} \leq Cc_0\left(\frac{t}{\lambda}\right)^{\frac{1}{2p}}.$$

Proof. The first inequality follows from the estimate for $X_{p,\lambda}$ in Theorem 1.2.1. For the second inequality, we observe that by interpolation in Besov spaces and Hölder inequality,

$$\|\partial_x v\|_{L^r(\dot{\mathbb{B}}^0_{p,1})} \lesssim \|\partial_x v\|^{1-\frac{1}{p}}_{L^1(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}\|\partial_x v\|^{\frac{1}{p}}_{L^2(\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1})} \quad \text{with} \quad \frac{1}{r} \triangleq 1 - \frac{1}{2p}.$$

[1] Our method actually allows us to investigate the connections between the compressible Euler system and the porous media equation, in the spirit of [62, 70, 103], cf Chapter 3

Since

$$\|\partial_x v\|_{L^1(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \|v\|^{\ell}_{L^1(\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1})} + \|v\|^{h}_{L^1(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})},$$

Theorem 1.2.1 gives us

$$\|\partial_x v\|_{L^r(\dot{\mathbb{B}}^0_{p,1})} \leq C c_0 \lambda^{-\frac{1}{2p}}.$$

Similarly, we have

$$\|\partial_x u\|_{L^{\tilde{r}}(\dot{\mathbb{B}}^0_{p,1})} \lesssim \|\partial_x u\|^{\frac{1}{2}-\frac{1}{2p}}_{L^1(\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1})} \|\partial_x u\|^{\frac{1}{2}+\frac{1}{2p}}_{L^\infty(\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1})} \quad \text{with} \quad \frac{1}{\tilde{r}} \triangleq \frac{1}{2} - \frac{1}{2p}.$$

Hence, using that the product maps $\dot{\mathbb{B}}^0_{p,1} \times \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ to $\dot{\mathbb{B}}^0_{p,1}$ and Theorem 1.2.1, we deduce that

$$\|v\partial_x u\|_{L^r(\dot{\mathbb{B}}^0_{p,1})} \lesssim \|v\|_{L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \|\partial_x u\|_{L^{\tilde{r}}(\dot{\mathbb{B}}^0_{p,1})} \leq C c_0 \lambda^{-\frac{1}{2p}}.$$

Since $\partial_t u = -\partial_x v - v\partial_x u$, we get the desired inequality for $u(t) - u_0$, by time integration and Hölder inequality. □

Remark 1.2.5. *In chapter 3 we derive a more advanced relaxation limit result in the multi-dimensional setting for the compressible Euler system.*

Our second main result concerns the optimal decay estimates of the solution constructed in the first theorem. For now, we only consider the case $p = 2$.

Theorem 1.2.2. *Under the hypotheses of Theorem 1.2.1 with $p = 2$, there exists a Lyapunov functional associated to the solution (u, v) constructed there, which is equivalent to $\|(u,v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$. If, additionally, $(u_0, v_0) \in \dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ then, there exists a constant C depending only on σ_1 and such that*

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0.$$

Furthermore, there exists a constant κ_0 depending only on σ_1, λ and on the norm of the data (and that may be taken equal to one in certain regimes, see the remark below) such that, if

$$\langle t \rangle \triangleq 1 + \kappa_0 t, \quad \alpha_2 \triangleq \sigma_1 + \frac{1}{2} \quad \text{and} \quad C_{0,\lambda} \triangleq \lambda^{1+\alpha_2} \|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}},$$

then we have the following decay estimates:

$$\begin{aligned}
\lambda^{\frac{3}{2}-\sigma} \|\langle \lambda t \rangle^{\alpha}(u,v)(t)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq C C_{0,\lambda}, \quad \sigma \in [-\sigma_1, 1/2], \quad \alpha \triangleq \frac{\sigma+\sigma_1}{2}, \\
\|\langle \lambda t \rangle^{\alpha_2}(u,v)(t)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,\infty}} &\leq C C_{0,\lambda}, \\
\lambda^{\sigma_1+\frac{3}{2}} \|\langle \lambda t \rangle^{\alpha_1} v\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} &\leq C C_{0,\lambda}, \qquad \alpha_1 \triangleq \frac{1}{2}\left(\frac{1}{2} + \sigma_1\right).
\end{aligned}$$

Remark 1.2.6. *Our proof reveals that*[2] *$\kappa_0 \approx 1$ whenever the first term of $C_{0,\lambda}$ is controlled by*

[2]The exact value is $\kappa_0 = \left(\frac{\lambda \|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{1/2}_{2,1}} + \|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{3/2}_{2,1}}}{\lambda^{1+\alpha_2} \|(u_0,v_0)\|^{\ell,J_\lambda}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u_0,v_0)\|^{h,J_\lambda}_{\dot{\mathbb{B}}^{3/2}_{2,1}}} \right)^{\frac{2}{\sigma_1+1/2}}$.

the second one (which amounts to saying that the low frequencies of the data are dominated by the high frequencies).

Remark 1.2.7. *The fact that v undergoes direct dissipation and not u explains why the decay of the low frequencies of v is stronger than that of u.*

Remark 1.2.8. *In light of the embedding $L^1 \hookrightarrow \dot{\mathbb{B}}^{-\frac{1}{2}}_{2,\infty}$, the above statement with $\sigma_1 = 1/2$ encompasses the classical L^1 condition of [76]. Actually, choosing suitable exponents allows to recover all the conditions used in [8] for getting decay estimates.*

1.3 The case $p = 2$

The present section is dedicated to the case $p = 2$ in Theorem 1.2.1 and to the proof of Theorem 1.2.2. The reason for looking first at $p = 2$ is that one can exhibit a Lyapunov functional for (TM_λ) that allows to treat the low and high frequencies of the solution together. Throughout this section, we focus on the proof of a priori estimates for smooth solutions to (TM_λ), the reader being referred to the next section for the rigorous proof of existence and uniqueness, in the general case.

Before starting, let us observe that (u, v) is a solution to (TM_λ) if and only if the couple $(\tilde{u}, \tilde{v})$ defined by

$$(u, v)(t, x) \triangleq (\tilde{u}, \tilde{v})(\lambda t, \lambda x) \tag{1.3}$$

satisfies (TM_1). Therefore, it suffices to establish Theorems 1.2.1 and 1.2.2 for $\lambda = 1$, scaling back giving the desired inequalities, owing to the use of *homogeneous* Besov norms.

In the rest of this section, and in the following one, we shall use the short notation (TM) to designate (TM_1).

1.3.1 Global a priori estimates for the linearized toy model

Here we are concerned with the proof of a priori estimates for the following linearization of (TM):

$$(LTM) \qquad \begin{cases} \partial_t u + w\partial_x u + \partial_x v = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}, \\ \partial_t v + w\partial_x v + \partial_x u + v = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}, \\ (u, v)|_{t=0} = (u_0, v_0) & \text{on} \quad \mathbb{R}, \end{cases}$$

where the given function $w : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is smooth.

In the following computations, we assume that we are given a smooth solution (u, v) of (LTM) on $[0, T] \times \mathbb{R}$, and denote, for all $j \in \mathbb{Z}$,

$$u_j \triangleq \dot{\Delta}_j u \quad \text{and} \quad v_j \triangleq \dot{\Delta}_j v.$$

Inspired by the work of the second author in [33, 35], we consider the following functional:

$$\mathcal{L}_j \triangleq \sqrt{\|u_j\|^2_{L^2} + \|v_j\|^2_{L^2} + \|\partial_x u_j\|^2_{L^2} + \|\partial_x v_j\|^2_{L^2} + \int_{\mathbb{R}} v_j\, \partial_x u_j}. \tag{1.4}$$

Applying operator $\dot{\Delta}_j$ to (LTM), simple computations lead to

$$\frac{1}{2}\frac{d}{dt}\|(u_j, v_j)\|_{L^2}^2 + \|v_j\|_{L^2}^2 + \int_{\mathbb{R}} \Big(\dot{\Delta}_j(w\partial_x u)\,\dot{\Delta}_j u + \dot{\Delta}_j(w\partial_x v)\,\dot{\Delta}_j v\Big) = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|(\partial_x u_j, \partial_x v_j)\|_{L^2}^2 + \|\partial_x v_j\|_{L^2}^2 + \int_{\mathbb{R}} \big(\dot{\Delta}_j\partial_x(w\partial_x u))\,\dot{\Delta}_j\partial_x u + \dot{\Delta}_j(\partial_x(w\partial_x v))\,\dot{\Delta}_j\partial_x v\Big) = 0,$$

$$\frac{d}{dt}\int_{\mathbb{R}} v_j\,\partial_x u_j + \int_{\mathbb{R}} v_j\,\partial_x u_j + \|\partial_x u_j\|_{L^2}^2 - \|\partial_x v_j\|_{L^2}^2 + \int_{\mathbb{R}} \Big(\dot{\Delta}_j\partial_x(w\partial_x u)\dot{\Delta}_j v + \dot{\Delta}_j(w\partial_x v)\partial_x\dot{\Delta}_j u\Big) = 0.$$

Using the fact that

$$\dot{\Delta}_j(w\partial_x z) = w\partial_x z_j + [\dot{\Delta}_j, w]\partial_x z \quad \text{for} \quad z = u, v \tag{1.5}$$

and integrating by parts, we see that

$$\int_{\mathbb{R}} \dot{\Delta}_j(w\partial_x z)\,\dot{\Delta}_j z = -\frac{1}{2}\int_{\mathbb{R}} \partial_x w\,|z_j|^2 + \int_{\mathbb{R}} [\dot{\Delta}_j, w]\partial_x z\, z_j.$$

Hence, using the classical commutator estimate (A.4) recalled in the Appendix and the embedding $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \hookrightarrow L^\infty$, we get an absolute constant $C > 0$ such that for all $j \in \mathbb{Z}$,

$$\left|\int_{\mathbb{R}} \dot{\Delta}_j(w\partial_x z)\,\dot{\Delta}_j z\right| \le C c_j 2^{-\frac{j}{2}} \|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\dot{\Delta}_j z\|_{L^2} \quad \text{with} \quad \sum_{j\in\mathbb{Z}} c_j = 1.$$

Likewise, we have, thanks to an integration by parts,

$$\int_{\mathbb{R}} \dot{\Delta}_j\partial_x(w\partial_x z)\,\dot{\Delta}_j\partial_x z = \frac{1}{2}\int_{\mathbb{R}} \partial_x w\,(\partial_x z_j)^2 + \int_{\mathbb{R}} \partial_x[\dot{\Delta}_j, w]\partial_x z\,\partial_x z_j.$$

Hence, using (A.4) and $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \hookrightarrow L^\infty$,

$$\left|\int_{\mathbb{R}} \dot{\Delta}_j\partial_x(w\partial_x z)\,\dot{\Delta}_j\partial_x z\right| \le C c_j 2^{-\frac{j}{2}} \|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x z_j\|_{L^2}.$$

Finally, integrating by parts reveals that

$$\int_{\mathbb{R}} \Big(\dot{\Delta}_j\partial_x(w\partial_x u)\dot{\Delta}_j v + \dot{\Delta}_j(w\partial_x v)\partial_x\dot{\Delta}_j u\Big) = \int_{\mathbb{R}} [\dot{\Delta}_j, w]\partial_x v\,\partial_x u_j - \int_{\mathbb{R}} [\dot{\Delta}_j, w]\partial_x u\,\partial_x v_j.$$

Hence, using (A.4),

$$\begin{aligned}\Bigg|\int_{\mathbb{R}} \Big(\dot{\Delta}_j\partial_x(w\partial_x u)\dot{\Delta}_j v + \dot{\Delta}_j(w\partial_x v)\partial_x\dot{\Delta}_j u\Big)\Bigg| \\ \le C c_j 2^{-\frac{j}{2}} \big(\|u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x v_j\|_{L^2} + \|v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x u_j\|_{L^2}\big)\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)\cdot\end{aligned}$$

In order to conclude the proof of estimates for $\mathcal{L}_j$, one can observe that there exist two absolute

constants C and c such that

$$C^{-1}\|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}^2 \le \mathcal{L}_j^2 \le C\|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}^2 \tag{1.6}$$
$$\text{and}\quad \|v_j\|_{L^2}^2 + \tfrac{1}{2}\Big(\|\partial_x u_j\|_{L^2}^2 + \|\partial_x v_j\|_{L^2}^2 + \textstyle\int_{\mathbb{R}} v_j\,\partial_x u_j\,dx\Big) \ge c\min(1,2^{2j})\mathcal{L}_j^2.$$

Consequently, putting together with the above inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + c\min(1,2^{2j})\mathcal{L}_j^2 \le Cc_j 2^{-\frac{j}{2}}\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\|(u,v,\partial_x u,\partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\mathcal{L}_j. \tag{1.7}$$

Then, integrating on [0,t] for $t\in[0,T]$ and using Lemma A.1.1 yields

$$\begin{aligned}2^{\frac{j}{2}}\|(u_j,v_j,\partial_x u_j,\partial_x v_j)(t)\|_{L^2} &+ \min(1,2^{2j})2^{\frac{j}{2}}\int_0^t \|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}\\ &\le C\Big(2^{\frac{j}{2}}\|(u_j,v_j,\partial_x u_j,\partial_x v_j)(0)\|_{L^2} + \int_0^t c_j\|(u,v,\partial_x u,\partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\Big).\end{aligned} \tag{1.8}$$

Since (direct) damping is present in the equation for v, one can expect v to have better decay and time integrability properties than u. In fact, as explained at the beginning of Section 1.4, it is even better to consider the function $z \triangleq v + \partial_x u$ that satisfies:

$$\partial_t z + z + w\partial_x z = -\partial_{xx}^2 v - \partial_x w\,\partial_x u.$$

Now, applying Operator $\dot{\Delta}_j$ to the above equation, then using the basic energy method gives:

$$\frac{1}{2}\frac{d}{dt}\|z_j\|_{L^2}^2 + \|z_j\|_{L^2}^2 = -\int_{\mathbb{R}} z_j\,\partial_{xx}^2 v_j - \int_{\mathbb{R}}\dot{\Delta}_j(w\partial_x z)\,z_j - \int_{\mathbb{R}}\dot{\Delta}_j(\partial_x w\,\partial_x u)z_j.$$

The last term may be handled thanks to a commutator decomposition, integration by parts (as above) and Inequality (A.4) with $s=1/2$. This gives

$$\frac{1}{2}\frac{d}{dt}\|z_j\|_{L^2}^2 + \|z_j\|_{L^2}^2 \le \|z_j\|_{L^2}\big(\|\partial_{xx}^2 v_j\|_{L^2} + C2^{-\frac{j}{2}}c_j\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|\dot{\Delta}_j(\partial_x w\,\partial_x u)\|_{L^2}\big).$$

After time integration (use Lemma A.1.1), we end up for all $t\in[0,T]$ with

$$\begin{aligned}2^{\frac{j}{2}}\|z_j(t)\|_{L^2} + 2^{\frac{j}{2}}\int_0^t \|z_j\|_{L^2} &\le 2^{\frac{j}{2}}\|z_j(0)\|_{L^2}\\ &+2^{\frac{j}{2}}\int_0^t \|\partial_{xx}^2 v_j\|_{L^2} + C\int_0^t c_j\big(\|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|\partial_x w\,\partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big).\end{aligned}$$

Hence, summing up on $j\le 0$ and using the stability of the space $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}$ by product yields

$$\|z(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \le \|z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + C\int_0^t \|v\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}} + C\int_0^t \|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big(\|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big). \tag{1.9}$$

Let us sum (1.8) on $j\in\mathbb{Z}$, then add (1.9) multiplied by a small enough constant. Using (1.6)

and denoting $X(t) \triangleq \|(u, v, \partial_x u, \partial_x v)(t)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$, we eventually get

$$X(t) + \int_0^t \big(\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}} + \|(u,v)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|v + \partial_x u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big) \leq C\bigg(X(0) + \int_0^t \|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} X\bigg). \tag{1.10}$$

Let us revert to our toy model, assuming that $w = v$. Then, denoting by $Y(t)$ the left-hand side of (1.10), we get

$$Y(t) \leq C\big(X(0) + Y^2(t)\big).$$

As $Y(0) = X(0)$, a continuity argument ensures that there exists $c_0 > 0$ such that if

$$X(0) \approx \|(u_0, v_0)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \leq c_0, \tag{1.11}$$

then we have

$$Y(t) \leq 2CX(0) \quad \text{for all} \quad t \in [0, T].$$

1.3.2 Proof of Theorem 1.2.2

For getting decay estimates *without any additional smallness condition*, the first step is to prove that the extra negative regularity for low frequencies is preserved through the time evolution. This is stated in the following lemma which is an adaptation to our setting of a result first proved by J. Xu and Z. Xin in [101] for the compressible Navier-Stokes system.

Lemma 1.3.1. *Let $\sigma_1 \in]-\frac{3}{2}, \frac{1}{2}]$. If, in addition to the hypotheses of Theorem 1.2.1, $\|(u_0, v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}$ is bounded then, for all $t \geq 0$, we have*

$$\|(u, v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\, \|(u_0, v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

Proof. Applying $\dot{\Delta}_j$ to (TM) yields

$$\begin{cases} \partial_t u_j + \partial_x v_j = -v\partial_x u_j + [v, \dot{\Delta}_j]\partial_x u, \\ \partial_t v_j + \partial_x u_j + v_j = -v\partial_x v_j + [v, \dot{\Delta}_j]\partial_x v. \end{cases}$$

Hence, an energy method, followed by time integration (use Lemma A.1.1) gives

$$\|(u_j, v_j)(t)\|_{L^2} + \int_0^t \|v_j\|_{L^2} \leq \|(u_j, v_j)(0)\|_{L^2} + \frac{1}{2}\int_0^t \|\partial_x v\|_{L^\infty}\|(u_j, v_j)\|_{L^2} + \int_0^t \|[v, \dot{\Delta}_j]\partial_x u\|_{L^2} + \int_0^t \|[v, \dot{\Delta}_j]\partial_x v\|_{L^2}.$$

Omitting the second term in the left-hand side, and using the commutator estimate (A.5) that is valid provided $-1/2 \leq -\sigma_1 < 3/2$, we get

$$\|(u, v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|(u_0, v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(u, v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

Hence, by Gronwall lemma,

$$\|(u, v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|(u_0, v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \exp\bigg(C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\bigg).$$

Since the term in the exponential is small (as $X(0)$ is small), we get the lemma. $\square$

The second ingredient is that one can work out from the computations we did in the previous paragraph, a *Lyapunov functional* that is equivalent to the norm of (u,v) in $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$. To proceed, observe that, on the one hand, Inequality (1.7) implies that for all $t\geq 0$,

$$\mathcal{L}(t)+c\int_0^t\mathcal{H}\leq\mathcal{L}(0)+C\int_0^t\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\mathcal{L}$$

$$\text{with}\quad \mathcal{L}\triangleq\sum_{j\in\mathbb{Z}}2^{\frac{j}{2}}\mathcal{L}_j\quad\text{and}\quad\mathcal{H}\triangleq\sum_{j\in\mathbb{Z}}2^{\frac{j}{2}}\min(1,2^{2j})\mathcal{L}_j$$

and that, on the other hand, (1.9) gives us

$$\|(v+\partial_x u)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}+\int_0^t\|v+\partial_x u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\leq\|v_0+\partial_x u_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}+C\int_0^t\|v\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}}+C\int_0^t\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\mathcal{L}.$$

Hence, there exist $\eta>0$ and $c'>0$ such that, denoting $\widetilde{\mathcal{L}}\triangleq\mathcal{L}+\eta\|v+\partial_x u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$, we have

$$\widetilde{\mathcal{L}}(t)+c'\int_0^t\widetilde{\mathcal{H}}\leq\widetilde{\mathcal{L}}(0)+C\int_0^t\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\mathcal{L}\quad\text{with}\quad\widetilde{\mathcal{H}}\triangleq\mathcal{H}+\eta\|v+\partial_x u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

Observe that $\widetilde{\mathcal{H}}\gtrsim\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$. Since the previous step ensures that $\mathcal{L}\lesssim X(0)$, one can conclude that the last term of the above inequality may be absorbed by the second term of the left-hand side provided $X(0)$ is small enough. So finally, taking c' smaller if need be, we discover that

$$\widetilde{\mathcal{L}}(t)+c'\int_0^t\widetilde{\mathcal{H}}\leq\widetilde{\mathcal{L}}(0).$$

Clearly, one can start the proof from any time $t_0\geq 0$ and get in a similar way:

$$\widetilde{\mathcal{L}}(t_0+h)+c'\int_{t_0}^{t_0+h}\widetilde{\mathcal{H}}\leq\widetilde{\mathcal{L}}(t_0),\quad h\geq 0.$$

This of course ensures that $\widetilde{\mathcal{L}}$ is nonincreasing on $\mathbb{R}^+$ (hence differentiable almost everywhere) and that for all $t_0\geq 0$ and $h>0$,

$$\frac{\widetilde{\mathcal{L}}(t_0+h)-\widetilde{\mathcal{L}}(t_0)}{h}+c'\frac{1}{h}\int_{t_0}^{t_0+h}\widetilde{\mathcal{H}}\leq 0.$$

Consequently, passing to the limit $h\to 0$ gives

$$\frac{d}{dt}\widetilde{\mathcal{L}}+c'\widetilde{\mathcal{H}}\leq 0\quad\text{a. e. on}\quad\mathbb{R}^+.\tag{1.12}$$

We thus come to the conclusion that:

Lemma 1.3.2. *There exist two functionals $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{D}}$ satisfying*

$$\widetilde{\mathcal{L}}\simeq\|(u,v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\quad\textit{and}\quad\widetilde{\mathcal{D}}\simeq\|u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}}+\|u\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\|v\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}},$$

and such that if $\|(u_0, v_0)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$ *is small enough then* (1.12) *is satisfied.*

One can now tackle the proof of decay estimates. Let us denote

$$C_0 \triangleq \|(u_0, v_0)\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u_0, v_0)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

As a first, observe that interpolation for homogeneous Besov norms gives us:

$$\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \Big(\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\Big)^{\theta_0}\Big(\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}}\Big)^{1-\theta_0} \quad \text{with} \quad \theta_0 \triangleq \frac{2}{5/2+\sigma_1}\cdot$$

Therefore, owing to Lemma 1.3.1, there exists $c > 0$ such that

$$\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}} \geq c\, C_0^{-\frac{\theta_0}{1-\theta_0}}\big(\|(u,v)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)^{\frac{1}{1-\theta_0}}.$$

Note that our definition of C_0 and the estimates we proved for (u, v) in the previous paragraph also ensure that

$$\|(u,v)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \gtrsim C_0^{-\frac{\theta_0}{1-\theta_0}}\big(\|(u,v)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big)^{\frac{1}{1-\theta_0}}.$$

Hence, thanks to the above lemma, we have,

$$\frac{d}{dt}\widetilde{\mathcal{L}} + cC_0^{-\frac{\theta_0}{1-\theta_0}}\widetilde{\mathcal{L}}^{\frac{1}{1-\theta_0}} \leq 0 \quad \text{with} \quad \widetilde{\mathcal{L}} \simeq \|(u,v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

Integrating, this gives us

$$\widetilde{\mathcal{L}}(t) \leq \big(1+\kappa_0\, t\big)^{1-\frac{1}{\theta_0}}\widetilde{\mathcal{L}}(0) \quad \text{with} \quad \kappa_0 \triangleq c\frac{\theta_0}{1-\theta_0}\left(\frac{\widetilde{\mathcal{L}}(0)}{C_0}\right)^{\frac{\theta_0}{1-\theta_0}}.$$

Rewriting θ_0 in terms of σ_1 and using that $\|(u_0, v_0)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim C_0$, we get

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \leq C(1+t)^{-\alpha_1}\|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \quad \text{with} \quad \alpha_1 \triangleq \frac{1}{2}\Big(\sigma_1+\frac{1}{2}\Big)\cdot \tag{1.13}$$

In order to get the decay rate in $\dot{\mathbb{B}}^{\sigma}_{2,1}$ for all $\sigma \in (-\sigma_1, 1/2)$, we just need the interpolation inequality

$$\|(u,v)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \leq \|(u,v)\|^{\theta_1}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\|(u,v)\|^{1-\theta_1}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \quad \text{with} \quad \theta_1 \triangleq \frac{1/2-\sigma}{1/2+\sigma_1} \in (0,1).$$

In the end, we get (since $\kappa_0 \lesssim 1$):

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \leq C(1+\kappa_0 t)^{-\frac{\sigma+\sigma_1}{2}}\|(u_0,v_0)\|^{\theta_1}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\|(u_0,v_0)\|^{1-\theta_1}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

In order to improve the decay for the damped component v, let us start from

$$\partial_t v + v = -\frac{1}{2}\partial_x(v^2) - \partial_x u.$$

As v_0^ℓ is in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in]-\frac{1}{2},\frac{1}{2}]$, we get

$$\|v(t)\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq e^{-t}\|v_0\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \int_0^t e^{-(t-\tau)}\left\|(\partial_x v^2, \partial_x u)\right\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} d\tau. \tag{1.14}$$

It is important to observe that, as $1-\sigma_1 \geq 1/2$,

$$\|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}. \tag{1.15}$$

Hence, multiplying (1.14) by $\langle t\rangle^{\alpha_1}$ and using the product laws in Besov spaces recalled in Proposition 1.7.2 yields:

$$\|\langle t\rangle^{\alpha_1} v(t)\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|v_0\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \int_0^t \langle t\rangle^{\alpha_1} e^{-(t-\tau)}\|u\|^\ell_{\dot{\mathbb{B}}^{1-\sigma_1}_{2,\infty}} d\tau + \int_0^t \langle t\rangle^{\alpha_1} e^{-(t-\tau)}\|v\|^2_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} d\tau,$$

and one can conclude as above that

$$\langle t\rangle^{\alpha_1}\|v(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

Let us finally exhibit the (optimal) decay rate of the high frequencies for the norm in $\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$. Recall that for $j \geq 0$, we have

$$\frac{d}{dt}\mathcal{L}_j^2 + c\mathcal{L}_j^2 \lesssim \mathcal{L}_j^2\|\partial_x v\|_{L^\infty} + c_j 2^{-\frac{j}{2}} X \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \mathcal{L}_j.$$

Hence, using Lemma A.1.1, multiplying by $2^{\frac{j}{2}}$, summing up on $j \geq 0$ and remembering that

$$\sum_{j\geq 0} 2^{\frac{j}{2}}\mathcal{L}_j \simeq \|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}},$$

we get

$$\|(u,v)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim e^{-ct}\|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t e^{-c(t-\tau)}\|v\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\|(u,v)\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}. \tag{1.16}$$

Multiplying both sides by $\langle t\rangle^{2\alpha_1}$, we get

$$\begin{aligned}\left\|\langle t\rangle^{2\alpha_1}(u,v)(t)\right\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} &\lesssim \langle t\rangle^{2\alpha_1} e^{-ct}\|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \\ &\quad + \int_0^t \left(\frac{\langle t\rangle}{\langle\tau\rangle}\right)^{2\alpha_1} e^{-c(t-\tau)}\left(\langle\tau\rangle^{\alpha_1}\|v\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\right)\left(\langle\tau\rangle^{\alpha_1}\|(u,v)\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\right).\end{aligned}$$

Taking advantage of (1.13) for bounding the norms in the time integral, one ends up with the desired decay estimate for $\|(u,v)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$. This completes the proof of Theorem 1.2.2.

Remark 1.3.1. *In the same way, making the slightly stronger assumption that $v_0^\ell \in \dot{\mathbb{B}}^{-\frac{1}{2}}_{2,1}$, we get*

$$\langle t\rangle^{\alpha_1}\|v(t)\|_{\dot{\mathbb{B}}^{-\frac{1}{2}}_{2,1}} \lesssim \|v_0\|_{\dot{\mathbb{B}}^{-\frac{1}{2}}_{2,1}} + \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

1.4 Proof of Theorem 1.2.1

An explicit computation in the Fourier space of the solution to (LTM) with $w = 0$ reveals that:

- In low frequencies, the matrix of the system corresponding to frequency ξ has two real eigenvalues that tend to be equal to 1 and to ξ^2, for ξ going to 0;
- In high frequencies, two complex conjugated eigenvalues coexist, that are, asymptotically, equal to $\frac{1}{2}(\xi^2 \pm i\xi)$.

Consequently, one can expect that the low frequency part of System (TM) is solvable in some L^p type functional framework with, possibly, $p \neq 2$, whereas going beyond the L^2 framework in high frequency is bound to fail. A similar dichotomy has been observed for the compressible Navier-Stokes equations (see in particular [23, 27, 56]) but the behavior of the low and high frequencies in our situation is exchanged.

In order to extend the results of the previous section to the L^p framework for low frequencies, we shall adapt [56] to our setting, introducing an 'effective velocity' that reads $z = v + \partial_x u$ and may be seen as an approximate dissipative eigenmode of the system, in the low frequency regime.

The bulk of the proof consists in establishing estimates in the functional framework of Theorem 1.2.1 for (LTM). This will be carried out in the first two subsections of this part. Then, we will prove the existence part of the theorem and, finally, the uniqueness of a solution.

1.4.1 Low frequencies estimates in L^p

The main result of this section reads as follows.

Proposition 1.4.1. *Let (u, v) be a smooth solution of (LTM) on $[0, T]$. Then, for all $1 \leq p < \infty$, we have*

$$\|(u,v)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}} + \int_0^t \|v + \partial_x u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \leq C\Big(\|(u_0,v_0)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|(u,v,\partial_x u)\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|w\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\Big). \tag{1.17}$$

Proof. Let us set $z \triangleq v + \partial_x u$. We observe that the couple (u, z) satisfies

$$\begin{cases} \partial_t u - \partial^2_{xx} u + w\partial_x u = -\partial_x z, \\ \partial_t z + z + w\partial_x z = \partial^3_{xxx} u - \partial^2_{xx} z - \partial_x w\, \partial_x u. \end{cases} \tag{1.18}$$

In low frequencies, we expect the linear terms of the right-hand side to be negligible, so that we will look at the first equation as a heat equation with a convection term, and at the second one as a damped transport equation.

Now, applying $\dot{\Delta}_j$ to the first equation of (1.18) yields

$$\begin{aligned} \partial_t u_j - \partial^2_{xx} u_j &= -\dot{\Delta}_j(w\partial_x u) - \partial_x z_j \\ &= -w\partial_x u_j - \partial_x z_j + [w, \dot{\Delta}_j]\partial_x u. \end{aligned}$$

Multiplying by $|u_j|^{p-2}u_j$ and integrating in space, we get

$$\frac{1}{p}\frac{d}{dt}\|u_j\|_{L^p}^p - \int_{\mathbb{R}} \partial_{xx}^2 u_j |u_j|^{p-2}u_j$$
$$= -\int_{\mathbb{R}} \partial_x z_j |u_j|^{p-2}u_j - \int_{\mathbb{R}} w\partial_x u_j\, |u_j|^{p-2}u_j + \int_{\mathbb{R}} [w,\dot\Delta_j]\partial_x u\, |u_j|^{p-2}u_j.$$

Hence, integrating by parts, using Cauchy-Schwarz inequality and Proposition 1.7.1 gives

$$\frac{1}{p}\frac{d}{dt}\|u_j\|_{L^p}^p + c_p 2^{2j}\|u_j\|_{L^p}^p \le \frac{1}{p}\|\partial_x w\|_{L^\infty}\|u_j\|_{L^p}^p + \left(\|\partial_x z_j\|_{L^p} + \left\|[w,\dot\Delta_j]\partial_x u\right\|_{L^p}\right)\|u_j\|_{L^p}^{p-1}.$$

Multiplying by $2^{\frac{j}{p}}$, summing up on $j \le J_0$ and using Lemma A.1.1, we obtain

$$\|u(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + c_p\int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}}$$
$$\le \|u_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} + \frac{1}{p}\int_0^t \|\partial_x w\|_{L^\infty}\|u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \sum_{j\le J_0} 2^{\frac{j}{p}}\int_0^t \left\|[w,\dot\Delta_j]\partial_x u\right\|_{L^p}.$$

The commutator term may be bounded according to Inequality (A.4) with $s = 1/p$. Hence, remembering that $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1} \hookrightarrow L^\infty$, we end up with

$$\|u(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + c_p\int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}} \le \|u_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} + C\int_0^t \|w\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\|u\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}. \tag{1.19}$$

Let us next look at the second equation of (1.18). We have for all $j \in \mathbb{Z}$,

$$\partial_t z_j + z_j + w\partial_x z_j = \partial_{xxx}^3 u_j - \partial_{xx}^2 z_j - \dot\Delta_j(\partial_x w\,\partial_x u) + [w,\dot\Delta_j]\partial_x z.$$

Multiplying by $z_j|z_j|^{p-2}$ and adapting what we did for the for the first equation of (LTM), we obtain

$$\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \le \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}} + \int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+3}_{p,1}} + \frac{1}{p}\int_0^t \|\partial_x w\|_{L^\infty}\|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}$$
$$+ \int_0^t \sum_{j\le J_0} 2^{\frac{j}{p}}\left\|[w,\dot\Delta_j]\partial_x z\right\|_{L^p} + \int_0^t \|\partial_x w\,\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}.$$

Combining Proposition A.2.3, the commutator estimate (A.4), the embedding $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1} \hookrightarrow L^\infty$ and Proposition 1.7.2, we discover that

$$\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \le \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}} + \int_0^t \|u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+3}_{p,1}}$$
$$+ \int_0^t \|w\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\|z\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|\partial_x w\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\|\partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}. \tag{1.20}$$

At this stage, the key observation is that, owing to Bernstein inequality, there exists an absolute

constant C such that for any couple $(\sigma,\sigma')\in\mathbb{R}^2$ with $\sigma\le\sigma'$, we have

$$\|f\|^{\ell}_{\dot{\mathbb{B}}^{\sigma'}_{p,1}}\le C2^{J_0(\sigma'-\sigma)}\|f\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{p,1}}. \tag{1.21}$$

Consequently, if J_0 is chosen small enough, then after adding up (1.19) and (1.20), we just get

$$\|(u,z)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\int_0^t\big(\|u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}}+\|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\big)\lesssim\|(u_0,z_0)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}$$
$$+\int_0^t\|w\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\big(\|u\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}+\|u\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\|z\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\big)\cdot$$

Because

$$\|z\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\lesssim\|u\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}+\|v\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\quad\text{and}\quad\|v\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\lesssim\|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\|u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}},$$

we conclude to the desired inequality. □

1.4.2 High frequencies estimates in L^2

Our second task is to bound the high frequencies of the solution of (LTM). Although the functional framework for high frequencies is the same as before, one cannot repeat exactly the computations therein since the terms $(w\partial_x u)^h$ and $(w\partial_x v)^h$ contain a little amount of low frequencies of w, u and v, that are only in spaces of the type $\dot{\mathbb{B}}^{s}_{p,1}$ with $p>2$ (and thus not in some $\dot{\mathbb{B}}^{s'}_{2,1}$). To overcome the difficulty, we have to study more carefully the commutators coming into play in the proof (see Lemma 1.7.1).

Proposition 1.4.2. *Let (u,v) be of solution of (LTM) with $u_0^\ell,v_0^\ell\in\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ and $u_0^h,v_0^h\in\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$ for some $2\le p\le 4$. Define p^* by the relation $1/p+1/p^*=1/2$. Then, the following a priori estimate holds:*

$$\|(u,v)(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\int_0^t\|(u,v)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\lesssim\|(u_0,v_0)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\int_0^t\big(\|\partial_x w\|_{L^\infty}\|(u,v)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$$
$$+\|w\|^{\ell}_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}\|(\partial_x u,\partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}+\|(\partial_x u,\partial_x v)\|_{L^\infty}\|w\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\|(\partial_x u,\partial_x v)\|^{\ell}_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}}\|\partial_x w\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\big)\cdot$$

Proof. We localize System (LTM) by means of $\dot{\Delta}_j$, getting

$$\begin{cases}\partial_t u_j+\partial_x v_j+\dot{S}_{j-1}w\,\partial_x u_j=R_j^1\\ \partial_t v_j+\partial_x u_j+v_j+\dot{S}_{j-1}w\,\partial_x v_j=R_j^2\end{cases}$$

with

$$R_j^1\triangleq\dot{S}_{j-1}w\,\partial_x u_j-\dot{\Delta}_j(w\,\partial_x u)\quad\text{and}\quad R_j^2\triangleq\dot{S}_{j-1}w\,\partial_x v_j-\dot{\Delta}_j(w\,\partial_x v).$$

The remainder terms R_j^1 and R_j^2 will be bounded according to Lemma 1.7.1. To handle the left-hand side of the above localized system, we introduce the following functional, designed for high frequencies:

$$\widetilde{\mathcal{L}}_j^2\triangleq\|(\partial_x u_j,\partial_x v_j)\|_{L^2}^2+\int_{\mathbb{R}}v_j\,\partial_x u_j,$$

and get

$$\frac{1}{2}\Big(\frac{d}{dt}\widetilde{\mathcal{L}}_j^2+\widetilde{\mathcal{L}}_j^2\Big)+\int_{\mathbb{R}}\partial_x\big((\dot{S}_{j-1}w\,\partial_x u_j)\partial_x u_j+\partial_x(\dot{S}_{j-1}w\,\partial_x v_j)\partial_x v_j\big)$$
$$+\int_{\mathbb{R}}\big(\dot{S}_{j-1}w\partial_x v_j\partial_x u_j+\partial_x(\dot{S}_{j-1}w\partial_x u_j)v_j\big)=\int_{\mathbb{R}}\big(\partial_x R_j^1\partial_x u_j+\partial_x R_j^2\partial_x v_j+\frac{1}{2}\big(\partial_x R_j^1 v_j+R_j^2\partial_x u_j\big)\big)\cdot$$

Using integration by parts and multiplying by 2 then yields

$$\frac{d}{dt}\widetilde{\mathcal{L}}_j^2+\widetilde{\mathcal{L}}_j^2+\int_{\mathbb{R}}\partial_x\dot{S}_{j-1}w\,\big((\partial_x u_j)^2+(\partial_x v_j)^2\big)=\int_{\mathbb{R}}\big(R_j^2\,\partial_x u_j-R_j^1\,\partial_x v_j-2R_j^1\,\partial_x^2 u_j-2R_j^2\,\partial_x^2 v_j\big)\cdot$$

From this, using Cauchy-Schwarz inequality, that $\widetilde{\mathcal{L}}_j\simeq\|(\partial_x u_j,\partial_x v_j)\|_{L^2}$ and Lemma A.1.1, we get for all $t\geq 0$ and $j\geq J_0$,

$$\widetilde{\mathcal{L}}_j(t)+\int_0^t\widetilde{\mathcal{L}}_j\leq C\Big(\int_0^t\|\partial_x w\|_{L^\infty}\widetilde{\mathcal{L}}_j+2^j\int_0^t\|R_j^1,R_j^2\|_{L^2}\Big),$$

with C depending only on J_0. Hence, multiplying by $2^{\frac{j}{2}}$ and summing up on $j\geq J_0$,

$$\|(u,v)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\int_0^t\|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\lesssim\|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$$
$$+\int_0^t\|\partial_x w\|_{L^\infty}\|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\int_0^t\sum_{j\geq J_0}2^{\frac{3}{2}j}\,\big\|(R_j^1,R_j^2)\big\|_{L^2}\,. \quad (1.22)$$

At this stage, taking advantage of Lemma 1.7.1 with $s=3/2$ to bound the sum, we conclude to the desired inequality. □

1.4.3 Global a priori estimates for the toy model

We are now ready to establish the following proposition which is the key to the proof of the existence part of the theorem.

Proposition 1.4.3. *Let (u,v) be a smooth solution of (TM) on $[0,T]$. Then, still assuming that $2\leq p\leq 4$, there exists a constant C and an integer J_0 (corresponding to the threshold between low and high frequencies) such that for all $t\in[0,T]$, we have*

$$X_p(t)\leq C\big(X_{p,0}+X_p^2(t)\big)$$

with $X_{p,0}\triangleq\|(u_0,v_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\|(u_0,v_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$ *and*

$$X_p(t)\triangleq\|(u,v)\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}+\|(u,v)\|^h_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}$$
$$+\|u\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})}+\|v+\partial_x u\|_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}+\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}+\|(u,v)\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}\,.$$

Proof. As a first, let us observe that $\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$ is dominated by the other terms of $X_p(t)$ (let us denote them by $\widetilde{X}_p(t)$). This is clearly the case of the high frequency part since, by Bernstein

inequality, Hölder inequality and the embedding $\dot{\mathbb{B}}^s_{2,1} \hookrightarrow \dot{\mathbb{B}}^{s+\frac{1}{p}-\frac{1}{2}}_{p,1}$ with $s = 3/2$,

$$\|v\|^h_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \|v\|^h_{L^2_t(\dot{\mathbb{B}}^{1+\frac{1}{p}}_{p,1})} \lesssim \|v\|^h_{L^2_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} \lesssim \sqrt{\|v\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} \|v\|^h_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}}.$$

For the low frequency part, we write that

$$\|v\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \leq \|\partial_x u\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|z\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \quad \text{with} \quad z \triangleq v + \partial_x u.$$

By Hölder inequality and interpolation, we have

$$\|\partial_x u\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \Big(\|u\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \|u\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})}\Big)^{\frac{1}{2}} \quad \text{and} \quad \|z\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \Big(\|z\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \|z\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}\Big)^{\frac{1}{2}}.$$

As the low frequencies of z in $L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$ may be bounded by $\widetilde{X}_p(t)$, we proved that

$$\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \widetilde{X}_p(t) \quad \text{for all} \quad t \in \mathbb{R}^+. \tag{1.23}$$

Let us also notice that by Sobolev embedding and Bernstein inequality,

$$\|v + \partial_x u\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \lesssim \|v\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|\partial_x u\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})}.$$

Therefore, adding up the inequalities from Propositions 1.4.1 and 1.4.2 for $w = v$ and observing that $2 \leq p \leq p^*$, we get

$$X_p(t) \lesssim X_{p,0} + \int_0^t \big(\|(u,v)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(u,v)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big)\big(\|v\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} + \|v\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big) + \int_0^t \|v\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|(\partial_x u, \partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}.$$

Since

$$\|v\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} \leq \|v + \partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} + C\|u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}},$$

we conclude that the inequality of Proposition 1.4.3 is satisfied. □

1.4.4 Proof of the existence part of Theorem 1.2.1

The proof relies on the following classical result about the local existence of strong solutions for hyperbolic symmetric systems of type

$$(QS) \qquad \begin{cases} \partial_t U + \sum_{k=1}^d A_k(U)\partial_k U + A_0(U) = 0, \\ U|_{t=0} = U_0, \end{cases}$$

where A_k, $k = 0, \ldots, d$ are smooth functions from $\mathbb{R}^n$ to the space of $n \times n$ matrices, that are symmetric if $k \neq 0$.

Theorem 1.4.1. *[3, Chap. 4] Let U_0 be in the nonhomogeneous Besov space $\mathbb{B}^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d;\mathbb{R}^n)$. Then, (QS) admits a unique maximal solution U in $\mathcal{C}([0,T^*[;\mathbb{B}^{\frac{d}{2}+1}_{2,1}) \cap \mathcal{C}^1([0,T^*[;\mathbb{B}^{\frac{d}{2}}_{2,1})$, and there*

exists a positive constant c such that

$$T^* \geq \frac{c}{\|U_0\|_{\mathbb{B}^{\frac{d}{2}+1}_{2,1}}}.$$

Furthermore,

$$T^* < \infty \Longrightarrow \int_0^{T^*} \|\nabla U\|_{L^\infty} = \infty.$$

The proof of the existence part of Theorem 1.2.1 is structured as follows. First, we multiply the low frequencies of the data by a cut-off function in order to have data in $\mathbb{B}^{\frac{3}{2}}_{2,1}$. One can then use the above theorem to construct a sequence of approximate solutions, that are shown to be global. We prove uniform estimates in the space E_p for those solutions, pass to the limit up to subsequence by means of compactness arguments, and finally check that the limit is a solution of (TM) with the required properties.

First step: Construction of approximate solutions

Let (u_0, v_0) be such that $u_0^\ell, v_0^\ell \in \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ and $u_0^h, v_0^h \in \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$. Since (u_0, v_0) need not be in $\mathbb{B}^{\frac{3}{2}}_{2,1}$, we set for all $n \geq 1$,

$$u_0^n \triangleq \chi_n \, \dot{S}_{J_0-5} u_0 + (\mathrm{Id} - \dot{S}_{J_0-5}) u_0 \quad \text{and} \quad v_0^n \triangleq \chi_n \, \dot{S}_{J_0-5} v_0 + (\mathrm{Id} - \dot{S}_{J_0-5}) v_0$$

with $\chi_n \triangleq \chi(n^{-1}\cdot)$, where χ stands (for instance) for the bump function of Section 1.2.

It is obvious that the sequences $(u_0^n)_{n\in\mathbb{N}}$ and $(v_0^n)_{n\in\mathbb{N}}$ tend to u_0 and v_0 in the sense of distributions, when n tends to infinity. Moreover, as u_0^ℓ and v_0^ℓ are in $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$, the low frequencies of the data are in L^∞, and the spatial truncation thus guarantees that $u_0^n, v_0^n \in \mathbb{B}^{\frac{3}{2}}_{2,1}$. Hence, Theorem 1.4.1 provides us with a unique maximal solution $(u^n, v^n) \in \mathcal{C}([0, T_n[; \mathbb{B}^{\frac{3}{2}}_{2,1}) \cap \mathcal{C}^1([0, T_n[; \mathbb{B}^{\frac{1}{2}}_{2,1})$.

We claim that we have for $z_0 = u_0, v_0$,

$$\|z_0^n\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|z_0^n\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|z_0\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}. \tag{1.24}$$

Indeed, since $\|\chi_n\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \simeq \|\chi\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} < \infty$, owing to the invariance of the norm in $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ by spatial dilation (see e.g. [3, Rem. 2.19]), we may write

$$\begin{aligned}
\|z_0^n\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} &\leq \Big\|\chi_n \, \dot{S}_{J_0-5} z_0\Big\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \Big\|(\mathrm{Id} - \dot{S}_{J_0-5}) z_0\Big\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \\
&\lesssim \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \, \|\chi_n\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|z_0\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \\
&\lesssim \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|z_0\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.
\end{aligned}$$

Next, we see that

$$\|z_0^n\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \leq \|\chi_n \, \dot{S}_{J_0-5} z_0\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|(\mathrm{Id} - \dot{S}_{J_0-5}) z_0^h\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

It is obvious that the last term may be bounded by $\|z_0\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$. For the other term, the important

observation is that for $j \geq J_0$, we have

$$\dot{\Delta}_j\big(\chi_n\, \dot{S}_{J_0-5} z_0\big) = \sum_{j' \geq j-3} \dot{\Delta}_j\big(\dot{S}_{j'+2} \dot{S}_{J_0-5} z_0\, \dot{\Delta}_{j'} \chi_n\big)\cdot$$

Hence, owing to the scaling properties of the space $\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}$,

$$\|\chi_n\, \dot{S}_{J_0-5} z_0\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \|\dot{S}_{J_0-5} z_0\|_{L^\infty} \|\chi_n\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim n^{-1} \|z_0\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\,,$$

which eventually yields (1.24).

Second step: Uniform estimates

Since, for all $T > 0$, the space $\mathcal{C}([0,T]; \mathbb{B}^{\frac{3}{2}}_{2,1}) \cap \mathcal{C}^1([0,T]; \mathbb{B}^{\frac{1}{2}}_{2,1})$ is included in our 'solution space' $E_p(T)$ (that is, E_p restricted to $[0,T]$), one can take advantage of Proposition 1.4.3 for bounding our sequence. From it and (1.24), we get, denoting X^n_p the function X_p pertaining to (u^n, v^n),

$$X^n_p \leq C\big(X_{p,0} + (X^n_p)^2\big)\cdot \tag{1.25}$$

It is clear that if

$$2CX^n_p(t) \leq 1, \tag{1.26}$$

then Inequality (1.25) implies that

$$X^n_p(t) \leq 2CX_{p,0}.$$

Then, thanks to a classical bootstrap argument, we can conclude that if $X_{p,0}$ is small enough then (1.26) is true as long as the solution exists. Hence, there exists a constant C such that

$$X^n_p(t) \leq CX_{p,0} \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad t \in [0, T_n[. \tag{1.27}$$

In order to show that the above inequality implies that the solution is global (namely that $T_n = \infty$), one can argue by contradiction, assuming that $T_n < \infty$, and use the blow-up criterion of Theorem 1.4.1. However, we first have to justify that the nonhomogeneous Besov norm $\mathbb{B}^{\frac{3}{2}}_{2,1}$ of the solution is under control *up to time* T_n. Now, applying the standard energy method to (TM) yields for all $t < T_n$,

$$\|(u^n, v^n)(t)\|^2_{L^2} \leq \|(u^n_0, v^n_0)\|^2_{L^2} + \int_0^t \|\partial_x v^n\|_{L^\infty} \|(u^n, v^n)\|^2_{L^2}\,.$$

Since (1.27) and the embedding of $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ and $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}$ in L^∞ ensure that $\partial_x v^n$ is in $L^1_{T_n}(L^\infty)$, using Gronwall lemma gives that (u^n, v^n) is in $L^\infty_{T_n}(L^2)$, and thus in $L^\infty_{T_n}(\mathbb{B}^{\frac{3}{2}}_{2,1})$ owing, again, to (1.27).

It is now easy to conclude : for all $t_{0,n} \in [0, T_n[$, Theorem 1.4.1 provides us with a solution of (TM) with the initial data $(u(t_{0,n}), v(t_{0,n}))$, on $[t_{0,n}, T + t_{0,n}]$ for some T that may be bounded from below independently of $t_{0,n}$. Consequently, choosing $t_{0,n}$ such that $t_{0,n} > T_n - T$, we see that the solution (u^n, v^n) can be extended beyond T_n, which contradicts the maximality of T_n. Hence $T_n = +\infty$ and the solution corresponding to the initial data (u^n_0, v^n_0) is global in time and satisfies (1.27) for all time.

Third step: Convergence

We have to show that $(u^n, v^n)_{n\in\mathbb{N}}$ tends, up to subsequence, to some $(u, v) \in E_p$ in the sense of distribution, that satisfies (TM).

The proof that we here propose rests on Ascoli Theorem and suitable compact embeddings. Let us explain how it goes for $(u^n)_{n\in\mathbb{N}}$, the convergence of $(v^n)_{n\in\mathbb{N}}$ being similar. From (1.27) and elementary embedding, we know that :

- $(\partial_x u^n)_{n\in\mathbb{N}}$ and $(\partial_x v^n)_{n\in\mathbb{N}}$ are bounded in $L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$,
- $(v^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$.

Hence, both $(v^n\partial_x u^n)_{n\in\mathbb{N}}$ and $(\partial_x v^n)_{n\in\mathbb{N}}$ are bounded in $L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$, which implies that $(\partial_t u^n)_{n\in\mathbb{N}}$ is bounded in $L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$. This means that $(u^n)_{n\in\mathbb{N}}$ viewed as a sequence of functions valued in $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ is locally equicontinuous on $\mathbb{R}^+$.

Moreover $(u^{n,h})_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})$, $(u^{n,\ell})_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$ and we know, thanks to a result of [3, Chap. 2], that the embedding from $F=\{u\in\mathcal{S}',\ u^\ell\in\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ and $u^h\in\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}\}$ to $\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}$ is locally compact. Therefore, one can combine Ascoli Theorem and the Cantor diagonal extraction process to deduce that there exists a distribution u such that, up to subsequence $(\phi u^n)_{n\in\mathbb{N}}$ converges to ϕu in $\mathcal{C}(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$ for all function ϕ compactly supported in $\mathbb{R}^+\times\mathbb{R}^n$. Then, using the Fatou property (cf. [3], chapter 2) we obtain that $u^\ell\in L^\infty(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})\cap L^1(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})$ and $u^h\in L^\infty(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})\cap L^1(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})$, with norms bounded by the right-hand side of (1.27). One can argue similarly for establishing the weak convergence of $(v^n)_{n\in\mathbb{N}}$ to some distribution v fulfilling the desired properties of regularity up to time continuity.

Finally, passing to the limit in (TM) is not an issue, since we have strong convergence (after localization) in norms with positive indices of regularity.

Last step: Proving that $(u,v)\in E_p$

The only property that misses is the time continuity. It may be obtained by looking at u and v as solutions of transport equations. Indeed, by construction, we have

$$\partial_t u+v\partial_x u=-\partial_x v \quad\text{and}\quad \partial_t v+v\partial_x v+v=-\partial_x u.$$

The properties we proved so far for u and v ensure that $\partial_x u$ and $\partial_x v$ belong to $L^2(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$. Hence, the standard properties for the transport equation (see e.g. [3, Chap. 3]) give us that $(u,v)\in\mathcal{C}(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$.

To show that $(u,v)^h\in\mathcal{C}(\mathbb{R}^+;\dot{B}^{\frac{3}{2}}_{2,1})$, one can mimic the proof for general symmetric hyperbolic systems, summing up only on high frequencies, as presented at [3, p.196] for instance.

In the end, we thus have proved that (u,v) is a global solution of (TM), that verifies the desired properties of regularity and $X_p(t)\le CX_{p,0}$ for all $t\in\mathbb{R}^+$.

1.4.5 Proof of uniqueness

Consider two solutions (u_1,v_1) and (u_2,v_2) of (TM) (not necessarily small) in the space E_p, that correspond to the same initial data (u_0,v_0). The proof of uniqueness will follow from stability estimates involving suitable norms. The difficulty is that our functional framework is not the standard one for the low frequency of the solution, so that one cannot follow the classical

proof for hyperbolic symmetric systems. Here we shall estimate $(\delta u, \delta v) := (u_2 - u_1, v_2 - v_1)$ in the space

$$F_p(T) \triangleq \Big\{ z \in \mathcal{C}([0,T]; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}) \,:\, z^h \in \mathcal{C}([0,T]; \dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}) \Big\}. \tag{1.28}$$

The reason for this choice is the usual loss of one derivative when proving stability estimates for quasilinear hyperbolic systems (hence the exponent 1/2 for high frequencies). The exponent for low frequencies looks to be the best one for controlling the nonlinearities. Before starting the proof, we introduce the notation

$$\delta U(t) \triangleq \|(\delta u, \delta v)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|(\delta u, \delta v)(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

Step 1: Proving that $(\delta u, \delta v) \in F_p(T)$

Remember that $\partial_t u_i = -\partial_x v_i - v_i \partial_x u_i$ for $i = 1, 2$. By interpolation in Besov spaces and Hölder inequality with respect to the time variable, since $\partial_x u_i^\ell$ and $\partial_x v_i^\ell$ are in $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1})$, we get

$$\partial_x u_i^\ell, \partial_x v_i^\ell \in L^r(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}) \quad \text{with} \quad \frac{1}{r} \triangleq \frac{1}{4} + \frac{1}{2p}. \tag{1.29}$$

It is clear that the same property holds for the high frequencies of $\partial_x u_i$ and $\partial_x v_i$, since they belong to $L^1(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}) \cap L^\infty(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$. We also know that v_i belongs to $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})$. Therefore, from the product laws in Besov spaces that have been recalled in Proposition 1.7.2, we gather that $\partial_x v_i$ and $v_i \partial_x u_i$ are in $L^r(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1})$. Hence, $\partial_t u_i$ is in $L^r(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1})$, and thus

$$(u_i - u_0) \in \mathcal{C}^{\frac{1}{r'}}_{loc}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}). \tag{1.30}$$

Proving the result for v_i is almost the same, except that we have to handle the damping term. To overcome it, we notice that

$$\partial_t(e^t v_i) = -e^t v_i \, \partial_x v_i - e^t \partial_x u_i.$$

Arguing as above, we get that $\partial_t(e^t v_i) \in L^r_{loc}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1})$, whence

$$(e^t v_i - v_0) \in \mathcal{C}^{\frac{1}{r'}}_{loc}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}). \tag{1.31}$$

From (1.30) and (1.31), we conclude that $(\delta u, \delta v) \in F_p(T)$ for all finite T.

Step 2: Estimates for the low frequencies

The system satisfied by $(\delta u, \delta v)$ reads:

$$\begin{cases} \partial_t \delta u + \partial_x \delta v = -\delta v \, \partial_x u_1 - v_2 \, \partial_x \delta u, \\ \partial_t \delta v + \delta v + \partial_x \delta u = -\delta v \, \partial_x v_1 - v_2 \, \partial_x \delta v. \end{cases} \tag{1.32}$$

Then, we follow the computations leading to Proposition 1.4.1 with $w = 0$, looking at $-\delta v \, \partial_x u_1 - v_2 \, \partial_x \delta u$ and $-\delta v \, \partial_x v_1 - v_2 \, \partial_x \delta v$ as source terms, and working at the level of regularity $2/p - 1/2$ instead of $1/p$ (since the left-hand side of (1.32) is linear with constant coefficients, this shift of regularity does not change the proof). Omitting the time integral in the left-hand side of the

Inequality given by Proposition 1.4.1, we find that

$$\|(\delta u,\delta v)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \int_0^t \big(\|\delta v\,\partial_x u_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|v_2\,\partial_x\delta u\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|\delta v\,\partial_x v_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|v_2\,\partial_x\delta v\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\big)\cdot$$

In order to bound the right-hand side, we may use that Proposition A.2.5 yields

$$\|a\,b\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\,\|b\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \tag{1.33}$$

as well as the following inequality that is a consequence of (A.8) in the appendix (take $s=\frac{2}{p}-\frac{1}{2}$ therein):

$$\|a\,b\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}\cap\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\,\|b\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{3}{2}}_{p,1}}\cdot \tag{1.34}$$

In the end, choosing $a=v_2$ and $b=\partial_x\delta u$ or $\partial_x\delta v$, we get

$$\|(\delta u,\delta v)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \int_0^t \big(\|(\partial_x u_1,\partial_x v_1)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\|\delta v\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|v_2\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}\cap\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}}\|(\delta u,\delta v)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\big)\cdot \tag{1.35}$$

Step 3: Estimates for the high frequencies

We look at the system satisfied by $(\delta u,\delta v)$ under the form:

$$\begin{cases}\partial_t\delta u + v_2\,\partial_x\delta u + \partial_x\delta v = -\delta v\,\partial_x u_1,\\ \partial_t\delta v + \delta v + v_2\,\partial_x\delta v + \partial_x\delta u = -\delta v\,\partial_x v_1.\end{cases}$$

This is System (LTM) except for the source terms in the right-hand side. Clearly, following the computations leading to (1.22), but using the index $1/2$ instead of $3/2$ gives

$$\begin{aligned}\|(\delta u,\delta v)(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\lesssim \int_0^t \|\partial_x v_2\|_{L^\infty}\|(\delta u,\delta v)\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\\ &+\int_0^t\sum_{j\geq J_0}2^{\frac{j}{2}}\big(\|[v_2,\dot\Delta_j]\partial_x\delta u\|_{L^2}+\|[v_2,\dot\Delta_j]\partial_x\delta v\|_{L^2}\big)+\int_0^t\big(\|\delta v\,\partial_x u_1\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}+\|\delta v\,\partial_x v_1\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)\cdot\end{aligned} \tag{1.36}$$

Let $p^*\triangleq 2p/(p-2)$. Lemma 1.7.1 tells us that, for $z=\delta u,\delta v$,

$$\begin{aligned}\sum_{j\geq J_0}2^{\frac{j}{2}}\Big\|[v_2,\dot\Delta_j]\partial_x z\Big\|_{L^2} &\lesssim \|\partial_x v_2\|_{L^\infty}\|z\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}+\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}\\ &\quad+\|\partial_x z\|_{\dot{\mathbb{B}}^{-1}_{\infty,\infty}}\|v_2\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}+\|\partial_x z\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p}}_{p,1}}\|\partial_x v_2\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\cdot\end{aligned}$$

Hence, using obvious embedding and the fact that

$$\|\partial_x z\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p}}_{p,1}}\lesssim\|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}},\quad \|v_2\|^{\ell}_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}\lesssim\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}\quad\text{and}\quad \|\partial_x v_2\|^{\ell}_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\lesssim\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}$$

yields for $z=\delta u,\,\delta v$,

$$\sum_{j\geq J_0}2^{\frac{j}{2}}\Big\|[v_2,\dot\Delta_j]\partial_x z\Big\|_{L^2}\lesssim\big(\|v_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}+\|v_2\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big)\big(\|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}+\|z\|^{h}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)\cdot \tag{1.37}$$

The last two terms of (1.36) may be bounded thanks to Inequality (1.78): we get for $z = u_1,\, v_1$,

$$\|\delta v\, \partial_x z\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \big(\|\delta v\|^\ell_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|\delta v\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)\big(\|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|\partial_x z\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big). \tag{1.38}$$

Plugging (1.37) and (1.38) in (1.36), and using also the fact that

$$\|\partial_x v_2\|_{L^\infty} \lesssim \|v_2\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|v_2\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}},$$

we end up with

$$\|(\delta u, \delta v)(t)\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \int_0^t \big(\|(u_1, v_1, v_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(u_1, v_1, v_2)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big)\delta\mathcal{U}. \tag{1.39}$$

Step 4: Conclusion

Putting (1.35) and (1.39) together yields

$$\delta\mathcal{U}(t) \le C \int_0^t \big(\|(u_1, v_1, v_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(u_1, v_1, v_2)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|(\partial_x u_1, \partial_x v_1)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\big)\delta\mathcal{U}\, d\tau.$$

Knowing that (u_1, v_1) and (u_2, v_2) are in $E_p(T)$ and remembering (1.29), we get,

$$\int_0^T \big(\|(u_1, v_1, v_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(u_1, v_1, v_2)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big) < \infty.$$

Hence, applying Gronwall lemma allows to conclude that $\delta\mathcal{U} \equiv 0$ on $[0, T]$. In other words, (u_1, v_1) and (u_2, v_2) coincide on $[0, T] \times \mathbb{R}$. Since T is arbitrary, uniqueness is proved. □

1.5 Compressible Euler system with damping

As pointed out in the introduction, System (TM) may be seen as an approximation of the damped isentropic compressible Euler system with pressure law $P(\rho) = \frac{\rho^2}{2}\cdot$ Here we want to adapt the method of the previous sections to the true damped compressible Euler system :

$$(E) \qquad \begin{cases} \partial_t \rho + \partial_x(V\rho) = 0, \\ \partial_t(\rho V) + \partial_x(\rho V^2) + \partial_x(P(\rho)) + \lambda\rho V = 0, \end{cases}$$

supplemented with initial data (ρ_0, V_0) that is a perturbation of some constant state $(\bar\rho, 0)$ with $\bar\rho > 0$. The (given) pressure function P is assumed to be smooth and such that :

$$\begin{cases} \text{Case } 2 < p \le 4 : P(\rho) = a\rho^\gamma \text{ for some positive } a,\ \gamma > 1 \text{ and } \rho \text{ in a neighborhood of } \bar\rho. \\ \text{Case } p = 2 : \text{just } \ P'(\bar\rho) > 0. \end{cases} \tag{1.40}$$

Note that, performing a suitable normalization reduces the study to the case $\bar\rho = P'(\bar\rho) = 1$ (hence $a = 1/\gamma$ in the first case), which will be assumed from now on.

Consider the new unknown

$$n(\rho) \triangleq \int_1^\rho \frac{P'(s)}{s}\, ds.$$

Since our assumptions on the pressure guarantee that $\rho \mapsto n(\rho)$ is a smooth diffeomorphism from a neighborhood of 1 to a neighborhood of 0, one can rewrite (E) under the form

$$\begin{cases} \partial_t n + V\partial_x n + \partial_x V + G(n)\partial_x V = 0, \\ \partial_t V + V\partial_x V + \partial_x n + \lambda V = 0, \end{cases} \tag{1.41}$$

where $G(n)$ is defined by the relation[3] $G(n(\rho)) \triangleq P'(\rho) - 1$.

Theorem 1.5.1. *Under hypothesis (1.40), there exist $k = k(p) \in \mathbb{Z}$ and $c_0 = c_0(p) > 0$ such that for $J_\lambda \triangleq \lfloor \log_2 \lambda \rfloor + k$, if we assume that (n_0, V_0) fulfills the same conditions as in Theorem 1.2.1, then System (1.41) admits a unique global solution (n, V) verifying the same properties as the solution therein. Furthermore, Corollary 1.2.1 and Theorem 1.2.2 hold true (with the same decay rate).*

Proof. Performing the rescaling (1.3) reduces the proof to the case $\lambda = 1$, and we are thus left with bounding for all $t \geq 0$, the functional

$$X_p(t) \triangleq \|(n,V)\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|(n,V)\|^h_{L^\infty_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|n\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})} + \|(n,V)\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} + \|V + \partial_x n\|_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} + \|V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$$

in terms of

$$X_{p,0} \triangleq \|(n_0,V_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} .$$

Remember that $\|V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$ and $\|V + \partial_x n\|^h_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$ are bounded by the first four terms of X_p (see (1.23)).

Low frequencies estimates

We follow the method we used for (TM), looking at $G(n)\partial_x V$ as a source term. Owing to Propositions 1.7.2 and A.2.5, we have

$$\|G(n)\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \lesssim \|n\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} .$$

Therefore, mimicking the proof of Proposition 1.4.1, we end up again for all $t \geq 0$ with

$$\|(n,V)(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|n\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1}} + \int_0^t \|V + \partial_x n\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \leq C\Big(\|(n_0,V_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \int_0^t \|(n,V,\partial_x n)\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|V\|_{\dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} \Big). \tag{1.42}$$

High frequencies estimates

One has the following proposition:

Proposition 1.5.1. *Let (n, V) be a smooth solution of (1.41) on the interval $[0, T]$, under assumption (1.40). Define p^* by the relation $1/p + 1/p^* = 1/2$. There exists a constant C*

[3] In what follows, we shall only use that G is a smooth function vanishing at 0.

depending only on the threshold J_0 between the low and high frequencies such that for all $t \in [0,T]$,

$$\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(\partial_x n, \partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$$
$$+ \int_0^t \Big(\|V\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|(\partial_x n, \partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|(\partial_x n, \partial_x V)\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}} \Big)$$
$$+ \int_0^t \|\partial_x V\|_{L^\infty} \|G(n)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \Big(\|(n,G(n))\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x G(n)\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}} \Big).$$

Proof. We localize System (1.41) by means of $\dot{\Delta}_j$, getting

$$\begin{cases} \partial_t n_j + \dot{S}_{j-1} V \partial_x n_j + \partial_x V_j + \dot{S}_{j-1} G(n) \partial_x V_j = R^1_j + R'^1_j, \\ \partial_t V_j + \dot{S}_{j-1} V \partial_x V_j + \partial_x n_j + V_j = R^2_j \end{cases} \tag{1.43}$$

where

$$R^1_j \triangleq \dot{S}_{j-1} V \partial_x n_j - \dot{\Delta}_j (V \partial_x n), \qquad R'^1_j \triangleq \dot{S}_{j-1} G(n) \partial_x V_j - \dot{\Delta}_j (G(n) \partial_x V)$$
$$\text{and} \quad R^2_j \triangleq \dot{S}_{j-1} V \partial_x V_j - \dot{\Delta}_j (V \partial_x V).$$

The only difference with (TM) is the appearance of $\dot{S}_{j-1} G(n) \partial_x V_j$ in the first equation and of the commutator R'^1_j. To handle the former term, one has to add a suitable weight in the definition of the functional we used for (TM): for $j \geq J_0$ and $\eta = \eta(J_0) > 0$ (to be chosen small enough), we set

$$\widetilde{\mathcal{L}}^2_j \triangleq \int_{\mathbb{R}} (\partial_x n_j)^2 + (1 + \dot{S}_{j-1} G(n))(\partial_x V_j)^2 + \eta \int_{\mathbb{R}} V_j \partial_x n_j.$$

Differentiating in time this quantity and performing several integration by parts yields:

$$\frac{d}{dt} \widetilde{\mathcal{L}}^2_j + \widetilde{\mathcal{H}}^2_j + \int_{\mathbb{R}} \partial_x \dot{S}_{j-1} V \left((\partial_x n_j)^2 + (1 + \dot{S}_{j-1} G(n))(\partial_x V_j)^2 \right)$$
$$- \int_{\mathbb{R}} \partial_x \dot{S}_{j-1} G(n)\, S_{j-1} V\, (\partial_x V_j)^2 = \int_{\mathbb{R}} (\partial_x V_j)^2\, \partial_t \dot{S}_{j-1} G(n)$$
$$+ 2 \int_{\mathbb{R}} \big(\partial_x (R^1_j + R'^1_j)\, \partial_x n_j + (1 + S_{j-1} G(n)) \partial_x R^2_j\, \partial_x V_j \big) + \eta \int_{\mathbb{R}} \big(\partial_x (R^1_j + R'^1_j)\, V_j + R^2_j\, \partial_x n_j \big). \tag{1.44}$$

with $\widetilde{\mathcal{H}}^2_j \triangleq \eta \|\partial_x n_j\|^2_{L^2} + (2-\eta) \int_{\mathbb{R}} (1 + \dot{S}_{j-1} G(n))(\partial_x V_j)^2 + \eta \int_{\mathbb{R}} V_j \partial_x n_j$.
To continue, let us assume that

$$\|n\|_{L^\infty} + \|V\|_{L^\infty} \ll 1 \quad \text{on} \quad [0,T]. \tag{1.45}$$

Then, since $G(0) = 0$, we have, using the mean value theorem and the uniform boundedness of operator $\dot{S}_{j-1}$ in all Lebesgue spaces:

$$\|\dot{S}_{j-1} G(n)\|_{L^\infty} \lesssim \|n\|_{L^\infty} \ll 1,$$

and thus, if η is small enough,

$$\widetilde{\mathcal{L}}^2_j \simeq \|(\partial_x n_j, \partial_x V_j)\|^2_{L^2} \quad \text{and} \quad \widetilde{\mathcal{H}}^2_j \simeq \|(\partial_x n_j, \partial_x V_j)\|^2_{L^2} \quad \text{for all} \quad j \geq J_0. \tag{1.46}$$

Let us also observe that

$$\begin{aligned}\partial_t G(n) &= G'(n)\partial_t n\\ &= -G'(n)(V\partial_x n + (1+G(n))\partial_x V).\end{aligned}$$

Owing to assumption (1.45) and to the mean value theorem, we thus get

$$\|\partial_t G(n)\|_{L^\infty} \lesssim \|\partial_x V\|_{L^\infty} + \|V\|_{L^\infty}\|\partial_x n\|_{L^\infty}. \tag{1.47}$$

Proceeding analogously, we obtain

$$\|\partial_x G(n)\|_{L^\infty} \lesssim \|\partial_x n\|_{L^\infty}. \tag{1.48}$$

Hence, from inequality (1.44) and (1.46), we get for some small enough c and large enough C,

$$\frac{d}{dt}\widetilde{\mathcal{L}}_j^2 + c\widetilde{\mathcal{L}}_j^2 \le C\big(\|(\partial_x V, \partial_x n)\|_{L^\infty}\widetilde{\mathcal{L}}_j + 2^j\|(R_j^1, R_j'^1, R_j^2)\|_{L^2}\big)\widetilde{\mathcal{L}}_j \quad \text{for all} \quad j \ge J_0. \tag{1.49}$$

At this point, taking advantage of Lemma A.1.1 yields

$$\widetilde{\mathcal{L}}_j(t) + c\int_0^t \widetilde{\mathcal{L}}_j \le \widetilde{\mathcal{L}}_j(0) + C\int_0^t \|(\partial_x V, \partial_x n)\|_{L^\infty}\mathcal{L}_j + C2^j\int_0^t \big\|(R_j^1, R_j'^1, R_j^2)\big\|_{L^2}. \tag{1.50}$$

Now, multiplying (1.50) by $2^{\frac{j}{2}}$, using (1.46) and summing up on $j \ge J_0$ gives us

$$\begin{aligned}\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} &\lesssim \|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\\ &+ \int_0^t \|(\partial_x V, \partial_x n)\|_{L^\infty}\|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \sum_{j\ge J_0} 2^{\frac{3j}{2}}\big\|(R_j^1, R_j'^1, R_j^2)\big\|_{L^2}.\end{aligned} \tag{1.51}$$

The terms R_j^1 and R_j^2 may be bounded exactly as in the proof Proposition 1.4.2. As regards $R_j'^1$, Lemma 1.7.1 gives us

$$\begin{aligned}\sum_{j\ge J_0} 2^{\frac{3}{2}j}\|R_j'^1\|_{L^2} \lesssim \|\partial_x G(n)\|_{L^\infty}\|\partial_x V\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &+ \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|G(n)\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}\\ &+\|\partial_x V\|_{L^\infty}\|G(n)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}}\|\partial_x G(n)\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}.\end{aligned}$$

Using (1.48) completes the proof of the proposition. □

Global-in-time a priori estimate

We claim that granted with Inequalities (1.42) and the above proposition, we have, whenever Condition (1.45) is satisfied on $[0,T]$,

$$X_p(t) \lesssim X_{p,0} + X_p^2(t) \quad \text{for all} \quad t \in [0,T]. \tag{1.52}$$

Inequality (1.42) is exactly the same as for (TM). Hence, the terms in $X_p(t)$ corresponding to the low frequencies of (n,V) are bounded by $X_p^2(t)$. Note also that $\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})}$ may be bounded according to (1.23), and thus eventually by $X_p^2(t)$.

In order to handle the high frequencies, we shall proceed differently depending on whether

$P(\rho) = \rho^\gamma/\gamma$ or P is a general pressure law with $P'(1) = 1$. In fact, to handle the latter case, we need to assume that $p = 2$.

1. Case $P(\rho) = \rho^\gamma/\gamma$ with $\gamma > 0$

Then, $G(n) = (\gamma - 1)n$ and the inequality of Proposition 1.5.1 reduces to

$$\begin{aligned}\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} &\lesssim \|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(\partial_x n, \partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \\ &+ \int_0^t \Big(\|V\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|(\partial_x n, \partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|(\partial_x n, \partial_x V)\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}} \Big) \\ &+ \int_0^t \Big(\|n\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x n\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}} \Big).\end{aligned}$$

Compared to our study of (TM), only the last line is new. However, one can use the fact that

$$\begin{aligned}&\int_0^t \|n\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} \lesssim \|n\|^\ell_{L^2_t(\dot{\mathbb{B}}^{1+\frac{1}{p}}_{p,1})} \|V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \\ &\int_0^t \|\partial_x V\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}} \|\partial_x n\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}} \lesssim \|V\|^\ell_{L^1_t(\dot{\mathbb{B}}^{1+\frac{1}{p}}_{p,1})} \|n\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1})} \qquad \text{as } 1 - \frac{1}{p^*} \geq \frac{1}{p}.\end{aligned}$$

The terms on the right may be bounded by $X^2_p(t)$. Hence we have (1.52).

2. Case of a general pressure law with $P'(1) = 1$

For $p = 2$, Proposition 1.5.1 together with the embeddings $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \hookrightarrow \dot{\mathbb{B}}^0_{\infty,1} \hookrightarrow L^\infty$ and $\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1} \hookrightarrow \dot{\mathbb{B}}^1_{\infty,1}$ give us

$$\begin{aligned}\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} &\lesssim \|(n_0,V_0)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(\partial_x n, \partial_x V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(n,V)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \\ &+ \int_0^t \|V\|^\ell_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|(n,V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|G(n)\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t \|(n,G(n))\|^\ell_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.\end{aligned}$$

Since, by Proposition A.2.5 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|G(n)\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|n\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \|\partial_x V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})} \|n\|_{L^2_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})}, \\ &\int_0^t \|(n,G(n))\|^\ell_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \|n\|_{L^2_t(\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1})} \|V\|_{L^2_t(\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})},\end{aligned}$$

one can conclude that (1.52) is satisfied.

Uniqueness

As for (TM), we look at the system satisfied by the difference $(\delta n, \delta V) := (n_2 - n_1, V_2 - V_1)$ between two solutions, namely:

$$\begin{cases} \partial_t \delta n + \partial_x \delta V + V_2\, \partial_x \delta n + G(n_2)\partial_x \delta V = -\delta V\, \partial_x n_1 - (G(n_2) - G(n_1))\partial_x V_1, \\ \partial_t \delta V + \delta V + \partial_x \delta n + V_2\, \partial_x \delta V = -\delta V\, \partial_x V_1, \end{cases} \tag{1.53}$$

and estimate $(\delta n, \delta V)$ for all $T > 0$ in the space $F_p(T)$ defined in (1.28). Compared to the proof of uniqueness for (TM) we have to handle the two terms containing the function G.

Let us first explain how to estimate the low frequencies. We have to bound the additional terms $G(n_2)\partial_x \delta V$ and $(G(n_2) - G(n_1))\partial_x V_1$ in $\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}$. Now, according to (1.33) and (1.34), we have

$$\|(G(n_2) - G(n_1))\partial_x V_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \|G(n_2) - G(n_1)\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \|\partial_x V_1\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}},$$

$$\|G(n_2)\partial_x \delta V\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \|n_2\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1} \cap \dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} \|\partial_x \delta V\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{3}{2}}_{p,1}}.$$

From the relation

$$G(n_2) - G(n_1) = \delta n \int_0^1 G'(n_1 + \tau \delta n)\, d\tau, \tag{1.54}$$

and Propositions 1.7.2 and A.2.5, we find out:

$$\|G(n_2) - G(n_1)\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} \lesssim \|\delta n\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}}.$$

Therefore, we eventually have

$$\begin{aligned} \|(\delta n, \delta V)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \lesssim \int_0^t \big(\|(\partial_x n_1, \partial_x V_1)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \|\delta V\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|V_2\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1} \cap \dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} \|(\delta n, \delta V)\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\big) \\ + \int_0^t \big(\|\partial_x V_1\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} \|\delta n\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1}} + \|n_2\|_{\dot{\mathbb{B}}^{\frac{1}{p}}_{p,1} \cap \dot{\mathbb{B}}^{\frac{1}{p}+1}_{p,1}} \|\delta V\|_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}\big). \end{aligned} \tag{1.55}$$

Let us next estimate the high frequencies of $(\delta n, \delta V)$ in $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}$. Applying operator $\dot{\Delta}_j$ to (1.53), we get for all $j \geq J_0$,

$$\begin{cases} \partial_t \delta n_j + \partial_x \delta V_j + \dot{S}_{j-1} V_2\, \partial_x \delta n_j + \dot{S}_{j-1} G(n_2)\partial_x \delta V_j \\ \qquad\qquad\qquad\qquad = -\dot{\Delta}_j\big(\delta V\, \partial_x n_1 + (G(n_2) - G(n_1))\partial_x V_1\big) + \delta R^1_j + \delta R'^1_j, \\ \partial_t \delta V_j + \delta V_j + \partial_x \delta n_j + \dot{S}_{j-1} V_2\, \partial_x \delta V_j = -\dot{\Delta}_j(\delta V\, \partial_x V_1) + \delta R^2_j, \end{cases}$$

with $\delta R^1_j \triangleq \dot{S}_{j-1} V_2 \partial_x \delta n_j - \dot{\Delta}_j(V_2 \partial_x \delta n)$, $\delta R'^1_j \triangleq \dot{S}_{j-1} G(n_2)\partial_x \delta V_j - \dot{\Delta}_j(G(n_2)\partial_x \delta V)$ and $\delta R^2_j \triangleq \dot{S}_{j-1} V_2 \partial_x \delta V_j - \dot{\Delta}_j(V_2 \partial_x \delta V)$.

Arguing as in the proof of Proposition 1.5.1, we consider the functional

$$\int_{\mathbb{R}} (\partial_x \delta n_j)^2 + (1 + \dot{S}_{j-1} G(n_2))(\partial_x \delta V_j)^2 + \eta \int_{\mathbb{R}} \delta V_j\, \partial_x \delta n_j$$

and follow the computations therein, with regularity exponent 1/2 instead of 3/2. We get

$$\|(\delta n,\delta V)(t)\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \int_0^t \|(\partial_x n_2,\partial_x V_2)\|_{L^\infty}\|(\delta n,\delta V)\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \sum_{j\geq J_0} 2^{\frac{j}{2}}\big(\|\delta R^1_j\|_{L^2}+\|\delta R'^1_j\|_{L^2}+\|\delta R^2_j\|_{L^2}\big)$$
$$+\int_0^t \big(\|\delta V\,\partial_x n_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\delta V\,\partial_x V_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|(G(n_2)-G(n_1))\partial_x V_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\cdot$$

The terms with δR^1_j and δR^2_j may be bounded as in the proof of uniqueness for (TM). Regarding $\delta R'^1_j$, we use Lemma 1.7.1 with $s=1/2$, and get

$$\sum_{j\geq J_0} 2^{\frac{j}{2}}\|\delta R'^1_j\|_{L^2} \lesssim \|\partial_x G(n_2)\|_{L^\infty}\|\delta V\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\partial_x\delta V\|_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}}\|G(n_2)\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}$$
$$+\|\partial_x\delta V\|_{L^\infty}\|G(n_2)\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\partial_x\delta V\|^\ell_{\dot{\mathbb{B}}^{1-\frac1p}_{p,1}}\|\partial_x G(n_2)\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}.$$

To continue the proof, we have two distinguish two cases depending on whether $P(\rho)=\rho^\gamma/\gamma$ and $2\leq p\leq 4$, or P is a general pressure law with $P'(1)=1$, and $p=2$. In the first case, we have $G(n)=(\gamma-1)n$, so that $G(n_2)-G(n_1)=(\gamma-1)\delta n$. Now, in light of (1.78), we have

$$\|\delta n\,\partial_x V_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \big(\|\delta n\|^\ell_{\dot{\mathbb{B}}^{\frac2p-\frac12}_{p,1}} + \|\delta n\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\big(\|\partial_x V_1\|^\ell_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}} + \|\partial_x V_1\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big)\cdot$$

As all the terms with $G(n_2)$ in the estimate for R'^1_j are proportional to n_2, we arrive at

$$\|(\delta n,\delta V)(t)\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}}$$
$$\lesssim \int_0^t \big(\|(n_1,n_2,V_1,V_2)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|(n_1,n_1,V_1,V_2)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|(\partial_x n_1,\partial_x V_1)\|_{\dot{\mathbb{B}}^{\frac2p-\frac12}_{p,1}}\big)\|(\delta n,\delta V)\|_{F_p}. \quad (1.56)$$

In the case $p=2$ with $P'(1)=1$, then one may proceed essentially as in the proof of Proposition 1.5.1 to bound the terms with $G(n_2)$ in the estimate for R'^1_j, and one can use Proposition A.2.5 combined with product laws and Relation (1.54) to eventually arrive at

$$\|(G(n_2)-G(n_1))\partial_x V_1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|\delta n\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_x V_1\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

Consequently, (1.56) still holds true.

In all cases, putting (1.55) and (1.56) together yields

$$\|(\delta n,\delta V)\|_{F_p(t)}$$
$$\leq C\int_0^t \big(\|(n_1,n_2,V_1,V_2)\|^\ell_{\dot{\mathbb{B}}^{\frac1p}_{p,1}} + \|(n_1,n_2,V_1,V_2)\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|(\partial_x n_1,\partial_x V_1)\|_{\dot{\mathbb{B}}^{\frac2p-\frac12}_{p,1}}\big)\|(\delta n,\delta V)\|_{F_p},$$

and using Gronwall lemma completes the proof of uniqueness.

Decay estimates

Here, we assume that $p=2$ and follow the same approach as for (TM).

Step 1: estimating the solution in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$

This is only a matter of handling the additional term $G(n)\partial_x v$. Applying $\dot{\Delta}_j$ to the system satisfied by (n, V) yields

$$\begin{cases} \partial_t n_j + \partial_x V_j = -v\partial_x n_j - \dot{\Delta}_j(G(n)\partial_x V) + [V, \dot{\Delta}_j]\partial_x n, \\ \partial_t V_j + \partial_x n_j + V_j = -V\partial_x V_j + [v, \dot{\Delta}_j]\partial_x V. \end{cases}$$

So, considering $G(n)\partial_x V$ as a source term, we get

$$\|(n_j, V_j)(t)\|_{L^2} + \int_0^t \|V_j\|_{L^2} \leq \|(n_j, V_j)(0)\|_{L^2} + \int_0^t \|\partial_x V\|_{L^\infty}\|(n_j, V_j)\|_{L^2}$$
$$+ \int_0^t \|[V, \dot{\Delta}_j]\partial_x n\|_{L^2} + \int_0^t \|[V, \dot{\Delta}_j]\partial_x V\|_{L^2} + \left\|\dot{\Delta}_j(G(n)\partial_x V)\right\|_{L^2} \|n_j\|_{L^2}.$$

We have

$$\|G(n)\partial_x V\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|G(n)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

In order to bound $G(n)$ in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$, one cannot readily use Proposition A.2.5 since $-\sigma_1$ may be negative. However, from Taylor formula, we know that there exists a smooth function H vanishing at 0 such that

$$G(n) = G'(0)\, n + H(n)\, n.$$

Hence, combining product and composition estimates gives

$$\|G(n)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|n\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \big(1 + \|n\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big).$$

In the regime we consider, $\|n\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$ is small. Hence we conclude that

$$\|(n, V)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|(n_0, V_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(n, V)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}},$$

which ensures after using Gronwall lemma and the bound of $\|\partial_x V\|_{L^1(\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})}$ in terms of $X_{2,0}$, that

$$\forall t \in \mathbb{R}^+,\ \|(n, V)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\|(n_0, V_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

Step 2: Lyapunov functional

We aim at exhibiting a Lyapunov functional that is equivalent to $\|(n, V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$. The high frequency part of the solution has already been treated efficiently with $\widetilde{\mathcal{L}}_j$. To bound the low frequency part, consider the evolution equation for $z \triangleq V + \partial_x n$:

$$\partial_t z + V\partial_x z + z = -\partial^2_{xx} V - \partial_x V \partial_x n - \partial_x(G(n)\partial_x V).$$

Following the computations we did for (TM) leads to

$$\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \leq \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|\partial^2_{xx} V\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$$
$$+ C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x n\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|\partial_x(G(n)\partial_x V)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

The last term may be bounded. by $\|G(n)\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$. Then, using Propositions 1.7.2 and A.2.5, one ends up with

$$\|z(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \leq \|z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|\partial^2_{xx} V\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(z,n,\partial_x n)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}. \quad (1.57)$$

Next, using the fact that

$$\partial_t n + V\partial_x n - \partial^2_{xx} n = -G(n)\partial_x V - \partial_x z,$$

we get

$$\|n(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}} \leq \|n_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|\partial_x z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|n\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}. \quad (1.58)$$

The high frequency part of the solution may be bounded according to (1.50). Hence, setting

$$\widetilde{\mathcal{L}} \triangleq \sum_{j\leq J_0} 2^{\frac{j}{2}} \|(\dot{\Delta}_j n, \dot{\Delta}_j z)\|_{L^2} + \sum_{j\geq J_0} 2^{\frac{j}{2}} \widetilde{\mathcal{L}}_j \quad \text{and} \quad \widetilde{\mathcal{H}} \triangleq \|V + \partial_x n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|V\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}} + \|n\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$$

and bounding R^1_j, R'^1_j and R^2_j as in the proof of Proposition 1.5.1, we discover that if taking J_0 negative enough, then all the linear terms in (1.57) and (1.58) may be absorbed by $\widetilde{\mathcal{H}}$, so that we have for some suitably small positive c,

$$\widetilde{\mathcal{L}}(t) + c\int_0^t \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(0) + C\int_0^t \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \mathcal{L} + C\int_0^t \|n\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}. \quad (1.59)$$

Above, we used that $\widetilde{\mathcal{L}} \simeq \|(n,V)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}$ and that $\widetilde{\mathcal{H}} \gtrsim \|\partial_x V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$. Now, since furthermore $\widetilde{\mathcal{H}} \gtrsim \|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$ and $\widetilde{\mathcal{L}} \gtrsim \|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}$, one may write

$$\begin{aligned}
\|n\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\leq \|n\|^2_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|n\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \\
&\lesssim \|n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}} + (\|n\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}})^2 + \|n\|^{\ell}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|n\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \\
&\lesssim \widetilde{\mathcal{L}}\widetilde{\mathcal{H}} + \widetilde{\mathcal{L}}\widetilde{\mathcal{H}} + \widetilde{\mathcal{L}}\widetilde{\mathcal{H}} + \widetilde{\mathcal{H}}\widetilde{\mathcal{L}}.
\end{aligned}$$

Hence, if $\widetilde{\mathcal{L}}(0)$ is small enough then, combining (1.59) with a bootstrap argument yields

$$\widetilde{\mathcal{L}}(t) + \frac{c}{2}\int_0^t \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(0) \quad \text{for all} \quad t \geq 0.$$

Step 3: Proof of decay estimates

From this point, one can repeat word for word the proof of decay estimates for the low frequencies of the solutions to (TM).

For the high frequencies, starting from (1.49), using Lemma 1.7.1 and integrating gives

$$\|(n,V)(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim e^{-ct}\|(n_0,V_0)\|^{h}_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \int_0^t e^{-c(t-\tau)}\big(\|V\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|(n,V)\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} + \|V\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|n\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\big)\cdot$$

Compared to (1.16), there is one more term. However, as for (TM), Steps 1 and 2 together imply that

$$\|(n,V)(t)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \langle t\rangle^{-\alpha_1}.$$

Hence, one may easily conclude that

$$\|(n,V)(t)\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \lesssim \langle t\rangle^{-2\alpha_1}.$$

This completes the proof of the theorem (up to the proof of existence, which is totally analogous as for (TM)). □

1.6 A more general 1D model

In this section, we consider a more general class of one dimensional systems, namely

$$\begin{cases} \partial_t u + \alpha\partial_x v + V^1\partial_x u + W^1\partial_x v = 0, \\ \partial_t v + \beta\partial_x u + V^2\partial_x u + W^2\partial_x v + \lambda v + \kappa\lambda v^q = 0 \end{cases} \tag{1.60}$$

where[4] κ is a real parameter, $q \geq 2$, an integer, $V^1 = V^1(v)$ and $V^2 = V^2(v)$ are smooth functions vanishing at 0, $W^1 = W^1(u,v)$ and $W^2 = W^2(u,v)$ are smooth functions vanishing at $(0,0)$, and α, β, λ are strictly positive constants. The study of this system is mainly motivated by academic reasons: we want to see whether the method we developed in the previous sections extend to a more general class of two-components and one-dimensional systems.

Theorem 1.6.1. *Let the data (u_0, v_0) satisfy the assumptions of Theorem 1.2.1 with $J_\lambda \triangleq \lfloor \log_2\lambda \rfloor$ and $p = 2$. Then, System (1.60) admits a unique global solution (u,v) verifying the same properties as the solution therein. Furthermore, Corollary 1.2.1 and Theorem 1.2.2 hold true.*

Remark 1.6.1. *If V^1, V^2, W^1 and W^2 are 'general' smooth functions, then it is unlikely that a L^p theory may be worked out. We need a very specific structure of the nonlinear terms in order that the L^p estimates of the low frequencies fit with the L^2 regularity of the high frequencies.*

Remark 1.6.2. *We do not know how to handle terms like $u\partial_x u$ in any equations of the system (this is the reason why we assumed that V^1 and V^2 only depend on v). In fact, although the system is locally well-posed if V^1 and V^2 also depend on u, the time integrability of u is not good enough for global estimates.*

Elements of proof

We just explain how to find a Lyapunov function and to control the norm in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ of a smooth solution (u,v) of (1.60) on $[0,T]$, in terms of the data. Proving existence and uniqueness is essentially the same as for the systems we treated before (uniqueness is easier somehow since we assumed $p = 2$). Although the system under consideration is no longer symmetric if $\alpha + W^1 \neq \beta + V^2$, it is symmetrizable (see [5, Chap. 10]).

Note that performing a suitable rescaling reduces our problem to the case

$$\alpha = \beta = \lambda = 1. \tag{1.61}$$

[4] In the case $q = 3$ and $\kappa > 0$, κv^q is a classical representation of a drag term.

Indeed, if we set

$$(u,v)(t,x) = (\sqrt{\alpha}\,\tilde{u}, \sqrt{\beta}\,\tilde{v})\Big(\lambda t, \frac{\lambda}{\sqrt{\alpha\beta}}\,x\Big),$$

then (u,v) satisfies (1.60) if and only if $(\tilde{u},\tilde{v})$ satisfies a similar system with (1.61), parameter $\kappa\beta^{\frac{q-1}{2}}$ and slightly modified functions V_1, V_2, W_1 and W_2 (the modification depending only on α and β). So we will assume (1.61) in the rest of this section.

A priori estimates

We adapt the method we used for (TM) *in the case* $p=2$. The terms $V^1\partial_x u$ and $W^2\partial_x v$ are a slight generalization of $v\partial_x u$ and $v\partial_x v$ and may be treated similarly. To handle $W^1\partial_x v$ and $V^2\partial_x u$, we need to introduce suitable weights in the definition of the Lyapunov. Finally, v^q may be seen as a harmless nonlinear source term.

Let us start the computations : we assume that we are given a smooth function (u,v) of (1.60) on some time interval $[0,T]$ such that for some suitably small $\eta>0$,

$$\sup_{t\in[0,T]} \|(u,v)(t)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \le \eta, \tag{1.62}$$

and, still denoting $u_j = \dot{\Delta}_j u$ and $v_j = \dot{\Delta}_j v$, we set for all $j\in\mathbb{Z}$,

$$\mathcal{L}_j \triangleq \Big(\|(u_j,v_j)\|_{L^2}^2 + \int_{\mathbb{R}} v_j\partial_x u_j + \int_{\mathbb{R}} (1+V^2)(\partial_x u_j)^2 + \int_{\mathbb{R}} (1+W^1)(\partial_x v_j)^2\Big)^{1/2}.$$

We shall use repeatedly that (1.62) implies that

$$\sup_{t\in[0,T]} \max\big(\|u(t)\|_{L^\infty},\, \|v(t)\|_{L^\infty},\, \|V^1(t)\|_{L^\infty},\, \|V^2(t)\|_{L^\infty},\, \|W^1(t)\|_{L^\infty},\, \|W^2(t)\|_{L^\infty}\big) \ll 1, \tag{1.63}$$

which in particular entails that

$$\mathcal{L}_j \simeq \|(u_j,v_j,\partial_x u_j,\partial_x v_j)\|_{L^2}. \tag{1.64}$$

Now, applying $\dot{\Delta}_j$ to (1.60)-(1.61) yields for all $j\in\mathbb{Z}$,

$$\begin{cases} \partial_t u_j + (1+W^1)\partial_x v_j + V^1\partial_x u_j = R_j^1, \\ \partial_t v_j + (1+V^2)\partial_x u_j + W^2\partial_x v_j + v_j = R_j^2 - \kappa\dot{\Delta}_j(v^q) \end{cases} \tag{1.65}$$

with

$$R_j^1 \triangleq [V^1,\dot{\Delta}_j]\partial_x u + [W^1,\dot{\Delta}_j]\partial_x v \quad\text{and}\quad R_j^2 \triangleq [V^2,\dot{\Delta}_j]\partial_x u + [W^2,\dot{\Delta}_j]\partial_x v.$$

In order to compute the time derivative of $\mathcal{L}_j^2$, we need the following obvious identities:

$$\frac{1}{2}\frac{d}{dt}\|(u_j,v_j)\|_{L^2}^2 + \|v_j\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}}\big((u_j)^2\partial_x V^1 + (v_j)^2\partial_x W^2\big) + \int_{\mathbb{R}}\big(W^1 u_j\partial_x v_j + V^2 v_j\partial_x u_j\big)$$
$$= \int_{\mathbb{R}}\big(R_j^1 u_j + R_j^2 v_j - \kappa\big(\dot{\Delta}_j v^q\big)v_j\big),$$

$$\frac{d}{dt}\int_{\mathbb{R}} v_j\partial_x u_j + \|\partial_x u_j\|_{L^2}^2 - \|\partial_x v_j\|_{L^2}^2 + \int_{\mathbb{R}} v_j\partial_x u_j$$
$$+\int_{\mathbb{R}}\big((W^2-V^1)\partial_x u_j\,\partial_x v_j + V^2(\partial_x u_j)^2 - W^1(\partial_x v_j)^2\big) = \int_{\mathbb{R}}\big(R_j^2\partial_x u_j - R_j^1\partial_x v_j - \kappa\big(\dot\Delta_j v^q\big)\partial_x u_j\big)\cdot$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(1+V^2)(\partial_x u_j)^2 + \int_{\mathbb{R}}(1+V^2)\partial_x u_j\partial_x(V^1\partial_x u_j) + \int_{\mathbb{R}}(1+V^2)\partial_x u_j\partial_x((1+W^1)\partial_x v_j)$$
$$= \int_{\mathbb{R}}(1+V^2)\partial_x u_j\partial_x R_j^1 + \frac{1}{2}\int_{\mathbb{R}}\partial_t V^2(\partial_x u_j)^2.$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(1+W^1)(\partial_x v_j)^2 + \int_{\mathbb{R}}(1+W^1)\partial_x v_j\partial_x(W^2\partial_x v_j) + \int_{\mathbb{R}}(1+W^1)\partial_x v_j\partial_x((1+V^2)\partial_x u_j)$$
$$+\int_{\mathbb{R}}(1+W^1)(\partial_x v_j)^2 = \int_{\mathbb{R}}(1+W^1)\partial_x v_j\big(\partial_x R_j^2 - \kappa\partial_x\dot\Delta_j v^q\big) + \frac{1}{2}\int_{\mathbb{R}}\partial_t W^1(\partial_x v_j)^2.$$

The fundamental observation that justifies our using those very weights in the definition of $\mathcal{L}_j$ is that the third integrals in the last two relations compensate. Consequently, denoting

$$\mathcal{H}_j^2 \triangleq \|v_j\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}} v_j\partial_x u_j + \frac{1}{2}\|\partial_x u_j\|_{L^2}^2 + \int_{\mathbb{R}}\Big(W^1+\frac{1}{2}\Big)(\partial_x v_j)^2,$$

and using the fact that

$$\int_{\mathbb{R}} V^2 v_j\partial_x u_j = -\int_{\mathbb{R}} V^2 u_j\partial_x v_j - \int_{\mathbb{R}} u_j v_j\partial_x V^2,$$

we arrive at

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + \mathcal{H}_j^2 = \frac{1}{2}\int((v_j)^2\partial_x W^2 + (u_j)^2\partial_x V^1) + \int_{\mathbb{R}} u_j v_j\partial_x V^2 + \int_{\mathbb{R}}(V^2-W^1)u_j\partial_x v_j$$
$$+\frac{1}{2}\int_{\mathbb{R}}\big((V^1-W^2)\partial_x u_j\,\partial_x v_j - V^2(\partial_x u_j)^2 + W^1(\partial_x v_j)^2\big) + \frac{1}{2}\int_{\mathbb{R}}(\partial_x u_j)^2\big(V^1\partial_x V^2 - (1+V^2)\partial_x V^1\big)$$
$$+\frac{1}{2}\int_{\mathbb{R}}(\partial_x v_j)^2\big(W^2\partial_x W^1 - (1+W^1)\partial_x W^2\big) + \frac{1}{2}\int_{\mathbb{R}}\big((\partial_x u_j)^2\partial_t V^2 + (\partial_x v_j)^2\partial_t W^1\big)$$
$$+\int_{\mathbb{R}}\big(u_j - \frac{1}{2}\partial_x v_j\big)R_j^1 + \int_{\mathbb{R}}\big(v_j + \frac{1}{2}\partial_x u_j\big)R_j^2 + \int_{\mathbb{R}}\big((1+V^2)\partial_x u_j\partial_x R_j^1 + (1+W^1)\partial_x v_j\partial_x R_j^2\big)$$
$$-\kappa\int_{\mathbb{R}}\Big(\big(v_j + \frac{1}{2}\partial_x u_j\big)\dot\Delta_j v^q + (1+W^1)\partial_x\dot\Delta_j v^q\partial_x v_j\Big)\cdot$$

Since

$$\partial_t V^2 = -(V^2)'\big((1+V^2)\partial_x u + W^2\partial_x v + v + \kappa v^q\big),$$

remembering (1.63), we have

$$\|\partial_t V^2\|_{L^\infty} \lesssim \|\partial_x u\|_{L^\infty} + \|(u,v)\|_{L^\infty}\|\partial_x v\|_{L^\infty} + \|v\|_{L^\infty}$$

and, similarly,

$$\|\partial_t W^1\|_{L^\infty} \lesssim \|\partial_x u\|_{L^\infty} + \|\partial_x v\|_{L^\infty} + \|v\|_{L^\infty}.$$

Observe also that

$$\|\partial_x V^i\|_{L^\infty} \lesssim \|\partial_x v\|_{L^\infty} \quad \text{and} \quad \|\partial_x W^i\|_{L^\infty} \lesssim \|(\partial_x u, \partial_x v)\|_{L^\infty} \quad \text{for} \quad i = 1, 2,$$

whence, in particular

$$\int_{\mathbb{R}} u_j v_j \partial_x V^2 \lesssim \|\partial_x v\|_{L^\infty} \|u_j\|_{L^2} \|v_j\|_{L^2}.$$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + \mathcal{H}_j^2 \lesssim{}& \|(u,v)\|_{L^\infty}\|\partial_x v_j\|_{L^2}\|(u_j, \partial_x u_j, \partial_x v_j)\|_{L^2} + \|v\|_{L^\infty}\|\partial_x u_j\|_{L^2}^2 \\ &+\|\partial_x u\|_{L^\infty}\|(v_j, \partial_x u_j, \partial_x v_j)\|_{L^2}^2 + \|\partial_x v\|_{L^\infty}\|(u_j, v_j, \partial_x u_j, \partial_x v_j)\|_{L^2}^2 \\ &+\|(R_j^1, R_j^2)\|_{L^2}\|(u_j, v_j, \partial_x u_j, \partial_x v_j)\|_{L^2} + \|(\partial_x R_j^1, \partial_x R_j^2)\|_{L^2}\|(\partial_x u_j, \partial_x v_j)\|_{L^2} \\ &+\|(v_j, \partial_x u_j)\|_{L^2}\|\dot{\Delta}_j v^q\|_{L^2} + \|\partial_x v_j\|_{L^2}\|\partial_x \dot{\Delta}_j v^q\|_{L^2}.\end{aligned}$$

Then, remembering (1.64) and using lemma A.1.1, we discover that for all $j \in \mathbb{Z}$ and $t \in [0, T]$,

$$\begin{aligned}&\mathcal{L}_j(t) + c\min(1, 2^{2j})\int_0^t \mathcal{L}_j \leq \mathcal{L}_j(0) \\ &+ C\int_0^t \big(\|\partial_x v\|_{L^\infty}\mathcal{L}_j + \|\partial_x u\|_{L^\infty}\|v_j\|_{L^2} + \|(v, \partial_x u)\|_{L^\infty}\|\partial_x u_j\|_{L^2} + \|(u, v, \partial_x u)\|_{L^\infty}\|\partial_x v_j\|_{L^2}\big) \\ &\qquad + \int_0^t \|(\dot{\Delta}_j v^q, \partial_x \dot{\Delta}_j v^q)\|_{L^2} + C\int_0^t \|(R_j^1, R_j^2, \partial_x R_j^1, \partial_x R_j^2)\|_{L^2}. \quad (1.66)\end{aligned}$$

To bound the commutator terms, let us use (A.4) that yields

$$\|R_j^1\|_{L^2} \lesssim c_j 2^{-\frac{j}{2}}\big(\|\partial_x V^1\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\|u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|\partial_x W^1\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\|v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big) \quad \text{for all} \quad j \in \mathbb{Z}.$$

Clearly, since v is small in $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}$, $V^1 = V^1(v)$ and $V^1(0) = 0$, Proposition A.2.5 entails that

$$\|\partial_x V^1\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}. \quad (1.67)$$

In order to bound the term with W^1, we use the fact that there exist two smooth functions $G = G(u, v)$ and $H = H(u, v)$ vanishing at $(0, 0)$ and such that

$$\partial_x W^1 = \partial_u W^1(0,0)\partial_x u + \partial_v W^1(0,0)\partial_x v + G(u,v)\partial_x u + H(u,v)\partial_x v.$$

Consequently, using the stability of the space $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}$ by product and results in [88, Section 5.5] for bounding $G(u, v)$ and $H(u, v)$, we get

$$\|\partial_x W^1\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \|(\partial_x u, \partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big(1 + \|(u,v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big). \quad (1.68)$$

So finally, remembering (1.62), we have

$$\|R_j^1\|_{L^2} \lesssim c_j 2^{-\frac{j}{2}}\big(\|u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\|(\partial_x u, \partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big). \quad (1.69)$$

Bounding R_j^2 works exactly the same. Next, in light of (A.4), we have

$$\|\partial_x R_j^1\|_{L^2} \lesssim c_j 2^{-\frac{j}{2}} \big(\|\partial_x V^1\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|\partial_x W^1\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big),$$

and a similar inequality for $\partial_x R_j^2$. Hence repeating the above arguments for bounding $\partial_x V^1$, $\partial_x V^2$, $\partial_x W^1$ and $\partial_x W^2$, we end up with

$$\|\partial_x R_j^i\|_{L^2} \lesssim c_j 2^{-\frac{j}{2}} \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(\partial_x u, \partial_x v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}, \quad i = 1, 2. \tag{1.70}$$

Now, reverting to (1.66), using the embedding $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1} \hookrightarrow L^\infty$ and that

$$\mathcal{H}_j \simeq \|(v_j, \partial_x u_j, \partial_x v_j)\|_{L^2} \tag{1.71}$$

we get, denoting

$$\mathcal{L} \triangleq \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \mathcal{L}_j,$$

two positive constants c and C such that

$$\begin{aligned}\mathcal{L}(t) + c\sum_j \min(1, 2^{2j}) 2^{\frac{j}{2}} \int_0^t \mathcal{L}_j \le \mathcal{L}(0) + C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \mathcal{L} \\ + C\int_0^t \|v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + C\int_0^t \|\partial_x u\|^2_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + C\int_0^t \|(v^q, \partial_x v^q)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.\end{aligned} \tag{1.72}$$

As for (TM), we need better properties of integrability for v in order to close the above estimate. The situation is a bit more complex since the second line above was not present. Nevertheless, it is still possible to exhibit a control of $z \triangleq v + \partial_x u$ in $L^1(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})$ (which, as we saw in (1.23) yields a bound for v in $L^2(\mathbb{R}_+; \dot{\mathbb{B}}^{\frac{1}{2}}_{2,1})$). Indeed, we have

$$\partial_t z + z + V^1 \partial_x z = (V^1 - W^2)\partial_x v - V^2 \partial_x u - \partial^2_{xx} v - \partial_x V^1\, \partial_x u - \partial_x(W^1 \partial_x v) - \kappa v^q \tag{1.73}$$

which, as in the proof of (1.9) leads to

$$\begin{aligned}\|z(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \le \|z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|\partial^2_{xx} v\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + C\int_0^t \|\partial_x V^1\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|z\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \\ + \int_0^t \|(V^1 - W^2)\partial_x v + V^2 \partial_x u + \partial_x V^1\, \partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \int_0^t \|\partial_x(W^1 \partial_x v)\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \kappa \int_0^t \|v^q\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.\end{aligned}$$

Using product and composition estimates and remembering (1.62), we get

$$\begin{aligned}\|(V^1 - W^2)\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\lesssim \|(u, v)\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}, \\ \|V^2 \partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\lesssim \|v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}, \\ \|\partial_x V^1\, \partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} &\lesssim \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.\end{aligned}$$

Since only low frequencies are involved, we have

$$\|\partial_x(W^1\partial_x v)\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|W^1\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|(u,v)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

Hence, using also (1.67), we get

$$\begin{aligned}\|z(t)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\le \|z_0\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|\partial^2_{xx} v\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \\ &+ C\int_0^t \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|(u,v,z,\partial_x u)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + C\int_0^t \|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \int_0^t \|v^q\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}. \end{aligned} \tag{1.74}$$

In order to close the estimates for the solution, it suffices to add up (1.72) to $\varepsilon\cdot$(1.74) with suitably small ε. More precisely, setting

$$\widetilde{\mathcal{L}} \triangleq \mathcal{L} + \varepsilon\|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \quad \text{and} \quad \widetilde{\mathcal{H}} \triangleq c\sum_{j\in\mathbb{Z}} \min(1,2^{2j})2^{\frac{j}{2}}\mathcal{L}_j + \varepsilon\|z\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}},$$

we get for all $t\in[0,T]$ if ε has been chosen small enough,

$$\begin{aligned}\widetilde{\mathcal{L}}(t) + \frac12\int_0^t \widetilde{\mathcal{H}} \le \widetilde{\mathcal{L}}(0) + C\int_0^t \big(\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\widetilde{\mathcal{L}} + \|\partial_x u\|^2_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big) \\ + C\int_0^t \|(v^q,\partial_x v^q)\|_{\dot{\mathbb{B}}^{\frac32}_{2,1}}. \end{aligned} \tag{1.75}$$

Let us emphasize that

$$\widetilde{\mathcal{L}} \simeq \|(u,v,\partial_x u,\partial_x v)\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \quad \text{and} \quad \widetilde{\mathcal{H}} \simeq \|u\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}} + \|u\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}} + \|v+\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|v\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}}.$$

Hence in particular, we have $\|\partial_x v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \widetilde{\mathcal{H}}$ and, as explained in the previous section (just replace n by u and V by v),

$$\|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \widetilde{\mathcal{L}}\widetilde{\mathcal{H}}. \tag{1.76}$$

Furthermore,

$$\begin{aligned}\|\partial_x u\|^2_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim (\|\partial_x u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}})^2 + (\|\partial_x u\|^h_{\dot{\mathbb{B}}^{\frac12}_{2,1}})^2 \\ &\lesssim \|u\|^\ell_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \|u\|^\ell_{\dot{\mathbb{B}}^{\frac52}_{2,1}} + (\|u\|^h_{\dot{\mathbb{B}}^{\frac32}_{2,1}})^2 \lesssim \widetilde{\mathcal{L}}\widetilde{\mathcal{H}}.\end{aligned}$$

Finally, Lemma 1.7.2 and (1.76) together ensure that

$$\begin{aligned}\|v^q\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|v\|^q_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \\ &\lesssim \|v\|^{q-1}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big(\|z\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\big) \\ &\lesssim \widetilde{\mathcal{L}}^{q-1}\widetilde{\mathcal{H}} + \widetilde{\mathcal{L}}^{q-2}\|v\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}\|\partial_x u\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \\ &\lesssim \widetilde{\mathcal{L}}^{q-1}\widetilde{\mathcal{H}}\end{aligned}$$

and

$$\|\partial_x v^q\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|v\|^{q-1}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \widetilde{\mathcal{L}}^{q-1}\widetilde{\mathcal{H}}.$$

Consequently, Inequality (1.75) reduces to

$$\widetilde{\mathcal{L}}(t) + \frac{1}{2}\int_0^t \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(0) + C\int_0^t (\widetilde{\mathcal{L}} + \widetilde{\mathcal{L}}^{q-1})\widetilde{\mathcal{H}}.$$

Now, applying a bootstrap argument, one may conclude that there exists a small constant η such that if $\widetilde{\mathcal{L}}(0) \leq \eta$, then

$$\forall t \in [0,T],\ \widetilde{\mathcal{L}}(t) + \frac{1}{4}\int_0^t \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(0). \tag{1.77}$$

This gives the desired control on the norm of the solution and, in addition, that $\widetilde{\mathcal{L}}$ is a Lyapunov functional.

Decay estimates

Granted with a Lyapunov functional that has the same properties as in the previous sections, in order to get the whole family of decay estimates, it suffices to establish a uniform in time bound in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for the solution. The starting point is that, for all $j \in \mathbb{Z}$,

$$\begin{cases} \partial_t u_j + \partial_x v_j + V^1\partial_x u_j = [V^1, \dot{\Delta}_j]\partial_x - \dot{\Delta}_j(W^1\partial_x v), \\ \partial_t v_j + \partial_x u_j + v_j = -\dot{\Delta}_j(W^2\partial_x v) - \dot{\Delta}_j(V^2\partial_x u) - \kappa\dot{\Delta}_j v^q. \end{cases}$$

Applying an energy method, using Lemma A.1.1 and Inequality (A.5) eventually delivers:

$$\begin{aligned} \|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + C\int_0^t \|\partial_x V^1\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \\ + \int_0^t \big(\|W^1\partial_x v\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|W^2\partial_x v\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|V^2\partial_x u\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \kappa\|v^q\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\big)\cdot \end{aligned}$$

Using for $i = 1, 2$, the decomposition

$$W^i(u,v) = \big(\partial_u W^i(0,0) + G^i(u,v)\big)u + \big(\partial_v W^i(0,0) + H^i(u,v)\big)v$$

where G^i and H^i are smooth functions vanishing at $(0,0)$, we get thanks to results in [88, Section 5.5] and Proposition 1.7.2

$$\|W^i\partial_x v\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|\partial_x v\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

Proposition 1.7.2 also implies that

$$\|v^q\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|v^2\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|v\|^{q-2}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}.$$

In order to estimate the term with v^2, we use that $v = z - \partial_x u$ and get the decomposition:

$$v^2 = v^h(v + v^\ell) + z^\ell(v^\ell - \partial_x u^\ell) + (\partial_x u^\ell)^2.$$

By Proposition 1.7.2 and interpolation, we thus have

$$\begin{aligned}\|v^2\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} &\lesssim \|v^h\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|v+v^\ell\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \big(\|v^\ell\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|\partial_x u^\ell\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\big)\|z^\ell\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|\partial_x u^\ell\|^2_{\dot{\mathbb{B}}^{\frac{1}{2}(\frac{1}{2}-\sigma_1)}_{2,1}}\\ &\lesssim \|v\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}} \|v\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} + \|u\|^\ell_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|u\|^\ell_{\dot{\mathbb{B}}^{\frac{5}{2}}_{2,1}}.\end{aligned}$$

Hence we have

$$\|v^q\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|v\|^{q-2}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \widetilde{\mathcal{H}}.$$

Finally, using the decomposition

$$V^2(v)\partial_x u = V^2(v)\partial_x u^h + V^2(z)\partial_x u^\ell - \Big(\int_0^1 V^2(z-\tau\partial_x u)\Big)\Big(\partial_x u^\ell \partial_x u^\ell + \partial_x u^\ell \partial_x u^h\Big),$$

we get by similar computations that

$$\|V^2\partial_x u\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \widetilde{\mathcal{H}}.$$

Therefore, in the end, we get

$$\|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \le \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + C\int_0^t \widetilde{\mathcal{H}}\big(1+\|v\|^{q-2}_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)\|(u,v)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}},$$

which, combined with (1.77) and Gronwall lemma implies that

$$\sup_{t\in[0,T]} \|(u,v)(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|(u_0,v_0)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

At this stage, completing the proof of decay estimates is left to the reader. □

1.7 Tool box

Here we gather a few technical results that have been used repeatedly in this chapter. The following result from [34] has been used in the proof of Proposition 1.4.1.

Proposition 1.7.1. *If Supp$(\mathcal{F}f) \subset \{\xi \in \mathbb{R}^d : R_1\lambda \le |\xi| \le R_2\lambda\}$ for some $0 < R_1 < R_2$ then, there exists $c = c(d, R_1, R_2) > 0$ such that for all $p \in [2,\infty[$, we have*

$$c\lambda^2\left(\frac{p-1}{p}\right)\int_{\mathbb{R}^d}|f|^p \le (p-1)\int_{\mathbb{R}^d}|\nabla f|^2|f|^{p-2} = -\int_{\mathbb{R}^d} f\Delta f|f|^{p-2}.$$

The following product laws in Besov spaces have been used several times.

Proposition 1.7.2. *In the case $d = 1$ and $2 \le p \le 4$, we have*

$$\|ab\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}} \lesssim \big(\|a\|^\ell_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}} + \|a\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big)\big(\|b\|^\ell_{\dot{\mathbb{B}}^{\frac{2}{p}-\frac{1}{2}}_{p,1}} + \|b\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\big). \tag{1.78}$$

Proof. The first two inequalities are direct consequences of the results stated in [3, Chap. 2]. To prove the third one, we need the following so-called Bony decomposition for the product of two

tempered distributions a and b (whenever it is defined):

$$ab = T_a b + T'_b a \quad \text{with} \quad T_a b \triangleq \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} a\, \dot{\Delta}_j b \quad \text{and} \quad T'_b a \triangleq \sum_{j\in\mathbb{Z}} \dot{S}_{j+2} b\, \dot{\Delta}_j a.$$

Now, using Bernstein inequality and the results of continuity for T and T' stated in [3, Chap. 2], we may write:

$$\|T_a b\|^{\ell}_{\dot{\mathbb{B}}^s_{p,1}} \lesssim \|T_a b\|_{\dot{\mathbb{B}}^{s-1}_{p,1}} \lesssim \|a\|_{L^\infty} \|b\|_{\dot{\mathbb{B}}^{s-1}_{p,1}}$$

and, provided, $s-1 \leq d/p$ and $s > -\min(d/p, d/p')$,

$$\|T'_b a\|_{\dot{\mathbb{B}}^s_{p,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|b\|_{\dot{\mathbb{B}}^{s-1}_{p,1}}.$$

This gives (A.8).

For proving (1.78), we combine Bony's decomposition and decomposition of a and b in low and high frequencies, writing

$$ab = T'_a b^\ell + T'_a b^h + T_{b^\ell} a^\ell + T_b a^h + T_{b^h} a^\ell.$$

All the terms in the right-hand side, except for the last one, may be bounded by means of the standard results of continuity for operators T and T' (see again [3, Chap. 2]). Setting $p^* = 2p/(p-2)$, we get:

$$\begin{aligned}
\|T'_a b^\ell\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|a\|_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}} \|b^\ell\|_{\dot{\mathbb{B}}^{\frac{1}{p^*}+1}_{p^*,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}} \|b\|^{\ell}_{\dot{\mathbb{B}}^{\frac1p+1}_{p,1}},\\
\|T'_a b^h\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|a\|_{L^\infty} \|b\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}},\\
\|T_{b^\ell} a^\ell\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|b^\ell\|_{L^{p^*}} \|a^\ell\|_{\dot{\mathbb{B}}^{\frac12}_{p,1}},\\
\|T_b a^h\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} &\lesssim \|b\|_{L^\infty} \|a^h\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.
\end{aligned}$$

Finally, since $a^\ell = \dot{S}_{J_0+1} a$ and $b^h = (\mathrm{Id} - \dot{S}_{J_0+1}) b$, we see that

$$T_{b^h} a^\ell = \dot{S}_{J_0} b^h\, \dot{\Delta}_{J_0+1} a^\ell.$$

Consequently,

$$\|T_{b^h} a^\ell\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|\dot{\Delta}_{J_0+1} a^\ell\|_{L^\infty} \|\dot{S}_{J_0} b^h\|_{L^2} \lesssim \|a\|_{L^\infty} \|b\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}}.$$

Adding up this latter inequality to the previous ones gives

$$\|ab\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} \lesssim \|a\|_{L^\infty} \|b\|^{h}_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|a\|_{\dot{\mathbb{B}}^{\frac1p-1}_{p,1}} \|b\|^{\ell}_{\dot{\mathbb{B}}^{\frac1p+1}_{p,1}} + \|b\|_{L^\infty} \|a^h\|_{\dot{\mathbb{B}}^{\frac12}_{2,1}} + \|b^\ell\|_{L^{p^*}} \|a^\ell\|_{\dot{\mathbb{B}}^{\frac12}_{p,1}}.$$

Then, using Bernstein inequality, $2/p - 1/2 \leq 1/p$ and the embeddings $\dot{\mathbb{B}}^{\frac12}_{2,1} \hookrightarrow L^\infty$ and $\dot{\mathbb{B}}^{\frac{2}{p}-\frac12}_{p,1} \hookrightarrow L^{p^*}$ completes the proof of (1.78). □

The following result is one of the key to Theorem 1.2.1 in the general case.

Lemma 1.7.1. *Assume that $d = 1$. Let $p \in [2,4]$ and $s \in [1/2, 3/2[$. Define $p^* \triangleq 2p/(p-2)$. For all $j \in \mathbb{Z}$, denote $\mathcal{R}_j \triangleq \dot{S}_{j-1} w\, \partial_x \dot{\Delta}_j z - \dot{\Delta}_j (w\, \partial_x z)$.*

There exists a constant C depending only on the threshold number J_0 between low and high frequencies and on s, such that

$$\sum_{j\geq J_0}\left(2^{js}\left\|\mathcal{R}_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x w\|_{L^\infty}\|\partial_x z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}+\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}$$
$$+\|\partial_x z\|_{\dot{\mathbb{B}}^{s-\frac{3}{2}}_{\infty,\infty}}\|w\|^h_{\dot{\mathbb{B}}^{s}_{2,1}}+\|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{s-\frac{1}{2}-\frac{1}{p}}_{p,1}}\|\partial_x w\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\Big).$$

In the case $s=3/2$, we have

$$\sum_{j\geq J_0}\left(2^{j\frac{3}{2}}\left\|\mathcal{R}_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x w\|_{L^\infty}\|\partial_x z\|^h_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}+\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}$$
$$+\|\partial_x z\|_{L^\infty}\|w\|^h_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}+\|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{1-\frac{1}{p}}_{p,1}}\|\partial_x w\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\Big).$$

Proof. From Bony decomposition recalled above, we deduce that

$$\begin{aligned}\mathcal{R}_j&=-\dot{\Delta}_j(T'_{\partial_x z}w)-\sum_{|j'-j|\leq 4}[\dot{\Delta}_j,\dot{S}_{j'-1}w]\partial_x\dot{\Delta}_{j'}z-\sum_{|j'-j|\leq 1}\left(\dot{S}_{j'-1}w-\dot{S}_{j-1}w\right)\dot{\Delta}_j\dot{\Delta}_{j'}\partial_x z\\&=\mathcal{R}^1_j+\mathcal{R}^2_j+\mathcal{R}^3_j.\end{aligned}$$

To estimate $\mathcal{R}^1_j$, we decompose w into low and high frequencies, getting

$$T'_{\partial_x z}w=T'_{\partial_x z}w^\ell+T'_{\partial_x z}w^h.$$

Because $1/p+1/p^*=1/2$, the classical results of continuity for paraproduct and remainder operators (see e.g. [3, Chap. 2]) ensure that

$$\|T'_{\partial_x z}w^\ell\|_{\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}}\lesssim\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\|w^\ell\|_{\dot{\mathbb{B}}^{\frac{1}{p^*}+1}_{p^*,1}},$$

and we have

$$\|T'_{\partial_x z}w^h\|_{\dot{\mathbb{B}}^{s}_{2,1}}\lesssim\|\partial_x z\|_{\dot{\mathbb{B}}^{s-\frac{3}{2}}_{\infty,\infty}}\|w^h\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\ \text{ if } 0<s<3/2,\quad\text{and}\quad\|T'_{\partial_x z}w^h\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\lesssim\|\partial_x z\|_{L^\infty}\|w^h\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}.$$

Observing that $T'_{\partial_x z}w^\ell$ contains only low frequencies so that its norm in $\dot{\mathbb{B}}^s_{2,1}$ is controlled by its norm in $\dot{\mathbb{B}}^{\frac{1}{2}}_{2,1}$ if $s\geq 1/2$, we deduce that

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\left(2^{js}\left\|\mathcal{R}^1_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\left\|w^\ell\right\|_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}+\|\partial_x z\|_{\dot{\mathbb{B}}^{s-\frac{3}{2}}_{\infty,\infty}}\left\|w^h\right\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\Big)\ \text{ if }\ 1/2\leq s<3/2,\\&\sum_{j\in\mathbb{Z}}\left(2^{\frac{3}{2}j}\left\|\mathcal{R}^1_j\right\|_{L^2}\right)\leq C\Big(\|\partial_x z\|_{\dot{\mathbb{B}}^{\frac{1}{p}-1}_{p,1}}\left\|w^\ell\right\|_{\dot{\mathbb{B}}^{1+\frac{1}{p^*}}_{p^*,1}}+\|\partial_x z\|_{L^\infty}\left\|w^h\right\|_{\dot{\mathbb{B}}^{\frac{3}{2}}_{2,1}}\Big).\end{aligned}\tag{1.79}$$

Next, taking advantage of Lemma A.2.2, we see that if $j'\geq J_0$ and $|j-j'|\leq 4$, then we have

$$2^{js}\|[\dot{\Delta}_j,\dot{S}_{j'-1}w]\partial_x\dot{\Delta}_{j'}z\|_{L^2}\lesssim\|\partial_x\dot{S}_{j'-1}w\|_{L^\infty}2^{j'(s-1)}\|\partial_x\dot{\Delta}_{j'}z\|_{L^2}$$

while, if $j' < J_0$, $j \geq J_0$ and $|j - j'| \leq 4$,

$$2^{js}\|[\dot{\Delta}_j, \dot{S}_{j'-1}w]\partial_x\dot{\Delta}_{j'}z\|_{L^2} \lesssim 2^{-\frac{j'}{p^*}}\|\partial_x\dot{S}_{j'-1}w\|_{L^{p^*}}\, 2^{j'(s-\frac{1}{2}-\frac{1}{p})}\|\partial_x\dot{\Delta}_{j'}z\|_{L^p}.$$

Therefore,

$$\sum_{j\geq J_0}\Big(2^{js}\left\|\mathcal{R}_j^2\right\|_{L^2}\Big) \leq C\Big(\|\partial_x w\|_{L^\infty}\|\partial_x z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}} + \|\partial_x z\|^\ell_{\dot{\mathbb{B}}^{s-\frac{1}{2}-\frac{1}{p}}_{p,1}}\|\partial_x w\|^\ell_{\dot{\mathbb{B}}^{-\frac{1}{p^*}}_{p^*,1}}\Big)\cdot \tag{1.80}$$

Finally, for all $j \geq J_0$ and $|j' - j| \leq 1$, we have

$$\begin{aligned}2^{js}\|(\dot{S}_{j'-1}w - \dot{S}_{j-1}w)\dot{\Delta}_j\dot{\Delta}_{j'}\partial_x z\|_{L^2} &\leq 2^j\|\dot{\Delta}_{j\pm1}w\|_{L^\infty}\, 2^{j(s-1)}\|\partial_x\dot{\Delta}_{j'}\dot{\Delta}_j z\|_{L^2}\\ &\leq C\|\dot{\Delta}_{j\pm1}\partial_x w\|_{L^\infty}\, 2^{j(s-1)}\|\partial_x\dot{\Delta}_j z\|_{L^2}.\end{aligned}$$

Hence

$$\sum_{j\geq J_0}\Big(2^{js}\left\|\mathcal{R}_j^3\right\|_{L^2}\Big) \leq C\|\partial_x w\|_{L^\infty}\|\partial_x z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}. \tag{1.81}$$

Putting (1.79), (1.80) and (1.81) together completes the proof. □

Chapitre 2

Partially dissipative hyperbolic systems in the critical regularity setting : the multi-dimensional case

This chapter is based on the paper [31].

Introduction

In this chapter, we are concerned with first order n-component systems in $\mathbb{R}^d$ of the type:

$$A^0(V)\frac{\partial V}{\partial t} + \sum_{k=1}^{d} A^k(V)\frac{\partial V}{\partial x_k} = H(V) \tag{2.1}$$

where the (smooth) matrix-valued functions A^k $(j = 0, \cdots, d)$ and vector-valued function H are defined on some open subset $\mathcal{O}_V$ of $\mathbb{R}^n$ and the unknown $V = V(t, x)$ depends on the time variable $t \in \mathbb{R}_+$ and on the space variable $x \in \mathbb{R}^d$ $(d \geq 2)$. We assume that the system is *symmetrizable* and satisfies additional structure assumptions that will be specified in the next section. System (2.1) is supplemented with initial data $V_0 \in \mathcal{O}_V$ at time $t = 0$ and we consider the case where V_0 is close to some constant state $\bar{V}$ such that $H(\bar{V}) = 0$.

Generalizing the method developed in the previous chapter, we establish the existence of global strong solutions in a critical regularity setting whenever the system under consideration satisfies the (SK) condition. More precisely we obtain the following main features:

- As in the previous chapter, compared to the recent papers by Kawashima and Xu [102, 105] devoted to similar issues, our use of *hybrid* Besov norms with different regularity exponents in low and high frequency enables us to pinpoint optimal smallness conditions for global well-posedness and to get more accurate information on the qualitative properties of the constructed solutions.
- A great part of our analysis relies on the study of a Lyapunov functional in the spirit of that of Beauchard and Zuazua in [4]. It avoids the use of a compensating function.
- We also exhibit a damped mode in the low frequencies regime with faster time decay than the whole solution. This plays a key role in order to close the a priori estimates.

The (SK) condition is not optimal in the sense that there are many systems for which it is not satisfied but still admit global strong solutions, see e.g. [6, 22, 86]. In [4], Beauchard and Zuazua developed a new and systematic approach that allows to establish global existence and to describe large time behavior of solutions to partially dissipative systems that need not satisfy this condition. Looking at the linearization of System (2.1) around a constant solution, namely (denoting from now on $\partial_t \triangleq \frac{\partial}{\partial t}$ and $\partial_k \triangleq \frac{\partial}{\partial x_j}$),

$$\partial_t Z + \sum_{k=1}^{d} A^k \partial_k Z = -LZ, \tag{2.2}$$

they show that the (SK) condition is equivalent to the Kalman maximal rank condition on the matrices A^k and L. More importantly, they introduced a Lyapunov functional equivalent to the L^2 norm that encodes enough information to recover dissipative properties of (2.2). Considering such a functional is motivated by the classical (linear) control theory of ODEs, and is also related to Villani's work in [97]. Back to the nonlinear system (2.1), Beauchard and Zuazua obtained the existence of global smooth solutions for perturbations of a constant equilibrium $\bar{V}$ that satisfies Condition (SK). Furthermore, using arguments borrowed from Coron's return method [28], they were able to achieve certain cases where (SK) does not hold.

Our aim here is to extend the results we obtained recently in the one-dimensional case [30] to multi-dimensional partially dissipative hyperbolic systems. More precisely, under Condition (SK), we shall develop Beauchard and Zuazua's approach as suggested by the second author in [35] and prove the global well-posedness of (2.1) supplemented with data that are close to $\bar{V}$ in an optimal critical regularity setting. As in the study of the compressible Navier-Stokes system and related models (see e.g. [23, 27, 33, 38]), it will appear naturally that, in order to get optimal results, one has to use functional spaces with *different* regularity exponents in low and high frequencies. Here, Beauchard and Zuazua's approach will give us the information that the low frequencies (resp. high frequencies) of the solution of the linearized system behave like the heat flow (resp. are exponentially damped). Furthermore, in order to improve our low frequency analysis, we will exhibit a damped mode with better decay properties than the whole solution. Thanks to it, we will end up with more accurate estimates and a weaker smallness condition than in [102] and refine the decay estimates that were obtained in [105]. In particular, our functional framework allows to keep track of the parameters of the system (if any).

This chapter is arranged as follows. In the first section, we specify the structure of the class of partially dissipative hyperbolic systems we aim at considering, and explain the construction of a Lyapunov functional that will be the key to our global results. In Section 2.2, we state the main results of the chapter. Section 2.3 is devoted to the proof of a first global existence result and time decay estimates for general partially dissipative systems satisfying the Shizuta-Kawashima condition. In Section 2.4, under additional structure assumptions (that are satisfied by the compressible Euler system with damping), we obtain a more accurate global existence result. Some technical results are proved or recalled in Appendix.

2.1 Hypotheses and method

In this section, we specify our assumptions on the system under consideration, and explain the main steps of our approach.

2.1.1 Friedrichs-symmetrizability

First, we fix some constant solution $\bar{V} \in \mathcal{O}_V$ of (2.1) (thus satisfying $H(\bar{V}) = 0$). To ensure the local well-posedness, we assume that (2.1) is *Friedrichs-symmetrizable*, namely that there exists a smooth function $S : V \mapsto S(V)$ defined on $\mathcal{O}_V$, valued in the set of symmetric and positively definite matrices such that for all $V \in \mathcal{O}_V$, the matrices $(SA^0)(V), \cdots, (SA^d)(V)$ are symmetric and, in addition, $(SA^0(V))$ is definite positive.

Denoting $\widetilde{H} \triangleq SH$ and $\widetilde{A}^k \triangleq SA^k$ for $j \in \{0, \cdots, \}$, System (2.1) rewrites

$$\widetilde{A}^0(V)\partial_t V + \sum_{k=1}^{d} \widetilde{A}^k(V)\partial_k V = \widetilde{H}(V).$$

Then, setting $Z \triangleq V - \bar{V}$, $L \triangleq -D_V\widetilde{H}(\bar{V})$ and $r(Z) \triangleq \widetilde{H}(\bar{V} + Z) + LZ$, we get

$$\widetilde{A}^0(V)\partial_t Z + \sum_{k=1}^{d} \widetilde{A}^k(V)\partial_k Z + LZ = r(Z). \tag{2.3}$$

By construction, the remainder r is at least quadratic with respect to Z.

Second, we assume that System (2.1) is *partially dissipative* in the following meaning:

(i) The whole space $\mathbb{R}^n$ may be decomposed into $\mathbb{R}^n = \mathcal{M} \bigoplus \mathcal{M}^\perp$ where

$$\mathcal{M} = \left\{\phi \in \mathbb{R}^n,\ \langle \phi, \widetilde{H}(V)\rangle = 0 \text{ for all } V \in \mathcal{O}_V\right\}\cdot$$

Hence, denoting by $\mathcal{P}$ the orthogonal projection on $\mathcal{M}$, we may write

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad \text{and} \quad H(V) = \begin{pmatrix} 0 \\ H_2(V) \end{pmatrix} \tag{2.4}$$

where $V_1 = \mathcal{P}V \in \mathbb{R}^{n_1}$, $V_2 = (I - \mathcal{P})V \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$.

(ii) The linear map $L \triangleq -D_V\widetilde{H}(\bar{V})$ is an isomorphism on $\mathcal{M}^\perp$ such that for some $c > 0$,

$$\forall \eta \in \mathbb{R}^n,\ (L\eta|\eta) \geq c|L\eta|^2. \tag{2.5}$$

(iii) System (2.1) has a *block structure* that is compatible with decomposition (2.4), namely for $1 \leq k \leq d$, $\widetilde{A}^k$ is a block matrix (first block being of size $n_1 \times n_1$ and second one of size $n_2 \times n_2$), $\widetilde{A}^0$ is diagonal by blocks and we have $r(Z_1, 0) = 0$ for all Z_1 close to 0. This entails that r is at least linear with respect to Z_2.

According to the above assumptions and introducing the decompositions:

$$\widetilde{A}^0 = \begin{pmatrix} \widetilde{A}^0_{1,1} & 0 \\ 0 & \widetilde{A}^0_{2,2} \end{pmatrix}, \quad \widetilde{A}^k = \begin{pmatrix} \widetilde{A}^k_{1,1} & \widetilde{A}^k_{1,2} \\ \widetilde{A}^k_{2,1} & \widetilde{A}^k_{2,2} \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ L_2 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} 0 \\ Q \end{pmatrix},$$

System (2.3) may thus be rewritten as:

$$\begin{cases} \widetilde{A}^0_{1,1}(V)\partial_t Z_1 + \sum_{k=1}^d \Big(\widetilde{A}^k_{1,1}(V)\partial_k Z_1 + \widetilde{A}^k_{1,2}(V)\partial_k Z_2\Big) = 0, \\ \widetilde{A}^0_{2,2}(V)\partial_t Z_2 + \sum_{k=1}^d \Big(\widetilde{A}^j_{2,1}(V)\partial_k Z_1 + \widetilde{A}^j_{2,2}(V)\partial_k Z_2\Big) + L_2 Z_2 = Q(Z). \end{cases} \tag{2.6}$$

As we shall see in Section 2.4, the compressible Euler equations with damping, rewritten in suitable variables, satisfies the above assumptions about any constant state with positive density and null velocity.

2.1.2 The Shizuta-Kawashima and Kalman rank conditions

In order to specify the supplementary conditions on the structure of the system ensuring global well-posedness and present the overall strategy, let us consider the linearization of (2.1) about $\bar{V}$, namely:

$$\bar{A}^0\partial_t Z + \sum_{k=1}^d \bar{A}^k\partial_k Z + LZ = G \quad \text{with} \quad \bar{A}^k := \widetilde{A}^k(\bar{V}) \quad \text{for} \quad j = 0, \cdots, d. \tag{2.7}$$

Then, owing to the symmetry of the matrices $\bar{A}^k$, the classical energy method leads to

$$\frac{1}{2}\frac{d}{dt}\|Z\|^2_{L^2_{\bar{A}_0}} + (LZ|Z) = 0 \quad \text{with} \quad \|Z\|^2_{L^2_{\bar{A}_0}} \triangleq (\bar{A}_0 Z|Z). \tag{2.8}$$

On the one hand, since the matrix $\bar{A}_0$ is symmetric and positive definite, we have

$$\|Z\|_{L^2_{\bar{A}_0}} \simeq \|Z\|_{L^2}. \tag{2.9}$$

On the other hand, (2.5) and the definition of Z_2 guarantee that there exists $\kappa_0 > 0$ such that

$$(LZ|Z) \geq \kappa_0\|Z_2\|^2_{L^2} \quad \text{for all} \quad Z \in L^2(\mathbb{R}^d;\mathbb{R}^n). \tag{2.10}$$

Hence, (2.8) yields L^2-in-time integrability on the components of Z experiencing direct dissipation, but not on the whole solution. To compensate this lack of coercivity, following Beauchard and Zuazua in [4], we are going to introduce a lower order corrector $\mathcal{I}$ to track the optimal dissipation of the solution to (2.7). Since it is more natural to define that corrector on the Fourier side, let us look at (2.7) in the Fourier space, that is, denoting by $\xi \in \mathbb{R}^d$ the Fourier variable,

$$\bar{A}^0\partial_t\widehat{Z} + i\sum_{k=1}^d \bar{A}^k\xi_j\widehat{Z} + L\widehat{Z} = \widehat{G}.$$

Let us write $\xi = \rho\omega$ with $\omega \in \mathbb{S}^{d-1}$ and $\rho = |\xi|$. Then, the above system rewrites

$$\partial_t\widehat{Z} + i\rho M_\omega\widehat{Z} + N\widehat{Z} = \bar{A}_0^{-1}\widehat{G} \quad \text{with} \quad M_\omega \triangleq \bar{A}_0^{-1}\sum_{k=1}^d \omega_j\bar{A}^k \quad \text{and} \quad N \triangleq \bar{A}_0^{-1}L. \tag{2.11}$$

Since $\bar{A}_0^{-1}$ is positive definite, we can write for all $\eta \in \mathbb{R}^n$,

$$L\eta \cdot \eta \lesssim (\bar{A}_0^{-1} L\eta) \cdot \eta = N\eta \cdot \eta. \tag{2.12}$$

Fix $n-1$ positive parameters $\varepsilon_1, \cdots \varepsilon_{n-1}$ (bound to be small), and set

$$\mathcal{I} \triangleq \Im \sum_{q=1}^{n-1} \varepsilon_q \big(N M_\omega^{q-1} \widehat{Z} \cdot N M_\omega^q \widehat{Z} \big) \tag{2.13}$$

where $\cdot$ designates the Hermitian scalar product in $\mathbb{C}^n$.

For expository purpose, assume that $G \equiv 0$. Then, differentiating $\mathcal{I}$ with respect to time and using (2.11) yields

$$\begin{aligned} \frac{d}{dt}\mathcal{I} + \sum_{q=1}^{n-1} \varepsilon_q \rho |N M_\omega^q \widehat{Z}|^2 &= -\Im \sum_{q=1}^{n-1} \varepsilon_q \big(N M_\omega^{q-1} N \widehat{Z} \cdot N M_\omega^q \widehat{Z} \big) \\ &\quad - \Re \sum_{q=1}^{n-1} \varepsilon_q \rho \big(N M_\omega^{q-1} \widehat{Z} \cdot N M_\omega^{q+1} \widehat{Z} \big) - \Im \sum_{q=1}^{n-1} \varepsilon_q \big(N M_\omega^{q-1} \widehat{Z} \cdot N M_\omega^q N \widehat{Z} \big) \cdot \end{aligned} \tag{2.14}$$

As pointed out in [4] (and recalled in Section 2.5 for the reader's convenience), it is possible to choose positive and arbitrarily small parameters $\varepsilon_1, \cdots, \varepsilon_{n-1}$ so that (2.14) implies for some $C > 0$,

$$\frac{d}{dt}\mathcal{I} + \frac{1}{2} \sum_{q=1}^{n-1} \varepsilon_q \rho |N M_\omega^q \widehat{Z}|^2 \leq \frac{\kappa_0}{2(2\pi)^d \rho} |N \widehat{Z}|^2 + C\varepsilon_1 |N \widehat{Z}|^2. \tag{2.15}$$

Setting $\varepsilon_0 = (2\pi)^{-d}\kappa_0/2$, taking ε_1 small enough, integrating on $\mathbb{R}^d$, using Fourier-Plancherel theorem and combining with (2.8), we end up with

$$\begin{aligned} \frac{d}{dt}\mathcal{L} + \mathcal{H} \leq 0 \quad \text{with} \quad \mathcal{H} &\triangleq \int_{\mathbb{R}^d} \sum_{q=0}^{n-1} \varepsilon_q \min(1, |\xi|^2) |N M_\omega^q \widehat{Z}(\xi)|^2 \, d\xi \\ \text{and} \quad \mathcal{L} &\triangleq \|Z\|^2_{L^2_{\bar{A}_0}} + \int_{\mathbb{R}^d} \min(|\xi|, |\xi|^{-1}) \mathcal{I}(\xi) \, d\xi. \end{aligned} \tag{2.16}$$

Clearly, if $\varepsilon_1, \cdots, \varepsilon_{n-1}$ are small enough, then $\mathcal{L} \simeq \|Z\|^2_{L^2}$. The question now is whether $\mathcal{H}$ may be compared to $\|Z\|^2_{L^2}$. The answer depends on the properties of the support of $\widehat{Z}_0$ and on the possible cancellation of the following quantity:

$$\mathcal{N}_{\bar{V}} := \inf \Big\{ \sum_{q=0}^{n-1} \varepsilon_q |N M_\omega^q x|^2; \; x \in \mathbb{S}^{n-1}, \, \omega \in \mathbb{S}^{d-1} \Big\}. \tag{2.17}$$

At this very point, the (SK) (for Shizuta and Kawashima) condition comes into play:

Definition 2.1.1. *System (2.1) verifies the (SK) condition at $\bar{V} \in \mathcal{M}$ if, for all $\omega \in \mathbb{S}^{d-1}$, we have at the same time $N\phi = 0$ and $\lambda\phi + M_\omega \phi = 0$ for some $\lambda \in \mathbb{R}$, if and only if $\phi = 0_{\mathbb{R}^n}$.*

It is clear that Condition (SK) at $\bar{V}$ is equivalent to:

$$\forall \omega \in \mathbb{S}^{d-1}, \quad \ker N \cap \{\text{eigenvectors of } M_\omega\} = \{0\}.$$

In order to pursue our analysis, we need the following key result (see the proof in e.g. [4]).

Proposition 2.1.1. *Let M and N be two matrices in $\mathcal{M}_n(\mathbb{R})$. The following assertions are equivalent:*

1. *$N\phi = 0$ and $\lambda\phi + M\phi = 0$ for some $\lambda \in \mathbb{R}$ implies $\phi = 0$;*
2. *For every $\varepsilon_0, \cdots, \varepsilon_{n-1} > 0$, the function*

$$y \longmapsto \sqrt{\sum_{k=0}^{n-1} \varepsilon_k |NM^k y|^2}$$

defines a norm on $\mathbb{R}^n$.

Thanks to the above proposition and observing that the unit sphere $\mathbb{S}^{d-1}$ is compact, one may conclude that Condition (SK) is satisfied by the pair (M_ω, N) for all $\omega \in \mathbb{S}^{d-1}$ if and only if $\mathcal{N}_{\bar{V}} > 0$. Furthermore, we note that:

- if $\widehat{Z}_0$ is compactly supported then $\mathcal{H} \gtrsim \|\nabla Z\|_{L^2}^2$, which reveals a parabolic behavior of all components of the solution;
- if the support of $\widehat{Z}_0$ is away from the origin, then $\mathcal{H} \gtrsim \|Z\|_{L^2}^2$, which corresponds to exponential decay.

The above analysis reveals that, in order to get optimal dissipative estimates, it is suitable to split the solution into low and high frequencies parts. This will be achieved by means of a Littlewood-Paley decomposition (introduced in the next section) that will enable us to localize (2.3) on the Fourier side. Then, an important part of our work will consist in studying the evolution of the functional $\mathcal{L}$ pertaining to each part.

2.1.3 The damped mode

Another important ingredient of our analysis is the use of a 'damped mode' that, somehow, corresponds to the part of the solution that experiences maximal dissipation in low frequencies. It is defined as follows :

$$W \triangleq -{L_2}^{-1}\widetilde{A}^0_{2,2}(V)\partial_t Z_2 = Z_2 + \sum_{k=1}^{d} {L_2}^{-1}\big(\widetilde{A}^k_{2,1}(V)\partial_k Z_1 + \widetilde{A}^j_{2,2}(V)\partial_k Z_2\big) - {L_2}^{-1}Q(Z)\cdot \quad (2.18)$$

Note that

$$\widetilde{A}^0_{2,2}(V)\partial_t W + L_2 W = \widetilde{A}^0_{2,2}(V){L_2}^{-1}\sum_{k=1}^{d}\partial_t\big(\widetilde{A}^k_{2,1}(V)\partial_k Z_1 + \widetilde{A}^j_{2,2}(V)\partial_k Z_2\big) - \widetilde{A}^0_{2,2}(V){L_2}^{-1}\partial_t Q(Z)\cdot \quad (2.19)$$

On the left-hand side, Property (2.5) ensures maximal dissipation on W. As the right-hand side of (2.19) contains only at least quadratic terms, or linear terms with one derivative, it can be expected to be negligible in low frequencies if Z is small enough, while (2.18) ensures that W is comparable to Z_2. Hence, in the end, we will get better integrability for Z_2 than for the whole solution Z.

2.2 Main results

Theorem 2.2.1. *Let $\bar{V}$ be an equilibrium state such that $H(\bar{V}) = 0$ and suppose that the structure assumptions of paragraph 2.1.1 and (SK) condition are satisfied. Then, there exists a positive constant α such that for all $Z_0 \in \dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ satisfying*

$$\mathcal{Z}_0 \triangleq \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq \alpha, \tag{2.20}$$

System (2.3) supplemented with initial data Z_0 admits a unique global-in-time solution Z in the space E defined by

$$Z \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \quad Z^h \in L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \quad Z^\ell_1 \in L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}) \quad \text{and} \quad W \in L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}),$$

with W defined according to (2.18).

Moreover, there exists a Lyapunov functional that is equivalent to $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$, and a constant C depending only on the matrices A^k and on H, such that

$$\mathcal{Z}(t) \leq C\mathcal{Z}_0 \quad \text{for all } t \geq 0 \tag{2.21}$$

where

$$\begin{aligned}\mathcal{Z}(t) \triangleq \|Z\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})} + \|Z\|^{h}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} \\ + \|W\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|Z_2\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}.\end{aligned} \tag{2.22}$$

Remark 2.2.1. *As is, the above theorem does not extend to the case $d = 1$. The reason why is that the low frequency regularity index then becomes negative, so that some nonlinear terms cannot be bounded in the proper spaces. For more details, the reader may refer to [30].*

Our second result concerns the time-decay estimates of the solution we constructed in Theorem 2.2.1.

Theorem 2.2.2. *Under the hypotheses of Theorem 2.2.1 and if, additionally, $Z_0 \in \dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1 \in \left]-\frac{d}{2}, \frac{d}{2}\right]$ then, there exists a constant C depending only on σ_1 and such that*

$$\|Z(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C\,\|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0. \tag{2.23}$$

Furthermore, if $\sigma_1 > 1 - d/2$ then, denoting

$$\langle t \rangle \triangleq \sqrt{1+t^2}, \quad \alpha_1 \triangleq \frac{\sigma_1 + \frac{d}{2} - 1}{2} \quad \text{and} \quad C_0 \triangleq \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}},$$

we have the following decay estimates:

$$\sup_{t\geq 0}\left\|\langle t\rangle^{\frac{\sigma+\sigma_1}{2}}Z(t)\right\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\leq CC_0 \quad \textit{if} \quad -\sigma_1<\sigma\leq d/2-1,$$
$$\sup_{t\geq 0}\left\|\langle t\rangle^{\frac{\sigma+\sigma_1}{2}+\frac{1}{2}}Z_2(t)\right\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\leq CC_0 \quad \textit{if} \quad -\sigma_1<\sigma\leq d/2-2,$$
$$\sup_{t\geq 0}\|\langle t\rangle^{\alpha_1}Z_2(t)\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\leq CC_0 \quad \textit{if} \quad \min(d/2-2,-\sigma_1)<\sigma\leq d/2-1$$
$$\textit{and} \quad \sup_{t\geq 0}\left\|\langle t\rangle^{2\alpha_1}Z(t)\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\leq CC_0.$$

Remark 2.2.2. *Since we have the embedding $L^1\hookrightarrow\dot{\mathbb{B}}^{-\frac{d}{2}}_{2,\infty}$, the above statement encompasses the classical integrability condition $Z_0\in L^1$ (see e.g. [76] in a slightly different context).*

Remark 2.2.3. *Owing to the presence of dissipation in the equation of Z_2, the decay rate of the low frequencies of Z_2 is stronger than the decay of the whole solution.*

If we assume in addition that:

$$\begin{cases}\text{for all } j\in\{1,\cdots,d\}, \quad A^k_{1,1}(\bar{V})=0 \quad\text{and}\quad D_{V_1}A^k_{1,1}(\bar{V})=0;\\ \text{for all } j\in\{1,\cdots,d\}, \quad D_{V_1}A^k_{2,1}(\bar{V})=0 \text{ (and thus also } \ D_{V_1}A^k_{1,2}(\bar{V})=0);\\ \text{the function } r \text{ is quadratic with respect to } Z_2 \text{ (i.e. } D^2_{V_i,V_m}r(0)=0 \text{ for } (i,m)\neq(2,2)),\end{cases} \tag{2.24}$$

then one can weaken the low frequency assumption, as we did in the first chapter dedicated to one-dimensional case. We get:

Theorem 2.2.3. *Let the assumptions of Theorem 2.2.1 concerning system (2.1) be in force and assume in addition that (2.24) holds true.*

Then, there exists a positive constant α such that for all $Z_0\in\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ satisfying

$$\mathcal{Z}'_0\triangleq\|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\leq\alpha, \tag{2.25}$$

System (2.3) supplemented with initial data Z_0 admits a unique global-in-time solution Z in the space F defined by

$$Z\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}),\quad Z^h\in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}),\quad Z^{\ell}_1\in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1}) \quad \textit{and} \quad W\in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}).$$

Moreover, there exists a Lyapunov functional that is equivalent to $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$, and we have the following a priori estimate:

$$\begin{aligned}\mathcal{Z}'(t)\leq C\mathcal{Z}'_0 \quad \textit{where} \quad \mathcal{Z}'(t)&\triangleq\|Z\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}+\|Z\|^{h}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}\\ &+\|Z_1\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1})}+\|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|Z_2\|^{\ell}_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}+\|Z\|^{h}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|W\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}.\end{aligned} \tag{2.26}$$

Finally, if, additionally, $Z_0\in\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1\in\left]-\frac{d}{2},\frac{d}{2}\right]$ then (2.23) is satisfied as well as the decay estimates that follow, up to $\sigma=d/2$ *for the first one, and with $d/2-1$ and $d/2$ instead of $d/2-2$ and $d/2-1$ for the next two ones, with α_1 replaced by $(\sigma_1+d/2)/2$.*

Remark 2.2.4. *The above result applies to the compressible Euler with damping (see Theorem 2.4.1).*

Remark 2.2.5. *In contrast with Theorem 2.2.1, the functional setting of Theorem 2.2.3 allows to obtain uniform estimates in the asymptotic $\lambda \to +\infty$ if the dissipative term is λH. This is the first step for the study of the high relaxation limit that we plan to carry out in Chapter 3.*

2.3 Proof of Theorems 2.2.1 and 2.2.2

This section is devoted to proving the global existence of strong solutions and decay estimates for System (2.1) supplemented with initial data that are close to the reference solution $\bar{V}$, in the general case where the structural assumptions listed in Subsection 2.1.1 and (SK) condition are satisfied.

The bulk of the proof consists in establishing a priori estimates, the other steps (proving existence and uniqueness) being more classical. As explained before, our strategy is to first work out a Lyapunov functional in Beauchard-Zuazua's style, that is equivalent to the norm that we aim at controlling, then to combine with the study of the damped mode W defined in (2.18) so as to close the estimates.

2.3.1 Establishing the a priori estimates

Throughout this part, we assume that we are given a smooth (and decaying) solution Z of (2.3) on $[0,T] \times \mathbb{R}^d$ with Z_0 as initial data, satisfying

$$\sup_{t\in[0,T]} \|Z(t)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \ll 1. \tag{2.27}$$

We shall use repeatedly that, owing to the embedding $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \hookrightarrow L^\infty$, we have also

$$\sup_{t\in[0,T]} \|Z(t)\|_{L^\infty} \ll 1. \tag{2.28}$$

From now on, $C > 0$ designates a generic harmless constant, the value of which depends on the context and we denote by $(c_j)_{q\in\mathbb{Z}}$ nonnegative sequences such that $\sum_{q\in\mathbb{Z}} c_j = 1$.

To start with, let us rewrite (2.3) as follows:

$$\bar{A}^0\partial_t Z + \sum_{k=1}^d \bar{A}^k\partial_k Z + LZ = G \tag{2.29}$$

with $G \triangleq G_1 + G_2 + G_3$ and

$$\begin{aligned}
G_1 &\triangleq -\sum_{k=1}^d \bar{A}^0 \left((\widetilde{A}^0(V))^{-1}\widetilde{A}^k(V) - (\bar{A}^0)^{-1}\bar{A}^k \right) \partial_k Z,\\
G_2 &\triangleq -\bar{A}^0 \left((\widetilde{A}^0(V))^{-1} - (\bar{A}^0)^{-1} \right) LZ,\\
G_3 &\triangleq \bar{A}^0(\widetilde{A}^0(V))^{-1} r(Z).
\end{aligned}$$

For $j \in \mathbb{Z}$, applying $\dot{\Delta}_j$ to (2.29) yields

$$\bar{A}^0 \partial_t Z_j + \sum_{k=1}^{d} \bar{A}^k \partial_k Z_j + L Z_j = \dot{\Delta}_j G \quad \text{with} \quad Z_j \triangleq \dot{\Delta}_j Z. \tag{2.30}$$

Our analysis will mainly consist in estimating for all $j \in \mathbb{Z}$ a functional $\mathcal{L}_j$ that is equivalent to the $L^2(\mathbb{R}^d;\mathbb{R}^n)$ norm of Z_j and encodes information on the dissipative properties of the system. That functional will be built from (2.16) and, since Condition (SK) is satisfied, the number $\mathcal{N}_{\bar{V}}$ defined in (2.17) will be positive. Furthermore, since the Fourier transform of Z_j is localized near the frequencies of magnitude 2^j, the corresponding dissipation term $\mathcal{H}_j$ will satisfy

$$\mathcal{H}_j \gtrsim \min(1, 2^{2j}) \mathcal{L}_j.$$

The prefactor $\min(1, 2^{2j})$ may be seen as a gain of two derivatives in low frequencies after time integration (like for the heat equation) whereas it corresponds to exponential decay for high frequencies. In our setting where the low and high frequencies of Z_0 belong to the spaces $\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}$ and $\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$, respectively, we thus have

$$\|Z(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \int_0^t \|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \int_0^t \|G\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}},$$
$$\|Z(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|G\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

A rapid examination reveals that the part G_1 of G may entail a loss of one derivative (since it is a combination of components of ∇Z) while G_2 and G_3 contain products of components of Z and Z_2. Overcoming the difficulty with G_1 will be achieved by exploiting the symmetrizable character of the system under consideration and changing slightly the weight $\bar{A}_0$ in the definition of $\mathcal{L}_j$ for the high frequencies: we shall take

$$\mathcal{L}_j \triangleq \|Z_j\|^2_{L^2_{\bar{A}_0(V)}} + 2^{-j}\mathcal{I}_j \quad \text{if} \quad j \geq 0, \tag{2.31}$$

with

$$\mathcal{I}_j \triangleq \int_{\mathbb{R}^d} \sum_{q=1}^{n-1} \varepsilon_q \Im \left((N M_\omega^{q-1} \widehat{Z_j}) \cdot (N M_\omega^{q} \widehat{Z_j}) \right), \tag{2.32}$$

where $\varepsilon_1, \cdots, \varepsilon_{n-1} > 0$ will be chosen small enough (according to the Appendix).

For the low frequencies, we shall keep the original definition that we proposed in the analysis of (2.7), that is to say, after integrating on the whole space and using Fourier-Plancherel theorem,

$$\mathcal{L}_j \triangleq \|Z_j\|^2_{L^2_{\bar{A}_0}} + 2^{j}\mathcal{I}_j \quad \text{if} \quad j < 0. \tag{2.33}$$

However, we will discover that the terms G_2 and G_3 cannot be controlled properly in the space $L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})$ because Z_2 is, somehow, too regular ! The way to overcome the difficulty is to look for an estimate of the low frequencies of the damped mode W, then to compare with Z_2.

We shall keep in mind all the time that if choosing the coefficients ε_q small enough, then

we have

$$\sum_{q=1}^{n-1} \varepsilon_q \big|((M_\omega)^t)^q N^t N M_\omega^{q-1}\big| \leq \frac{1}{2}\frac{1}{(2\pi)^d},$$

whence, owing to Fourier-Plancherel theorem,

$$|\mathcal{I}_j| \leq \frac{1}{2}\|Z_j\|_{L^2}.$$

Furthermore, as $\bar{A}_0 = A_0(\bar{V})$ is positive definite and $V \mapsto \widetilde{A}_0(V)$, continuous, Condition (2.28) ensures that $\|Z_j\|_{L^2_{\bar{A}_0}} \simeq \|Z_j\|_{L^2}$ and $\|Z_j\|_{L^2_{\widetilde{A}_0(V)}} \simeq \|Z_j\|_{L^2}$. Therefore, we have

$$\mathcal{L}_j \simeq \|Z_j\|_{L^2}^2 \ \text{ for all } \ q \in \mathbb{Z}. \tag{2.34}$$

2.3.1.1 Basic energy estimates

The first step is devoted to studying the time evolution of $\|Z_j\|^2_{L^2_{\widetilde{A}_0(V)}}$ and $\|Z_j\|^2_{L^2_{\bar{A}_0}}$. The outcome is given in the following proposition.

Proposition 2.3.1. *Let Z be a smooth solution of (2.3) on $[0,T]\times\mathbb{R}^d$ satisfying (2.27). Then, for all $s \in \left[\frac{d}{2}, \frac{d}{2}+1\right]$ and $j \geq 0$, we have:*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z_j\|^2_{L^2_{\widetilde{A}_0(V)}} + \kappa_0\|Z_{2,j}\|^2_{L^2} \lesssim{}& \|(\nabla Z, Z_2)\|_{L^\infty}\|Z_j\|^2_{L^2} + c_j 2^{-js}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^s_{2,1}}\|Z_j\|_{L^2}\\ &+ c_j 2^{-js}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{s-1}_{2,1}}\|Z_j\|_{L^2}\\ &+ c_j 2^{-js}\big(\|Z\|_{\dot{\mathbb{B}}^s_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^s_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)\|Z_j\|_{L^2}.\end{aligned} \tag{2.35}$$

Furthermore, for all $s' \in \left[\frac{d}{2}-1, \frac{d}{2}\right]$ and $j \leq 0$, we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z_j\|^2_{L^2_{\bar{A}_0}} + \kappa_0\|Z_{2,j}\|^2_{L^2} \lesssim{}& \|\nabla Z\|_{L^\infty}\|Z_j\|^2_{L^2} + c_j 2^{-js'}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{s'}_{2,1}}\|Z_j\|_{L^2}\\ &+ c_j 2^{-js'}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{s'}_{2,1}}\|Z_j\|_{L^2} + c_j 2^{-js'}\|Z_2\|_{\dot{\mathbb{B}}^{s'}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_j\|_{L^2}.\end{aligned} \tag{2.36}$$

Proof. It relies on an energy method implemented on (2.3) after localization in the Fourier space, and on classical commutator estimates.

In order to prove (2.35), apply operator $\dot{\Delta}_j$ to (2.3) to get:

$$\widetilde{A}^0(V)\partial_t Z_j + \sum_{k=1}^d \widetilde{A}^k(V)\partial_k Z_j + LZ_j = R_j^1 + R_j^2 + \dot{\Delta}_j(r(Z))$$

with $R_j^1 \triangleq \sum_{k=1}^d [\widetilde{A}^k(V), \dot{\Delta}_j]\partial_k Z$ and $R_j^2 \triangleq [\widetilde{A}^0(V), \dot{\Delta}_j]\partial_t Z$.

Taking the $L^2(\mathbb{R}^d;\mathbb{R}^n)$ scalar product with Z_j, integrating by parts in the second term and

using the fact that $\widetilde{A}^k(V)$ is symmetric yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\widetilde{A}_0(V)Z_j\cdot Z_j+\int_{\mathbb{R}^d}LZ_j\cdot Z_j=\frac{1}{2}\int_{\mathbb{R}^d}\Big(\partial_t\tilde{A}^0(V)+\sum_j\partial_k(\widetilde{A}^k(V))\Big)Z_j\cdot Z_j$$
$$+\int_{\mathbb{R}^d}(R_j^1+R_j^2)\cdot Z_j+\int_{\mathbb{R}^d}\dot{\Delta}_jr(Z)\cdot Z_j.$$

Hence, thanks to Property (2.12), we obtain

$$\frac{1}{2}\frac{d}{dt}\|Z_j\|^2_{L^2_{\widetilde{A}^0(V)}}+\kappa_0\|NZ_j\|^2_{L^2}\leq\frac{1}{2}\int_{\mathbb{R}^d}\Big(\partial_t\widetilde{A}^0(V)+\sum_j\partial_k(\widetilde{A}^k(V))\Big)Z_j\cdot Z_j$$
$$+\int_{\mathbb{R}^d}(R_j^1+R_j^2)\cdot Z_j+\int_{\mathbb{R}^d}\dot{\Delta}_jr(Z)\cdot Z_j. \quad (2.37)$$

For the first term in the right-hand side, we have

$$\int_{\mathbb{R}^d}\partial_t(\widetilde{A}^0(V))Z_j\cdot Z_j\lesssim\|\partial_tZ\|_{L^\infty}\|Z_j\|^2_{L^2}.$$

Hence, using the fact that

$$\partial_tZ=(\widetilde{A}^0(V))^{-1}\Big(\widetilde{H}(\bar{V}+Z)-\sum_{k=1}^d\widetilde{A}^k(V)\partial_kZ\Big), \quad (2.38)$$

the smallness condition (2.28) and the structure of $\widetilde{H}$, we get

$$\int_{\mathbb{R}^d}\partial_t(\widetilde{A}^0(V))Z_j\cdot Z_j\lesssim\|(\nabla Z,Z_2)\|_{L^\infty}\|Z_j\|^2_{L^2}. \quad (2.39)$$

For the second term in the right-hand side of (2.37), we may write

$$\int_{\mathbb{R}^d}\sum_{k=1}^d\partial_k(\widetilde{A}^k(V))Z_j\cdot Z_j\lesssim\|\nabla Z\|_{L^\infty}\|Z_j\|^2_{L^2}. \quad (2.40)$$

Bounding the commutators terms in (2.37) relies on Cauchy-Schwarz inequality and Inequality (A.4) that give

$$\int_{\mathbb{R}^d}(R_j^1+R_j^2)\cdot Z_j\lesssim c_j2^{-js}\Big(\sum_j\big\|\nabla(\widetilde{A}^k(V))\big\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{s-1}_{2,1}}+\big\|\nabla(\widetilde{A}^0(V))\big\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\partial_tZ\|_{\dot{\mathbb{B}}^{s-1}_{2,1}}\Big)\|Z_j\|_{L^2}$$
$$\lesssim c_j2^{-js}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big(\|Z\|_{\dot{\mathbb{B}}^s_{2,1}}+\|\partial_tZ\|_{\dot{\mathbb{B}}^{s-1}_{2,1}}\big)\|Z_j\|_{L^2}.$$

To bound ∂_tZ, we need the following lemma.

Lemma 2.3.1. *Under assumption (2.27), we have for all $\sigma\in]-d/2,d/2]$,*

$$\|\partial_tZ\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\lesssim\|\nabla Z\|_{\dot{\mathbb{B}}^\sigma_{2,1}}+\min\big(\|W\|_{\dot{\mathbb{B}}^\sigma_{2,1}},\|Z_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\big).$$

Proof. Using (2.38), Propositions A.2.4, A.2.5 and 2.5.1 yields

$$\begin{aligned}\|\partial_t Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} &\leq \Big\|\sum_{k=1}^{d}\widetilde{A}^k(V)\partial_k Z\Big\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|LZ\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|r(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\\ &\lesssim \big(1+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)\|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}.\end{aligned}$$

Since we assumed that $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$ is small, we have

$$\|\partial_t Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}.$$

Note that, actually, $\partial_t Z_1$ can be bounded by just ∇Z and that we have $\partial_t Z_2 = -(\widetilde{A}^0_{2,2}(V))^{-1}L_2 W$ by definition of W, whence the final result. □

Finally, Proposition 2.5.1 ensures that

$$\begin{aligned}\int_{\mathbb{R}^d}\dot{\Delta}_j r(Z)\cdot Z_j &\lesssim c_j 2^{-js}\|r(Z)\|_{\dot{\mathbb{B}}^{s}_{2,1}}\|Z_j\|_{L^2}\\ &\lesssim c_j 2^{-js}\Big(\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{s}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{s}_{2,1}}\Big)\|Z_j\|_{L^2}.\end{aligned}$$

Putting all the above estimates together completes the proof of (2.35).

For proving (2.36), since we do not know how to control $\partial_t Z$ in $L^1_T(\dot{\mathbb{B}}^{s'-1}_{2,1})$ for $s' = d/2-1$ (which is the value that we will take eventually), we proceed slightly differently, writing the equation satisfied by Z_j as follows:

$$\bar{A}^0\partial_t Z_j + \sum_{k=1}^{d}\widetilde{A}^k(V)\partial_k Z_j + LZ_j = R^1_j + R^3_j + \dot{\Delta}_j(r(Z)) \quad\text{with}\quad R^3_j \triangleq \dot{\Delta}_j\Big(\big(\bar{A}^0 - \widetilde{A}^0(V)\big)\partial_t Z\Big).$$

Arguing as for proving (2.37), we now get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z_j\|^2_{L^2_{\bar{A}^0}} + \kappa_0\|NZ_j\|^2_{L^2} &\leq \frac{1}{2}\int_{\mathbb{R}^d}\Big(\sum_j \partial_k(\widetilde{A}^k(V))\Big)Z_j\cdot Z_j\\ &\quad + \int_{\mathbb{R}^d}(R^1_j + R^3_j)\cdot Z_j + \int_{\mathbb{R}^d}\dot{\Delta}_j r(Z)\cdot Z_j. \qquad (2.41)\end{aligned}$$

The term R^1_q may be estimated as above (with s' instead of s), and $\dot{\Delta}_j(r(Z))$ may be bounded by means of (2.95). As regards R^3_j, we write that

$$\|R^3_j\|_{L^2} \lesssim c_j 2^{-js'}\Big\|\widetilde{A}^0(V) - \widetilde{A}^0(\bar{V})\Big\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\partial_t Z\|_{\dot{\mathbb{B}}^{s'}_{2,1}}.$$

Thus, using composition, product estimates and Lemma 2.3.1, we obtain

$$\Big|\int_{\mathbb{R}^d} R^3_j\cdot Z_j\Big| \lesssim c_j 2^{-js'}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{s'}_{2,1}}\|Z_j\|_{L^2},$$

which leads to the desired estimate. □

2.3.1.2 Cross estimates

Proposition 2.3.1 only allows to exhibit the integrability properties of the components of Z experiencing direct dissipation. To recover the dissipation for all the components, we have to look at the time derivative of the quantity $\mathcal{I}_j$ defined in (2.32). To achieve it, we apply to (2.30) the method that has been explained in Section 2.1 and leads to (2.15). The only change lies in the (harmless) additional source term G_j. In the end, integrating on $\mathbb{R}^d$ the obtained identity, then using the fact that $\operatorname{Supp}\widehat{Z}_j \subset \{3\cdot 2^j/4 \le |\xi| \le 8\cdot 2^j/3\}$ yields

$$\frac{d}{dt}\mathcal{I}_j + \frac{2^j}{2}\sum_{q=1}^{n-1}\varepsilon_q\int_{\mathbb{R}^d}|NM_\omega^q\widehat{Z}_j|^2\,d\xi \le \frac{2^{-j}\kappa_0}{2}\|NZ_j\|_{L^2}^2 + C\|\dot{\Delta}_j G\|_{L^2}\|Z_j\|_{L^2}.$$

The last term may be bounded by means of Propositions A.2.4 and A.2.5 (keeping all the time in mind that (2.27) is satisfied). More precisely, for G_1, we have for all $\sigma \in]-d/2, d/2]$,

$$\begin{aligned}\|G_1\|_{\dot{\mathbb{B}}^\sigma_{2,1}} &\lesssim \sum_{q=1}^{d}\big\|\big(\widetilde{A}^0(V)^{-1}\widetilde{A}^q(V) - (\bar{A}^0)^{-1}\bar{A}^q\big)\partial_k Z\big\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\\ &\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^\sigma_{2,1}}.\end{aligned} \tag{2.42}$$

Similarly,

$$\|G_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim \Big\|\Big((\widetilde{A}^0(V))^{-1} - (\bar{A}_0)^{-1}\Big)LZ\Big\|_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}} \tag{2.43}$$

and, using Proposition 2.5.1,

$$\|G_3\|_{\dot{\mathbb{B}}^\sigma_{2,1}} = \Big\|\bar{A}^0\widetilde{A}^0(V)^{-1}r(Z)\Big\|_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^\sigma_{2,1}}. \tag{2.44}$$

Hence, one can conclude that for all $\sigma \in]-d/2, d/2]$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_j + \frac{2^j}{2}\sum_{q=1}^{n-1}\varepsilon_q\int_{\mathbb{R}^d}|NM_\omega^q\widehat{Z}_j|^2\,d\xi\\ \le \frac{2^{-j}\kappa_0}{2}\|NZ_j\|_{L^2}^2 + Cc_j2^{-j\sigma}\|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_j\|_{L^2}.\end{aligned} \tag{2.45}$$

2.3.1.3 Closure of the estimates : a first attempt

Remember that since Condition (SK) is satisfied, the quantity $\mathcal{N}_{\bar{V}}$ defined in (2.17) is positive for any choice of positive parameters $\varepsilon_0, \cdots, \varepsilon_{n-1}$. Consequently, if we set

$$\mathcal{H}_j := \frac{\kappa_0}{2}\|NZ_j\|^2 + \min(1, 2^{2j})\sum_{q=1}^{n-1}\varepsilon_q\int_{\mathbb{R}^d}|NM_\omega^q\widehat{Z}_j|^2\,d\xi$$

and use Fourier-Plancherel theorem and the equivalence (2.34), we see that (up to a change of κ_0), we have for all $j \in \mathbb{Z}$,

$$\mathcal{H}_j \ge \kappa_0\min(1, 2^{2j})\mathcal{L}_j. \tag{2.46}$$

Our goal is to use this inequality to bound the quantity $\mathcal{Z}$ defined in (2.22) in terms of $\mathcal{Z}_0$ only.

Let us start with the bounds for the low frequencies. Putting together Inequality (2.36) with $s' = d/2 - 1$ and the cross estimate (2.45) then, using (2.46), we get for all $j < 0$,

$$\frac{d}{dt}\mathcal{L}_j + \kappa_0 2^{2j}\mathcal{L}_j \lesssim \|\nabla Z\|_{L^\infty}\|Z_j\|_{L^2}^2 + c_j 2^{-j(\frac{d}{2}-1)}\Big(\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Big)\|Z_j\|_{L^2}.$$

Hence, using (2.34), applying Lemma A.1.1, multiplying by $2^{j(\frac{d}{2}-1)}$, using the embedding $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \hookrightarrow L^\infty$ and summing up on $j < 0$ gives

$$\|Z(t)\|^\ell_{\dot{\mathcal{B}}^{\frac{d}{2}-1}_{2,1}} + \kappa_0\int_0^t \|Z\|^\ell_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}}\,d\tau \le \|Z_0\|^\ell_{\dot{\mathcal{B}}^{\frac{d}{2}-1}_{2,1}} + \int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Big),$$

where we used the notation

$$\|Z\|^\ell_{\dot{\mathcal{B}}^\sigma_{2,1}} \triangleq \sum_{j<0} 2^{j\sigma}\sqrt{\mathcal{L}_j}. \tag{2.47}$$

To handle the high frequencies, we combine Inequality (2.35) with $s = d/2 + 1$, the cross estimate (2.45) and (2.46), to get for all $j \ge 0$,

$$\frac{d}{dt}\mathcal{L}_j + \kappa_0\mathcal{L}_j \lesssim \|(\nabla Z, Z_2)\|_{L^\infty}\|Z_j\|^2_{L^2} + c_j 2^{-j(\frac{d}{2}+1)}\Big(\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Big)\|Z_j\|_{L^2}. \tag{2.48}$$

Hence, using the equivalence (2.34), Lemma A.1.1, multiplying by $2^{j(\frac{d}{2}+1)}$ and summing up on $j \ge 0$ gives

$$\|Z(t)\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} + \kappa_0\int_0^t \|Z\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} \le \|Z_0\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t\Big(\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Big)$$

where we used the notation

$$\|Z\|^h_{\dot{\mathcal{B}}^\sigma_{2,1}} \triangleq \sum_{q\ge 0} 2^{j\sigma}\sqrt{\mathcal{L}_j}.$$

Let us introduce the functional

$$\mathcal{L} \triangleq \|Z\|^\ell_{\dot{\mathcal{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} \tag{2.49}$$

which, in light of (2.34), is equivalent to $\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$, and thus to $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$.

Adding up the above inequalities for the low and high frequencies, we get up to a change

of κ_0 and for all $t \in [0,T]$,

$$\mathcal{L}(t) + \kappa_0 \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq \mathcal{L}(0) + C \int_0^t \Big(\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \Big).$$

Hence, using the interpolation inequality

$$\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \sqrt{\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}} \lesssim \mathcal{L}, \tag{2.50}$$

and eliminating some redundant terms, we end up with

$$\mathcal{L}(t) + \kappa_0 \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq \mathcal{L}(0) + C \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \mathcal{L} + C \int_0^t \|Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \tag{2.51}$$

As we start from small data, we expect $\mathcal{L}$ to be small as well so that the first term in the first integral in the right-hand side may be absorbed by the second term on the left. However, at this stage, we have no proper control on $\|Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}$. Studying the evolution of the damped mode W, which is the aim of the next section, will enable us to overcome the difficulty.

2.3.1.4 The damped mode

As underlined in the introduction, the function

$$W \triangleq -L_2{}^{-1} A^0_{2,2}(V)\partial_t Z_2 = Z_2 + \sum_{k=1}^d L_2{}^{-1}\big(\widetilde{A}^k_{2,1}(V)\partial_k Z_1 + \widetilde{A}^j_{2,2}(V)\partial_k Z_2\big) - L_2{}^{-1} Q(Z)\cdot$$

is expected to have *better* integrability properties in low frequencies than the whole solution. This will be a consequence of the following proposition.

Proposition 2.3.2. *Let Z be a smooth solution of (2.3) on $[0,T]\times\mathbb{R}^d$ satisfying (2.27), and denote $\bar{A}^0_{2,2} \triangleq A^0_{2,2}(\bar{V})$. Assume that $\sigma \in]-d/2, d/2]$. Then we have for all $j<0$,*

$$\frac{1}{2}\frac{d}{dt}\|W_j\|^2_{L^2_{\bar{A}^0_{2,2}}} + \kappa_0 \|W_j\|^2_{L^2} \lesssim \Big(\|\nabla^2 Z_j\|_{L^2} + \|\nabla W_j\|_{L^2}\Big)\|W_j\|_{L^2}$$
$$+ c_j 2^{-j\sigma}\|(W,Z_2)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|W_j\|_{L^2} + c_j 2^{-j\min(\sigma,\frac{d}{2}-1)} \|(\nabla Z, W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\min(\sigma+1,\frac{d}{2})}_{2,1}} \|W_j\|_{L^2}.$$

Proof. From (2.19), we gather that

$$\bar{A}^0_{2,2}\partial_t W + L_2 W = h \tag{2.52}$$

with $h \triangleq h_1 + \bar{A}^0_{2,2} {L_2}^{-1}(h_2+h_3)$ and

$$\begin{aligned}
h_1 &\triangleq \big(\mathrm{Id} - \bar{A}^0_{2,2}(A^0_{2,2}(V))^{-1}\big)L_2W,\\
h_2 &\triangleq \sum_{k=1}^d \partial_t\big(A^k_{2,1}(V)\partial_k Z_1 + A^j_{2,2}(V)\partial_k Z_2\big),\\
h_3 &= -\partial_t Q(Z)\cdot
\end{aligned}$$

Applying $\dot\Delta_j$ to (2.52) and taking the scalar product with $W_j \triangleq \dot\Delta_j W$ yields, thanks to (2.10),

$$\frac{1}{2}\frac{d}{dt}\|W_j\|^2_{L^2_{\bar{A}^0_{2,2}}} + \kappa_0\|W_j\|^2_{L^2} \le \big(\|\dot\Delta_j h_1\|_{L^2} + C\|\dot\Delta_j h_2\|_{L^2} + C\|\dot\Delta_j h_3\|_{L^2}\big)\|W_j\|_{L^2}. \tag{2.53}$$

As (2.27) is satisfied, Composition estimates readily give that for all $\sigma \in]-d/2, d/2]$,

$$\|h_1\|_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|W\|_{\dot{\mathbb{B}}^\sigma_{2,1}}. \tag{2.54}$$

For bounding h_2, we use that for all $j \in \{1,\cdots,d\}$,

$$\begin{aligned}
&\partial_t(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2) = D_V A^k_{2,1}(V)\partial_t Z\partial_k Z_1 + \bar{A}^k_{2,1}\partial_t\partial_k Z_1\\
&+\big(A^k_{2,1}(V) - A^k_{2,1}(\bar{V})\big)\partial_t\partial_k Z_1 + D_V A^k_{2,2}(V)\partial_t Z\partial_k Z_2 + \bar{A}^k_{2,2}\partial_t\partial_k Z_2 + \big(A^k_{2,2}(V) - A^k_{2,2}(\bar{V})\big)\partial_t\partial_k Z_2.
\end{aligned}$$

For $m = 1, 2$, we have, according to product and composition laws, and Lemma 2.3.1,

$$\begin{aligned}
\|D_V A^k_{2,m}(V)\partial_t Z\partial_k Z_k\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} &\lesssim \|\partial_t Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\\
&\lesssim \|(\nabla Z, W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}
\end{aligned}$$

as well as (provided we also have $\sigma \le d/2 - 1$):

$$\begin{aligned}
\|\big(A^k_{2,m}(V) - A^k_{2,m}(\bar{V})\big)\partial_t\partial_k Z_k\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} &\lesssim \|\partial_t\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}\\
&\lesssim \|(\nabla Z, W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}.
\end{aligned}$$

Multiplying the first equation of (2.6) (on the left) by the matrix $(\widetilde{A}^0_{1,1}(V))^{-1}$ then differentiating with respect to x_j, we discover that $\partial_t\partial_k Z_1$ is a combination of terms of type $A(Z)\,D^2Z$ and $B(Z)\,DZ\otimes DZ$. Consequently, we have for all $j \le 0$ (still if $\sigma \le d/2 - 1$):

$$\|\dot\Delta_j(\partial_t\partial_k Z_1)\|_{L^2} \lesssim \|D^2 Z_j\|_{L^2} + c_j 2^{-j\sigma}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Note that if $\sigma \in]d/2-1, d/2]$, then the above inequalities are valid (owing to (A.2)) if we change $\sigma+1$ to $d/2$.

Finally, we have

$$\partial_t\partial_k Z_2 = -\partial_k\big((\bar{A}^0_{2,2})^{-1}L_2W\big) + \partial_k\big((\bar{A}^0_{2,2})^{-1} - \big(\widetilde{A}^0_{2,2}(V)\big)^{-1}\big)L_2W.$$

Hence, for all $j \leq 0$, and thanks to (A.2),

$$\begin{aligned}\|\dot{\Delta}_j(\partial_t\partial_k Z_2)\|_{L^2} &\lesssim \|\nabla W_j\|_{L^2} + c_j 2^{-j\sigma}\|\big((\bar{A}^0_{2,2})^{-1} - \big(\widetilde{A}^0_{2,2}(V)\big)^{-1}\big)L_2 W\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\\ &\lesssim \|\nabla W_j\|_{L^2} + c_j 2^{-j\sigma}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|W\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}.\end{aligned}$$

To bound h_3, we use the fact that $\partial_t Q(Z) = D_Z Q(Z)\partial_t Z$. Hence, as $Q(Z)$ is at least quadratic, we easily obtain from Propositions A.2.4 and A.2.5 that

$$\|h_3\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|(\nabla Z, W)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}$$

which concludes the proof. □

It is now easy to obtain dissipative estimates for the low frequencies of W. Indeed, starting from the inequality of Proposition 2.3.2, taking advantage of Lemma A.1.1, multiplying the resulting inequality with $2^{j\sigma}$ and summing up on $j < 0$, we get whenever $\sigma \in]-d/2, d/2]$,

$$\begin{aligned}\mathcal{W}^{\sigma}(t) + \kappa_0\int_0^t \|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} &\leq \mathcal{W}^{\sigma}(0) + C\int_0^t \|(\nabla^2 Z, \nabla W)\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\\ &\quad + C\int_0^t \|(W, Z_2)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + C\int_0^t \|(\nabla Z, W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\min(\sigma+1,\frac{d}{2})}_{2,1}}\end{aligned} \tag{2.55}$$

with $\mathcal{W}^{\sigma} \triangleq \sum_{j<0} 2^{j\sigma}\|\dot{\Delta}_j W\|_{L^2_{\bar{A}^0_{2,2}}}$.

Let us first apply (2.55) with $\sigma = d/2$. Then we get (discarding the redundant terms):

$$\begin{aligned}\mathcal{W}^{\frac{d}{2}}(t) + \kappa_0\int_0^t \|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} &\leq \mathcal{W}^{\frac{d}{2}}(0)\\ &\quad + C\int_0^t \|(\nabla^2 Z, \nabla W)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + C\int_0^t \|(\nabla Z, Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.\end{aligned} \tag{2.56}$$

In order to close the estimates, we also need the inequality corresponding to $\sigma = d/2 - 1$, namely

$$\begin{aligned}\mathcal{W}^{\frac{d}{2}-1}(t) + \kappa_0\int_0^t \|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} &\leq \mathcal{W}^{\frac{d}{2}-1}(0) + C\int_0^t \|(\nabla Z, W)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\\ &\quad + C\int_0^t \|(W, \nabla Z)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + C\int_0^t \|(Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.\end{aligned} \tag{2.57}$$

Since

$$Z_2 = W - \sum_{k=1}^{d} L_2{}^{-1}\big(A^k_{2,1}(V)\partial_k Z_1 + A^j_{2,2}(V)\partial_k Z_2\big) + L_2{}^{-1}Q(Z) \tag{2.58}$$

and $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$ is small, we have for all $\sigma \in]-d/2, d/2]$,

$$\|W - Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|\nabla Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}. \tag{2.59}$$

Hence, W may be omitted in the last term of Inequality (2.56), and (2.57) becomes

$$\mathcal{W}^{\frac{d}{2}-1}(t)+\kappa_0\int_0^t\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\leq\mathcal{W}^{\frac{d}{2}-1}(0)+C\int_0^t\|(\nabla Z,W)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$$
$$+C\int_0^t\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \quad (2.60)$$

2.3.1.5 Global a priori estimates

We are now ready to establish the following proposition which will be the key to the proof of the existence part of Theorem 2.2.1.

Proposition 2.3.3. *Let Z be a smooth solution of (2.3) on $[0,T]$ satisfying the smallness condition (2.27). Then, there exist three (small) positive parameters κ_0, ε and ε' such that*

$$\widetilde{\mathcal{L}}\triangleq\mathcal{L}+\varepsilon\mathcal{W}^{\frac{d}{2}}+\varepsilon'\mathcal{W}^{\frac{d}{2}-1}$$

with $\mathcal{L}$ and $\mathcal{W}^{\sigma}$ defined in (2.51) and (2.55), respectively, satisfies for all $0\leq t_0\leq t\leq T$,

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_{t_0}^t\big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\big)\leq\widetilde{\mathcal{L}}(t_0). \quad (2.61)$$

Furthermore, there exists a positive constant C such that

$$\mathcal{Z}(t)\leq C\mathcal{Z}_0\quad\text{for all}\quad t\in[0,T], \quad (2.62)$$

where $\mathcal{Z}_0$ and $\mathcal{Z}$ have been defined in (2.20) and (2.22), respectively.

Proof. From (2.51), (2.56), (2.60) and (2.58), we get after a few simplifications,

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\Big)\leq\widetilde{\mathcal{L}}(0)+C(\varepsilon+\varepsilon')\int_0^t\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$$
$$+C\varepsilon'\int_0^t\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\mathcal{L}+C\int_0^t\|Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Hence, choosing (positive) ε and ε' so that

$$2C\varepsilon'\leq\kappa_0\varepsilon\quad\text{and}\quad 2C(\varepsilon+\varepsilon')\leq\kappa_0,$$

using again (2.59) and the interpolation inequality (2.50) eventually yields:

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\Big)\leq\widetilde{\mathcal{L}}(0)$$
$$+C\int_0^t\big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\big)\mathcal{L}. \quad (2.63)$$

Let us denote

$$T_0\triangleq\sup\big\{t\in[0,T],\ \sup_{\tau\in[0,t]}\widetilde{\mathcal{L}}(\tau)\leq 2\widetilde{\mathcal{L}}(0)\big\}.$$

Discarding the trivial case $\widetilde{\mathcal{L}}(0)=0$ (corresponding to the stationary solution $\bar{V}$), the continuity of $\widetilde{\mathcal{L}}$ ensures that $T_0>0$. Now, for all $t\in[0,T_0]$, Inequality (2.63) ensures that

$$\widetilde{\mathcal{L}}(t)+\kappa_0\int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\Big)\le\widetilde{\mathcal{L}}(0)+2C\widetilde{\mathcal{L}}(0)\int_0^t\big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\big).$$

Consequently, if the initial data are so small that $4C\widetilde{\mathcal{L}}(0)\le\varepsilon'\kappa_0$, then we deduce that

$$\widetilde{\mathcal{L}}(t)+\frac{\kappa_0}{2}\int_0^t\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\varepsilon\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\varepsilon'\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\Big)\le\widetilde{\mathcal{L}}(0),$$

and thus $T_0=T$. Hence (2.61) holds (with $\kappa_0/2$) on $[0,T]$. Clearly, the argument may be started from any time $t_0\in[0,T]$, which gives (2.61) in full generality.

Let us finally establish (2.62). First, since $\mathcal{L}$ is equivalent to $\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}+\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}$, it is easy to see that, under Assumption (2.27), we also have $\widetilde{\mathcal{L}}\simeq\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}+\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$. Combining with (2.61), we thus already get

$$\|Z\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}+\|Z\|^h_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|Z\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|W\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\le C\mathcal{Z}_0\quad\text{for all}\quad t\in[0,T].$$

Combining with (2.59), we discover that

$$\int_0^t\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\le\int_0^t\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+C\int_0^t\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\mathcal{Z}_0$$

and

$$\begin{aligned}\|Z_2\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\le\|W\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}+C\|\nabla Z\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}&\\ +C\|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|Z_2\|^h_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}+C\|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}&.\end{aligned}$$

Owing to (2.27), the last term may be absorbed by the left-hand side. Furthermore, one can bound the last but one thanks to (A.2) and, by Hölder inequality, interpolation and (2.61),

$$\begin{aligned}\|\nabla Z\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}&\lesssim\sqrt{\|Z\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\|Z\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}}\lesssim\mathcal{Z}_0\\ \|W\|^\ell_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}&\le\sqrt{\|W\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}\|W\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1})}}\lesssim\mathcal{Z}_0,\end{aligned}$$

which completes the proof of the proposition. □

2.3.2 Proof of Theorem 2.2.1

The starting point of the proof of existence is the following local well-posedness result that may be found in [102].

Proposition 2.3.4. *For any data Z_0 in the nonhomogeneous Besov space $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$, the following results hold true:*

1. *Existence: there exists a positive time T_1, depending only the coefficients of the matrices A^k, on H and on $\|Z_0\|_{\mathbb{B}^{\frac{d}{2}+1}_{2,1}}$ such that System (2.3) has a unique classical solution Z with*

$$Z \in \mathcal{C}^1([0,T_1]\times\mathbb{R}^d) \quad \textit{and} \quad Z \in \mathcal{C}([0,T_1];\mathbb{B}^{\frac{d}{2}+1}_{2,1}) \cap \mathcal{C}^1([0,T_1];\mathbb{B}^{\frac{d}{2}}_{2,1}).$$

2. *Blow-up criterion: if T^* is finite, then*

$$\int_0^{T^*} \|\nabla Z\|_{L^\infty}\, dt = \infty.$$

The proof of the existence part of Theorem 2.2.1 is structured as follows. First, we truncate the low frequencies of the data and use the above theorem to construct a sequence $(Z^n)_{n\in\mathbb{N}}$ of (a priori local) approximate solutions. Then we use the previous part to establish that those solutions are actually global and uniformly bounded in E. In order to pass to the limit, we show that $(Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T];\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})$ for all $T>0$. Then, we eventually check that the limit is indeed a solution of (2.3) and has the required regularity.

First step: construction of approximate solutions

Fix some initial data $Z_0 \in \dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ satisfying (2.20) and approximate it by

$$Z_0^n = (\mathrm{Id} - \dot{S}_n)Z_0, \qquad n \geq 1.$$

By construction, $Z_{0,n}$ belongs to $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$. Consequently, Theorem 2.3.4 provides us with a unique maximal solution $Z^n \in \mathcal{C}([0,T_n[;\mathbb{B}^{\frac{d}{2}+1}_{2,1}) \cap \mathcal{C}^1([0,T_n[;\mathbb{B}^{\frac{d}{2}}_{2,1})$.

Second step: uniform estimates

Taking advantage of Proposition 2.3.3 and denoting by $\mathcal{Z}^n$ the function $\mathcal{Z}$ pertaining to Z^n, we get $\mathcal{Z}^n \leq C\mathcal{Z}_0^n$ as long as Z^n satisfies the smallness condition (2.27). Owing to the definition of Z_0^n, we have $\mathcal{Z}_0^n \leq \mathcal{Z}_0$ and we clearly have $\|Z^n(t)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,\infty}} \lesssim \mathcal{Z}^n(t)$. Hence using a classical bootstrap argument, one can conclude that, if $\mathcal{Z}_0$ is small enough, then

$$\mathcal{Z}^n(t) \leq C\mathcal{Z}_0, \quad \text{for all} \quad t \in [0,T_n[. \tag{2.64}$$

In order to show that the solution Z^n is global (that is $T_n = +\infty$), one can use the blow-up criterion of Theorem 2.3.4. However, we first have to justify that the nonhomogeneous Besov norm $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$ of the solution is under control *up to time* T_n. Indeed, using the classical energy method for (2.3), then the Gronwall lemma, we discover that for all $t < T_n$,

$$\|Z^n(t)\|_{L^2} \leq C\,\|Z_0^n\|_{L^2} \exp\Big(C\int_0^t \|\nabla Z^n\|_{L^\infty}\Big).$$

Now, (2.64) and the embedding of $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$ in L^∞ ensure that $\nabla\mathcal{Z}^n$ is in $L^1_{T_n}(L^\infty)$, from which we deduce that Z^n is in $L^\infty_{T_n}(L^2)$, and thus in $L^\infty_{T_n}(\mathbb{B}^{\frac{d}{2}+1}_{2,1})$ owing, again, to (2.64). It is now easy to conclude : we have ∇Z^n in $L^1_{T_n}(L^\infty)$, and Z^n is in $\mathcal{C}([0,T];\mathbb{B}^{\frac{d}{2}+1}_{2,1}) \cap \mathcal{C}^1([0,T];\mathbb{B}^{\frac{d}{2}}_{2,1})$ for all $T < T_n$. Hence $T_n = +\infty$ and (2.64) is satisfied for all time.

Third step: convergence

The following stability result will ensure both the convergence of $(Z^n)_{n\in\mathbb{N}}$ and the uniqueness of our solution.

Proposition 2.3.5. *Let $\widetilde{Z} = Z^1 - Z^2$ where Z^1 and Z^2 are two solutions of (2.3), having respectively Z^1_0 and Z^2_0 as initial data, and belonging to the space E. There exists a constant c such that if both $\|Z^1\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}$ and $\|Z^2\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}$ are smaller than c, then we have for all $t\in[0,T]$,*

$$\|\widetilde{Z}\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \lesssim \|\widetilde{Z}_0\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \left(\|(Z^1,Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(Z^1,Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \right) \left\|\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \tag{2.65}$$

Proof. Let $V^1 \triangleq \bar{V} + Z^1$ and $V^2 \triangleq \bar{V} + Z^2$. Observe that $\widetilde{Z}$ is a solution of

$$\begin{aligned}\widetilde{A}^0(V^1)\partial_t\widetilde{Z} + \sum_{k=1}^d \widetilde{A}^k(V^1)\partial_k\widetilde{Z} \\ = -\widetilde{A}^0(V^1)\sum_{k=1}^d\big(\widetilde{A}^0(V^1)^{-1}\widetilde{A}^k(V^1) - \widetilde{A}^0(V^2)^{-1}\widetilde{A}^k(V^2)\big)\partial_k Z^2 - L\widetilde{Z} + r(Z^1) - r(Z^2).\end{aligned}$$

Applying $\dot{\Delta}_j$, taking the scalar product with $\widetilde{Z}_j$, integrating on $\mathbb{R}_+\times\mathbb{R}^d$ and using Lemma A.1.1, we get for all $j\in\mathbb{Z}$,

$$\begin{aligned}\left\|\widetilde{Z}_j\right\|_{L^2_{\widetilde{A}^0(V^1)}} + \kappa_0\int_0^t \left\|L\widetilde{Z}_j\right\|_{L^2} \le \left\|\widetilde{Z}_{0,j}\right\|_{L^2} + \int_0^t \left\|\nabla\widetilde{A}^k(V^1)\right\|_{L^\infty}\left\|\widetilde{Z}_j\right\|_{L^2} \\ + \int_0^t \left\|\dot{\Delta}_j\sum_{k=1}^d\big(\widetilde{A}^0(V^1)^{-1}\widetilde{A}^k(V^1) - \widetilde{A}^0(V^2)^{-1}\widetilde{A}^k(V^2)\big)\partial_k Z^2\right\|_{L^2} \\ + \int_0^t \left\|\dot{\Delta}_j(r(Z^1)-r(Z^2))\right\|_{L^2} + \sum_j \|[\dot{\Delta}_j,\widetilde{A}_j(V^1)]\partial_k\widetilde{Z}\|_{L^2}.\end{aligned}$$

Multiplying this inequality by $2^{j\frac{d}{2}}$ and using commutator estimates, we get

$$\begin{aligned}2^{j\frac{d}{2}}\left\|\widetilde{Z}_j\right\|_{L^2} \lesssim 2^{j\frac{d}{2}}\left\|\widetilde{Z}_{0,j}\right\|_{L^2} + \int_0^t \|\nabla Z^1\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} 2^{j\frac{d}{2}}\|\widetilde{Z}_j\|_{L^2} \\ + \int_0^t 2^{j\frac{d}{2}}\left\|\dot{\Delta}_j\sum_{k=1}^d\left(\widetilde{A}^0(V^1)^{-1}\widetilde{A}^k(V^1) - \widetilde{A}^0(V^2)^{-1}\widetilde{A}^k(V^2)\right)\partial_k Z^2\right\|_{L^2} \\ + \int_0^t 2^{j\frac{d}{2}}\left\|\dot{\Delta}_j(r(Z^1)-r(Z^2))\right\|_{L^2}.\end{aligned} \tag{2.66}$$

Thanks to Propositions A.2.4 and Inequality (A.9),

$$\left\|\left(\widetilde{A}^0(V^1)^{-1}\widetilde{A}^k(V^1) - \widetilde{A}^0(V^2)^{-1}\widetilde{A}^k(V^2)\right)\partial_k Z^2\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z^2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$$

and, according to Inequality (2.97), we have

$$\|r(Z^1)-r(Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|(Z^1,Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Hence, summing (2.66) on $j \in \mathbb{Z}$, we end up with

$$\left\|\widetilde{Z}\right\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \lesssim \left\|\widetilde{Z}_0\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \|Z^1\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \left\|\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \left\|\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z^2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \left\|\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|(Z^1, Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Splitting in low and high frequencies yields the desired estimate. □

The above lemma combined with the fact that $(Z_0^n)_{n\in\mathbb{N}}$ converges to Z_0 in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$ ensures that $(Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})$ and thus has a limit Z in that space, and passing to the limit in (2.3) is straightforward. Furthermore, using the Fatou property of Besov spaces, we obtain that $Z^\ell \in L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}) \cap L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})$ and $Z^h \in L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}) \cap L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})$ for all $T > 0$, together with the desired bounds. Time continuity of the solution may be obtained by adapting the arguments of [3, Chap. 4].

Fourth step: uniqueness

Knowing that Z^1 and Z^2 are in E, we have for all $T > 0$,

$$\int_0^T \left(\|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|(Z^1, Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\right) < \infty.$$

Furthermore, one can assume with no loss of generality that Z^1 is the solution that we constructed before and thus satisfies the smallness assumption (2.27). Owing to time continuity and since $Z^2(0) = Z^1(0)$, the solution Z^2 also satisfies (2.27) on some nontrivial time interval $[0, T]$, and combining Inequality (2.65) with Gronwall lemma allows to conclude that Z^1 and Z^2 coincide on $[0, T]$. A bootstrap argument then yields uniqueness on the whole half-line $\mathbb{R}_+$. □

2.3.3 Proof of Theorem 2.2.2

The overall strategy is taken from the work by Guo and Wang in [53] and Xin and Xu in [101].

First step : uniform bound in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$

In order to establish (2.23), one can look at System (2.3) as

$$\bar{A}^0\partial_t Z + \sum_{k=1}^d \bar{A}^k\partial_k Z + LZ = f + g + h$$

$$\text{with} \quad f \triangleq \sum_{k=1}^d \big(\bar{A}^k - \widetilde{A}^k(V)\big)\partial_k Z, \quad g \triangleq r(Z) \quad \text{and} \quad h \triangleq \big(\bar{A}^0 - \widetilde{A}^0(V)\big)\partial_t Z,$$

then apply $\dot{\Delta}_j$ and perform L^2 estimates for each Z_j.

After using Lemma A.1.1, multiplying by $2^{-j\sigma_1}$ then taking the supremum on $\mathbb{Z}$, we end up (omitting the term coming from L that has the 'good' sign) with

$$\|Z(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \int_0^t \|(f, g, h)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}. \tag{2.67}$$

Setting $f_k = \left(\widetilde{A}^k(V) - \bar{A}^k\right)\partial_k Z$ and using Inequality (A.8) yields

$$\|f_k\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \left\|\widetilde{A}^k(V) - \bar{A}^k\right\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|\partial_k Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} .$$

In order to bound $\widetilde{A}^k(V) - \bar{A}^k$ in $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$, one cannot use directly Proposition A.2.5 as $-\sigma_1$ may be negative. However, applying Taylor formula, product laws and a composition estimate (see the details in the proof of [30, Th. 4.1]), we can still obtain if (2.27) is satisfied,

$$\left\|\widetilde{A}^k(V) - \bar{A}^k\right\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} ,$$

whence

$$\|f\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} .$$

For g, using similar arguments as in Proposition 2.5.1 combined with Inequality (A.8) yield

$$\|g\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} .$$

Concerning h, we have, keeping Lemma 2.3.1 in mind, that

$$\begin{aligned}\|h\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} &\lesssim \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|\partial_t Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \\ &\lesssim \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} .\end{aligned}$$

Thus, regrouping all those estimates, we obtain

$$\|Z(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \lesssim \|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \int_0^t \|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} , \qquad t \geq 0. \tag{2.68}$$

Since, as pointed out before, we have

$$\int_0^t \|(\nabla Z, Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \mathcal{Z}(t) \lesssim \mathcal{Z}_0$$

and because the smallness condition (2.20) is satisfied, applying Gronwall inequality completes the proof of (2.23).

For the sake of completeness, one has to justify that if Z_0 is in $\dot{B}^{-\sigma_1}_{2,\infty}$ (in addition to (2.20)), then the solution constructed in Theorem 2.2.1 is in $\dot{B}^{-\sigma_1}_{2,\infty}$ for all time. This may be checked by following the construction scheme of the previous subsection. Indeed, recall that the approximated solutions Z^n are in $\mathcal{C}^1(\mathbb{R}_+; \mathbb{B}^{\frac{d}{2}}_{2,1})$. Then, discarding the linear term LZ^n (that may be handled by suitable conjugation), we get $\partial_t Z^n \in \mathcal{C}(\mathbb{R}_+; L^1)$. As $L^1 \hookrightarrow \dot{B}^{-\frac{d}{2}}_{2,\infty}$ and $\sigma_1 \geq d/2$, the low frequencies of $\partial_t Z^n$ (and thus the whole $\partial_t Z^n$) are in $\mathcal{C}(\mathbb{R}_+; \dot{B}^{-\sigma_1}_{2,\infty})$ As Z^n_0 itself is in $\dot{B}^{-\sigma_1}_{2,\infty}$ (since $\mathcal{F}(Z^n_0)$ is supported away from 0), we have $Z^n \in \mathcal{C}^1(\mathbb{R}_+; \dot{B}^{-\sigma_1}_{2,\infty})$. Consequently, (2.23) holds for Z^n and, passing to the limit, ensures that it holds for Z, too.

Second step : proof of generic decay estimates

According to Proposition 2.3.3, the functional $\widetilde{\mathcal{L}}$ introduced therein is nonincreasing and

equivalent to $\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$. Furthermore, there exist positive κ_0, ε and ε' such that denoting $\widetilde{\mathcal{H}} \triangleq \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \varepsilon\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \varepsilon'\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}$, we have

$$\widetilde{\mathcal{L}}(t) + \kappa_0 \int_{t_0}^{t} \widetilde{\mathcal{H}} \leq \widetilde{\mathcal{L}}(t_0) \quad \text{for all} \quad 0 \leq t_0 \leq t.$$

Hence and one may conclude as in [30] that $\widetilde{\mathcal{L}}$ is differentiable almost everywhere and satisfies

$$\frac{d}{dt}\widetilde{\mathcal{L}} + c'\widetilde{\mathcal{H}} \leq 0 \quad \text{a. e. on} \quad \mathbb{R}^+. \tag{2.69}$$

Granted with this information and (2.23), one can prove the first decay estimate of Theorem 2.2.2 by following the general argument of [101]. The starting point is that, provided $-\sigma_1 < d/2 - 1$,

$$\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} \lesssim \left(\|Z\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}\right)^{\theta_0} \left(\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\right)^{(1-\theta_0)} \quad \text{with} \quad \theta_0 = \frac{2}{d/2+1+\sigma_1}.$$

Inequality (2.23) thus implies that

$$\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \gtrsim \left(\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\right)^{\frac{1}{1-\theta_0}} \|Z_0\|^{-\frac{\theta_0}{1-\theta_0}}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}.$$

For the high frequencies term, using the estimate of Theorem 2.2.1, one can just write:

$$\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \gtrsim \left(\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\right)^{\frac{1}{1-\theta_0}} \|Z_0\|^{-\frac{\theta_0}{1-\theta_0}}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

Hence, there exists a (small) constant c such that

$$\frac{d}{dt}\widetilde{\mathcal{L}} + cC_0^{-\frac{\theta_0}{1-\theta_0}}\widetilde{\mathcal{L}}^{\frac{1}{1-\theta_0}} \leq 0 \quad \text{with} \quad C_0 \triangleq \|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

Integrating, this gives us

$$\widetilde{\mathcal{L}}(t) \leq \left(1 + c\,\frac{\theta_0}{1-\theta_0}\left(\frac{\widetilde{\mathcal{L}}(0)}{C_0}\right)^{\frac{\theta_0}{1-\theta_0}} t\right)^{1-\frac{1}{\theta_0}} \widetilde{\mathcal{L}}(0)$$

whence, since $\widetilde{\mathcal{L}} \leq \widetilde{\mathcal{L}}(0) \lesssim \mathcal{Z}_0 \lesssim C_0$,

$$\|Z(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|Z(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim (1+t)^{-\alpha_1}\mathcal{Z}_0 \quad \text{with} \quad \alpha_1 = \frac{d/2-1+\sigma_1}{2}. \tag{2.70}$$

The decay rates in $\dot{\mathbb{B}}^{\sigma}_{2,1}$ for all $\sigma \in]-\sigma_1, d/2-1]$ follow from Inequalities (2.23) and (2.70), and interpolation inequalities.

Third step: decay enhancement for the damped mode

From (2.53) and Lemma A.1.1, one can get for all $\sigma \in]-\sigma_1, d/2-1]$,

$$\mathcal{W}^{\sigma}(t) \leq e^{-ct}\mathcal{W}^{\sigma}(0) + C\int_0^t e^{-c(t-\tau)}\,\|h(\tau)\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\,d\tau.$$

Hence, since $\mathcal{W}^\sigma \approx \|W\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}$ and using the estimates of h pointed out in the proof of Proposition 2.3.2, we get

$$\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim e^{-t}\|W_0\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} + \int_0^t e^{-(t-\tau)}\Big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|(Z_2,W)\|_{\dot{\mathbb{B}}^\sigma_{2,1}} + \|(\nabla Z,W)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}} + \|(\nabla^2 Z,\nabla W)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\Big),$$

whence

$$\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim e^{-t}\|W_0\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} + \int_0^t e^{-(t-\tau)}\Big(\|(\nabla Z,Z_2)\|_{\dot{\mathbb{B}}^\sigma_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|(\nabla Z,Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma+1}_{2,1}} + \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}\Big).$$

In light of the previous step, the worst decay comes from the last term. In order to be allowed to use the corresponding estimate however, we need $\sigma+1 \le d/2-1$. If that condition is satisfied then, setting $\beta = (\sigma+\sigma_1+1)/2$, the above inequality implies that

$$\langle t\rangle^\beta\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim \langle t\rangle^\beta e^{-ct}\|W_0\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} + C_0\int_0^t \frac{\langle t\rangle^\beta}{\langle\tau\rangle^\beta}\, e^{-c(t-\tau)}\, d\tau \lesssim C_0.$$

In the case $\sigma+1 > d/2-1$, one can use the fact that $\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim \|W(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}-2}_{2,1}}$, and the above argument thus just implies that

$$\|W(t)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim (1+t)^{-\alpha_1}.$$

Keeping in mind (2.59), one can conclude that $\|Z_2\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}$ satisfies the same decay estimates as W.

Last step : high frequencies decay

Let us start from (2.48). The usual method based on Lemma A.1.1 leads after multiplying by $\langle t\rangle^{2\alpha_1}$ (where α_1 comes from (2.70)) yields

$$\big\|\langle t\rangle^{2\alpha_1}Z(t)\big\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \le e^{-ct}\,\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \langle t\rangle^{2\alpha_1}e^{-c(t-\tau)}\,\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\Big(\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Big)d\tau + \int_0^t \langle t\rangle^{2\alpha_1}e^{-c(t-\tau)}\,\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\,\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\,d\tau. \quad (2.71)$$

Thanks to (2.70), the first quadratic term may be bounded as follows:

$$\int_0^t \langle t\rangle^{2\alpha_1}e^{-c(t-\tau)}\,\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\,\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \le \int_0^t \left(\frac{\langle t\rangle}{\langle\tau\rangle}\right)^{2\alpha_1} e^{-c(t-\tau)}\big(\langle\tau\rangle^{\alpha_1}\,\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)^2 d\tau \lesssim C_0,$$

and the other terms of the right-hand side of (2.71) may be bounded similarly. This completes the proof of Theorem 2.2.2.

2.4 Proof of Theorem 2.2.3 and application to the Euler system

This section is devoted to the proof of Theorem 2.2.3, that is to say to a refinement of Theorem 2.2.1 corresponding to the case where System (2.1) satisfies the extra conditions listed in (2.24). As an application, we shall obtain a global existence statement for the compressible Euler with damping, in a new functional framework, and will specify the dependency of the estimates with respect to the relaxation (or damping) parameter.

2.4.1 Proof of Theorem 2.2.3

Proving existence and uniqueness being very similar to what we did before, we focus on establishing a priori estimates for a smooth solution Z of (2.3) on $[0,T]\times\mathbb{R}^d$, satisfying the smallness condition (2.27). The general strategy is the same as in the previous section, and we shall mainly underline the places where having the structure (2.24) comes into play.

The first difference is in the following refinement of Lemma 2.3.1

Lemma 2.4.1. *Under hypotheses (2.24) and (2.27), we have for all $\sigma\in]-d/2,d/2]$,*

$$\|\partial_t Z_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|\nabla Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}},$$
$$\|\partial_t Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|W\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}.$$

Proof. The second inequality has been proved before (see Lemma 2.3.1). The first one relies on the decomposition

$$\partial_t Z_1 = -\sum_{k=1}^{d} (\widetilde{A}^0_{1,1}(V))^{-1}\left(\widetilde{A}^k_{1,1}(V)\partial_k Z_1 + \widetilde{A}^k_{1,2}(V)\partial_k Z_2\right). \tag{2.72}$$

As the function $V\mapsto \widetilde{(A^0_{1,1}(V))}^{-1}\widetilde{A}^k_{1,1}(V)$ vanishes at $\bar{V}$ and is linear with respect to Z_2, Propositions A.2.4, A.2.5 and Condition (2.27) guarantee the desired inequality.

□

2.4.1.1 Basic energy estimates

As for Theorem 2.2.1, the first step consists in proving estimates for $\|Z_j\|^2_{L^2_{\widetilde{A}_0(V)}}$ and $\|Z_j\|^2_{L^2_{\widetilde{A}_0}}$.

Proposition 2.4.1. *Let Z be a smooth solution (2.3) on $[0,T]$ satisfying (2.27). Then, under Condition (2.24), we have for all $j\geq 0$,*

$$\frac{1}{2}\frac{d}{dt}\|Z_j\|^2_{L^2_{\widetilde{A}_0(V)}} + \kappa_0\|Z_{2,j}\|^2_{L^2} \lesssim c_j 2^{-j(\frac{d}{2}+1)}\|(W,\nabla Z)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z_j\|_{L^2}, \tag{2.73}$$

and for all $j\leq 0$,

$$\frac{1}{2}\frac{d}{dt}\|Z_j\|^2_{L^2_{\widetilde{A}_0}} + \kappa_0\|Z_{2,j}\|^2_{L^2}$$
$$\lesssim c_j 2^{-j\frac{d}{2}}\big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|(W,\nabla Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)\|Z_j\|_{L^2}. \tag{2.74}$$

Proof. The starting point is still (2.37) but we now take advantage of Lemma 2.4.1 and refine the estimates for R_q^1 and (2.40). More precisely, we have

$$R_q^1 = \sum_{k=1}^{d} \begin{pmatrix} [\widetilde{A}_{1,1}^k(V), \dot{\Delta}_j]\partial_k Z_1 + [\widetilde{A}_{1,2}^k(V), \dot{\Delta}_j]\partial_k Z_2 \\ [\widetilde{A}_{2,1}^k(V), \dot{\Delta}_j]\partial_k Z_1 + [\widetilde{A}_{2,2}^k(V), \dot{\Delta}_j]\partial_k Z_2 \end{pmatrix}.$$

Hence, using Inequality (A.4), we get for all $\sigma \in]-d/2, d/2+1]$,

$$\|R_q^1\|_{L^2} \lesssim c_j 2^{-j\sigma}\Big(\|\nabla(\widetilde{A}_{1,1}^k(V)), \nabla(\widetilde{A}_{2,1}^k(V))\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|Z_1\|_{\dot{\mathbb{B}}_{2,1}^{\sigma}} + \|\nabla(\widetilde{A}_{1,2}^k(V)), \nabla(\widetilde{A}_{2,2}^k(V))\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\sigma}}\Big).$$

At this point, one can use that (2.24) ensures that for all $j \in \{1, \cdots, d\}$ and $m \in \{1, 2\}$, there exist a linear map h and a smooth map F such that $\widetilde{A}_{m,1}^k(V) - \widetilde{A}_{m,1}^k(\bar{V}) = h(Z_2)F(Z)$. Consequently, product laws and composition estimates give us

$$\begin{aligned} \|\nabla(\widetilde{A}_{m,1}^k(V))\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} &\lesssim \|\nabla(h(Z_2)) \otimes F(Z)\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} + \|h(Z_2) \otimes \nabla(F(Z))\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \\ &\lesssim \|\nabla Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} (1 + \|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}) + \|Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|\nabla Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \\ &\lesssim \|\nabla Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} + \|Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|\nabla Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}, \end{aligned}$$

whence

$$\|R_q^1\|_{L^2} \lesssim c_j 2^{-j\sigma}\Big(\|\nabla Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} + \|Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|\nabla Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}\Big)\|Z_1\|_{\dot{\mathbb{B}}_{2,1}^{\sigma}} + \|\nabla Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\sigma}}. \tag{2.75}$$

Let us also observe that Inequality (2.99) of Proposition 2.5.1 gives us

$$\|r(Z)\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}} \lesssim \|Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}} \|Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} + \|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}} \|Z_2\|^2_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}.$$

Remembering that

$$\|R_q^2\|_{L^2} \lesssim c_j 2^{-j(\frac{d}{2}+1)} \|\nabla Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|\partial_t Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}},$$

and using Lemma 2.4.1 as well as (2.40) and (2.39), we eventually get (2.73). For proving (2.74), the starting point is (2.41). The term corresponding to R_q^1 (resp. $r(Z)$) can be bounded according to (2.75) (resp. (2.99)) with $\sigma = d/2$. In order to bound the term corresponding to R_q^3, we observe that, in light of Lemma 2.4.1,

$$\begin{aligned} \|(\widetilde{A}_{1,1}^0(V) - \widetilde{A}_{1,1}^0(\bar{V}))\partial_t Z_1\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} &\lesssim \|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|\partial_t Z_1\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \\ &\lesssim \|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \big(\|\nabla Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} + \|Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|\nabla Z_1\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}\big), \\ \|(\widetilde{A}_{2,2}^0(V) - \widetilde{A}_{2,2}^0(\bar{V}))\partial_t Z_2\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} &\lesssim \|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}} \|W\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

Finally, we have to refine Inequality (2.40). To this end, we use the decomposition

$$\int_{\mathbb{R}^d}\sum_{k=1}^d \partial_k(\widetilde{A}^k(V))Z_j\cdot Z_j = \sum_{k=1}^d\int_{\mathbb{R}^d}\Big(\partial_k\big(\widetilde{A}^k_{1,1}(V)\big)Z_{1,j}\cdot Z_{1,j}+\partial_k\big(\widetilde{A}^k_{2,1}(V)\big)Z_{2,j}\cdot Z_{1,j}$$
$$+\partial_k\big(\widetilde{A}^k_{1,2}(V)\big)Z_{1,j}\cdot Z_{2,j}+\partial_k\big(\widetilde{A}^k_{2,2}(V)\big)Z_{2,j}\cdot Z_{2,j}\Big).$$

The structure assumptions (2.24) and the symmetry of the system ensure that

$$\|\partial_k\big(\widetilde{A}^k_{1,1}(V)\big)\|_{L^\infty}+\|\partial_k\big(\widetilde{A}^k_{1,2}(V)\big)\|_{L^\infty}+\|\partial_k\big(\widetilde{A}^k_{2,1}(V)\big)\|_{L^\infty}\lesssim\|Z_2\|_{L^\infty}\|\nabla Z\|_{L^\infty}+\|\nabla Z_2\|_{L^\infty}.$$

Hence, remembering (2.28),

$$\int_{\mathbb{R}^d}\sum_{k=1}^d \partial_k(\widetilde{A}^k(V))Z_j\cdot Z_j\lesssim\big(\|Z_2\|_{L^\infty}\|\nabla Z\|_{L^\infty}+\|\nabla Z_2\|_{L^\infty}\big)\|Z_{1,j}\|^2_{L^2}+\|\nabla Z\|_{L^\infty}\|Z_{2,j}\|^2_{L^2}.$$

Plugging all the above inequalities in (2.41), we end up with (2.74). □

2.4.1.2 Cross estimates

Remember that for all $j\in\mathbb{Z}$, we have

$$\frac{d}{dt}\mathcal{I}_j+\frac{2^j}{2}\sum_{q=1}^{n-1}\varepsilon_q\int_{\mathbb{R}^d}|NM^q_\omega\widehat{Z}_j|^2\,d\xi\le\frac{2^{-j}\kappa_0}{2}\|NZ_j\|^2_{L^2}+C\|\dot{\Delta}_jG\|_{L^2}\|Z_j\|_{L^2}. \tag{2.76}$$

In our new regularity context, we have to add up $2^j\mathcal{I}_j$ if $j<0$ (resp. $2^{-j}\mathcal{I}_j$ if $j\ge 0$) to $\mathcal{L}_j$, then to multiply by $2^{j\frac{d}{2}}$ (resp. $2^{j(\frac{d}{2}+1)}$). This amounts to bounding $\|G\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$ and $\|G\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$. To this end, we have to refine the estimates (2.42), (2.43) and (2.44) taking our structure assumption (2.24) into account.

As a first, we see that (A.2) and Proposition 2.5.1 ensure that

$$\|G_3\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\|G_3\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\|G_3\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \tag{2.77}$$

Next, we have, thanks to Propositions A.2.4 and 2.5.1,

$$\|G_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\|G_2\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\|G_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\lesssim\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \tag{2.78}$$

In order to improve the estimate for G_1, we use that $\bar{A}_0^{-1}G_1$ is the sum for $j=1$ to d of

$$\begin{pmatrix}((\widetilde{A}^0_{1,1}(V))^{-1}\widetilde{A}^k_{1,1}(V)-(\bar{A}^0_{1,1})^{-1}\bar{A}^k_{1,1})\partial_kZ_1+\big((\widetilde{A}^0_{1,1}(V))^{-1}\widetilde{A}^k_{1,2}(V)-(\bar{A}^0_{1,1})^{-1}\bar{A}^k_{1,2}\big)\partial_kZ_2\\ ((\widetilde{A}^0_{2,2}(V))^{-1}\widetilde{A}^k_{2,1}(V)-(\bar{A}^0_{2,2})^{-1}\bar{A}^k_{2,1})\partial_kZ_1+\big((\widetilde{A}^0_{2,2}(V))^{-1}\widetilde{A}^k_{2,2}(V)-(\bar{A}^0_{2,2})^{-1}\bar{A}^k_{2,2}\big)\partial_kZ_2\end{pmatrix}.$$

Hence, owing to (2.24), we just have

$$\|G_1\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\lesssim\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Together with (2.77) and (2.78), we can conclude that

$$\|G\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|G\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}. \tag{2.79}$$

2.4.1.3 Provisional assessment

Let $\mathcal{L}' \triangleq \sum_{j<0} 2^{j\frac{d}{2}}\sqrt{\mathcal{L}_j} + \sum_{q\geq 0} 2^{j(\frac{d}{2}+1)}\sqrt{\mathcal{L}_j}$. Putting together Inequalities (2.73), (2.74), (2.76) and (2.79), using Lemma A.1.1 and discarding the redundant terms, we end up with

$$\begin{aligned}
&\mathcal{L}'(t) + \kappa_0 \int_0^t \big(\|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1}} + \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big) \leq \mathcal{L}'(0) \\
&+C\int_0^t \big(\|(W,\nabla Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)\mathcal{L}' + C\int_0^t \big(\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)\cdot
\end{aligned} \tag{2.80}$$

In order to close the estimates, we need to exhibit the L^1-in-time integrability of W and ∇Z_2 in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$ and the L^2-in-time integrability of Z_2 in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$.

2.4.1.4 Bounds for the damped mode

With the notations we used to prove (2.53), remember that

$$\frac{1}{2}\frac{d}{dt}\|W_j\|^2_{L^2_{\bar{A}^0_{2,2}}} + \kappa_0\|W_j\|^2_{L^2} \leq \big(\|\dot{\Delta}_j h_1\|_{L^2} + C\|\dot{\Delta}_j h_2\|_{L^2} + C\|\dot{\Delta}_j h_3\|_{L^2}\big)\|W_j\|_{L^2}. \tag{2.81}$$

From Lemma 2.4.1, Propositions A.2.4, A.2.5 and 2.5.1 and, since

$$\partial_t Q(Z) = D_{Z_1}Q(Z)\partial_t Z_1 + D_{Z_2}Q(Z)\partial_t Z_2,$$

we readily get

$$\begin{aligned}
\|h_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|h_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} &\lesssim \|h_1\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|W\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}, \\
\|h_3\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|h_3\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} &\lesssim \|h_3\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|W\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.
\end{aligned}$$

For bounding h_2, we need to refine the decomposition we did in the previous section. More precisely, we now write that for all $k \in \{1,\cdots,d\}$,

$$\begin{aligned}
\partial_t(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2) &= D_{V_1}A^k_{2,1}(V)\partial_t Z_1\partial_k Z_1 + D_{V_2}A^k_{2,1}(V)\partial_t Z_2\partial_k Z_1 \\
&\quad + A^k_{2,1}(\bar{V})\partial_t\partial_k Z_1 + \big(A^k_{2,1}(V) - A^k_{2,1}(\bar{V})\big)\partial_t\partial_k Z_1 \\
&\quad + D_V A^k_{2,2}(V)\partial_t Z\partial_k Z_2 + A^k_{2,2}(\bar{V})\partial_t\partial_k Z_2 + \big(A^k_{2,2}(V) - A^k_{2,2}(\bar{V})\big)\partial_t\partial_k Z_2.
\end{aligned}$$

Since $A^k_{2,1}$ is linear with respect to V_2, we get after using (2.72) and Lemma 2.4.1 that

$$\|\dot\Delta_j h_2\|_{L^2} \lesssim \|\nabla^2 Z_{2,j}\|_{L^2} + \|\nabla W_j\|_{L^2} + c_j 2^{-j\frac{d}{2}}\Big(\|\nabla Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|W\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|(Z,\nabla Z)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Big)\cdot$$

Hence, reverting to (2.81), using Lemma A.1.1 and keeping the notation (2.55), we end up for $\sigma \in [d/2, d/2+1]$ with

$$\mathcal{W}^\sigma(t) + \kappa_0 \int_0^t \|W\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}} \le \mathcal{W}^\sigma(0) + C\int_0^t \|(\nabla^2 Z_2, \nabla W)\|^\ell_{\dot{\mathbb{B}}^\sigma_{2,1}}$$
$$+C\int_0^t \Big(\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\nabla Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|W\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|(Z,\nabla Z)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\Big)\cdot \tag{2.82}$$

In order to compare W with Z_2, one can use the decomposition:

$$W - Z_2 = L_2{}^{-1}\sum_{k=1}^d \Big(\bar A^k_{2,1}\partial_k Z_1 + \big(\widetilde A^k_{2,1}(V) - \bar A^k_{2,1}\big)\partial_k Z_1 + \bar A^k_{2,2}\partial_k Z_2 + \big(A^k_{2,2}(V) - \bar A^k_{2,2}\big)\partial_k Z_2\Big) - L_2{}^{-1}Q(Z),$$

which implies that

$$\|W - Z_2\|^h_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \tag{2.83}$$

and that, for all $s \ge d/2$,

$$\|W - Z_2\|^\ell_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^s_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \tag{2.84}$$

2.4.1.5 Closure of the estimates

As in the previous section, if we set

$$\widetilde{\mathcal{L}}' \triangleq \mathcal{L}' + \varepsilon\mathcal{W}^{\frac{d}{2}+1} + \varepsilon'\mathcal{W}^{\frac{d}{2}} \quad \text{and} \quad \widetilde{\mathcal{H}}' \triangleq \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \varepsilon\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \varepsilon'\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$$

with suitable ε and ε', then putting together (2.80) and (2.82) yields

$$\widetilde{\mathcal{L}}'(t) + \kappa_0\int_0^t \widetilde{\mathcal{H}}' \le \widetilde{\mathcal{L}}'(0) + C\int_0^t \big(\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|W\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)\mathcal{L}'$$
$$+ C\int_0^t \big(\|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)\cdot \tag{2.85}$$

Note that we have

$$\big(\|Z_2\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)^2 \lesssim \big(\|Z_2\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)^2 \lesssim \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\mathcal{L}'$$

and

$$\begin{aligned}\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z_2\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} &\lesssim \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \\ &\lesssim \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1}} \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \big(\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)^2 \\ &\lesssim \big(\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)\mathcal{L}'. \end{aligned} \tag{2.86}$$

Hence, using also (2.83), and (2.84) with $s = d/2 + 1$, we see that (2.85) becomes just

$$\widetilde{\mathcal{L}}'(t) + \kappa_0 \int_0^t \widetilde{\mathcal{H}}' \leq \widetilde{\mathcal{L}}'(0) + C \int_0^t \widetilde{\mathcal{H}}'\mathcal{L}' + \int_0^t \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \big(\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)\cdot$$

To handle the last integral, let us write that, by virtue of (2.84) with $s = d/2$, we have

$$(\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}})^2 \lesssim \big(\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)^2 + \|Z_2\|^4_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \big(\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)^2 + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_1\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\nabla Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

The last three terms of the right-hand side may be bounded (owing to (2.86) and to (2.27)) by $\widetilde{\mathcal{H}}'\mathcal{L}'$, and we have

$$\big(\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\big)^2 \lesssim \widetilde{\mathcal{H}}'\widetilde{\mathcal{L}}'$$

$$\|Z_2\|^4_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim (\mathcal{L}')^2 \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Finally, we have

$$\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

Hence there exists a constant C (that may depend on ε and ε' but not on the solution) such that for all $t \in [0, T]$, we have

$$\widetilde{\mathcal{L}}'(t) + \kappa_0 \int_0^t \widetilde{\mathcal{H}}' \leq \widetilde{\mathcal{L}}'(0) + C \int_0^t \widetilde{\mathcal{H}}'\widetilde{\mathcal{L}}' + C \int_0^t (\mathcal{L}' + (\mathcal{L}')^2) \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}.$$

Then, one can conclude exactly has in the previous section that if $\widetilde{\mathcal{L}}'(0)$ (or, equivalently, $\mathcal{Z}'_0$) is small enough, then $\widetilde{\mathcal{L}}'$ is a Lyapunov functional such that for some (new) positive real numbers κ_0 and C,

$$\widetilde{\mathcal{L}}'(t) + \kappa_0 \int_0^t \widetilde{\mathcal{H}}' \leq \widetilde{\mathcal{L}}'(0) + C\widetilde{\mathcal{L}}'(0) \int_0^t \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \tag{2.87}$$

Furthermore, (2.84) with $s = d/2$ ensures that

$$\begin{aligned}\|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} &\lesssim \|W\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|\nabla Z\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|Z_2\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \|\nabla Z\|_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \\ &\quad + \|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \|\nabla Z_2\|_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|Z_2\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \|Z_2\|^h_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} + \|Z_2\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}.\end{aligned}$$

Inequality (2.87) combined with (2.27) and an obvious interpolation inequality thus yields

$$\|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} \lesssim \mathcal{Z}'(0). \tag{2.88}$$

Similarly, using again (2.84) but with $s = d/2+1$, we see that

$$\|Z_2\|^{\ell}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} \lesssim \|W\|^{\ell}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|\nabla Z\|^{\ell}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z_2\|_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|\nabla Z\|_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}$$
$$+\|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|Z_2\|^{h}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|Z_2\|^{\ell}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}.$$

In light of (2.27), the last term may be absorbed by the left-hand side and all the other terms may be bounded either through (2.87) or through (2.88).

From this point, the rest of the proof of this theorem essentially follows the lines of the previous section. □

2.4.2 The isentropic compressible Euler System with damping

We consider

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla P + \lambda\rho u = 0, \end{cases} \tag{2.89}$$

with $\lambda > 0$ and where P is a (smooth) pressure law satisfying[1]

$$P'(\rho) > 0 \quad \text{for} \quad \rho \quad \text{close to} \quad 1 \quad \text{and} \quad P'(1) = 1. \tag{2.90}$$

Considering the new unknown $n(\rho) = \displaystyle\int_1^\rho \frac{P'(s)}{s}\,ds$, we can rewrite (2.89) under the form

$$\begin{cases} \partial_t n + u\cdot\nabla n + \operatorname{div} u + G(n)\operatorname{div} u = 0, \\ \partial_t u + u\cdot\nabla u + \nabla n + \lambda u = 0, \end{cases} \tag{2.91}$$

where $G(n)$ is defined by the relation[2] $G(n(\rho)) = P'(\rho) - 1$.

In order to state our global existence for (2.91), we need to introduce the following notations:

$$z^{\ell,\lambda} \triangleq \sum_{2^j\le\lambda}\dot{\Delta}_j z, \qquad z^{h,\lambda} \triangleq \sum_{2^j>\lambda}\dot{\Delta}_j z,$$
$$\|z\|^{\ell,\lambda}_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{2^j\le\lambda} 2^{js}\|\dot{\Delta}_j z\|_{L^2} \quad \text{and} \quad \|z\|^{h,\lambda}_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{2^j>\lambda} 2^{js}\|\dot{\Delta}_j z\|_{L^2}.$$

Theorem 2.4.1. *Let (n_0, u_0) be in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ and $\lambda > 0$. Then, there exist two positive constants c and C depending only on G and on d, such that if*

$$\|(n_0,u_0)\|^{\ell,\lambda}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \lambda^{-1}\|(n_0,u_0)\|^{h,\lambda}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \le c,$$

then System (2.91) supplemented with initial data (n_0, u_0) admits a unique global-in-time solution

[1] For simplicity we assume that the reference density is 1 so that the steady state is $\bar{V} = (1, 0)$.
[2] Observe that $\rho \mapsto n(\rho)$ is a smooth diffeomorphism from a neighborhood of 1 to a neighborhood of 0.

(n,u) in the space defined by

$$(n,u)\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}\cap\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}),\quad (n^{h,\lambda},u^{h,\lambda})\in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}),\quad n^{\ell,\lambda}\in L^1(\mathbb{R}^+,\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1}),$$
$$u^{\ell,\lambda}\in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}),\quad u\in L^2(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})\quad \textit{and}\quad \nabla n+\lambda u\in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}).$$

Moreover we have the following a priori estimate:

$$\mathcal{Z}_\lambda(t)\lesssim \|(n_0,u_0)\|^{\ell,\lambda}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\lambda^{-1}\|(n_0,u_0)\|^{h,\lambda}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\quad \textit{for all } t\geq 0 \tag{2.92}$$

where

$$\mathcal{Z}_\lambda(T)\triangleq \|(n,u)\|^{\ell,\lambda}_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}+\lambda^{-1}\|(n,u)\|^{h,\lambda}_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}$$
$$+\lambda^{-1}\|n\|^{\ell,\lambda}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+2}_{2,1})}+\|(n,u)\|^{h,\lambda}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|u\|^{\ell,\lambda}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\lambda^{1/2}\|u\|^{\ell,\lambda}_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}+\|\nabla n+\lambda u\|^{\ell,\lambda}_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}.$$

If furthermore, (n_0,u_0) belongs to $\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ for some $\sigma_1\in]-d/2,d/2]$, then the solution (n,u) satisfies (2.23) and the decay estimates mentioned at the end of Theorem 2.2.3 hold true.

Proof. Performing the rescaling

$$(n,u)(t,x)\triangleq(\widetilde{n},\widetilde{u})(\lambda t,\lambda x)$$

reduces the proof to $\lambda=1$ (and the inverse scaling will eventually give the desired dependency with respect to λ in the above statement). Then, the whole result is a corollary of Theorem 2.2.3 provided System (2.91) satisfies the structural assumption (2.24) at 0. Indeed, one can take as a symmetrizer the matrix $\begin{pmatrix}(1+G(n))^{-1} & 0\\ 0 & I_d\end{pmatrix}$ where the first diagonal block is of size 1×1 and the second one, of size $d\times d$. The blocks of type $A^k_{1,1}$ and $A^k_{2,1}$ depend only (and linearly) on u, which is indeed the damped component. Finally, the damped mode (in the case $\lambda=1$) is $W=u+\nabla n+u\cdot\nabla u$. Now, by virtue of (2.92),

$$\|W-(u+\nabla n)\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\lesssim\|u\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|\nabla u\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\lesssim\big(\|(n_0,u_0)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}+\|(n_0,u_0)\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big)^2,$$

hence $u+\nabla n$ satisfies the same estimates as W, which completes the proof. □

2.5 Tool Box

Here we gather a few technical results that have been used repeatedly in this chapter.

The first one is the justification that one may choose arbitrarily small positive parameters $\varepsilon_1,\cdots,\varepsilon_{n-1}$ so that, whenever $\widehat{Z}$ satisfies (2.11), Inequality (2.15) holds true. The proof just consists in bounding suitably the terms of the right-hand side of (2.14).

- Terms $\mathcal{I}^1_q\triangleq\big(NM^{q-1}_\omega N\widehat{Z}\cdot NM^q_\omega\widehat{Z}\big)$ with $q\in\{1,\cdots,n-1\}$.

  Since matrices M_ω are bounded on $\mathbb{S}^{d-1}$, we may write

$$\varepsilon_q|\mathcal{I}^1_q|\lesssim\varepsilon_q|N\widehat{Z}||NM^q_\omega\widehat{Z}|\leq\frac{\varepsilon_0}{4n\rho}|N\widehat{Z}|^2+C\rho\frac{\varepsilon_q^2}{\varepsilon_0}|NM^q_\omega\widehat{Z}|^2.$$

- Terms $\varepsilon_q\big(NM_\omega^{q-1}\widehat{Z}\cdot NM_\omega^q N\widehat{Z}\big)$ with $q\in\{2,\cdots,n-1\}$ may be bounded similarly.
- We have $\varepsilon_1|\big(N\widehat{Z}\cdot NM_\omega N\widehat{Z}\big)|\le C\varepsilon_1|N\widehat{Z}|^2$.
- Terms $\mathcal{I}_q^2\triangleq\rho\big(NM_\omega^{q-1}\widehat{Z}\cdot NM_\omega^{q+1}\widehat{Z}\big)$ with $q\in\{1,\cdots,n-2\}$. We have

$$\begin{aligned}\varepsilon_q|\mathcal{I}_q^2|&\le\varepsilon_{q-1}\rho|NM_\omega^{q-1}\widehat{Z}|\,|NM_\omega^{q+1}\widehat{Z}|\\&\le\frac{1}{4}\rho|NM_\omega^{q-1}\widehat{Z}|^2+C\rho\frac{\varepsilon_q^2}{\varepsilon_{q-1}}|NM_\omega^{q+1}\widehat{Z}|^2.\end{aligned}$$

  As we want the two terms to be absorbed by the left-hand side of (2.14), we take ε_q so that

$$4\varepsilon_q^2\le\varepsilon_{q-1}\varepsilon_{q+1}. \tag{2.93}$$

  We keep in mind that ε_0 has been set to $(2\pi)^{-d}\kappa_0$ (but can be taken smaller if needed).

- Term $\mathcal{I}_{n-1}^2\triangleq\varepsilon_{n-1}\rho\big(NM_\omega^{n-2}\widehat{Z}\cdot NM_\omega^n\widehat{Z}\big)$. We start with the observation that, owing to Cayley-Hamilton theorem, there exist coefficients c_ω^j (that are uniformly bounded on $\mathbb{S}^{d-1}$, such that

$$M_\omega^n=\sum_{q=0}^{n-1}c_\omega^q M_\omega^q.$$

  Consequently, one may write

$$\begin{aligned}|\mathcal{I}_{n-1}^2|&\lesssim\varepsilon_{n-1}\rho\sum_{q=0}^{n-1}|NM_\omega^{n-2}\widehat{Z}||NM_\omega^q\widehat{Z}|\\&\le\frac{C\varepsilon_{n-1}^2\rho}{\varepsilon_j}|NM_\omega^q\widehat{Z}|^2+\frac{\varepsilon_j\rho}{4}|NM_\omega^q\widehat{Z}|^2.\end{aligned}$$

  Therefore one needs to assume in addition that

$$4C\varepsilon_{n-1}^2\le\varepsilon_q\varepsilon_{n-2},\qquad q=0,\cdots,n-1. \tag{2.94}$$

Clearly, one is done if it is possible to find $\varepsilon_1,\cdots,\varepsilon_{n-1}$ fulfilling (2.93) and (2.94). One can take for instance $\varepsilon_q=\varepsilon^{m_q}$ with ε small enough and $m_1,\cdots,m_{n-1}$ satisfying for some $\delta>0$ (that can be taken arbitrarily small):

$$m_q\ge\frac{m_{q-1}+m_{q+1}}{2}+\delta\quad\text{and}\quad m_{n-1}\ge\frac{m_q+m_{n-2}}{2}+\delta,\quad q=1,\cdots,n-2.$$

We used the following result to estimate the remainder of the dissipative term.

Proposition 2.5.1. *Let $\bar{V}\in\mathcal{M}$ and $Z\triangleq V-\bar{V}$. Define $r(Z)\triangleq\widetilde{H}(\bar{V}+Z)+LZ$, $L\triangleq-D_V\widetilde{H}(\bar{V})$ and $Z_2\triangleq(I_d-\mathcal{P})Z$, and assume that $r(Z_1,0)=0$ for Z_1 in a neighborhood of 0. Then, provided $\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$ is sufficiently small, the following inequalities hold true:*

$$\|r(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\quad\textit{for}\quad\sigma\in]-d/2,d/2] \tag{2.95}$$

and, for $\sigma>d/2$,

$$\|r(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\lesssim\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}+\|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}. \tag{2.96}$$

Furthermore, if both Z^1 and Z^2 are sufficiently small in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$ then we have the following estimate for $\widetilde{Z} := Z^1 - Z^2$:

$$\|r(Z^1) - r(Z^2)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|Z^1\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\widetilde{Z}_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} + \|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \|Z^2_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}, \qquad \sigma \in]0, d/2]. \tag{2.97}$$

Finally, if r is at least quadratic with respect to Z_2 (that is $D^2_{V_i,V_j} r(0) = 0$ for $(i,j) \neq (2,2)$), then we have

$$\|r(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \qquad \text{for } \sigma \in]-d/2, d/2] \tag{2.98}$$

$$\text{and} \quad \|r(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \|Z_2\|^2_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \qquad \text{for } \sigma > d/2. \tag{2.99}$$

Proof. Since $r(Z_1, 0) = 0$ for Z_1 close to 0, the mean value formula gives

$$r(Z_1, Z_2) = \int_0^1 D_{Z_2} r(Z_1, \tau Z_2) \cdot Z_2 \, d\tau.$$

Furthermore, we have $Dr(0) = 0$ and thus $D_{Z_2} r(0) = 0$. Hence there exists a smooth function F defined near 0 and such that $D_{Z_2} r(Z) = F(Z) \cdot Z$. Consequently, there exists a smooth function G vanishing at 0, and such that

$$r(Z_1, Z_2) = G(Z) \cdot Z_2.$$

Granted with the above decomposition, the first two inequalities readily follow from product laws and composition lemma.

To prove (2.97), we use the decomposition

$$r(Z^1) - r(Z^2) = G(Z^1) \cdot (Z^2_2 - Z^1_2) + \big(G(Z^2) - G(Z^1)\big) \cdot Z^2_2,$$

then products laws and composition lemma, combined with Corollary 2.66 from [3].

Finally, if r is quadratic with respect to Z_2 then there exists a quadratic form $\widetilde{Q}$ and a smooth function F such that $r(Z) = \widetilde{Q}(Z_2) F(Z)$, whence

$$r(Z) = F(0)\widetilde{Q}(Z_2) + G(Z)\widetilde{Q}(Z_2) \quad \text{with} \quad G(Z) \triangleq F(Z) - F(0).$$

In the case $\sigma \in]-d/2, d/2]$, thanks to composition and products law lemmas we can write

$$\|r(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|\widetilde{Q}(Z_2)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} (1 + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}) \lesssim \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}}$$

while, if $\sigma > d/2$,

$$\|r(Z)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \lesssim \|\widetilde{Q}(Z_2)\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} (1 + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}) + \|Z\|_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \|\widetilde{Q}(Z_2)\|_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}},$$

whence the last inequality. □

Chapitre 3

Global existence for partially dissipative hyperbolic systems in the L^p framework, and relaxation limit

This chapter is partially based on the ongoing work [32].

Introduction

In this chapter, we are concerned with multi-dimensional first order n-component systems in $\mathbb{R}^d$ of the type:

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{d} A^k(V)\frac{\partial V}{\partial x_k} + \frac{LV}{\varepsilon} = 0 \tag{3.1}$$

where ε stands for the (positive) relaxation parameter, the (smooth) symmetric matrices valued functions A^k $(k = 0, \cdots, d)$ and vector valued function H are defined on some open subset $\mathcal{O}_V$ of $\mathbb{R}^n$ and the unknown $V = V(t, x)$ depends on the time variable $t \in \mathbb{R}_+$ and on the space variable $x \in \mathbb{R}^d$ $(d \geq 2)$. We assume that *partial* dissipation occurs, that is to say, the term LV acts only on a part of the solution. This can be written:

$$LV = \begin{pmatrix} 0 \\ L_2V_2 \end{pmatrix} \tag{3.2}$$

where $0 \in \mathbb{R}^{n_1}$, $L_2V_2 \in \mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$ and $n_1 + n_2 = n$.

The system under consideration here is a simplification of the one we studied in the previous chapter, the main modifications are

- We do not consider a remainder term, i.e. $r \equiv 0$.
- The time derivatives matrix satisfy $A^0(V) = I_d$.
- For every $1 \leq k \leq d$, the matrices $A^k(V)$ are symmetric and linear with respect to V.

Due to difficulties arising from the L^p framework, these assumptions are necessary to obtain a result with our current methods. Still, the class of systems we are investigating here encompasses

the compressible Euler system with damping and our aim here is to study its global well-posedness in a more general framework than in the previous chapter, and its relaxation limit.

We supplement System (3.1) with initial data $V_0 \in \mathcal{O}_V$ at time $t = 0$. We are concerned with the existence of global strong solutions in the case where V_0 is close to some constant state $\bar{V}$ such that $H(\bar{V}) = 0$. As in the previous chapters, to guarantee the global existence, we assume that (3.1) satisfies the (SK) condition.

Here are the main features of this chapter:

- We extend the analysis of the second chapter to a functional framework where the low frequencies of the solutions are only bounded in some L^p type spaces with p potentially larger than 2. This enables us to pinpoint weaker smallness conditions for global well-posedness and to get more accurate information on the qualitative properties of the constructed solutions.
- As a corollary, our existence theorem provides for free bounds that are independent of the relaxation parameter, which permits us to justify the relaxation limit of the damped isentropic compressible Euler system and to exhibit explicit rates of (strong) convergence.

To have a better understanding of the succession of improvements concerning these systems, one must have in mind the following embeddings (where $s > 1 + d/2$ and $p \geq 2$):

$$H^s \hookrightarrow B^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{p}}_{p,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1} \hookrightarrow \mathcal{C}^1.$$

We recall that the case H^s with $s > 1 + d/2$ was investigated by Yong in [108] and that the non-homogeneous Besov spaces $B^{\frac{d}{2}+1}_{2,1}$ framework was studied by Kawashima and Xu in [102] (the other two functional settings being investigated in the first three chapter of this manuscript).

The idea of well-posedness with one of the two frequencies regimes in L^p spaces comes from the study of the compressible Navier-Stokes system in [23, 27] where the authors proved a global well-posedness result with the high frequencies in homogeneous Besov spaces based on L^p space with $p \geq 2$ while the low frequencies stay at the energy level. Essentially, this is possible because the high frequencies' eigenvalues are asymptotically real. In our case, the situation is reversed, as only the low frequencies' eigenvalues are asymptotically real, and therefore we can only employ L^p-type argument in low frequencies.

To handle the low frequencies, as done in [30], we will exhibit a damped mode (which is analog to the effective velocity introduced by Hoff and Haspot) with better decay properties than the whole solution and use it as a new unknown. It allows to decouple the system into a purely damped equation and a parabolic one. Then, we establish a link between the (SK) condition and the ellipticity of a certain operator so as to be able to obtain a priori estimates in the L^p framework. As for the high frequencies, we shall use Beauchard and Zuazua's approach developed in [31], and take advantage of new commutator and product laws to handle the low frequencies part of the nonlinearities that do not belong to L^2.

Once we obtain our existence theorem, we move on to the study of the relaxation limit for the compressible Euler with damping to the porous media equation.

The study of relaxation problems associated to systems of conservation laws can be tracked back to the work of Chen et al [26]. More recently, Giovangigli and Yong in [48, 49] studied

a relaxation-limit problem arising in the dynamics of perfect gases out of thermodynamical-equilibrium. At a mathematical level, they dealt with dissipative and diffusive systems of conservation laws of the form (3.1) with an additional diffusive term coming from the Chapman-Enskog expansion. Roughly speaking, assuming the existence of an entropy compatible with the diffusion and dissipation operators, they proved the local existence of solutions for the Cauchy problem as well as error estimates between solutions at fixed ε and the solution of the limit system. We mention that as their results hold only locally in time, the (SK) condition is not relevant in their work (the dissipative term being responsible for the large-time behaviour of the solution).

In [62], Junca and Rascle were able to justify the relaxation process from the damped Euler equation to the porous media equation in the one-dimensional setting for large global-in-time BV solution and to provide an explicit convergence rate. Their approach is based on stream function technique which is related to the Lagrangian mass coordinates. Then Coulombel and Goudon in [29] and Coulombel and Lin [70] proved the strong convergence locally-in-space in the multi-dimensional setting in Sobolev spaces with regularity index $s > \frac{d}{2} + 1$. One of their main ingredient, that will be a key tool for us, is the use of a suitable rescaling in the slow time variable. One must also mention the result of Xu and Wang [107] who justified the relaxation limit in non-homonegenous critical Besov spaces.

More recently, following the approach of [62], Peng, Li and Zhao in [69] justified the convergence of partially dissipative hyperbolic systems to parabolic systems globally-in-time in one space dimension and derived a convergence rate of the relaxation process. Using similar techniques, Liang and Shuai in [109] obtained a convergence rate of the relaxation process in the periodic setting (in the torus $\mathbb{T}^3$). The main reason why they cannot obtain a result in the whole space is that they need to use the Poincaré inequality to control some mixed partial derivative terms that do not appear in the one dimensional case.

Here, we focus on the strong smooth solutions and aim at deriving an explicit convergence rate. The main part of the job is actually done in our existence result where we obtain uniform bounds (with respect to ε) on the solution. With theses bounds it will be possible to study the relaxation process as $\varepsilon \to 0$ which, in reality, is not so intuitive. Indeed, at first sight, when the damping parameter increases, we expect the dissipation phenomenon to be more and more dominant. However, the so-called *overdamping* phenomenon occurs : the decay rates do not always increase when the damping coefficient increases but it behaves more like $\min(\varepsilon, 1/\varepsilon)$. This can be seen from a spectral analysis of the Euler system for example (see also [110] for the case of the damped harmonic oscillator. In the Fourier side, the linearized Euler system with relaxation parameter equation to ε has the following matrix:

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \dfrac{1}{\varepsilon} \end{pmatrix}.$$

Then, it is straightforward to see that

- the discriminant of the characteristic polynomial being equal to $\dfrac{1}{\varepsilon^2} - 4\xi^2$, the threshold between low and high frequencies should be at the level of $\dfrac{1}{2\varepsilon}$.
- in low frequencies (i.e. $|\xi| \ll \varepsilon^{-1}$), this matrix has two real eigenvalue asymptotically equal to $\dfrac{1}{\varepsilon}$ and $\varepsilon\xi^2$ as ξ tends to 0;

- in high frequencies (i.e. $|\xi| \gg \varepsilon^{-1}$), it has two complex conjugated eigenvalues whose real part are asymptotically equal to $\dfrac{1}{2\varepsilon}$

In the previous approaches, all the frequencies were considered together and, therefore, the dissipative aspect of the low frequencies was not fully used because the overall behavior that emerged was always the worst one. In the approach that we offer here, we use the uniform bounds on the damped mode (which is directly related to the eigenvalue asymptotically equal to $1/\varepsilon$ in low frequencies) and the dependence of the threshold between high and low frequencies on ε. Indeed, if for $\varepsilon < 1$ we do the exact same computations but with the threshold independent of ε, then we would end up with a coefficient ε in front of most of the L^1-in-time norms of the solution, which will annihilate all hope of studying the relaxation phenomenon. More precisely, our threshold $J_\varepsilon = \lfloor -\log_2 \varepsilon \rfloor + k_p$, corresponds[1] to the place where the 0-order terms and the 1-order terms have the same weight (parameter included). As $\varepsilon \to 0$, heuristically, it implies that the low frequencies *invade* the whole space of frequencies. This can be, in a way, justified by seeing that the behavior of the limit system is purely parabolic just like the initial system in the low frequency regime.

Moreover, it turns out that the damped mode is a key quantity to justify the relaxation limit. Indeed, performing a slow-time variable and diffusive rescaling as in [29, 70], the damped mode almost directly provides the suitable decay when estimating the difference between the Euler System (3.15) and the limit system (the Porous media equation). This allows us to prove the strong convergence (globally-in-space) and to derive a convergence rate of the relaxation process, which, to the best of our knowledge, is new.

This chapter is arranged as follows. In the first section, we specify the structure of the class of partially dissipative hyperbolic systems we aim at considering, introduce the damped mode that will be the key to the study of the low frequencies and explain the construction of a Lyapunov functional that will be essential to study the high frequencies. In the second section, we state the main results of this chapter. Section 3.3 is devoted to the proof of a global existence result for general partially dissipative systems satisfying the Shizuta-Kawashima condition. In section 3.4, we justify rigorously the relaxation limit of the compressible Euler system with damping and derive an explicit convergence rate of the process. Some technical results are proved or recalled in Appendix.

3.1 Hypotheses and method

Let $\bar{V}$ be a constant state such that $L\bar{V} = 0$. Setting $Z \triangleq V - \bar{V}$, we get

$$\partial_t Z + \sum_{k=1}^{d} A^k(V)\partial_k Z + \frac{LZ}{\varepsilon} = 0\cdot \tag{3.3}$$

As in the previous section, we assume that the system is *partially dissipative* in the following meaning:

(i) The whole space $\mathbb{R}^n$ may be decomposed into $\mathbb{R}^n = \mathcal{M} \bigoplus \mathcal{M}^\perp$ where

$$\mathcal{M} = \big\{\phi \in \mathbb{R}^n,\ \langle \phi, H(V)\rangle = 0 \text{ for all } V \text{ close to } \bar{V}\big\}\cdot$$

[1] The constant k_p is due to the low frequencies analysis, we need the threshold to be small enough in the computations

Hence, denoting by $\mathcal{P}$ the orthogonal projection on $\mathcal{M}$, we may write

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad \text{and} \quad LV = \begin{pmatrix} 0 \\ L_2 V_2 \end{pmatrix} \tag{3.4}$$

where $V_1 = \mathcal{P}V \in \mathbb{R}^{n_1}$, $V_2 = (I - \mathcal{P})V \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$.

(ii) The linear map L is an isomorphism on $\mathcal{M}^\perp$ such that for some $c > 0$,

$$\forall \eta \in \mathbb{R}^n,\ (L\eta|\eta) \geq c|L\eta|^2. \tag{3.5}$$

Introducing the decomposition:

$$A^k = \begin{pmatrix} A^k_{1,1} & A^k_{1,2} \\ A^j_{2,1} & A^k_{2,2} \end{pmatrix}$$

System (3.1) may thus be rewritten as follows:

$$\begin{cases} \partial_t Z_1 + \displaystyle\sum_{k=1}^d \Big(A^k_{1,1}(V)\partial_k Z_1 + A^k_{1,2}(V)\partial_k Z_2 \Big) = 0, \\ \partial_t Z_2 + \displaystyle\sum_{k=1}^d \Big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2 \Big) + \dfrac{L_2 Z_2}{\varepsilon} = 0. \end{cases} \tag{3.6}$$

As we shall see in the last section, the compressible Euler equations with damping, rewritten in suitable variables, satisfy the above assumptions about any constant state with positive density and null velocity.

3.1.1 The damped mode

Another important ingredient of our analysis is the use of a 'damped mode' that may be seen as an eigenmode corresponding to the part of the solution that experiences maximal dissipation in low frequencies. In the case $\varepsilon = 1$, it is defined as follows :

$$W \triangleq -L_2^{-1}\partial_t Z_2 = Z_2 + \sum_{k=1}^d L_2^{-1}\big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2\big)\cdot \tag{3.7}$$

Hence W satisfies

$$\partial_t W + L_2 W = L_2^{-1} \sum_{k=1}^d \partial_t\big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2\big)\cdot \tag{3.8}$$

On the left-hand side, Property (3.5) ensures maximal dissipation on W. As the right-hand side of (3.8) contains only at least quadratic terms, or linear terms with one derivative, it can be expected to be negligible in low frequencies. Furthermore, the second equality of (3.7) reveals that W is comparable to Z_2 in low frequencies and if Z is small enough. This will ensure better integrability for Z_2 than for the whole solution Z. Finally, expressing Z_2 in terms of W in the

first equation of (3.6), we obtain the following equation for Z_1:

$$\partial_t Z_1 + \sum_{k=1}^{d} \bar{A}^k_{1,1}\partial_k Z_1 - \sum_{k=1}^{d}\sum_{\ell=1}^{d} \bar{A}^k_{1,2} L_2^{-1} \bar{A}^\ell_{2,1}\partial_k\partial_\ell Z_1 = f_1 + f_2 + f_3 + f_4 + f_5, \tag{3.9}$$

where $\bar{A}^k_{p,m} \triangleq A^k_{p,m}(\bar{V})$, and

$$\begin{aligned}
f_1 &= \sum_{k=1}^{d}\sum_{\ell=1}^{d} A^k_{1,2}(V)\partial_k\big(L_2^{-1}A^\ell_{2,2}(V)\partial_\ell Z_2\big),\\
f_2 &= -\sum_{k=1}^{d} A^k_{1,2}(V)\partial_k W,\\
f_3 &= \sum_{k=1}^{d}\sum_{\ell=1}^{d} A^k_{1,2}(V)\partial_k(L_2^{-1}(A^\ell_{2,1}(V) - A^\ell_{2,1}(\bar{V}))\partial_\ell Z_1),\\
f_4 &= \sum_{k=1}^{d}\sum_{\ell=1}^{d}(A^k_{1,2}(V) - A^k_{1,2}(\bar{V}))L_2^{-1}A^\ell_{2,1}(\bar{V})\partial_k\partial_\ell Z_1,\\
f_5 &= \sum_{k=1}^{d}(A^k_{1,1}(\bar{V}) - A^k_{1,1}(V))\partial_k Z_1.
\end{aligned}$$

In order to ensure the parabolic behavior of the unknown Z_1 in the low frequencies regime, we will need the following assumption:

$$\forall\ k \in \{1,\cdots,d\},\ \bar{A}^k_{1,1} = 0 \quad \text{and} \quad \text{operator } -\sum_{k=1}^{d}\sum_{\ell=1}^{d} \bar{A}^k_{1,2}L_2^{-1}\bar{A}^\ell_{2,1}\partial_k\partial_\ell \text{ is strongly elliptic.}$$

Remark 3.1.1. *In the context of fluid mechanics, the first condition is satisfied up to a Galilean change of frame. Indeed Z_1 corresponds to the density (or, more generally, to conserved quantities like e.g. the entropy) and $\sum_{k=1}^{d} A^k_{1,1}(V)\partial_k Z_1$ represents the corresponding transport term by the velocity field.*

Further remark that even if the matrices $\bar{A}^k_{1,1}$ are nonzero, the above sum has no contribution in the energy arguments, and can be treated in a purely L^2 framework (cf previous chapter).

It turns out that we have the following relation between the (SK) condition and the strong ellipticity condition.

Lemma 3.1.1. *Assume that $\forall\ k \in \{1,\cdots,d\}$, $\bar{A}^k_{1,1} = 0$. Then the following assertions are equivalent:*

- *System (3.1) satisfies the condition (SK) in $\bar{V}$;*
- *the operator $\mathcal{A} \triangleq -\sum_{k=1}^{d}\sum_{\ell=1}^{d} \bar{A}^k_{1,2}L_2^{-1}\bar{A}^\ell_{2,1}\partial_k\partial_\ell$ is strongly elliptic.*

Moreover, if one of the assertions is satisfied then if $Supp(\mathcal{F}Z_1) \subset \{\xi \in \mathbb{R}^d : R_1\lambda \le |\xi| \le R_2\lambda\}$ for some $0 < R_1 < R_2$ then, for all $p \in [2,\infty[$, there exists $c = c(p,d,R_1,R_2) > 0$ such that

$$\int_{\mathbb{R}^d} \langle \sum_{k=1}^{d}\sum_{\ell=1}^{d} \bar{A}^k_{1,2}L_2^{-1}\bar{A}^\ell_{2,1}\partial_k\partial_\ell Z_1, |Z_1|^{p-2}Z_1\rangle \ge c\lambda^2\|Z_1\|^p_{L^p}. \tag{3.10}$$

Proof. The direct implication was proved in [83, 100, 108]. For the converse implication, let us still denote by L_2 the $n_2 \times n_2$ (invertible) matrix of L_2 and set

$$\mathcal{A}_{\ell,m}(\xi) \triangleq \sum_{k=1}^{d} \bar{A}^k_{\ell,m}\xi_k, \quad 1 \leq \ell, m \leq 2.$$

Our assumptions ensure that, on the one hand, for all $\xi \neq 0$, the matrix $\mathcal{A}_{1,2}(\xi)L_2^{-1}\mathcal{A}_{2,1}(\xi)$ is positive definite and that, on the other hand the matrix L_2 is positive definite (hence so does L_2^{-1}). This in particular implies that the ranks of $\mathcal{A}_{1,2}(\xi)$ and $\mathcal{A}_{2,1}(\xi)$ must be equal to n_1 and thus, so does the rank of $L_2\mathcal{A}_{2,1}(\xi)$. Now, the matrices of L and of $L\mathcal{A}(\xi)$ can be written by blocks as follows:

$$L = \begin{pmatrix} 0 & 0 \\ 0 & L_2 \end{pmatrix} \quad \text{and} \quad L\mathcal{A}(\xi) = \begin{pmatrix} 0 & 0 \\ L_2\mathcal{A}_{2,1}(\xi) & L_2\mathcal{A}_{2,2}(\xi) \end{pmatrix}.$$

Hence the rank of $\begin{pmatrix} L \\ L\mathcal{A}(\xi) \end{pmatrix}$ is $n_1 + n_2 = n$, and (SK) condition is thus satisfied and the equivalence is proved.

To prove Inequality (3.10), one has to include the ellipticity property of $\mathcal{A}$: there exists a constant $c > 0$ such that

$$\sum_{k=1}^{d}\sum_{\ell=1}^{d} \bar{A}^k_{1,2}L_2^{-1}\bar{A}^\ell_{2,1}\xi_k\xi_\ell \geq c|\xi|^2$$

in the proof of Proposition 1.7.1 (from the first Chapter) done in [34]. □

To obtain our results, we will need the following structural assumptions:

$$\text{For all } k \in \{1, \cdots, d\}, \quad \bar{A}^k_{1,1} = 0 \quad \text{and} \quad Z \mapsto A^k_{1,1}(\bar{V}+Z) \quad \text{is linear with respect to } Z_2. \tag{3.11}$$

$$\text{For all } k \in \{1, \cdots, d\}, \quad Z \mapsto A^k(\bar{V} + Z) \text{ is linear with respect to } Z. \tag{3.12}$$

Remark 3.1.2. *Compared to the structure assumption we used in [31] for Theorem 2.3, the consideration of W as an unknown replacing Z_2 in the low frequencies regime allows us to avoid additional structure conditions on $A^k_{2,1}$.*

Remark 3.1.3. *We believe the condition of linearity for $A^k(V)$ to be purely technical. It comes from the fact that, so far, we do not know how to handle the low frequencies part of more than quadratic nonlinearities when performing estimates in the high frequencies regime.*

3.2 Main results

In this chapter we use the notation defined at the beginning of section 1 of the first chapter. We assume throughout that the dimension is $d \geq 2$.

Let us first state our main global existence result for System (3.1), rewritten as (3.3), where

$$W_\varepsilon \triangleq \frac{Z_2}{\varepsilon} + \sum_{k=1}^{d} L_2^{-1}\big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2\big).$$

Theorem 3.2.1. *Let $d \geq 2$, $\varepsilon > 0$ and $p \in [2, \min(4, \frac{2d}{d-2})]$ and assume that the (SK) condition and (3.11)-(3.12) are satisfied. There exist $k_p \in \mathbb{Z}$ and $c_0 = c_0(p) > 0$ such that for $J_\varepsilon \triangleq \lfloor -\log_2 \varepsilon \rfloor + k_p$, if we assume that $Z_0^{\ell,J_\varepsilon} \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ and $Z_0^{h,J_\varepsilon} \in \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ with*

$$\|Z_0\|^{\ell,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

then System (3.3) admits a unique global solution Z in the space $E_p^{J_\varepsilon}$ defined by

$$\begin{aligned} &Z_1^{\ell,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}), \quad Z^{h,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}) \cap L^1(\mathbb{R}^+, \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}), \\ &Z_2^{\ell,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}), \quad W_\varepsilon \in L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \quad \text{and} \quad Z_2 \in L^2(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}). \end{aligned}$$

Moreover, we have the following a priori bound:

$$X_{p,\varepsilon}(t) \lesssim \|Z_0\|^{\ell,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \quad \text{for all } t \geq 0,$$

where

$$\begin{aligned} X_{p,\varepsilon}(t) \triangleq \|Z\|^{\ell,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon \|Z\|^{h,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \varepsilon \|Z_1\|^{\ell,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{1}{p}+2}_{p,1})} + \|Z\|^{h,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \\ + \|W_\varepsilon\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon^{-\frac{1}{2}} \|Z_2\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}. \end{aligned}$$

Remark 3.2.1. *As explained in the introduction, the particular choice of the threshold J_ε depending on ε allows us to obtain key bounds to treat the relaxation limit.*

As an application, consider the isentropic compressible Euler equation with damping:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P + \dfrac{1}{\varepsilon} \rho v = 0, \end{cases} \tag{3.13}$$

with $\varepsilon > 0$ and for a pressure law P satisfying

$$P(\rho) = A\rho^\gamma \quad \text{for } \gamma > 1 \quad \text{and} \quad A > 0. \tag{3.14}$$

Clearly, System (3.13) is not symmetric, hence Theorem 3.2.1 cannot be directly. In order to fall in the range of Theorem 3.2.1, differently from the previous chapters, we consider the new unknown function

$$c = \frac{2}{\gamma - 1} \sqrt{\frac{\partial P}{\partial \rho}} = \frac{(4\gamma A)^{\frac{1}{2}}}{\gamma - 1} \rho^{\frac{\gamma-1}{2}},$$

called the sound speed. The reason behind the consideration of a different unknown comes from the fact we need the condition (3.12) to be satisfied. Defining $\tilde{\gamma} = \dfrac{\gamma - 1}{2}$ and $\widetilde{c} = c - \bar{c}$ for a positive constant $\bar{c}$ related to $\bar{\rho}$ through the defintion of c, we can rewrite (3.13) under the form :

$$\begin{cases} \partial_t \widetilde{c} + v \cdot \nabla \widetilde{c} + \tilde{\gamma}(\widetilde{c} + \bar{c}) \operatorname{div} v = 0, \\ \partial_t v + v \cdot \nabla v + \tilde{\gamma}(\widetilde{c} + \bar{c}) \nabla \widetilde{c} + \dfrac{1}{\varepsilon} v = 0. \end{cases} \tag{3.15}$$

The above system is symmetric and satisfies both Conditions (SK) and (3.11). Therefore, Theorem 3.2.1 can be applied to (3.15) with

$$\check{W} = v\cdot\nabla v + \frac{v}{\varepsilon} + \tilde{\gamma}(\widetilde{c}+\bar{c})\nabla\widetilde{c} = v\cdot\nabla v + \frac{v}{\varepsilon} + \tilde{\gamma}c\nabla c$$

as damped mode. This gives the following theorem.

Theorem 3.2.2. *Fix a positive constant $\bar{c}$. Let $d\geq 2$, $\varepsilon>0$ and $2\leq p\leq \min(4,\frac{2d}{d-2})$. There exist $k_p\in\mathbb{Z}$ and $c_0=c_0(p,P)>0$ such that for $J_\varepsilon \triangleq \lfloor -\log_2\varepsilon\rfloor + k_p$, if we assume that $(c-\bar{c})^{\ell,J_\varepsilon}, v_0^{\ell,J_\varepsilon}\in\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ and $(c-\bar{c})^{h,J_\varepsilon}, v_0^{h,J_\varepsilon}\in\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ with*

$$\|(c-\bar{c},v_0)\|^{\ell,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon\,\|(c-\bar{c},v_0)\|^{h,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

then System (3.15)-(3.14) admits a unique global solution $(c-\bar{c},v)$ in the space $E_p^{J_\varepsilon}$ defined by

$$\begin{aligned}
&(c-\bar{c})^{\ell,J_\varepsilon}\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})\cap L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}),\quad (c-\bar{c})^{h,J_\varepsilon}\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})\cap L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}),\\
&v^{\ell,J_\varepsilon}\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})\cap L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}),\quad v^{h,J_\varepsilon}\in\mathcal{C}_b(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})\cap L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})\\
&\frac{v}{\varepsilon}+\tilde{\gamma}c\nabla c\in L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})\quad \text{and}\quad v\in L^2(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}).
\end{aligned}$$

Moreover, we have the following a priori bound:

$$X_{p,\varepsilon}(t)\lesssim \|(c-\bar{c},v_0)\|^{\ell,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon\,\|(c-\bar{c},v_0)\|^{h,J_\varepsilon}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\quad \text{for all } t\geq 0,$$

where

$$\begin{aligned}
X_{p,\varepsilon}(t)\triangleq{}& \|(c-\bar{c},v)\|^{\ell,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon\,\|(c-\bar{c},v)\|^{h,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \varepsilon\,\|c-\bar{c}\|^{\ell,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})} + \|v\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}\\
&+\|(c-\bar{c},v)\|^{h,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \left\|\frac{v}{\varepsilon}+\tilde{\gamma}c\nabla c\right\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon^{-\frac{1}{2}}\,\|v\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}.
\end{aligned}$$

Remark 3.2.2. *Strictly speaking, if we define the damped mode according to Theorem 3.2.1, then we find that for System (3.13), we have $\check{W}=v\cdot\nabla v+\frac{v}{\varepsilon}+\tilde{\gamma}c\nabla c$. However, the above estimate combined with product laws ensure that $\|v\cdot\nabla v\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\leq c_0$ hence is negligible compared to $\varepsilon^{-1}v$, and $W'=\frac{v}{\varepsilon}+\tilde{\gamma}c\nabla c$ is thus also a damped quantity. We shall see that this latter function is more adapted to study the relaxation limit.*

This theorem allows us to recover a solution $(\rho-\bar{\rho},v)$ from the initial system (3.13) but with weaker regularity properties than the one satisfied by $(c-\bar{c},v)$. This is described by Lemma 3.4.1. Still, the uniform estimates we obtained from Theorem 3.2.2 enable us to obtain the following result pertaining to the relaxation limit of the compressible Euler system.

Theorem 3.2.3. *Let $d\geq 2$, $\varepsilon>0$ and $2\leq p\leq\min(4,\frac{2d}{d-2})$. Let $\bar{\rho}$ be a positive constant density and $(\rho-\bar{\rho},v)$ be the solution of the Cauchy problem (3.13)-(3.14) with $(\rho_0-\bar{\rho})^\ell, v_0^\ell\in\dot{B}^{\frac{d}{p}}_{p,1}$ and $(\rho_0-\bar{\rho})^h, v_0^h\in\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ obtained thanks to Theorem 3.2.2. Let the positive function $\mathcal{N}_0$ be such that*

$\mathcal{N}_0 - \bar{\rho}$ is small enough in $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$, and let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{p}+2}_{p,1})$ be the corresponding global solution of the following Cauchy problem

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \end{cases} \tag{3.16}$$

given by Proposition 3.5.1. Let $(\widetilde{\rho}^\varepsilon, \widetilde{v})(t,x) \triangleq (\rho, \varepsilon^{-1} v)(\varepsilon^{-1} t, x)$ and assume that

$$\|\widetilde{\rho}^\varepsilon_0 - \mathcal{N}_0\|_{\mathbb{B}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon.$$

Then, as $\varepsilon \to 0$, we have

$$\widetilde{\rho}^\varepsilon - \mathcal{N} \longrightarrow 0 \quad \textit{strongly in} \quad L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}),$$

and

$$\widetilde{v}^\varepsilon + \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} \longrightarrow 0 \quad \textit{strongly in} \quad L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}).$$

Moreover, we have the following quantitative estimate:

$$\|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \widetilde{v}^\varepsilon \right\|_{L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

Remark 3.2.3. *Compared to the previous works on these topics [70, 83, 107], here we obtain the strong convergence* globally in space and time with an explicit convergence rate. *The reason why is that our proof is based on a priori estimates of appropriate norms for the difference of the solutions rather than on compactness methods like Aubin-Lions lemma. In our proof, having estimates on the damped mode at hand plays a key role.*

3.3 Global existence in the L^p framework

This section is devoted to proving the global existence of strong solutions for System (3.1) supplemented with critical regularity initial data that are close to the reference solution $\bar{V}$, in the case where the structural assumptions listed in Theorem 3.2.1 and (SK) condition are satisfied. We shall focus on the case $\varepsilon = 1$, which is not restrictive owing to the rescaling:

$$Z(t,x) \triangleq \check{Z}\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\Big). \tag{3.17}$$

Indeed, Z satisfies (3.1) with relaxation parameter ε if and only if $\check{Z}$ satisfies (3.1) with relaxation parameter 1, and performing the inverse scaling gives us the desired dependency with respect to ε in Theorem 3.2.1. Indeed, it is well known (see e.g. [3, Chap.2]) that for all $s \in \mathbb{R}$ and $p \in [1, \infty]$, we have

$$\|z(\varepsilon\cdot)\|_{\dot{\mathbb{B}}^s_{p,1}} \simeq \varepsilon^{s-d/p} \|z\|_{\dot{\mathbb{B}}^s_{p,1}} \tag{3.18}$$

and we have a similar relation for the norms restricted to low or high frequencies, provided we shift the threshold between the two, by a factor ε^{-1}.

The bulk of the proof consists in establishing a priori estimates, then proving the global existence is classical. Concerning the uniqueness, one has to be careful because of the L^p framework. We follow an approach similar to what we did in [30].

Our strategy is to estimate separately the low and high frequencies of the solution. For the low frequencies, we consider the equations for Z_1 – of parabolic type – and for W defined in (3.7) – of damped type, pointing out that they are coupled by at least quadratic terms, or by linear terms of higher order (in terms of derivatives) that are thus negligible provided that the threshold J_0 between low and high frequencies is negative enough. To handle the high frequencies, we shall work out a Lyapunov functional in Beauchard-Zuazua's style, that is equivalent to the norm that we aim at controlling. We shall proceed as in [31], the functional framework being a bit more involved, though.

3.3.1 Establishing a priori estimates

Throughout this part we set the threshold between low and high frequencies at some integer k_p the value of which will be chosen during the computations. For better readability, we omit the index k_p on the Besov norms. We assume that we are given a smooth (and decaying) solution Z of (3.3) on $[0,T]\times\mathbb{R}^d$ with Z_0 as initial data, satisfying

$$\sup_{t\in[0,T]}\|Z(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}+\sup_{t\in[0,T]}\|Z(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\ll 1. \tag{3.19}$$

We shall use repeatedly that, owing to the embedding $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}\hookrightarrow L^\infty$, we have

$$\sup_{t\in[0,T]}\|Z(t)\|_{L^\infty}\ll 1 \tag{3.20}$$

and also that for all $\sigma\in\mathbb{R}$, $q\in[1,\infty]$ and $\alpha\geq 0$,

$$\|z\|^{\ell}_{\dot{\mathbb{B}}^{\sigma+\alpha}_{q,1}}\lesssim\|z\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{q,1}}\quad\text{and}\quad\|z\|^{h}_{\dot{\mathbb{B}}^{\sigma-\alpha}_{q,1}}\lesssim\|z\|^{h}_{\dot{\mathbb{B}}^{\sigma}_{q,1}}. \tag{3.21}$$

From now on, $C>0$ designates a generic harmless constant, the value of which depends on the context and we denote by $(c_j)_{j\in\mathbb{Z}}$ various nonnegative sequences such that $\sum_{j\in\mathbb{Z}}c_j=1$.

3.3.1.1 Low frequencies analysis in L^p spaces

In this section we fix $j\leq J_0$ where J_0 stands for a negative integer the value of which will be determined later.

The main result of this section reads as follows.

Proposition 3.3.1. *Let Z be a smooth solution of (3.3) on $[0,T]$ satisfying (3.19). Then, for all $1\leq p<\infty$, we have*

$$\begin{aligned}\|(Z_1,Z_2,W)(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}&+\kappa_0\int_0^t\left(\|Z_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}}+\|Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}+\|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\right)\\
&\lesssim\|(Z_{1,0},W_0)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}+\int_0^t\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}+\int_0^t\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}\\
&\qquad+\int_0^t\|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}+\int_0^t\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}+\int_0^t\|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.\end{aligned}$$

First, here is a result that we will use on several occasions.

Lemma 3.3.1. *Under hypotheses* (3.11) *and* (3.19)*, we have for all* $\sigma \in]-\min(d/p, d/p'), d/p]$,

$$\|\partial_t Z_1\|_{\dot{\mathbb{B}}^{\sigma}_{p,1}} \lesssim \|\nabla Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\sigma}_{p,1}},$$
$$\|\partial_t Z_2\|_{\dot{\mathbb{B}}^{\sigma}_{p,1}} \lesssim \|W\|_{\dot{\mathbb{B}}^{\sigma}_{p,1}}.$$

Proof. The proof relies on the explicit expressions of $\partial_t Z_1$ and $\partial_t Z_2$ given in (3.3) and is a mere adaptation of the corresponding one in Chapter 2. As regards Z_1, since $\bar{A}^k_{1,1}$ for all $k \in \{1, \cdots, d\}$, we have

$$\partial_t Z_1 + \sum_{k=1}^{d} \bar{A}^k_{1,2} \partial_k Z_2 = -\sum_{k=1}^{d} \Big(A^k_{1,1}(V)\partial_k Z_1 + \big(A^k_{1,2}(V) - \bar{A}^k_{1,2}\big)\partial_k Z_2 \Big). \tag{3.22}$$

All the terms of the right-hand side are at least quadratic, and (3.11), (3.19) thus ensure the desired inequality for $\partial_t Z_1$.

Next, the definition of W in (3.7) ensures that

$$\partial_t Z_2 = -L_2 W, \tag{3.23}$$

whence the second inequality. □

We now turn to the proof of Proposition 3.3.1.

Step 1: Estimates for W.

We have the following statement.

Proposition 3.3.2. *Let Z be a smooth solution of* (3.3) *on* $[0,T] \times \mathbb{R}^d$ *satisfying* (3.19)*. Then,*

$$\|W(t)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \kappa_0 \int_0^t \|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \leq \|W_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + C \int_0^t \|(\nabla Z_2, W)\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}$$
$$+ C \int_0^t \|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + C \int_0^t \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}. \tag{3.24}$$

Proof. From (3.8), we gather that

$$\partial_t W + L_2 W = h \tag{3.25}$$

with $h \triangleq {L_2}^{-1} h_1$ and

$$h_1 \triangleq \sum_{k=1}^{d} \partial_t \big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2 \big).$$

Applying $\dot{\Delta}_j$ to (3.25) and taking the scalar product with $|W_j|^{p-2}W_j$ yields, thanks to (3.5),

$$\frac{1}{p}\frac{d}{dt}\|W_j\|^p_{L^p} + \kappa_0 \|W_j\|^p_{L^p} \leq \|\dot{\Delta}_j h_1\|_{L^p} \|W_j\|^{p-1}_{L^p}. \tag{3.26}$$

For bounding h_1, we use that for all $k \in \{1, \cdots, d\}$,

$$\partial_t (A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2) = D_V A^k_{2,1}(V)\partial_t Z \partial_k Z_1 + \bar{A}^k_{2,1}\partial_t\partial_k Z_1$$
$$+ \big(A^k_{2,1}(V) - A^k_{2,1}(\bar{V})\big)\partial_t\partial_k Z_1 + D_V A^k_{2,2}(V)\partial_t Z \partial_k Z_2 + \bar{A}^k_{2,2}\partial_t\partial_k Z_2 + \big(A^k_{2,2}(V) - A^k_{2,2}(\bar{V})\big)\partial_t\partial_k Z_2.$$

For $m = 1, 2$, we have, according to product laws and Lemma 3.3.1,

$$\begin{aligned}\|D_V A^k_{2,m}(V)\partial_t Z\partial_k Z_m\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} &\lesssim \|D_V A^k_{2,m}(V)\partial_t Z\partial_k Z_m\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}}\\ &\lesssim \|\partial_t Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}}\\ &\lesssim (\|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}})\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\end{aligned}$$

as well as

$$\begin{aligned}\|\big(A^k_{2,m}(V) - A^k_{2,m}(\bar{V})\big)\partial_t\partial_k Z_m\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} &\lesssim \|\big(A^k_{2,m}(V) - A^k_{2,m}(\bar{V})\big)\partial_t\partial_k Z_m\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}}\\ &\lesssim \|\partial_t\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\\ &\lesssim (\|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}})\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.\end{aligned}$$

From (3.22), product laws and (3.21), one can get

$$\|(\partial_t\nabla Z_1, \partial_t\nabla Z_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|\nabla Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.$$

Hence, using Lemma A.1.1, multiplying the resulting inequality with $2^{j\frac{d}{p}}$, summing up on $j \le J_0$, and using (3.19), we get

$$\begin{aligned}\|W(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \kappa_0\int_0^t \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} &\le \|W_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + C\int_0^t \|(\nabla Z_2, W)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}\\ &\quad + C\int_0^t \|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + C\int_0^t \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}},\end{aligned} \tag{3.27}$$

which is the desired estimate. □

Step 2: Estimates for Z_1

We have the following proposition.

Proposition 3.3.3. *Let Z be a smooth solution of (3.3) on $[0,T]\times\mathbb{R}^d$ satisfying (3.19). We have*

$$\begin{aligned}\|Z_1(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \kappa_0\int_0^t \|Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} &\le \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t \|(W, \nabla Z_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}\\ &\quad + \int_0^t \big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}\big)\\ &\quad + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.\end{aligned} \tag{3.28}$$

Proof. Applying $\dot{\Delta}_j$ to (3.9), taking the scalar product with $|Z_{1,j}|^{p-2}Z_{1,j}$ and using (3.10) gives that there exists a κ_0 such that

$$\frac{1}{p}\frac{d}{dt}\|Z_{1,j}\|^p_{L^p} + c_p\kappa_0 2^{2j}\|Z_{1,j}\|^p_{L^p} \lesssim \|(f_{1,j}, f_{2,j}, f_{3,j}, f_{4,j}, f_{5,j})\|_{L^p}\|Z_{1,j}\|^{p-1}_{L^p}.$$

This easily leads to

$$\|Z_1(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \kappa_0 \int_0^t \|Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} \lesssim \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|(f_1, f_2, f_3, f_4, f_5)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \tag{3.29}$$

where the f_i are defined in the first section of the chapter.

Using (3.19) and Inequality (3.21) yields

$$\begin{aligned}\|f_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} &\lesssim \|\sum_{k=1}^d \bar{A}^k_{1,2}\partial_k W^\ell\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|\sum_{k=1}^d \big(A^k_{1,2}(V) - \bar{A}^k_{1,2}\big)\partial_k W\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} \\ &\lesssim \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.\end{aligned}$$

In a similar way, since

$$\begin{aligned}f_1 = \sum_{k=1}^d \Big(&\bar{A}^k_{1,2} L_2^{-1} \bar{A}^k_{2,2}\partial_k\partial_\ell Z_2 + \bar{A}^k_{1,2}\partial_k\big(L_2^{-1}(A^\ell_{2,2}(V) - \bar{A}^\ell_{2,2})\partial_\ell Z_2\big) \\ &+\big(A^k_{1,2}(V) - \bar{A}^k_{1,2}\big)L_2^{-1}\bar{A}^\ell_{2,2}\partial_k\partial_\ell Z_2 + \big(A^k_{1,2}(V) - \bar{A}^k_{1,2}\big)\partial_k\big(L_2^{-1}(A^\ell_{2,2}(V) - \bar{A}^\ell_{2,2})\partial_\ell Z_2\big)\Big),\end{aligned}$$

we obtain

$$\|f_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}},$$

$$\begin{aligned}\|f_3\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} &= \|\sum_{k=1}^d\sum_{\ell=1}^d A^\ell_{1,2}(V)\partial_\ell(L_2^{-1}(A^k_{2,1}(V) - \bar{A}^k_{2,1})\partial_k Z_1)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \\ &\lesssim \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \big(\|\nabla^2 Z_1\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \|\nabla^2 Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\big),\end{aligned}$$

$$\begin{aligned}\|f_4\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} &\lesssim \|\sum_{k=1}^d\sum_{\ell=1}^d (A^k_{1,2}(V) - \bar{A}^k_{1,2})L_2^{-1}A^\ell_{2,1}(\bar{V}))\partial_k\partial_\ell Z_1)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \\ &\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \big(\|\nabla^2 Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|\nabla^2 Z_1\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}}\big)\cdot\end{aligned}$$

Taking advantage of the structure condition (3.11) and of the smallness assumption (3.19), we get

$$\|f_5\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.$$

Inserting the estimates pertaining to f_1, f_2, f_3, f_4, f_5 in (3.29) and using (3.19) completes the proof of the proposition. □

Now, choosing J_0 small enough and putting Propositions 3.3.2 and 3.3.3 together yields

$$\begin{aligned}\|(Z_1,W)(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} &+ \int_0^t \big(\|Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\big) \lesssim \|(Z_{1,0},W_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \\ &+ \int_0^t \|(W,\nabla Z_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \big(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}\big) \\ &+ \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}. \end{aligned} \tag{3.30}$$

Step 3: Recovering information for Z_2

In order to recover information on Z_2 from W one has to use the identity:

$$W - Z_2 = L_2^{-1} \sum_{k=1}^d \Big(\bar{A}^k_{2,1}\partial_k Z_1 + \big(A^k_{2,1}(V) - \bar{A}^k_{2,1}\big)\partial_k Z_1 + \bar{A}^k_{2,2}\partial_k Z_2 + \big(A^k_{2,2}(V) - \bar{A}^k_{2,2}\big)\partial_k Z_2\Big). \tag{3.31}$$

Hence,

$$\|W - Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}, \tag{3.32}$$

which yields, owing to (3.19) and provided that J_0 is negative enough,

$$\|W - Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|\nabla Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}. \tag{3.33}$$

This already ensures that one can add $\|Z_2(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$ in the left-hand side of (3.30) (without changing the right-hand side).

Furthermore, since for $m = 1, 2$, owing to (3.21) and the linearity of A^k, it holds that

$$\begin{aligned}\|(A^m_{2,1}(V) - \bar{A}^m_{2,1})\partial_k Z_m\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} &\lesssim \|(A^k_{2,m}(V) - \bar{A}^k_{2,m})\partial_k Z^\ell_m\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|(A^k_{2,m}(V) - \bar{A}^k_{2,m})\partial_k Z^h_m\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \\ &\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z^\ell_m\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|\nabla Z^\ell_m\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z^h_m\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\end{aligned}$$

we have

$$\|W - Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \lesssim \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} (1 + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}) + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}. \tag{3.34}$$

Hence, one can also include $\|Z_2\|^\ell_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}$ in the left-hand side of (3.30), which completes the proof of Proposition 3.3.1.

3.3.1.2 High frequencies analysis

Although the functional framework for high frequencies is the same as in the previous chapter, one cannot repeat exactly the computations therein since the non-linear terms contain a little amount of low frequencies of Z that are only in spaces of the type $\dot{\mathbb{B}}^s_{p,1}$ for some $p > 2$ (and thus not in some $\dot{\mathbb{B}}^{s'}_{2,1}$). To overcome the difficulty, we have to study more carefully the composition operators and the commutators in our 'hybrid' functional framework (see details in

Appendix).

As in the previous chapter, the starting point is to differentiate in time the following functional:

$$\mathcal{L}_j \triangleq \|Z_j\|_{L^2}^2 + 2^{-j}\mathcal{I}_j, \qquad j \geq J_0, \tag{3.35}$$

with

$$\mathcal{I}_j \triangleq \int_{\mathbb{R}^d} \sum_{q=1}^{n-1} \varepsilon_q \Im\left((NM_\omega^{q-1}\widehat{Z_j})\cdot(NM_\omega^q\widehat{Z_j})\right), \tag{3.36}$$

where $\varepsilon_1, \cdots, \varepsilon_{n-1} > 0$ are chosen small enough.

Energy estimates

One has the following proposition:

Proposition 3.3.4. *Let Z be of solution of (3.3) satisfying (3.19). Define p^* by the relation $1/p + 1/p^* = 1/2$. Then, the following a priori estimate holds:*

$$\|Z(t)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \left(\|(Z,\nabla Z)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^\ell_{\mathbb{B}^{\frac{d}{p}+2}_{p,1}} \|Z\|_{\mathbb{B}^{\frac{d}{p}}_{p,1}}\right)\cdot$$

Proof. Localizing in frequencies System (3.3) (in its compact form) by means of $\dot{\Delta}_j$, we get

$$\partial_t Z_j + \sum_{k=1}^d \dot{S}_{j-1}A^k(V)\partial_k Z_j + LZ_j = R_j^1$$

with $R_j^1 \triangleq \sum_{k=1}^d \dot{S}_{j-1}A^k(V)\partial_k Z_j - \dot{\Delta}_j(A^k(V)\partial_k Z)$.

For $j \geq J_0$, similar computations as in the previous chapters lead to:

$$\frac{1}{2}\frac{d}{dt}\|Z_j\|_{L^2}^2 + \kappa_0\|Z_{2,j}\|_{L^2}^2 \lesssim \|\nabla Z\|_{L^\infty}\|Z_j\|_{L^2}^2 + \left\|R_j^1\right\|_{L^2}\|Z_j\|_{L^2}. \tag{3.37}$$

For the remainder term R_j^1, using Lemma 3.5.1 with $w = A^k(V) - \bar{A}^k$, $z = \partial_k Z$, $k = 0$, $\sigma_1 = \frac{d}{p} + 2$ and $\sigma_2 = \frac{d}{p} + 1$, we have

$$\begin{aligned}\left\|R_j^1\right\|_{L^2} \leq Cc_j 2^{-j(\frac{d}{2}+1)} \sum_{k=1}^d \Big(&\left\|\nabla A^k(V)\right\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \left\|A^k(V)\right\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} \\ &+ \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \left\|A^k(V)\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \left\|\nabla A^k(V)\right\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}}\Big),\end{aligned}$$

whence, using the linearity of $A^k(V)$,

$$\left\|R_j^1\right\|_{L^2} \leq Cc_j 2^{-j(\frac{d}{2}+1)}\left(\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\right)\cdot \tag{3.38}$$

□

Cross estimates

Proposition 3.3.4 only allows to exhibit the integrability properties of the components of Z experiencing direct dissipation. To recover the dissipation for all the components, we have to look at the time derivative of $\mathcal{I}_j$ defined in (3.36).

To start with, let us rewrite (3.3) as follows:

$$\partial_t Z + \sum_{k=1}^{d} \bar{A}^k \partial_k Z + LZ = G \quad \text{with} \quad G \triangleq -\sum_{k=1}^{d} \left(A^k(V) - \bar{A}^k \right) \partial_k Z. \tag{3.39}$$

Following the exact same computations as what we did in the previous chapter leads to

$$\frac{d}{dt}\mathcal{I}_j + \frac{2^j}{2} \sum_{k=1}^{n-1} \varepsilon_k \int_{\mathbb{R}^d} |N M_\omega^k \widehat{Z}_j|^2 \, d\xi \leq \frac{2^{-j}\kappa_0}{2} \|N Z_j\|_{L^2}^2 + C \|\dot{\Delta}_j G\|_{L^2} \|Z_j\|_{L^2}. \tag{3.40}$$

Writing $G_j = (A^k(V) - \bar{A}^k)\partial_k Z_j + R_j$, where R_j is the associated commutator, from Lemma 3.5.1 with $k = 0$, $\sigma_1 = d/p + 2$ and $\sigma_2 = \frac{d}{p} + 1$ and the linearity of A^k for $k = 1, \cdots, d$, we infer that

$$\|G\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}},$$

whence, observing that $p^* \geq d$, Bernstein inequality implies

$$\|\nabla Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \lesssim \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}},$$

we get

$$\|G\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} (\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}) + \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}}. \tag{3.41}$$

Remember that since the (SK) condition is satisfied, the quantity $\mathcal{N}_{\bar{V}}$ defined in Chapter 2 is positive for any choice of positive parameters $\varepsilon_0, \cdots, \varepsilon_{n-1}$. Consequently, if we set

$$\mathcal{H}_j := \frac{\kappa_0}{2} \|N Z_j\|^2 + \sum_{q=1}^{n-1} \varepsilon_q \int_{\mathbb{R}^d} |N M_\omega^q \widehat{Z}_j|^2 \, d\xi$$

and use Fourier-Plancherel theorem and that $\mathcal{L}_j \sim \|Z_j\|_{L^2}$, we see that (up to a change of κ_0), we have for all $j \in \mathbb{Z}$,

$$\mathcal{H}_j \geq \kappa_0 \mathcal{L}_j. \tag{3.42}$$

Combining Inequalities (3.37) and (3.38), the cross estimate (3.40), (3.41) and (3.42), we get for all $j \geq J_0$,

$$\frac{d}{dt} 2^{j(\frac{d}{2}+1)} \mathcal{L}_j + \kappa_0 2^{j(\frac{d}{2}+1)} \mathcal{L}_j \lesssim c_j \|(\nabla Z, Z)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z_j\|_{L^2} + c_j \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} \|Z_j\|_{L^2}.$$

Hence, using that $\mathcal{L}_j \sim \|Z_j\|_{L^2}$ and Lemma A.1.1, and summing up on $j \geq 0$ yields

$$\|Z(t)\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} + \kappa_0 \int_0^t \|Z\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|Z_0\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \Big((\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}) \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} \Big) \tag{3.43}$$

where we used the notation

$$\|Z\|^h_{\dot{\mathcal{B}}^{\sigma}_{2,1}} \triangleq \sum_{j \geq J_0} 2^{j\sigma} \sqrt{\mathcal{L}_j}.$$

The final a priori estimate

Let us introduce the functionals

$$\mathcal{L} \triangleq \|(Z,W)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathcal{B}}^{\frac{d}{2}+1}_{2,1}} \quad \text{and} \quad \mathcal{H} \triangleq \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}. \tag{3.44}$$

As seen in (3.33), $\mathcal{L}$ is equivalent to $\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$.

Adding up the inequality of Proposition 3.3.1 with Inequality (3.43), remembering (3.19) and using several times the fact that

$$\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \lesssim \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}},$$

we get for all $t \in [0,T]$,

$$\mathcal{L}(t) + \int_0^t \mathcal{H} \lesssim \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(W_0,Z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t (\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}) \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.$$

Using the definition of $\mathcal{L}$ an obvious embedding, the above inequality may be simplified into:

$$\mathcal{L}(t) + \int_0^t \mathcal{H} \lesssim \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(W_0,Z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t \big(\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \big) \mathcal{L} + \int_0^t \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|W\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}. \tag{3.45}$$

On the one hand, (3.33) guarantees that

$$\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \lesssim \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|\nabla Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}.$$

On the other hand, thanks to product laws, (3.21) and to (3.31), we have

$$\begin{aligned}\|W-Z_2\|^h_{\dot B^{\frac dp}_{p,1}} &\lesssim \|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac dp}_{p,1}} + \sum_{m=1}^{2}\big(\|(A^k_{2,m}(V)-\bar A^k_{2,m})\partial_k Z^\ell_m\|_{\dot{\mathbb{B}}^{\frac dp+1}_{p,1}} + \|(A^k_{2,1}(V)-\bar A^k_{2,1})\partial_k Z^h_m\|_{\dot{\mathbb{B}}^{\frac dp}_{p,1}}\big)\\ &\lesssim \|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac dp}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac dp}_{p,1}}\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac dp+2}_{p,1}} + \|\nabla Z\|^2_{\dot{\mathbb{B}}^{\frac dp}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac dp}_{p,1}}\|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac dp}_{p,1}}.\end{aligned}$$

Hence, using once more (3.19), we conclude that Inequality (3.45) reduces to

$$\mathcal{L}(t)+\int_0^t \mathcal{H} \le C\Big(\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} + \|(W_0,Z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac dp}_{p,1}} + \int_0^t \mathcal{H}\mathcal{L}\Big).$$

It is now clear that if $\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} + \|(W_0,Z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac dp}_{p,1}}$ or, equivalently, $\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} + \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac dp}_{p,1}}$ is small enough, then we have

$$\mathcal{L}(t)+\int_0^t \mathcal{H} \lesssim \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac d2+1}_{2,1}} + \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac dp}_{p,1}} \quad \text{for all } \ t\in[0,T].$$

In order to complete the proof of the estimate in Theorem 3.2.1, it suffices to observe that, in light of (3.33), one can recover a L^2-in-time control of Z_2 as follows:

$$\begin{aligned}\|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac dp}_{2,1})} &\lesssim \|W\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac dp}_{p,1})} + \|\nabla Z_1\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac dp}_{p,1})} + \|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac dp}_{p,1})}\|\nabla Z\|^h_{L^2_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\\ &\lesssim \big(\|W\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac dp}_{p,1})}\|W\|^\ell_{L^\infty_T(\dot{\mathbb{B}}^{\frac dp}_{p,1})}\big)^{1/2} + \big(\|Z_1\|^\ell_{L^\infty_T(\dot{\mathbb{B}}^{\frac dp}_{p,1})}\|Z_1\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac dp+2}_{p,1})}\big)^{1/2}\\ &\qquad + \|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac dp}_{p,1})}\big(\|\nabla Z\|^h_{L^\infty_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\|\nabla Z\|^h_{L^1_T(\dot{\mathbb{B}}^{\frac d2}_{2,1})}\big)^{1/2}.\end{aligned}$$

3.3.2 Proof of the existence part of Theorem 3.2.1

The proof relies on the following classical result about the local existence of strong solutions for hyperbolic symmetric systems of type

$$(QS)\qquad \begin{cases}\partial_t U + \sum_{k=1}^d A_k(U)\partial_k U + A_0(U) = 0,\\ U|_{t=0} = U_0,\end{cases}$$

where A_k, $k=0,\dots,d$ are smooth functions from $\mathbb{R}^n$ to the space of $n\times n$ matrices, that are symmetric if $k\neq 0$.

Theorem 3.3.1. *[3, Chap. 4] Let U_0 be in the nonhomogeneous Besov space $\mathbb{B}^{\frac d2+1}_{2,1}(\mathbb{R}^d;\mathbb{R}^n)$. Then, (QS) admits a unique maximal solution U in $\mathcal{C}([0,T^*[;\mathbb{B}^{\frac d2+1}_{2,1})\cap\mathcal{C}^1([0,T^*[;\mathbb{B}^{\frac d2}_{2,1})$, and there exists a positive constant c such that*

$$T^* \ge \frac{c}{\|U_0\|_{\mathbb{B}^{\frac d2+1}_{2,1}}}.$$

Furthermore,

$$T^*<\infty \Longrightarrow \int_0^{T^*}\|\nabla U\|_{L^\infty} = \infty.$$

The proof of the existence part of Theorem 3.2.1 is an adaptation of the proof in [30] to the multi-dimensional case. First, we multiply the low frequencies of the data by a cut-off function in order to have data in $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$. One can then use the above theorem to construct a sequence $(Z^n)_{n\in\mathbb{N}}$ of solutions to (3.3) with $\varepsilon = 1$ and smoothed out initial data, that are shown to be global. We prove uniform estimates in the space E_p (that is $E_p^{J_\varepsilon}$ with $\varepsilon = 1$) for those solutions, pass to the limit up to subsequence by means of compactness arguments, and finally check that the limit is a global solution of (3.3) supplemented with initial data Z_0 with the required properties.

First step: Construction of approximate solutions

Let Z_0 be such that $Z_0^\ell \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ and $Z_0^h \in \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$. Since Z_0 need not be in $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$, we set for all $n \geq 1$,

$$Z_0^n \triangleq \chi_n \, \dot{S}_{J_0-5} Z_0 + (\mathrm{Id} - \dot{S}_{J_0-5}) Z_0 \quad \text{with} \quad \chi_n \triangleq \chi(n^{-1}\cdot),$$

where χ stands (for instance) for a smooth function with range in $[0,1]$, supported in $]-4/3, 4/3[$ and such that $\chi \equiv 1$ on $[-3/4, 3/4]$.

It is obvious that the sequence $(Z_0^n)_{n\in\mathbb{N}}$ tends to Z_0 in the sense of distributions, when n tends to infinity. Moreover, as Z_0^ℓ is in $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$, the low frequencies of the data are in L^∞, and the spatial truncation thus guarantees that $Z_0^n \in \mathbb{B}^{\frac{d}{2}+1}_{2,1}$. Hence, for all $n \geq 1$, Theorem 3.3.1 provides us with a unique maximal solution Z^n in $\mathcal{C}([0,T_n[;\mathbb{B}^{\frac{d}{2}+1}_{2,1}) \cap \mathcal{C}^1([0,T_n[;\mathbb{B}^{\frac{d}{2}}_{2,1})$.

We claim that we have

$$\|Z_0^n\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_0^n\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}. \tag{3.46}$$

Indeed, since $\|\chi_n\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \simeq \|\chi\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} < \infty$, owing to the invariance of the norm in $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ by spatial dilation (see e.g. [3, Rem. 2.19]), we may write

$$\begin{aligned}
\|Z_0^n\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} &\leq \Big\|\chi_n \, \dot{S}_{J_0-5} Z_0\Big\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \Big\|(\mathrm{Id} - \dot{S}_{J_0-5}) Z_0\Big\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \\
&\lesssim \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \, \|\chi_n\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \\
&\lesssim \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.
\end{aligned}$$

Next, we see that

$$\|Z_0^n\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq \|\chi_n \, \dot{S}_{J_0-5} Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(\mathrm{Id} - \dot{S}_{J_0-5}) Z_0^h\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

It is obvious that the last term may be bounded by $\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$. For the other term, the important observation is that for $j \geq J_0$, we have

$$\dot{\Delta}_j\big(\chi_n \, \dot{S}_{J_0-5} Z_0\big) = \sum_{j' \geq j-3} \dot{\Delta}_j\big(\dot{S}_{j'+2} \dot{S}_{J_0-5} Z_0 \, \dot{\Delta}_{j'} \chi_n\big).$$

Hence, owing to the scaling properties of the space $\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$,

$$\|\chi_n \dot{S}_{J_0-5} Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|\dot{S}_{J_0-5} Z_0\|_{L^\infty} \|\chi_n\|_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim n^{-1} \|z_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}},$$

which eventually yields (3.46).

Second step: Uniform estimates

Since, for all $T > 0$, the space $\mathcal{C}([0,T]; \mathbb{B}^{\frac{d}{2}+1}_{2,1}) \cap \mathcal{C}^1([0,T]; \mathbb{B}^{\frac{d}{2}}_{2,1})$ is included in our 'solution space' $E_p(T)$ (that is, E_p restricted to $[0,T]$), one can take advantage of the computations for the previous sequence to bound our sequence. From it and (3.46), we get, denoting by X^n_p the function X_p pertaining to Z^n

$$X^n_p \leq C\big(X_{p,0} + (X^n_p)^2\big). \tag{3.47}$$

It is clear that, if there exists T_0 in $[0,T]$ such that

$$2CX^n_p \leq 1 \quad \text{on} \quad [0,T_0], \tag{3.48}$$

then Inequality (3.47) implies that

$$X^n_p \leq 2CX_{p,0} \quad \text{on} \quad [0,T_0].$$

Then, thanks to a classical bootstrap argument, we can conclude that if $X_{p,0}$ is small enough then (3.48) is true as long as the solution exists. In other words, there exists a constant C such that for all $n \geq 1$,

$$X^n_p \leq CX_{p,0} \quad \text{on} \quad [0,T_n[. \tag{3.49}$$

In order to show that the above inequality implies that the solution is global (namely that $T_n = \infty$), one can argue by contradiction, assuming that $T_n < \infty$, and use the blow-up criterion of Theorem 3.3.1. However, we first have to justify that the nonhomogeneous Besov norm $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$ of the solution is under control *up to time* T_n. Using the classical energy method for (3.3) and the Gronwall lemma, we get that for all $t < T_n$,

$$\|Z^n(t)\|_{L^2} \leq C\,\|Z^n_0\|_{L^2} \exp\Big(C \int_0^t \|\nabla Z^n\|_{L^\infty}\Big).$$

Since (3.49) and the embedding of $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ and $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}$ in L^∞ ensure that ∇Z^n is in $L^1_{T_n}(L^\infty)$, using Gronwall lemma gives that Z^n is in $L^\infty_{T_n}(L^2)$, and thus in $L^\infty_{T_n}(\mathbb{B}^{\frac{d}{2}+1}_{2,1})$ owing, again, to (3.49).

It is now easy to conclude : for all $t_{0,n} \in [0,T_n[$, Theorem 3.3.1 provides us with a solution of (TM) with the initial data $Z(t_{0,n}$ on $[t_{0,n}, T + t_{0,n}]$ for some T that may be bounded from below independently of $t_{0,n}$. Consequently, choosing $t_{0,n}$ such that $t_{0,n} > T_n - T$, we see that the solution Z^n can be extended beyond T_n, which contradicts the maximality of T_n. Hence $T_n = +\infty$ and the solution corresponding to the initial data Z^n_0 is global in time and satisfies (3.49) for all time.

Third step: Convergence

Then we have to show that $(Z^n)_{n\in\mathbb{N}}$ tends, up to subsequence, to some $Z \in E_p$, in the sense of distributions, that satisfies (3.3). This can be done using Ascoli Theorem and suitable compact embeddings in a similar fashion as in the Section 1.4.4 of Chapter 1. We omit the details here.

3.3.3 Uniqueness

The difficulty is that our functional framework is not the standard one for the low frequencies of the solution, so that one cannot follow the classical proof for hyperbolic symmetric systems we used in [31]. Here we shall estimate $\widetilde{Z} := Z^1 - Z^2$ in the space

$$F_p(T) \triangleq \Big\{ Z^\ell \in \mathcal{C}([0,T]; \dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}) \,:\, Z^h \in \mathcal{C}([0,T]; \dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}) \Big\}. \tag{3.50}$$

The reason for the exponent $d/2$ for high frequencies is the usual loss of one derivative when proving stability estimates for quasilinear hyperbolic systems. The exponent for low frequencies looks to be the best one for controlling the nonlinearities. To prove the uniqueness, we will need to following lemma.

Lemma 3.3.2. *Let Z^1 and Z^2 be two solutions of (3.3) on $[0,T]$ supplemented by initial data Z^1_0 and Z^2_0, respectively. Then, $\widetilde{Z} \triangleq Z^1 - Z^2$ satisfies the following a priori estimate for all $0 \le t \le T$:*

$$\begin{aligned}\big\|\widetilde{Z}\big\|^\ell_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1})} + \big\|\widetilde{Z}\big\|^h_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} &\lesssim \big\|\widetilde{Z}_0\big\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} + \big\|\widetilde{Z}_0\big\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \\ &+ \int_0^t \Big(\|(Z^1,Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|(\nabla Z^1, \nabla Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \Big) \big(\|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} + \|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \big).\end{aligned}$$

Proof. Here also, we shall proceed differently for estimating the low and the high frequencies of $\widetilde{Z}$.

Step 1: Estimates for the low frequencies. Let $V^1 \triangleq \bar{V} + Z^1$ and $V^2 \triangleq \bar{V} + Z^2$. Observe that $\widetilde{Z}$ is a solution of

$$\partial_t \widetilde{Z} + L\widetilde{Z} + \sum_{k=1}^d A^k(V^1)\partial_k \widetilde{Z} = \sum_{k=1}^d \big(A^k(V^2) - A^k(V^1)\big)\partial_k Z^2.$$

Applying $\dot{\Delta}_j$, taking the scalar product with $|\widetilde{Z}_j|^{p-2}\widetilde{Z}_j$, integrating on $\mathbb{R}_+ \times \mathbb{R}^d$ and using Lemma A.1.1, we get for all $j \in \mathbb{Z}$,

$$\begin{aligned}\big\|\widetilde{Z}_j\big\|_{L^p} + \kappa_0 \int_0^t \big\|L\widetilde{Z}_j\big\|_{L^p} &\le \big\|\widetilde{Z}_{0,j}\big\|_{L^p} + \sum_{k=1}^d \int_0^t \big\|\nabla A^k(V^1)\big\|_{L^\infty} \big\|\widetilde{Z}_j\big\|_{L^p} \\ &+ \int_0^t \Big\|\dot{\Delta}_j \sum_{k=1}^d \big(A^k(V^2) - A^k(V^1)\big)\partial_k Z^2\Big\|_{L^p} + \int_0^t \sum_k \|[\dot{\Delta}_j, A^k(V^1)]\partial_k \widetilde{Z}\|_{L^p}.\end{aligned}$$

Multiplying this inequality by $2^{j(\frac{d}{p}-\frac{d}{p*})}$ and using the embedding $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \hookrightarrow L^\infty$ as well as the commutator estimate (A.4), we get

$$\begin{aligned}2^{j(\frac{d}{p}-\frac{d}{p*})} \big\|\widetilde{Z}_j\big\|_{L^p} &\lesssim 2^{j(\frac{d}{p}-\frac{d}{p*})} \big\|\widetilde{Z}_{0,j}\big\|_{L^p} + c_j \int_0^t \|\nabla Z^1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \big\|\widetilde{Z}\big\|_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \\ &+ \int_0^t 2^{j(\frac{d}{p}-\frac{d}{p*})} \Big\|\dot{\Delta}_j \sum_{k=1}^d \Big(A^k(V^2) - A^k(V^1)\Big)\partial_k Z^2\Big\|_{L^p}.\end{aligned} \tag{3.51}$$

Therefore, summing up on $j \le J_0$, we arrive at

$$\left\|\widetilde{Z}\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \lesssim \left\|\widetilde{Z}_0\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} + \int_0^t \|\nabla Z^1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \left\|\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} + \int_0^t \left\|\sum_{k=1}^d \left(A^k(V^2) - A^k(V^1)\right)\partial_k Z^2\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}}. \tag{3.52}$$

In order to bound the right-hand side, we may use Inequality (A.7) with $s = d/p - d/p^*$, namely

$$\|a\,b\|_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|b\|_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \tag{3.53}$$

Remembering that $V \mapsto A^k(V)$ is linear, we conclude that

$$\left\|\widetilde{Z}\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \lesssim \left\|\widetilde{Z}_0\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} + \int_0^t \|(\nabla Z^1, \nabla Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \left\|\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}}. \tag{3.54}$$

Step 2: Estimates for the high frequencies. As a first, let us observe that, for all $j \in \mathbb{Z}$, $\widetilde{Z}_j$ satisfies

$$\partial_t \widetilde{Z}_j + \sum_{k=1}^d \dot{S}_{j-1} A^k(V^1)\partial_k \widetilde{Z}_j + L\widetilde{Z}_j = \dot{\Delta}_j\left(\sum_{k=1}^d \big(A^k(V^2) - A^k(V^1)\big)\partial_k Z^2\right) + \sum_{k=1}^d R_k$$

with

$$R_k \triangleq \dot{S}_{j-1}\left(A^k(V^1) - \bar{A}^k\right)\dot{\Delta}_j \partial_k \widetilde{Z} - \dot{\Delta}_j\left((A^k(V^1) - \bar{A}^k)\partial_k \widetilde{Z}\right).$$

Hence, performing the classical procedure, we end up with

$$\left\|\widetilde{Z}\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \left\|\widetilde{Z}_0\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \|\nabla Z^1\|_{L^\infty} \left\|\widetilde{Z}\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \left\|\sum_{k=1}^d \left(A^k(V^2) - A^k(V^1)\right)\partial_k Z^2\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \sum_{j\ge J_0} 2^{j\frac{d}{2}} \|\sum_{k=1}^d R_k\|_{L^2}.$$

Applying Lemma 3.5.1 to $w = A^k(V^1) - \bar{A}^k$ and $z = \partial_k \widetilde{Z}$ with $s = \frac{d}{2}$, $k = 1$, $\sigma_1 = \frac{d}{p} + 2$ and $\sigma_2 = \frac{d}{p} + 1$, and remembering that all the maps $V \mapsto A^k(V)$ are linear, we get

$$\sum_{j\ge J_0}\left(2^{j\frac{d}{2}}\|\sum_{k=1}^d R_k\|_{L^2}\right) \lesssim \|\nabla Z^1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \left\|\nabla\widetilde{Z}\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \left\|\nabla\widetilde{Z}\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \left\|Z^1\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \left\|\nabla\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} \left\|Z^1\right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \left\|\nabla\widetilde{Z}\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \left\|\nabla Z^1\right\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}}.$$

Using (3.73) with $a = \partial_k Z^2$, $b = A^k(V^2) - A^k(V^1)$, $s = \frac{d}{2}$ and $\sigma = \frac{d}{p} + 1$ yields

$$\begin{aligned}
&\|\sum_{k=1}^{d}\Big(A^k(V^2) - A^k(V^1)\Big)\partial_k Z^2\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\\
&\lesssim \|\nabla Z^2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z^2\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\nabla Z^2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|\nabla Z^2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}.
\end{aligned}$$

Gathering the above estimates and using once more the embedding $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \hookrightarrow L^\infty$, we obtain

$$\begin{aligned}
\|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \|\widetilde{Z}_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} &+ \int_0^t \|(\nabla Z^1, \nabla Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \|(\nabla Z^1, \nabla Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\\
&+ \int_0^t \|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}}
\end{aligned} \tag{3.55}$$

Step 3: Conclusion. Summing (3.54) and (3.55) together, we get

$$\begin{aligned}
\|\widetilde{Z}\|^\ell_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1})} + \|\widetilde{Z}\|^h_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})} &\lesssim \|\widetilde{Z}_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} + \|\widetilde{Z}_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\\
&+ \int_0^t \|(\nabla Z^1, \nabla Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}(\|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} + \|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}) + \int_0^t \|(\nabla Z^1, \nabla Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\\
&+ \int_0^t \|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}},
\end{aligned}$$

which yields the desired estimate of Lemma 3.3.2. □

Consider two solutions Z^1 and Z^2 of (3.3) (not necessarily small) in the space E_p, that correspond to the same initial data Z_0. The proof of uniqueness follows from the stability estimates of the Lemma 3.3.2, *provided we prove that the difference between the two solutions belongs to $F_p(T)$ for all $T > 0$.*

Just denoting by Z one of those two solutions, we have

$$\partial_t Z = -\sum_{k=1}^{d} A^k(V)\partial_k Z - LZ\cdot \tag{3.56}$$

By interpolation in Besov spaces and Hölder inequality with respect to the time variable, since Z^ℓ is in $L^\infty(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$, we get

$$\nabla Z^\ell \in L^r(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}) \quad \text{with} \quad \frac{1}{r} \triangleq \frac{1}{2} - \frac{d}{4} + \frac{d}{2p}\cdot \tag{3.57}$$

It is clear that the same property holds for the high frequencies of Z since it belongs to $L^1(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^\infty(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$. We also know that $A^k(V)$ belongs to $L^\infty(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$. Therefore, from the product laws in Besov spaces that have been recalled in Proposition 3.5.2, we have that $\partial_t Z_1$ is in $L^r(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1})$,

and thus

$$Z_1 - Z_{1,0} \in \mathcal{C}^{\frac{1}{r'}}_{loc}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}). \tag{3.58}$$

We conclude that $Z_1 - Z_{1,0}$ is in $F_p(T)$ for all finite T.

Owing to the 0-th order term LZ in the equation, in order to justify that $(Z_2 - Z_{2,0}) \in F_p(T)$, we have to proceed slightly differently. Now, we notice that

$$\partial_t(e^{tL_2}Z_2) = -e^{tL_2}\left(\sum_{k=1}^{d} A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2\right).$$

Arguing as above, we see that the right-hand side is in $L^r(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1})$, which, as above, allows to conclude that $Z_2 - Z_{2,0} \in \mathcal{C}^{\frac{1}{r'}}_{loc}(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1})$.

Back to our two solutions Z_1 and Z_2, since they coincide initially, the above arguments ensure that $Z_1 - Z_2$ is in $F_p(T)$. Hence, combining Lemma 3.3.2, Gronwall lemma and the fact that the low frequencies of Z^1 and Z^2 (resp. the high frequencies of ∇Z^1 and ∇Z^2) are bounded in $L^1(0,T; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$ (resp. in $L^1(0,T; \dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})$) for all $T > 0$ completes the proof of uniqueness.

3.4 Relaxation limit for the compressible Euler system

In this section we prove Theorem 3.2.3. We shall often use without mentioning it that, as a combination of Bernstein inequality and of the definition of J_ε, we have

$$\|f\|^{\ell,J_\varepsilon}_{L^1_T(\dot{\mathbb{B}}^s_{p,1})} \leq \frac{C}{\varepsilon}\|f\|^{\ell,J_\varepsilon}_{L^1_T(\dot{\mathbb{B}}^{s-1}_{p,1})} \quad \text{and} \quad \|f\|^{h,J_\varepsilon}_{L^1_T(\dot{\mathbb{B}}^s_{2,1})} \leq C\varepsilon\|f\|^{h,J_\varepsilon}_{L^1_T(\dot{\mathbb{B}}^{s+1}_{2,1})}. \tag{3.59}$$

3.4.1 Reformulation and recovering estimates for ρ

Let $(c - \bar{c}, v)$ be the solution from Theorem 3.2.2. As in [29, 70], we introduce the rescaled time variable $\tau = \varepsilon t$ and define

$$(\tilde{c}^\varepsilon, \tilde{v}^\varepsilon)(\tau, x) = (c, \frac{v}{\varepsilon})(t, x).$$

The couple $(\widetilde{c}^\varepsilon, \widetilde{v}^\varepsilon)$ satisfies:

$$\begin{cases} \partial_t \tilde{c}^\varepsilon + \operatorname{div}(\widetilde{c}^\varepsilon \tilde{v}^\varepsilon) = 0, \\ \varepsilon^2(\partial_t \tilde{v}^\varepsilon + \tilde{v}^\varepsilon \cdot \nabla \tilde{v}^\varepsilon) + \tilde{\gamma}\widetilde{c}^\varepsilon \nabla \widetilde{c}^\varepsilon + \tilde{v}^\varepsilon = 0. \end{cases} \tag{3.60}$$

To be able to study the relaxation limit, we need the following reformulation of Theorem 3.2.2, corresponding to the scale of System (3.60):

Corollary 3.4.1. *We have the following uniform bounds for* $(\widetilde{c}^\varepsilon, \widetilde{v}^\varepsilon)$ *:*

$$\|\widetilde{c}^\varepsilon-\bar{c}\|^{\ell,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}+\varepsilon\|\widetilde{v}^\varepsilon\|^{\ell,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}+\varepsilon\|\widetilde{c}^\varepsilon-\bar{c}\|^{h,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\varepsilon^2\|\widetilde{v}^\varepsilon\|^{h,J_\varepsilon}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|\widetilde{c}^\varepsilon-\bar{c}\|^{\ell,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})}$$
$$+\frac{1}{\varepsilon}\|\widetilde{c}^\varepsilon-\bar{c}\|^{h,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|\widetilde{v}^\varepsilon\|^{h,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})}+\|\widetilde{v}^\varepsilon\|^{\ell,J_\varepsilon}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}+\|\widetilde{c}^\varepsilon-\bar{c}\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}+\|\widetilde{v}^\varepsilon\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}$$
$$+\frac{1}{\varepsilon}\left\|\widetilde{W}^\varepsilon\right\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\leq Cc_0 \tag{3.61}$$

where $\widetilde{W}^\varepsilon=\tilde{\gamma}\widetilde{c}^\varepsilon\nabla\widetilde{c}^\varepsilon+\tilde{v}^\varepsilon$ *and the threshold is* $J_\varepsilon=-\log_2(\varepsilon)+k$.

Proof. One just has to use the scaling property of the homogeneous Besov norms to get the desired bounds from the bounds of Theorem 3.2.2. The new crucial properties $\|\widetilde{c}^\varepsilon-\bar{c}\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}$ can easily be deduced from the other bounds. □

First, we need to recover estimates on ρ from Theorem 3.2.2 to be able to prove the relaxation limit for System (3.13). We have the following lemma.

Lemma 3.4.1. *Let* (c,v) *be the solution of System* (3.15) *from Theorem 3.2.2. Then, for all* $t\geq 0$, $\widetilde{\rho}^\varepsilon$ *satisfies the following properties:*

$$\|\widetilde{\rho}^\varepsilon-\bar{\rho}\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}+\|\widetilde{\rho}^\varepsilon-\bar{\rho}\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}+\frac{1}{\varepsilon}\left\|\widetilde{v}^\varepsilon+\frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon}\right\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\leq c_0.$$

Proof. First, noticing that

$$\tilde{\gamma}\widetilde{c}^\varepsilon\nabla\widetilde{c}^\varepsilon=\frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon},$$

from the estimates of Corollary 3.4.1, we immediately get

$$\frac{1}{\varepsilon}\left\|\widetilde{v}^\varepsilon+\frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon}\right\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\leq c_0.$$

Then, to recover the other properties on ρ, we have to use the composition lemma on norms that do not depend on the frequency regime. Using (3.59) and the embedding $\mathbb{B}^{\frac{d}{2}}_{2,1}\hookrightarrow\mathbb{B}^{\frac{d}{p}}_{p,1}$, it is easy to see that

$$\|\widetilde{c}^\varepsilon-\bar{c}\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}+\|\widetilde{c}^\varepsilon-\bar{c}\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}\leq c_0.$$

One must now use that there exists a function G such that $G(0)=0$ and

$$\rho-\bar{\rho}=G(c-\bar{c}).$$

Therefore, using Proposition A.2.5 related to composition operator gives

$$\|\widetilde{\rho}^\varepsilon-\bar{\rho}\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}+\|\widetilde{\rho}^\varepsilon-\bar{\rho}\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}\leq c_0.$$

This concludes the proof of the lemma. □

3.4.2 Deriving the limit system

Here we start from the following system:

$$\begin{cases} \partial_t \widetilde{\rho}^\varepsilon + \operatorname{div}(\widetilde{\rho}^\varepsilon \widetilde{v}^\varepsilon) = 0, \\ \varepsilon^2 (\partial_t \tilde{v}^\varepsilon + \tilde{v}^\varepsilon \cdot \nabla \tilde{v}^\varepsilon) + \dfrac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \tilde{v}^\varepsilon = 0. \end{cases} \tag{3.62}$$

Owing to the above corollary, $\varepsilon v^\varepsilon$ and ∇v^ε are uniformly bounded in the spaces $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$ and $L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$, which implies that

$$\varepsilon^2 v^\varepsilon \cdot \nabla v^\varepsilon = \mathcal{O}(\varepsilon) \quad \text{in} \quad L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}).$$

Corollary 3.4.1 also implies that $\varepsilon^2 \partial_t v^\varepsilon$ tends to 0 in the sense of distributions. Plugging this information in the second equation of (3.62), one may conclude that

$$\tilde{v}^\varepsilon + \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d). \tag{3.63}$$

Since Lemma 3.4.1 implies that $\widetilde{\rho}^\varepsilon - \bar{\rho}$ is uniformly bounded in $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$, there exists $\mathcal{N}$ in $\bar{\rho} + L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$ such that, up to subsequence,

$$\widetilde{\rho}^\varepsilon - \bar{\rho} \overset{*}{\rightharpoonup} \mathcal{N} - \bar{\rho} \quad \text{in} \quad L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}). \tag{3.64}$$

Now, using that $W^\varepsilon = \dfrac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \tilde{v}^\varepsilon$, the first equation of (3.62) may be written

$$\partial_t \widetilde{\rho}^\varepsilon - \Delta P(\widetilde{\rho}^\varepsilon) = \operatorname{div}(\widetilde{\rho}^\varepsilon \widetilde{W}^\varepsilon).$$

Hence, combining (3.63) and (3.64), one can hint that $\mathcal{M}$ satisfies

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0, \\ \mathcal{N}|_{t=0} = \tilde{\rho}_0^\varepsilon. \end{cases}$$

Now that we have derived the limit system, we will justify the relaxation process rigorously.

3.4.3 Strong convergence and rate of convergence

First, one needs to prove that the limit system admits solutions in suitable spaces.

We consider the following Cauchy problem

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0, \\ \mathcal{N}|_{t=0} = \mathcal{N}_0. \end{cases} \tag{3.65}$$

From Proposition 3.5.1 below, we know that if $\mathcal{N}_0 - \bar{\mathcal{N}} \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ is small enough and if (3.14) is satisfied then (3.65) has a unique global solution $\mathcal{N}$ such that $\mathcal{N} - \bar{\mathcal{N}} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$.

We can now prove Theorem 3.2.3. In what follows, it is always assumed that $\varepsilon > 0$ is small. Assume that

$$\|\widetilde{\rho}_0^\varepsilon - \mathcal{N}_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} \leq \varepsilon. \tag{3.66}$$

To derive the convergence rate we will estimate the difference of the solutions to the following two equations:

$$\partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \tag{3.67}$$

and

$$\partial_t \tilde{\rho}^\varepsilon + \operatorname{div}(\widetilde{\rho}^\varepsilon \tilde{v}^\varepsilon) = 0 \tag{3.68}$$

Again, using the damped mode $\tilde{W}^\varepsilon$, we have

$$\partial_t \widetilde{\rho}^\varepsilon - \Delta P(\widetilde{\rho}^\varepsilon) = S^\varepsilon \quad \text{with} \quad S^\varepsilon = -\operatorname{div}(\widetilde{\rho}^\varepsilon \widetilde{W}^\varepsilon)$$

Lemma 3.4.1 tells us that

$$\|\widetilde{W}^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} = \mathcal{O}(\varepsilon) \quad \text{and} \quad \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} = \mathcal{O}(1),$$

we expect S^ε to be $\mathcal{O}(\varepsilon)$ in $L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})$. Indeed, basic product laws and composition inequality give us:

$$\|S^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} \lesssim \|\widetilde{W}^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \Big(\bar{\rho} + \|\widetilde{\rho}^\varepsilon\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\Big).$$

So finally, we do have, uniformly in $T \geq 0$,

$$\|S^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} \leq C\varepsilon. \tag{3.69}$$

Next, we see that $\delta D^\varepsilon \triangleq \widetilde{\rho}^\varepsilon - \mathcal{N}$ satisfies

$$\partial_t \delta D^\varepsilon - \Delta(P(\widetilde{\rho}^\varepsilon) - P(\mathcal{N})) = S^\varepsilon$$

From Taylor formula, we know that there exists a smooth function H_1 vanishing at $\bar{\rho} = \bar{\mathcal{N}}$ such that

$$P(\widetilde{\rho}^\varepsilon) - P(\bar{\rho}) = P'(\bar{\rho})\,(\widetilde{\rho}^\varepsilon - \bar{\rho}) + H_1(\widetilde{\rho}^\varepsilon)\,(\widetilde{\rho}^\varepsilon - \bar{\rho})$$

and

$$P(\mathcal{N}) - P(\bar{\mathcal{N}}) = P'(\bar{\mathcal{N}})\,(\mathcal{N} - \bar{\mathcal{N}}) + H_1(\mathcal{N})\,(\mathcal{N} - \bar{\mathcal{N}}).$$

Assuming for simplicity that $P'(\bar{\rho}) = P'(\bar{\mathcal{N}}) = 1$, we have

$$\partial_t \delta D^\varepsilon - \Delta \delta D^\varepsilon = \Delta\,(\delta D^\varepsilon\, H_1(\widetilde{\rho}^\varepsilon)) + \Delta\,((H_1(\widetilde{\rho}^\varepsilon) - H_1(\mathcal{N}))\mathcal{N}) + S^\varepsilon$$

Therefore, using standard estimates for the heat equation as well as product laws and composition

lemma, we get for all $T \geq 0$,

$$\begin{aligned}
\|\delta D^\varepsilon\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\delta D^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} &\lesssim \|\delta D^\varepsilon_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} + \|S^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} \\
&\quad + \|\delta D^\varepsilon\,(H_1(\widetilde{\rho}^\varepsilon) - H_1(\bar{\rho}))\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \\
&\quad + \|(H_1(\widetilde{\rho}^\varepsilon) - H_1(\mathcal{N}))(\mathcal{N} - \bar{\mathcal{N}})\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \\
&\lesssim \|\delta D^\varepsilon_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} + \|S^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} \\
&\quad + \|\delta D^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \|(\widetilde{\rho}^\varepsilon - \bar{\rho}, \mathcal{N} - \bar{\mathcal{N}})\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \\
&\quad + \|\delta D^\varepsilon\|_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \|(\widetilde{\rho}^\varepsilon - \bar{\rho}, \mathcal{N} - \bar{\mathcal{N}})\|_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}.
\end{aligned}$$

Hence, using that thanks to Lemma 3.4.1 and Proposition 3.5.1, we have

$$\|(\widetilde{\rho}^\varepsilon - \bar{\rho}, \mathcal{N} - \bar{\mathcal{N}})\|_{L^\infty_T(\mathbb{B}^{\frac{d}{p}}_{p,1})} + \|(\widetilde{\rho}^\varepsilon - \bar{\rho}, \mathcal{N} - \bar{\mathcal{N}})\|_{L^2_T(\mathbb{B}^{\frac{d}{p}+1}_{p,1})} \leq c_0 \ll 1,$$

(3.66) and (3.69) yields for all $T \geq 0$,

$$\|\delta D^\varepsilon\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\delta D^\varepsilon\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \lesssim \varepsilon,$$

and Theorem 3.2.3 is proved.

3.5 Appendix

We now prove an existence result for (3.65).

Proposition 3.5.1. *Let $\mathcal{N}_0 - \bar{\mathcal{N}} \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ with $\bar{\mathcal{N}} > 0$. There exists a constant $c_0 > 0$ such that if*

$$\|\mathcal{N}_0 - \bar{\mathcal{N}}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \leq c_0 \tag{3.70}$$

then System (3.65) with P defined in (3.14) has a unique global solution $\mathcal{N}$ such that $\mathcal{N} - \bar{\mathcal{N}} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$.

Proof. Assume that we have a smooth solution $\mathcal{N}$ of:

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0, \\ \mathcal{N}|_{t=0} = \mathcal{N}_0. \end{cases} \tag{3.71}$$

There exists a function H_1 vanishing at $\bar{\mathcal{N}}$ such that:

$$P(\mathcal{N}) - P(\bar{\mathcal{N}}) = P'(\bar{\mathcal{N}})\,(\mathcal{N} - \bar{\mathcal{N}}) + H_1(\mathcal{N})\,(\mathcal{N} - \bar{\mathcal{N}}).$$

Therefore one can rewrite the first equation of (3.71) as

$$\partial_t \mathcal{N} - P'(\bar{\mathcal{N}})\Delta\mathcal{N} = \Delta\Big(H_1(\mathcal{N})\,(\mathcal{N} - \bar{\mathcal{N}})\Big).$$

Hence, using the standard estimates for the heat equation, we get for all $T > 0$,

$$\|\mathcal{N} - \bar{\mathcal{N}}\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \|\mathcal{N} - \bar{\mathcal{N}}\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})} \lesssim \|\mathcal{N}_0 - \bar{\mathcal{N}}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|H_1(\mathcal{N})\,(\mathcal{N} - \bar{\mathcal{N}})\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})}. \tag{3.72}$$

Combining product and composition estimates, we get

$$\|H_1(\mathcal{N})\,(\mathcal{N} - \bar{\mathcal{N}})\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})} \lesssim \|\mathcal{N} - \bar{\mathcal{N}}\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \|\mathcal{N} - \bar{\mathcal{N}}\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})}.$$

Hence the left-hand side of (3.72) may be bounded for all $T > 0$ in terms of the data provided (3.70) is satisfied with a small enough c_0. From that point, it is easy to deduce the existence of a small solution for (3.71). □

The following product laws in Besov spaces have been used several times.

Proposition 3.5.2. *For $2 \le p \le 4$ and $\sigma \ge s > 0$,*

$$\|ab\|^h_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|a\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|b\|^h_{\dot{\mathbb{B}}^s_{2,1}} + \|b\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|a\|^h_{\dot{\mathbb{B}}^s_{2,1}} + \|a\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \|b\|^\ell_{\dot{\mathbb{B}}^\sigma_{p,1}} + \|b\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}} \|a\|^\ell_{\dot{\mathbb{B}}^\sigma_{p,1}} \tag{3.73}$$

where $p^* := \dfrac{2p}{p-2} \ge p$.

Proof. We combine Bony's decomposition and the decomposition of a and b in low and high frequencies, writing

$$ab = T_{a^\ell} b^\ell + T'_{b^\ell} a^\ell + T'_b a^h + T_a b^h + T'_{b^h} a^\ell + T_{a^h} b^\ell.$$

All the terms in the right-hand side, except for the last two ones, may be bounded by means of the standard results of continuity for operator T (see again [3, Chap. 2]): provided $\sigma \ge s > 0$, we get,

$$\|T'_{b^\ell} a^\ell\|^h_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|T'_{b^\ell} a^\ell\|^h_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim \|b^\ell\|_{L^{p^*}} \|a^\ell\|_{\dot{\mathbb{B}}^\sigma_{p,1}},$$
$$\|T_b a^h\|_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|b\|_{L^\infty} \|a^h\|_{\dot{\mathbb{B}}^s_{2,1}}.$$

Finally, since $a^\ell = \dot{S}_{J_0+1} a$ and $b^h = (\mathrm{Id} - \dot{S}_{J_0+1}) b$, we see that

$$T'_{b^h} a^\ell = \dot{S}_{J_0+2} b^h \, \dot{\Delta}_{J_0+1} a^\ell.$$

Consequently, as $\dot{S}_{J_0+2} b^h = (\dot{\Delta}_{J_0-1} + \dot{\Delta}_{J_0} + \dot{\Delta}_{J_0+1}) b^h$,

$$\|T_{b^h} a^\ell\|_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|\dot{\Delta}_{J_0+1} a^\ell\|_{L^\infty} \|\dot{S}_{J_0+2} b^h\|_{L^2} \lesssim \|a\|_{L^\infty} \|b\|^h_{\dot{\mathbb{B}}^s_{2,1}}.$$

Adding up this latter inequality to the previous one and the symmetric ones (with just operator T instead of T'), using Bernstein inequality, and the embeddings $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \hookrightarrow L^\infty$ and, as $p \le p*$, $\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1} \hookrightarrow L^{p^*}$ completes the proof of (3.73). □

To handle commutator in high frequencies, we will need the following lemma.

Lemma 3.5.1. *Let $p \in [2,4]$ and $s > 0$. Define $p^* \triangleq 2p/(p-2)$. For all $j \in \mathbb{Z}$, denote $\mathfrak{R}_j \triangleq \dot{S}_{j-1} w \, \dot{\Delta}_j z - \dot{\Delta}_j (wz)$.*

There exists a constant C depending only on the threshold number J_0 between low and high frequencies and on s, p, d, such that

$$\sum_{j\geq J_0}\left(2^{js}\left\|\mathfrak{R}_j\right\|_{L^2}\right)\leq C\Big(\|\nabla w\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}+\|z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{\sigma_1}_{p,1}}$$
$$+\|z\|_{\dot{\mathbb{B}}^{\frac{d}{p}-k}_{p,1}}\|w\|^h_{\dot{\mathbb{B}}^{s+k}_{2,1}}+\|z\|^\ell_{\dot{\mathbb{B}}^{\sigma_2}_{p,1}}\|\nabla w\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\Big),$$

for any $k\geq 0$, $\sigma_1\geq s$ and $\sigma_2\in\mathbb{R}$.

Proof. From Bony's decomposition recalled above, we deduce that

$$\begin{aligned}\mathfrak{R}_j&=-\dot{\Delta}_j(T'_z w)-\sum_{|j'-j|\leq 4}[\dot{\Delta}_j,\dot{S}_{j'-1}w]\dot{\Delta}_{j'}z-\sum_{|j'-j|\leq 1}\left(\dot{S}_{j'-1}w-\dot{S}_{j-1}w\right)\dot{\Delta}_j\dot{\Delta}_{j'}z\\&\triangleq\mathcal{R}^1_j+\mathcal{R}^2_j+\mathcal{R}^3_j.\end{aligned}$$

To estimate $\mathcal{R}^1_j$, we use the decomposition

$$T'_z w=T'_{z^\ell}w^\ell+T'_{z^h}w^\ell+T'_z w^h$$

and proceed as in the proof of Proposition 3.5.2. In the end, we get

$$\|T'_z w\|^h_{\dot{\mathbb{B}}^s_{2,1}}\lesssim\|z\|_{\dot{\mathbb{B}}^{-k}_{\infty,1}}\|w\|^h_{\dot{\mathbb{B}}^{s+k}_{2,1}}+\|z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{\sigma_1}_{p,1}}.$$

Therefore, since $\dot{\mathbb{B}}^{\frac{d}{p}-k}_{p,1}\hookrightarrow\dot{\mathbb{B}}^{-k}_{\infty,1}$,

$$\sum_{j\in\mathbb{Z}}\left(2^{js}\left\|\mathcal{R}^1_j\right\|_{L^2}\right)\leq C\Big(\|z\|_{\dot{\mathbb{B}}^{\frac{d}{p}-k}_{p,1}}\|w\|^h_{\dot{\mathbb{B}}^{s+k}_{2,1}}+\|z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{\sigma_1}_{p,1}}\Big).\tag{3.74}$$

Next, taking advantage of Proposition A.2.3, we see that if $j'\geq J_0$ and $|j-j'|\leq 4$, then we have

$$2^{js}\|[\dot{\Delta}_j,\dot{S}_{j'-1}w]\dot{\Delta}_{j'}z\|_{L^2}\lesssim\|\nabla\dot{S}_{j'-1}w\|_{L^\infty}\,2^{j'(s-1)}\|\dot{\Delta}_{j'}z\|_{L^2}$$

while, if $j'<J_0$, $j\geq J_0$ and $|j-j'|\leq 4$,

$$\begin{aligned}2^{js}\|[\dot{\Delta}_j,\dot{S}_{j'-1}w]\dot{\Delta}_{j'}z\|_{L^2}&\lesssim 2^{J_0(s-\sigma_2-1)}2^{j(\sigma_2+1)}\|[\dot{\Delta}_j,\dot{S}_{j'-1}w]\dot{\Delta}_{j'}z\|_{L^2}\\&\lesssim 2^{J_0(s-\sigma_2-1)}\|\nabla\dot{S}_{j'-1}w\|_{L^{p^*}}\,2^{j'\sigma_2}\|\dot{\Delta}_{j'}z\|_{L^p}.\end{aligned}$$

Therefore,

$$\sum_{j\geq J_0}\left(2^{js}\left\|\mathcal{R}^2_j\right\|_{L^2}\right)\;\leq\;\|z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}\|\nabla w\|_{L^\infty}+\|z\|^\ell_{\dot{\mathbb{B}}^{\sigma_2}_{p,1}}\|\nabla w\|^\ell_{L^{p^*}}.\tag{3.75}$$

Then, as $p\leq p*$, with suitable embeddings one gets

$$\sum_{j\geq J_0}\left(2^{js}\left\|\mathcal{R}^2_j\right\|_{L^2}\right)\;\leq\;\|z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}\|\nabla w\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}+\|z\|^\ell_{\dot{\mathbb{B}}^{\sigma_2}_{p,1}}\|\nabla w\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}.\tag{3.76}$$

Finally, for all $j \geq J_0$ and $|j' - j| \leq 1$, we have

$$\begin{aligned} 2^{js}\|(\dot{S}_{j'-1}w - \dot{S}_{j-1}w)\dot{\Delta}_j\dot{\Delta}_{j'}z\|_{L^2} &\leq 2^j\|\dot{\Delta}_{j\pm1}w\|_{L^\infty}\, 2^{j(s-1)}\|\dot{\Delta}_{j'}\dot{\Delta}_j z\|_{L^2} \\ &\leq C\|\dot{\Delta}_{j\pm1}\nabla w\|_{L^\infty}\, 2^{j(s-1)}\|\dot{\Delta}_j z\|_{L^2}. \end{aligned}$$

Hence

$$\sum_{j\geq J_0}\left(2^{js}\left\|\mathcal{R}_j^3\right\|_{L^2}\right) \leq C\|\nabla w\|_{L^\infty}\|z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}}. \tag{3.77}$$

Putting (3.74), (3.76) and (3.77) together yields the desired estimate.

□

Chapitre 4

Relaxation limit for a damped one-velocity Baer-Nunziato model to a Kapila model

This chapter is based on the paper [21].

4.1 Introduction

In this chapter we study a damped Baer-Nunziato bi-fluid model in $\mathbb{R}^d$ which reads:

$$\begin{cases} \partial_t \alpha_\pm + u \cdot \nabla \alpha_\pm = \pm \dfrac{\alpha_+ \alpha_-}{2\mu + \lambda} \left(P_+ \left(\rho_+ \right) - P_- \left(\rho_- \right) \right), \\ \partial_t \left(\alpha_\pm \rho_\pm \right) + \operatorname{div} \left(\alpha_\pm \rho_\pm u \right) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{A}_{\mu,\lambda} u + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = \alpha_+ P_+ \left(\rho_+ \right) + \alpha_- P_- \left(\rho_- \right) \end{cases} \tag{BN}$$

It is the multi-dimensional version of the system derived by Bresch, Burtea and Lagoutière in [17] to which we added a dissipation term to be able to study its singular limit as $\mu, \lambda \to 0$ in the context of partially dissipative first order quasilinear systems. Here are the main features of this chapter:

- Even if the System (BN) does not satisfy the (SK) condition nor is symmetric, we are able to prove a global well-posedness result for System (BN) in a similar framework as developed in the previous chapters.
- Moreover, the uniformity with respect to μ, λ of the estimates on the solution obtained from the existence theorem allows us to consider the relaxation limit. We justify the strong convergence from system (BN) to the so-called Kapila system and derive an explicit convergence rate of the relaxation process.

4.1.1 Context

Multiphase flows are ubiquitous in real world applications ranging from engineering to biological systems. The term multiphase includes flows that are topologically very different. As

it is explained in [61], we distinguish mixtures with separated phases flows (film flows, jet flows), mixed or transitional phase flows (gas pockets in liquids) and dispersed phase flows (bubbly flows, sprays). Understanding the mathematical qualitative properties of the different governing models is important, for instance, in order to construct more pertinent numerical schemes which would increase the predictive power of these models.

In order to describe *dispersed two-phase flows*, besides the classical variables like densities, velocities and state laws, one need to introduce two extra ones called volume fractions, measuring *how much space* does one phase occupy at a given position in space. It goes without saying that in order to have a closed system, besides the equations expressing conservation of mass and momentum, two extra equations are needed.

In [2], Baer and Nunziato proposed a model for which the volume fractions verify

$$\partial_t \alpha_\pm + v_I \cdot \nabla \alpha_\pm = \frac{P_\pm - P_\mp}{\varepsilon}, \tag{4.1}$$

where ε can be seen as a time relaxation parameter and, in practice, is chosen to be very small. The unknown v_I is interpreted as an interface velocity which depends on the densities, volumes fractions and phase velocities. In [2], the interface velocity coincides with one of the phase velocities but different choices have also been used in the literature, see for instance [57] and [90].

From the Baer-Nunziato model, Kapila et al. [63] proposed the following equations

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ P_+ = P_-. \end{cases} \tag{4.2}$$

Of course, one might wonder what is the link between (4.1) and (4.2). First of all, from (4.1) one immediately infers that

$$\partial_t \left(\alpha_+ + \alpha_-\right) + v_I \cdot \nabla(\alpha_+ + \alpha_-) = 0.$$

Hence

$$\alpha_+ + \alpha_- = 1$$

is recovered provided it is true initially. Moreover, one expects to obtain the second equation of (4.2) in the limit $\varepsilon \to 0$. The same closure equations (4.2) are obtained using the so-called averaging methods [61] or variational methods [20, 45]. In this framework, one can interpret the Baer-Nunziato model as a relaxation model for the Kapila model.

Let us mention that in the mathematical community [13, 15], equations (4.1) are refereed to as PDEs closure laws while (4.2) are called algebraic closure laws.

Multiphase models received a lot of attention from the mathematical community recently. In the context of weak solutions we mention the results of Novotny [78], Novotny and Pokorny [79], Bresch, Mucha and Zatorska [16], Vasseur, Wen and Yu [96]. About applications in biology see Gwiazda et al [84] and Debiec et al [40]. We shall bring up that in all the above papers, viscosity plays a crucial role.

In [17], constructing upon previous works [12–14], the authors showed that it is possible to obtain a viscous Baer-Nunziato system following a homogenization procedure. The basic assumption is that if we zoom in the mixture, we arrive at a mesoscale where the two phases are separated. Assuming that each phase verifies the Navier-Stokes equations in their own domain, the authors were able to write a closed system for the mixture. When going back to the macroscopic scale, loosely speaking, the density of a fluid mixture is assumed to wildly oscillate

between two reference densities. The propagation of oscillations is quantified through Young measures and it is shown that if these measures are convex combinations of Dirac masses at initial time, then this structure is preserved for later times. *At this point, it is important to note that in these papers, the authors obtained equations for the volume fractions $\alpha_\pm$ of the form* (4.1) *while the relaxation time ε is proportional to the mean viscosity of the two phases.*

4.1.2 Presentation of the models

We consider a mixture of two compressible fluids filling the ambient space $\mathbb{R}^d$ with $d \geq 2$. The characteristic state function of the two phases will be denoted separately by $+$, $-$. We suppose, as in many applications, that the flow of the mixture is animated by a single velocity vector field:

$$u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d.$$

We denote the two mass densities of the phases $\pm$ by

$$\rho_\pm : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$$

and we introduce the volume fractions of the fluid $\pm$

$$\alpha_\pm : \mathbb{R}_+ \times \mathbb{R}^d \to [0,1].$$

The multidimensional version of the system obtained in [17] reads:

$$\begin{cases} \partial_t \alpha_\pm + u \cdot \nabla \alpha_\pm = \pm \dfrac{\alpha_+ \alpha_-}{2\mu + \lambda} \left(P_+ (\rho_+) - P_- (\rho_-) \right), \\ \partial_t \left(\alpha_\pm \rho_\pm \right) + \operatorname{div} \left(\alpha_\pm \rho_\pm u \right) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{A}_{\mu,\lambda} u + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = \alpha_+ P_+ (\rho_+) + \alpha_- P_- (\rho_-) \end{cases} \tag{BN}$$

where we added a damping term in the equation of the velocity with a parameter $\eta \geq 1$. This terms models elastic-type drag forces slowing down the fluid and it is a crucial in our mathematical analysis since it allows to use techniques coming from the theory of partially dissipative hyperbolic systems. The relevant model with common pressure is obtained in [12]. Above, $\mathcal{A}_{\mu,\lambda}$ stands for the Lamé operator:

$$\mathcal{A}_{\mu,\lambda} u = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u$$

with μ, λ given constants verifying

$$\mu \geq 0, \ \lambda + \mu \geq 0 \quad \text{and} \quad \nu = \lambda + 2\mu \leq 1.$$

The functions P_+ and P_- model the internal barotropic pressures for each fluid. We will assume that they take the following explicit form

$$P_\pm (s) = A_\pm s^{\gamma_\pm} \quad \text{for all } s \geq 0, \tag{4.3}$$

where $\gamma_\pm \geq 1$, $A_\pm > 0$ are given constants. Moreover, without loss of generality, we will suppose that

$$\gamma_+ > \gamma_-. \tag{4.4}$$

The density and pressure of the mixture are denoted by

$$\rho = \alpha_+\rho_+ + \alpha_-\rho_- \quad \text{and} \quad P = \alpha_+ P_+(\rho_+) + \alpha_- P_-(\rho_-).$$

We are concerned with solutions that satisfy

$$\alpha_\pm(t,x) \to \bar{\alpha}_\pm, \quad \rho_\pm(t,x) \to \bar{\rho}_\pm, \quad u(t,x) \to 0_{\mathbb{R}^d} \quad \text{as } |x| \to \infty, \tag{4.5}$$

where $0 < \bar{\alpha}_\pm \leq 1, 0 < \bar{\rho}_\pm$ are given constants and

$$\bar{\alpha}_+ + \bar{\alpha}_- = 1. \tag{4.6}$$

We denote by $(\alpha_{\pm 0}, \rho_{\pm 0}, u_0)$ the initial condition:

$$\alpha_\pm(t,x)|_{t=0} = \alpha_{\pm 0}, \quad \rho_\pm(t,x)|_{t=0} = \rho_{\pm 0}, \quad u(t,x)|_{t=0} = u_0, \quad x \in \mathbb{R}^d. \tag{4.7}$$

Notice that when $\nu := 2\mu + \lambda$ tends to 0, System (BN) formally converges to the following system:

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ \partial_t(\alpha_\pm\rho_\pm) + \operatorname{div}(\alpha_\pm\rho_\pm u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P + \eta\rho u = 0, \\ \rho = \alpha_+\rho_+ + \alpha_-\rho_-, \\ P = P_+(\rho_+) = P_-(\rho_-). \end{cases} \tag{K}$$

Our main goal is to justify the relaxation/inviscid limit on a solid mathematical background, at least when the initial data $(\alpha_{\pm 0}, \rho_{\pm 0}, u_0)$ are close to the constant equilibrium $(\bar{\alpha}_\pm, \bar{\rho}_\pm, 0)$. Inasmuch as we expect the limiting pressures to agree in the vanishing viscosity limit, and in order to avoid initial time layers, we will suppose that the pressures are at equilibrium at infinity:

$$P_+(\bar{\rho}_+) = P_-(\bar{\rho}_-) \overset{not.}{=} \bar{P}. \tag{4.8}$$

Throughout the chapter, C stands for a "harmless" constant and we use different notations from the previous chapters concerning the Besov spaces: we fix a homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ that is defined by

$$\dot{\Delta}_j \triangleq \varphi(2^{-j}D)$$

where φ stands for a smooth function supported in $\mathcal{C} = \{\xi \in \mathbb{R}^d,\ 5/6 \leq |\xi| \leq 12/5\}$ such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \text{ for } \xi \neq 0,$$

which is a slightly different annulus compared to the one used in the appendix. Then, we introduce the homogeneous Besov semi-norms:

$$\|u\|_{B^s} \triangleq \left\| 2^{js} \|\dot{\Delta}_j u\|_{L^2(\mathbb{R}^d)} \right\|_{\ell^1(\mathbb{Z})}.$$

Notice that, since we will only work with homogeneous Besov spaces based with second index equal to 2 and third index equal to 1, we omitted those index in the definition of the norm. This modification in the definition does not change any of the properties satisfied by these spaces compared to the one defined in the appendix and will just affect the coefficients appearing in the

computations.

4.2 Main results

We are now in the position of stating our main results. First we state our uniform (with respect to μ and λ) global existence result for the System (*BN*).

Theorem 4.2.1. *Let $d \geq 2$ and assume that the parameters satisfy $\mu \geq 0, \lambda + \mu \geq 0, \nu \leq 1$ and $\eta \geq 1$. Let constants $\bar{\alpha}_\pm \in (0,1)\,, \bar{\rho}_\pm > 0$ satisfy (4.6) and (4.8). There exists a constant $c_1 > 0$ independent of the viscosity coefficients μ, λ such that for any initial data $(\alpha_{+0}, \alpha_{-0}, \rho_{+0}, \rho_{-0}, u_0)$ verifying*

$$\|(\alpha_{\pm 0} - \bar{\alpha}_\pm, \rho_{\pm 0} - \bar{\rho}_\pm, u_0)\|_{B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}} \leq c_1,$$

then System (BN) admits a unique global-in-time solution $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$ such that

$$\begin{cases} (\alpha_\pm - \bar{\alpha}_\pm, \rho_\pm - \bar{\rho}_\pm, u) \in \mathcal{C}_b(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}), \\ \dfrac{P_+(\rho_+) - P_-(\rho_-)}{2\mu + \lambda} \in L^1(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}}) \quad \text{and} \quad u \in L^1(\mathbb{R}_+; B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}). \end{cases}$$

Moreover, the following estimate holds true uniformly with respect to the viscosity coefficients μ and λ:

$$\|(\alpha_\pm - \bar{\alpha}_\pm, \rho_\pm - \bar{\rho}_\pm, u)\|_{L^\infty(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} + \|u\|_{L^1(\mathbb{R}_+; B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1})} \\ \frac{1}{2\mu + \lambda} \|P_+(\rho_+) - P_-(\rho_-)\|_{L^1(\mathbb{R}_+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}})} \leq C c_1,$$

with a universal constant $C > 0$.

Remark 4.2.1. *The same result is valid for the quasilinear first order system associated to (BN), in other words, the viscosity plays no role in the mathematical analysis, the same result is valid if $\mathcal{A}_{\mu,\lambda} \equiv 0$.*

As a consequence of relaxation limit arguments, we obtain the following theorem regarding System (*K*).

Theorem 4.2.2. *Let $d \geq 2$ and $\eta \geq 1$. Let constants $\bar{\alpha}_\pm \in (0,1)\,, \bar{\rho}_\pm > 0$ satisfy (4.6) and (4.8). There exists a constant $c_2 > 0$ depending on $\eta, \gamma_\pm, A_\pm$ and d such that for any initial data $(\alpha_{+0}, \alpha_{-0}, \rho_{+0}, \rho_{-0}, u_0)$ verifying*

$$\|(\alpha_{\pm 0} - \bar{\alpha}_\pm, \rho_{\pm 0} - \bar{\rho}_\pm, u_0)\|_{B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}} \leq c_2,$$

then System (K) admits a unique global-in-time solution $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$ such that

$$\begin{cases} (\alpha_\pm - \bar{\alpha}_\pm, \rho_\pm - \bar{\rho}_\pm, u) \in \mathcal{C}_b(\mathbb{R}_+, B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}), \\ \rho_\pm - \bar{\rho}_\pm \in L^1(\mathbb{R}_+; B^{\frac{d}{2}+1}) \quad \text{and} \quad u \in L^1(\mathbb{R}_+; B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}). \end{cases}$$

It turns out that we can further obtain a convergence rate of solutions of System (*BN*) towards solutions of System (*K*).

Theorem 4.2.3. *Let $d \geq 3$ and assume the same hypothesis on the parameters as in Theorem 4.2.1. Let $(\alpha_+^\nu, \alpha_-^\nu, \rho_+^\nu, \rho_+^\nu, u^\nu)$ (resp. $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$) be the solution to the Cauchy problem*

(BN), associated with the initial data $(\alpha^\nu_{+0}, \alpha^\nu_{-0}, \rho^\nu_{+0}, \rho^\nu_{-0}, u^\nu_0)$, from Theorem 4.2.1 (resp. (K)-(4.7) from Theorem 4.2.2) such that

$$\|(\frac{\alpha^\nu_{+0}\rho^\nu_{+0}}{\alpha^\nu_{+0}\rho^\nu_{+0}+\alpha^\nu_{0-}\rho^\nu_{-0}} - \frac{\alpha_{+0}\rho_{+0}}{\alpha_{+0}\rho_{+0}+\alpha_{-0}\rho_{-0}}, P^\nu_{\pm 0} - P_{\pm 0}, u^\nu_0 - u_0)\|_{B^{\frac{d}{2}-\frac{3}{2}}\cap B^{\frac{d}{2}-\frac{1}{2}}}$$
$$+ \|P^\nu_{+0} - \frac{\gamma_+\alpha^\nu_- P^\nu_+}{\gamma_+\alpha^\nu_- P^\nu_+ + \gamma_-\alpha^\nu_+ P^\nu_-}(P^\nu_{+0} - P^\nu_{-0}) - P_{+0}\|_{B^{\frac{d}{2}-\frac{3}{2}}\cap B^{\frac{d}{2}-\frac{1}{2}}} \leq C\sqrt{\nu}$$

Then there exists a constant $C > 0$ independent of ν such that $(\alpha^\nu_+, \alpha^\nu_-, \rho^\nu_+, \rho^\nu_+, u^\nu)$ converges toward $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$ in the following sense

$$\|(\alpha^\nu_\pm - \alpha_\pm, \rho^\nu_\pm - \rho_\pm, \rho^\nu_- - \rho_-, u^\nu - u)\|_{L^\infty(B^{\frac{d}{2}-\frac{1}{2}})} + \|\rho^\nu_\pm - \rho_\pm\|_{L^2(B^{\frac{d}{2}-\frac{1}{2}})}$$
$$+ \|u^\nu - u\|_{L^1(B^{\frac{d}{2}-\frac{1}{2}})} \leq C\sqrt{\nu}.$$

Remark 4.2.2. *The rather strange condition on the difference of initial data is due to a technical limitation on the composition arguments, see Proposition A.2.6 for more details. In Section 6, we state a more comprehensive theorem concerning stability between the two systems, which immediately implies Theorem 4.2.3.*

4.2.1 Strategy of the proof

Our problem is not covered by the papers mentioned previously and it is not completely clear if and how the general theories from [4, 31, 46–49] could be adapted to study (BN) and the associated relaxation limit. The first obvious reason is that (BN) is not a system of conservation laws because the equations of the volume fractions cannot be put in conservative form. The second reason is that the entropy that is naturally associated with this system is not positive definite since it is linear with respect to the volume fractions. Concerning global existence results, we remark that the associated quasilinear system does not satisfy the (SK) condition as it admits the eigenvalue 0. It turns out that the situation is not too degenerate in the sense that the eigenspace associated to the eigenvalue 0 is of dimension 1 and that, roughly speaking, the non-degenerate part (i.e. the part associated to non-zero eigenvalues) fulfils the (SK) condition. Thus we will be able to isolate the undamped mode, and rewrite the remaining system as a partially dissipative quasilinear system satisfying the (SK) condition while the undamped mode will be seen as a parameter and always appears in nonlinear terms as a prefactor of a function of the damped variable.

More precisely, after observing that there exist four main unknowns in the system, namely, α_+, ρ_+, ρ_- and u we consider a change of variables that leaves the velocity u invariant:

$$(y, w, r) = \Phi(\alpha_+, \rho_+, \rho_-)$$

where the variable w is proportional to $P_+ - P_-$ and r is like an effective pressure. The system verified by the new variables is of the form

$$\begin{cases} \partial_t y + u\cdot\nabla y = 0, \\ \partial_t w + u\cdot\nabla w + \big(\bar{H}_1 + H_1(w,r,y)\big)\operatorname{div} u + \big(\bar{H}_2 + H_2(w,r,y)\big)\dfrac{w}{\nu} = 0, \\ \partial_t r + u\cdot\nabla r + \big(\bar{H}_3 + H_3(w,r,y)\big)\operatorname{div} u = \big(\bar{H}_4 + H_4(w,r,y)\big)\dfrac{w^2}{\nu}, \\ \partial_t u + u\cdot\nabla u - \dfrac{1}{\rho}\mathcal{A}_{\mu,\lambda}u + \eta u + \dfrac{1}{\rho}\nabla r + (\gamma_+ - \gamma_-)\dfrac{1}{\rho}\nabla w = 0, \end{cases} \tag{4.9}$$

where $\bar{H}_i$ and H_i (for $i = 1, 2, 3, 4$) stands for the constant part and perturbation part, respectively. The very specific form of the nonlinear part appearing in the above system is crucial to close our estimates. Obviously there is no hope to recover time decay properties for y and in fact we only need L^∞_T integrability on y, therefore we will treat this transport equation separately. Considering the system satisfied by the three unknowns (w, r, u) and by adapting similar ideas developed in [4, 30, 31] to this sub-system, we will obtain the necessary integrability on all the components of the solution, which will allow us to obtain uniform a priori estimates with respect to λ and μ.

One of the main difference with the previous chapters is that we cannot perform a rescaling to keep track of the coefficients as what they did, because we are not able to treat the low frequencies in $B^{d/2}$ as ρ lacks of time integrability since the complex form of the pressure. Moreover, it is important to point out that the System (4.9) does not verify the (SK) condition and it is not symmetric. However, the subsystem formed by the last three equations of (4.9) satisfies the (SK) condition and the coupling with the first equation is achieved via lower-order terms. In order to deal with the lack of symmetry, we construct a nonlinear energy-functional in order to derive a priori estimates.

Another technical difficulty that we encounter is that one cannot recover uniform dissipation with respect to $1/\nu$ for w at the higher energy level (i.e. $d/2 + 1$). It is only possible to recover such a strong decay effect for the $d/2-$energy level. Therefore, estimating nonlinear terms which are proportional to $1/\nu$ are delicate. It turns out that the quadratic form of the non-linearity appearing in the equation of r is crucial, indeed some other reformulation we performed led to non-linearity that couldn't be controlled.

Moving on to the justification of the relaxation limit, the fact that the solutions to the Kapila system (K) are obtained as limits of the (BN) system is a consequence of the uniform estimates and classical weak-compactness arguments. In order to obtain a convergence rate, we estimates the difference of the solutions of the two systems. Since we are not able to a obtain decay rate for $\partial_t(P^\nu_+ - P^\nu_-)$ in any space, we define a new unknown to avoid treating this term as a source term. Under a smallness assumption depending on ν and on the difference of the initial data we are able to obtain decay rate of $\sqrt{\nu}$ in $L^\infty(B^{\frac{d}{2}-\frac{3}{2}} \cap B^{\frac{d}{2}-\frac{1}{2}})$ which allows us to recover decay rate for the source terms involving the unknown $\alpha_\pm$. But still, it seems hard to control $\alpha_\pm$ in the spaces $L^1(B^s)$ for any s, to our knowledge, this is the reason why we end up with a convergence rate equal to $\sqrt{\nu}$.

The rest of the chapter unfolds as follows.

In Section 3, we rewrite the original (BN)-system into good unknowns and give a theorem for the reformulated system, that is Theorem 4.3.1.

In Section 4 we derive a priori estimates uniform with respect to the viscosity parameters for a mixed linear partially dissipative hyperbolic system and prove Theorem 4.3.1.

The fifth section is devoted to the stability estimates between (BN)-system and (K)-system, which implies Theorem 4.2.3.

Finally in the Appendix, we show some basic estimates for several classical linear problem.

4.3 A reformulation of the System (BN) and sketch of the proof

The first part of this section will concern the reformulation of the System (BN) so it is in the range of application of recent developments about partially dissipative hyperbolic system.

Then we state a global existence result which contains the statement of Theorem 4.2.1.

4.3.1 Change of unknowns

Here, we propose new unknowns that are more appropriate to obtain uniform a priori estimates. First, observe that by adding the equations of α_+ and α_- together we get

$$D_t(\alpha_+ + \alpha_-) = 0,$$

where D_t is the material derivative defined by

$$D_t := \partial_t + u \cdot \nabla. \tag{4.10}$$

Thus, at least formally, if the initial data considered satisfies

$$\alpha_{+0} + \alpha_{-0} = 1,$$

then this property remains true for latter times. Thus, the number of independent unknowns for system (BN) is reduced to four: α_+, ρ_+, ρ_- and u.

We now consider the change of unknowns from $(\alpha_+, \rho_+, \rho_-)$ to (w, R, Y) given by

$$(w, R, Y) := \Phi(\alpha_+, \rho_+, \rho_-) = (\Phi_1(\alpha_+, \rho_+, \rho_-), \Phi_2(\alpha_+, \rho_+, \rho_-), \Phi_3(\alpha_+, \rho_+, \rho_-)), \tag{4.11}$$

with

$$\begin{cases} \Phi_1(\alpha_+, \rho_+, \rho_-) = \dfrac{P_+(\rho_+) - P_-(\rho_-)}{\dfrac{\gamma_+}{\alpha_+} + \dfrac{\gamma_-}{\alpha_-}}, \\ \Phi_2(\alpha_+, \rho_+, \rho_-) = \alpha_+ P_+(\rho_+) + \alpha_- P_-(\rho_-) - (\gamma_+ - \gamma_-) \dfrac{P_+(\rho_+) - P_-(\rho_-)}{\dfrac{\gamma_+}{\alpha_+} + \dfrac{\gamma_-}{\alpha_-}}, \\ \Phi_3(\alpha_+, \rho_+, \rho_-) = \dfrac{\alpha_+ \rho_+}{\alpha_+ \rho_+ + \alpha_- \rho_-}. \end{cases} \tag{4.12}$$

We see its equilibrium state $(\bar{w}, \bar{R}, \bar{Y})$ will be $(0, \bar{P}, \frac{\bar{\alpha}_+ \bar{\rho}_+}{\bar{\alpha}_+ \bar{\rho}_+ + \bar{\alpha}_- \bar{\rho}_-})$. The differential of the transformation computed at $(\bar{\alpha}_+, \bar{\rho}_+, \bar{\rho}_-)$ is

$$\begin{pmatrix} 0 & \dfrac{P'_+(\bar{\rho}_+)}{\dfrac{\gamma_+}{\bar{\alpha}_+} + \dfrac{\gamma_-}{\bar{\alpha}_-}} & \dfrac{-P'_-(\bar{\rho}_-)}{\dfrac{\gamma_+}{\bar{\alpha}_+} + \dfrac{\gamma_-}{\bar{\alpha}_-}} \\ 0 & \bar{\alpha}_+ P'_+(\bar{\rho}_+) - (\gamma_+ - \gamma_-) \dfrac{P'_+(\bar{\rho}_+)}{\dfrac{\gamma_+}{\bar{\alpha}_+} + \dfrac{\gamma_-}{\bar{\alpha}_-}} & \bar{\alpha}_- P'_-(\bar{\rho}_-) + (\gamma_+ - \gamma_-) \dfrac{P'_-(\bar{\rho}_-)}{\dfrac{\gamma_+}{\bar{\alpha}_+} + \dfrac{\gamma_-}{\bar{\alpha}_-}} \\ \dfrac{\bar{\rho}_+ \bar{\rho}_-}{\bar{\rho}^2} & \dfrac{\bar{\alpha}_+ \bar{\alpha}_- \bar{\rho}_-}{\bar{\rho}^2} & -\dfrac{\bar{\alpha}_+ \bar{\alpha}_- \bar{\rho}_+}{\bar{\rho}^2} \end{pmatrix},$$

such that the Jacobian of the transformation computed at $(\bar{\alpha}_+, \bar{\rho}_+, \bar{\rho}_-)$ is

$$J_{|(\bar{\alpha}_+, \bar{\rho}_+, \bar{\rho}_-)} = \frac{\bar{\rho}_+ \bar{\rho}_-}{\bar{\rho}^2} \cdot \frac{P'_+(\bar{\rho}_+) P'_-(\bar{\rho}_-)}{\dfrac{\gamma_+}{\bar{\alpha}_+} + \dfrac{\gamma_-}{\bar{\alpha}_-}} > 0.$$

Thus, owing to the Inverse Function Theorem there exists constants δ_1, δ_2 and a function

$$\begin{aligned}\Psi : B_{\delta_2}(\bar{w}, \bar{R}, \bar{Y}) &\to B_{\delta_1}(\bar{\alpha}_+, \bar{\rho}_+, \bar{\rho}_-)\\ (w, R, Y) &\mapsto (\alpha_+, \rho_+, \rho_-)\end{aligned}$$

such that Ψ is one-to-one and Ψ is the inverse of the restriction of Φ to the ball $B_{\delta_1}(\bar{\alpha}_+, \bar{\rho}_+, \bar{\rho}_-)$. Thus, Ψ is smooth on the ball $B_{\delta_2}(\bar{w}, \bar{R}, \bar{Y}) \subset \mathbb{R}^3$. An extension theorem in the book of Evans [42] (p. 254) enables us to assume that Ψ is smooth in $\mathbb{R}^3$ without loss of generality, this extension of Ψ will be useful later to use a composition lemma.

Consider $(w, R, Y, u)\,(0, x)$ such that

$$\|(w, R, Y, u)(0, \cdot) - (0, \bar{P}, \frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_+ + \bar{\alpha}_-\bar{\rho}_-}, 0)\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}} \leq c \tag{4.13}$$

where c is chosen such that $c \leq \frac{\delta_2}{4}$. Furthermore, define

$$T_{\max} = \sup\left\{T > 0 : \|(w, R, Y, u)(t, \cdot) - (0, \bar{P}, \frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_+ + \bar{\alpha}_-\bar{\rho}_-}, 0)\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}} \leq 2c\right\}. \tag{4.14}$$

Obviously, owing to the embedding $B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1} \hookrightarrow L^\infty_{t,x}$, it is clear that $(w, R, Y, u)(t, x)$ lies in a ball centered in $(0, \bar{P}, \frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_+ + \bar{\alpha}_-\bar{\rho}_-}, 0)$ with radius depending on δ_2, on the time interval $[0, T_{\max})$. Theorem 4.3.1 in the next subsection shows that for c chosen sufficiently small then $T_{\max} = +\infty$, and thus $(\alpha_+, \rho_+, \rho_-) = \Psi(w, R, Y)$ for all time.

Let us now derive the equations of (w, R, Y, u). We observe from the first and the second equations of System (BN) that

$$\alpha_\pm(\partial_t\rho_\pm + \operatorname{div}(\rho_\pm u)) = \mp\frac{\alpha_+\alpha_-\rho_\pm}{\nu}(P_+(\rho_+) - P_-(\rho_-)).$$

Since we work with the power laws $P_\pm(\rho_\pm) = A_\pm\rho_\pm^{\gamma_\pm}$ (which we simply represent by $P_\pm$), by multiplying the two equations above by $P'_\pm$ respectively, we obtain that

$$D_tP_\pm + \gamma_\pm P_\pm \operatorname{div} u = \mp\frac{\gamma_\pm\alpha_\mp P_\pm}{\nu}(P_+ - P_-).$$

Then, taking into consideration the equations of $\alpha_\pm$, we get that

$$D_tP + (\gamma_+\alpha_+P_+ + \gamma_-\alpha_-P_-)\operatorname{div} u + \frac{\alpha_+\alpha_-}{\nu}(P_+ - P_-)\big((\gamma_+ - 1)\,P_+ - (\gamma_- - 1)\,P_-\big) = 0, \tag{4.15}$$

which we further put under the form

$$D_tP+(\gamma_+\alpha_+P_++\gamma_-\alpha_-P_-)\operatorname{div} u+(\gamma_+ - 1)\,\frac{\alpha_+\alpha_-}{\nu}(P_+-P_-)^2+(\gamma_+ - \gamma_-)\,\frac{\alpha_+\alpha_-}{\nu}(P_+-P_-)P_- = 0. \tag{4.16}$$

Next, observe that

$$D_t\,(P_+ - P_-) + (\gamma_+P_+ - \gamma_-P_-)\operatorname{div} u + (\frac{\gamma_+P_+}{\alpha_+} + \frac{\gamma_-P_-}{\alpha_-})\frac{\alpha_+\alpha_-}{\nu}\,(P_+ - P_-) = 0, \tag{4.17}$$

which we further put under the form

$$D_t\,(P_+ - P_-) + (\gamma_+ P_+ - \gamma_- P_-)\,\mathrm{div}\,u + \frac{\gamma_+\alpha_-}{\nu}\,(P_+ - P_-)^2 + \left(\frac{\gamma_+}{\alpha_+} + \frac{\gamma_-}{\alpha_-}\right)\frac{\alpha_+\alpha_-}{\nu}\,(P_+ - P_-)\,P_- = 0.$$

Notice that we chose to define the effective pressure R by:

$$R = P - (\gamma_+ - \gamma_-)\,\frac{P_+ - P_-}{\frac{\gamma_+}{\alpha_+} + \frac{\gamma_-}{\alpha_-}} = \frac{\gamma_-\alpha_+}{\gamma_+\alpha_- + \gamma_-\alpha_+}P_+ + \frac{\gamma_+\alpha_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}P_-$$

to cancel the coupling between $P_+ - P_-$ and P_- in (4.16), and the definition of w:

$$w = \frac{P_+ - P_-}{\frac{\gamma_+}{\alpha_+} + \frac{\gamma_-}{\alpha_-}} = \frac{\alpha_+\alpha_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}(P_+ - P_-)$$

enables us to rewrite the pressure in a simple form: $P = R + (\gamma_+ - \gamma_-)w$. The unknown w can be comprehended as the *damped part* of the pressure. A straightforward computation yields

$$D_t\Big(\frac{\gamma_+}{\alpha_+} + \frac{\gamma_-}{\alpha_-}\Big)^{-1} = \frac{\gamma_+\alpha_-^2 - \gamma_-\alpha_+^2}{\gamma_+\alpha_- + \gamma_-\alpha_+}\frac{w}{\nu}$$

and

$$\begin{cases} P_+ = R + \dfrac{\gamma_+}{\alpha_+}w, \\ P_- = R - \dfrac{\gamma_-}{\alpha_-}w. \end{cases}$$

Then the equation of w reads

$$D_t w + F_1 \mathrm{div}\,u + F_2\frac{w}{\nu} = 0 \tag{4.18}$$

with

$$\begin{cases} F_1 := \dfrac{(\gamma_+ - \gamma_-)\alpha_+\alpha_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}R + \dfrac{\gamma_+^2\alpha_- + \gamma_-^2\alpha_+}{\gamma_+\alpha_- + \gamma_-\alpha_+}w, \\ F_2 := (\gamma_+\alpha_- + \gamma_-\alpha_+)R - \dfrac{(\gamma_+ - \gamma_+^2)\alpha_-^2 - (\gamma_- - \gamma_-^2)\alpha_+^2}{\alpha_+\alpha_-}w, \end{cases}$$

and the equation of R reads

$$D_t R + F_3\,\mathrm{div}\,u = F_4\frac{w^2}{\nu} \tag{4.19}$$

with

$$\begin{cases} F_3 := \dfrac{\gamma_+\gamma_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}\big(R + (\gamma_+ - \gamma_-)w\big), \\ F_4 := \dfrac{\gamma_+\gamma_-}{\alpha_+\alpha_-}\big(1 - (\gamma_+\alpha_- + \gamma_-\alpha_+)\big). \end{cases}$$

Moreover, we have

$$\alpha_+ = \Psi_1(w, R, Y), \quad \alpha_- = 1 - \Psi_1(w, R, Y). \tag{4.20}$$

Next, as

$$Y = \frac{\alpha_+\rho_+}{\rho},$$

we observe that

$$D_t Y = 0. \tag{4.21}$$

We denote the perturbations of Y and R by

$$y := Y - \frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_+ + \bar{\alpha}_-\bar{\rho}_-}, \ r = R - \bar{P}. \tag{4.22}$$

We see that there exists a function G_0 of the unknowns (w, r, y) such that

$$\frac{1}{\rho} = \frac{1}{\alpha_+\rho_+ + \alpha_-\rho_-} = \bar{F}_0 + G_0(w, r, y), \quad \text{with} \ \ \bar{F}_0 := \frac{1}{\bar{\alpha}_+\bar{\rho}_+ + \bar{\alpha}_-\bar{\rho}_-}.$$

Gathering the equations (4.18), (4.19), (4.21) and the equation of u together, we obtain the following system in terms of the unknowns (y, w, r, u):

$$\begin{cases} D_t y = 0, \\ D_t w + \big(\bar{F}_1 + G_1\big)\mathrm{div}\, u + \big(\bar{F}_2 + G_2\big)\dfrac{w}{\nu} = 0, \\ D_t r + \big(\bar{F}_3 + G_3\big)\mathrm{div}\, u = F_4\dfrac{w^2}{\nu}, \\ D_t u - \big(\bar{F}_0 + G_0\big)\mathcal{A}_{\mu,\lambda}u + \eta u + \big(\bar{F}_0 + G_0\big)\nabla r + (\gamma_+ - \gamma_-)\,\big(\bar{F}_0 + G_0\big)\nabla w = 0 \end{cases} \tag{4.23}$$

where

$$\begin{cases} \bar{F}_1 := \dfrac{(\gamma_+ - \gamma_-)\bar{\alpha}_+\bar{\alpha}_-}{\gamma_+\bar{\alpha}_- + \gamma_-\bar{\alpha}_+}\bar{P} > 0, \\ \bar{F}_2 := (\gamma_+\bar{\alpha}_- + \gamma_-\bar{\alpha}_+)\bar{P} > 0, \\ \bar{F}_3 := \dfrac{\gamma_+\gamma_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}\bar{P} > 0, \\ G_i(w, r, y) := F_i - \bar{F}_i \quad \text{for each} \quad i = 1, \cdots, 3. \end{cases} \tag{4.24}$$

Note that by virtue of (4.20) and (4.22), the G_i (for $i = 0, 1, 2, 3$) can be written as smooth functions of the unknowns (w, r, y) which vanish at origin.

4.3.2 Elements of proof for Theorem 4.2.1

Let us observe that the first equation of (4.23) is a pure transport equation and that there is a linear coupling between the second to fourth equations. As we are considering viscosity vanishing limit, we need to get appropriate estimates independent of μ and ν. This fact motivates us to study the following mixed linear system:

$$\begin{cases} \partial_t w + v\cdot\nabla w + \big(h_1 + H_1\big)\mathrm{div}\, u + (h_2 + H_2)\dfrac{w}{\nu} = S_2, \\ \partial_t r + v\cdot\nabla r + \big(h_3 + H_3\big)\mathrm{div}\, u = S_3, \\ \partial_t u + v\cdot\nabla u - \big(h_4 + H_4\big)\mathcal{A}_{\mu,\lambda}u + \eta u + \big(h_5 + H_5\big)\nabla r + \big(h_6 + H_6\big)\nabla w = S_4 \end{cases} \tag{4.25}$$

where S_2, S_3, S_4 and $H_1, H_2, \cdots, H_6$ are given functions of (t, x), and $h_1, \cdots, h_6$ are given positive constants.

Based on the analysis done in the previous chapters, we expect that System (4.25) behaves like the heat equation in the low frequencies regime and we expect a damping effect for the high frequencies if the functions $H_1, \cdots, H_6$ are small enough. Moreover we expect to recover better integrability properties for w and u in the low frequency regime because they undergo *direct* damping.

Precisely, we obtain the following *a priori* estimates uniformly with respect to μ, λ, ν.

Proposition 4.3.1. *Let $d \geq 2$ and let parameters satisfy $\mu, \lambda + \mu \geq 0$, $\nu = 2\mu + \lambda \leq 1$ and*

$\eta \geq 1$. *Assume that*

$$H_1, H_2, \cdots, H_6, v \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{S}(\mathbb{R}^d)), \qquad \|H_i\|_{L^\infty_{t,x}} \leq \frac{1}{2} h_i, \qquad i = 1, 2, \cdots 6. \tag{4.26}$$

Let $-\frac{d}{2} < s_1 \leq \frac{d}{2} - 1$ *and* $s_1 \leq s_2 - 1 \leq s_1 + 1$. *Let* (w, r, u) *be a solution of system* (4.25) *on the time interval* $[0, T)$. *There exists a positive constant* c_0 *independent of* μ, λ *such that if*

$$\sum_{i=1}^{6} \|H_i\|_{L^\infty_t(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} \leq c_0, \tag{4.27}$$

then the following estimate holds on $[0, T)$:

$$\begin{aligned}
&\|(w,r,u)\|^\ell_{\widetilde{L}^\infty_t(B^{s_1})} + \|(w,r,u)\|^h_{\widetilde{L}^\infty_t(B^{s_2})} + \kappa \Big(\|(w,r,u)\|^\ell_{L^1_t(B^{s_1+2})} + \|(w,r,u)\|^h_{L^1_t(B^{s_2})} \Big) \\
&+ \int_0^t (\|(\partial_t w, \frac{w}{\nu})\|^\ell_{B^{s_1}} + \|(\partial_t w, \frac{w}{\nu})\|^h_{B^{s_2-1}}) + \int_0^t (\|(\partial_t u, \eta u, \partial_t r)\|^\ell_{B^{s_1+1}} + \|(\partial_t u, \eta u, \partial_t r)\|^h_{B^{s_2-1}}) \\
&\qquad + \int_0^t (\|\mu \Delta u, (\mu+\lambda)\nabla \mathrm{div}\, u\|^\ell_{B^{s_1+1}} + \|\mu \Delta u, (\mu+\lambda)\nabla \mathrm{div}\, u\|^h_{B^{s_2-1}}) \\
&\qquad \leq \exp\Big(C\big(H(t) + V(t)\big)\Big) \Big(\|(w_0, r_0, u_0)\|_{B^{s_1} \cap B^{s_2}} + \int_0^t \|(S_2, S_3, S_4)\|_{B^{s_1} \cap B^{s_2}} \Big).
\end{aligned} \tag{4.28}$$

where $V(t) := \int_0^t \|v\|_{B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}}$, $H(t) := \sum_{i=1}^{6} \|\partial_t H_i(t)\|_{B^{\frac{d}{2}}}$ *and* κ *is a constant defined by* (4.58).

Remark 4.3.1. *It turns out that we have to choose* $c_0 << \min\{\eta, \frac{1}{\eta}\}$, *therefore we are not able to take into account the case when* $\eta \to \infty$ *in the same time as the relaxation parameter* $\nu \to 0$. *This is due to the overdamping phenomena that is described in the introduction.*

We will now state a global-in-time existence result for (4.23). First, let us introduce the functional spaces which appear in the global existence theorem and the rest of the chapter :

$$\begin{aligned}
E_T^{s_1,s_2} \triangleq \{&(y,w,r,u) \in C_b([0,T], B^{s_1} \cap B^{s_2}),\ \ (w,r,u)^h \in L^1([0,T], B^{s_2}),\ \ r^\ell \in L^1([0,T], B^{s_1+2}), \\
&(\frac{w}{\nu}, \partial_t w)^\ell \in L^1([0,T], B^{s_1}),\ (\frac{w}{\nu}, \partial_t w)^h \in L^1([0,T], B^{s_2-1}),\ (\eta u, \partial_t r, \partial_t u)^\ell \in L^1([0,T], B^{s_1+1}) \\
&(\eta u, \partial_t r, \partial_t u)^h \in L^1([0,T], B^{s_2-1}),\ \ (\mu\Delta u, (\mu+\lambda)\nabla \mathrm{div}\, u)^\ell \in L^1([0,T], B^{s_1+1} \\
&\text{and } (\mu\Delta u, (\mu+\lambda)\nabla \mathrm{div}\, u)^h \in L^1([0,T], B^{s_2-1}\}.
\end{aligned}$$

We define $\|\cdot\|_{E_T^{s_1,s_2}}$ as the norm associated to $E_T^{s_1,s_2}$ and if $T = +\infty$, we use the notation E^{s_1,s_2} and replace the interval $[0, T]$ by $\mathbb{R}_+$.

We have the following theorem.

Theorem 4.3.1. *Let* $d \geq 2$ *and assume that the parameters satisfy* $\mu \geq 0, \lambda + \mu \geq 0, 0 < \nu \leq 1$ *and* $\eta \geq 1$. *Let the constants* $\bar{\alpha}_\pm \in (0,1), \bar{\rho}_\pm > 0$ *satisfying* (4.6) *and* (4.8). *There exists a constant* $c > 0$ *independent of the viscosity coefficients* μ, λ *such that for any initial data such that*

$$\|(y_0, w_0, r_0, u_0)\|_{B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}} \leq c$$

then System (4.23) *admit a unique global-in-time solution* (y, w, r, u) *in the space* $E^{\frac{d}{2}-1, \frac{d}{2}+1}$. *Moreover, the following estimate holds true uniformly w.r.t. the viscosity coefficients* μ *and* λ:

$$\|(y,w,r,u)\|_{L^\infty(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})}+\|(w,r,u)\|_{L^1(B^{\frac{d}{2}+1})}$$
$$+\|\frac{w}{\nu}\|_{L^1(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}})}+\|\eta u\|_{L^1(B^{\frac{d}{2}})}\leq Cc. \quad (4.29)$$

For $d\geq 3$, using Proposition A.2.6 related to composition operator, Theorem 4.3.1 directly implies Theorem 4.2.1. In the two-dimensional setting, however, the Proposition A.2.6 fails to work as the regularity index is equal to 0 and therefore one must be careful when trying to recover the regularity properties for the original unknowns. Let us explain how to proceed to this issue. For example, concerning $\alpha_+-\bar{\alpha}_+$, we have

$$\alpha_+-\bar{\alpha}_+=\Psi_1(w,R,Y)-\Psi_1(0,\bar{P},\frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_++\bar{\alpha}_-\bar{\rho}_-}).$$

Using the decomposition

$$\begin{aligned}&\Psi_1(w,R,Y)-\Psi_1(0,\bar{P},\frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_++\bar{\alpha}_-\bar{\rho}_-})\\ =&\big(\partial_w\Psi_1(0,\bar{P},\frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_++\bar{\alpha}_-\bar{\rho}_-})+G^1(w,r,y)\big)w+\big(\partial_R\Psi_1(0,\bar{P},\frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_++\bar{\alpha}_-\bar{\rho}_-})+G^2(w,r,y)\big)r\\ &+\big(\partial_Y\Psi_1(0,\bar{P},\frac{\bar{\alpha}_+\bar{\rho}_+}{\bar{\alpha}_+\bar{\rho}_++\bar{\alpha}_-\bar{\rho}_-})+G^3(w,r,y)\big)y,\end{aligned}$$

where G^1,G^2,G^3 are smooth functions vanishing at $(0,0,0)$. We get, thanks to product law (A.7) that

$$\|\alpha_+-\bar{\alpha}_+\|_{B^0}\leq C(1+\|(G^1,G^2,G^3)(w,r,y)\|_{B^1})\|w,r,y\|_{B^0}.$$

And by Proposition A.2.6 we see $\|(G^1,G^2,G^3)(w,r,y)\|_{B^1}$ is under control, thus we can recover estimate for $\alpha_+-\bar{\alpha}_+$ in the space $\mathcal{C}_b(\mathbb{R}_+;B^0)$. Doing similar arguments for α_-,ρ_+ and ρ_-, one can finally deduce Theorem 4.2.1. Note that these arguments are also useful to prove Theorem 4.2.2 in the case $d=2$.

4.4 Analysis of the linear system (4.25)

This section is devoted to the proof of Proposition 4.3.1 for the linear system (4.25). We first localize (4.25) in frequencies thanks to the Littlewood-Paley decomposition, then use a renormalized energy method to estimate each dyadic block. In the following computations, assume that we are given a smooth solution (y,w,r,u) of (4.25) on $[0,T)\times\mathbb{R}^d$. And $(q_j)_{j\in\mathbb{Z}}$ is a generic sequence such that

$$\left\|(q_j)_{j\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})}\leq 1.$$

We will consider the following energy functionals,

$$\mathcal{L}_j(t):=\sqrt{\int_{\mathbb{R}^d}\Big(\frac{h_6}{h_1}w_j^2+\frac{h_5}{h_3}r_j^2+|u_j|^2+2\varepsilon_\ell u_j\cdot\nabla r_j\Big)}\quad\text{for}\quad j\leq 0$$

and

$$\mathcal{L}_j(t) := \sqrt{\int_{\mathbb{R}^d}\Big(\frac{h_6+H_6}{h_1+H_1}w_j^2+\frac{h_5+H_5}{h_3+H_3}r_j^2+|u_j|^2+2\varepsilon_h 2^{-2j}u_j\cdot\nabla r_j\Big)} \quad \text{for} \quad j>0$$

where $\varepsilon_h, \varepsilon_l > 0$ are constants that will be fixed later on such that

$$|\mathcal{L}_j(t)|^2 \sim \|(w_j, r_j, u_j)(t)\|_{L^2}^2. \tag{4.30}$$

The next two steps will explain how we constructed those two functionals to derive a priori estimates in low and high frequencies, respectively.

4.4.1 Low frequencies analysis

Throughout this part, we shall suppose that $j \leq 0$. Using spectral localization properties, that is

$$\|\nabla r_j\|_{L^2} \leq \frac{12}{5}2^j\|r_j\|_{L^2} \leq \frac{12}{5}\|r_j\|_{L^2},$$

we easily obtain that as soon as $\varepsilon_\ell \leq \min\{\frac{5h_5}{24h_3}, \frac{5}{24}\}$ then (4.30) is satisfied in the following way

$$\frac{C_1^2}{4}\|(w_j, r_j, u_j)(t)\|_{L^2}^2 \leq \mathcal{L}_j^2(t) \leq 4C_2^2\|(w_j, r_j, u_j)(t)\|_{L^2}^2, \tag{4.31}$$

where $C_1 = \min\{\frac{h_6}{h_1}, \frac{h_5}{h_3}, 1\}$, $C_2 = \max\{\frac{h_6}{h_1}, \frac{h_5}{h_3}, 1\}$. indeed, we have

$$\begin{aligned}
\mathcal{L}_j^2 &= \int_{\mathbb{R}^d}\Big(\frac{h_6}{h_1}w_j^2+\frac{h_5}{h_3}r_j^2+|u_j|^2+2\varepsilon_\ell u_j\cdot\nabla r_j\Big)\\
&\geq \frac{h_6}{h_1}\|w_j\|_{L^2}^2+\frac{h_5}{h_3}\|r_j\|_{L^2}^2+\|u_j\|^2-\frac{12}{5}\varepsilon_\ell(\|u_j\|_{L^2}^2+\|r_j\|_{L^2}^2)\\
&\geq \min\{\frac{h_6}{h_1}, \frac{h_5}{2h_3}, \frac{1}{2}\}\|(w_j, r_j, u_j)\|_{L^2}^2 \geq \frac{C_1^2}{4}\|(w_j, r_j, u_j)\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_j^2 &\leq \frac{h_6}{h_1}\|w_j\|_{L^2}^2+\frac{h_5}{h_3}\|r_j\|_{L^2}^2+\|u_j\|^2+\frac{12}{5}\varepsilon_\ell(\|u_j\|_{L^2}^2+\|r_j\|_{L^2}^2)\\
&\leq \max\{\frac{h_6}{h_1}, \frac{3h_5}{2h_3}, \frac{3}{2}\}\|(w_j, r_j, u_j)\|_{L^2}^2 \leq 4C_2^2\|(w_j, r_j, u_j)\|_{L^2}^2.
\end{aligned}$$

In the sequel we will use C_1 and C_2 frequently, note that

$$C_1 \leq \frac{h_6}{h_1}, \frac{h_5}{h_3}, 1 \leq C_2. \tag{4.32}$$

Applying the operator $\dot{\Delta}_j$ to the three equations of (4.25) yields

$$\begin{cases} \partial_t w_j + v\cdot\nabla w_j + h_1 \operatorname{div} u_j + h_2 \dfrac{w_j}{\nu} = \dot\Delta_j S_2 - K_j^1, \\ \partial_t r_j + v\cdot\nabla r_j + h_3 \operatorname{div} u_j = \dot\Delta_j S_3 - K_j^2, \\ \partial_t u_j + v\cdot\nabla u_j - h_4 \mathcal{A}_{\mu,\lambda} u_j + \eta u_j + h_5 \nabla r_j + h_6 \nabla w_j = \dot\Delta_j S_4 - K_j^3, \end{cases} \tag{4.33}$$

where K_j^1, K_j^2, K_j^3 will act as source terms

$$\begin{cases} K_j^1 = [\dot\Delta_j, v]\nabla w + \dot\Delta_j(H_1 \operatorname{div} u) + \dfrac{1}{\nu}\dot\Delta_j(H_2\, w) \\ K_j^2 = [\dot\Delta_j, v]\nabla r + \dot\Delta_j(H_3 \operatorname{div} u), \\ K_j^3 = [\dot\Delta_j, v]\nabla u - \dot\Delta_j(H_4\, \mathcal{A}_{\mu,\lambda}\, u) + \dot\Delta_j(H_5\, \nabla r) + \dot\Delta_j(H_6\, \nabla w). \end{cases}$$

Multiplying the first equation of (4.33) with $\dfrac{h_6}{h_1} w_j$, the second equation with $\dfrac{h_5}{h_3} r_j$, the last one with u_j, respectively, we get

$$\begin{aligned} \frac{1}{2}\frac{h_6}{h_1}\frac{d}{dt}\|w_j\|_{L^2}^2 + h_6 \int_{\mathbb{R}^d} w_j \operatorname{div} u_j + \frac{h_2 h_6}{h_1}\frac{\|w_j\|_{L^2}^2}{\nu} &\le \frac{h_6}{h_1}\|w_j\|_{L^2}(\|\dot\Delta_j S_2\|_{L^2} + \|K_j^1\|_{L^2}) \\ &\quad + \frac{h_6}{h_1}\|w_j\|_{L^2}^2 \|\operatorname{div} v\|_{L^\infty}, \end{aligned}$$

$$\frac{1}{2}\frac{h_5}{h_3}\frac{d}{dt}\|r_j\|_{L^2}^2 + h_5 \int_{\mathbb{R}^d} r_j \operatorname{div} u_j \le \frac{h_5}{h_3}\|r_j\|_{L^2}(\|\dot\Delta_j S_3\|_{L^2} + \|K_j^2\|_{L^2}) + \frac{h_5}{h_3}\|r_j\|_{L^2}^2\|\operatorname{div} v\|_{L^\infty}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_j\|_{L^2}^2 + h_4\big(\mu\|\nabla u_j\|_{L^2}^2 + (\mu+\lambda)\|\operatorname{div} u_j\|_{L^2}^2\big) + \eta\|u_j\|_{L^2}^2 - h_5\int_{\mathbb{R}^d} r_j \operatorname{div} u_j - h_6 \int_{\mathbb{R}^d} w_j \operatorname{div} u_j \\ &\quad \le \|u_j\|_{L^2}(\|\dot\Delta_j S_4\|_{L^2} + \|K_j^3\|_{L^2}) + \|u_j\|_{L^2}^2\|\operatorname{div} v\|_{L^\infty}. \end{aligned}$$

Summing up the above inequalities together, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\frac{h_6}{h_1}\|w_j\|_{L^2}^2 + \frac{h_5}{h_3}\|r_j\|_{L^2}^2 + \|u_j\|^2\Big) + \frac{h_2 h_6}{h_1}\frac{\|w_j\|_{L^2}^2}{\nu} + h_4\big(\mu\|\nabla u_j\|_{L^2}^2 + (\mu+\lambda)\|\operatorname{div} u_j\|_{L^2}^2\big) \\ &\quad + \eta\|u_j\|_{L^2}^2 \le C_2\|w_j\|_{L^2}(\|\dot\Delta_j S_2\|_{L^2} + \|K_j^1\|_{L^2}) + C_2\|r_j\|_{L^2}(\|\dot\Delta_j S_3\|_{L^2} + \|K_j^2\|_{L^2}) \\ &\quad + \|u_j\|_{L^2}(\|\dot\Delta_j S_4\|_{L^2} + \|K_j^3\|_{L^2}) + C_2\|(w_j, r_j, u_j)\|_{L^2}^2\|\operatorname{div} v\|_{L^\infty}. \end{aligned} \tag{4.34}$$

We observe that the left-hand side of (4.34) does not provide any decay information for r. Recovering such information relies on differentiating in time the second part of the Lyapunov functional and is the objective of the following lines. Taking the gradient of the second equation in (4.33), one find that ∇r_j verifies

$$\partial_t \nabla r_j + h_3 \nabla \operatorname{div} u_j = -\nabla(v\cdot\nabla r_j) + \nabla(\dot\Delta_j S_3 - K_j^2).$$

Multiplying this equation by u_j and the equation of u_j from (4.33) by ∇r_j and summing up the results we end up with

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u_j\cdot\nabla r_j)+h_5\|\nabla r_j\|_{L^2}^2\\
&\quad\le h_3\|\mathrm{div}\,u_j\|_{L^2}^2+\|\mathrm{div}\,u_j\|_{L^2}\|v\cdot\nabla r_j\|_{L^2}+h_4\|\mathcal{A}_{\mu,\lambda}u_j\|_{L^2}\|\nabla r_j\|_{L^2}\\
&\qquad+\eta\|u_j\|_{L^2}\|\nabla r_j\|_{L^2}+h_6\|\nabla w_j\|_{L^2}\|\nabla r_j\|_{L^2}+\|v\cdot\nabla u_j\|_{L^2}\|\nabla r_j\|_{L^2}\\
&\qquad+\|\mathrm{div}\,u_j\|_{L^2}(\|\dot\Delta_j S_3\|_{L^2}+\|K_j^2\|_{L^2})+\|\nabla r_j\|_{L^2}(\|\dot\Delta_j S_4\|_{L^2}+\|K_j^3\|_{L^2}).
\end{aligned}
\tag{4.35}
$$

Recalling the definition of $\mathcal{L}_j$ and combining (4.34) and (4.35), one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2+\frac{h_2h_6}{h_1}\frac{\|w_j\|_{L^2}^2}{\nu}+h_5\varepsilon_\ell\|\nabla r_j\|_{L^2}^2+\eta\|u_j\|_{L^2}^2+h_4\big(\mu\|\nabla u_j\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}\,u_j\|_{L^2}^2\big)\\
&\le C_2\|w_j\|_{L^2}(\|\dot\Delta_j S_2\|_{L^2}+\|K_j^1\|_{L^2})+C_2(\|r_j\|_{L^2}+\varepsilon_\ell\|\mathrm{div}\,u_j\|_{L^2})(\|\dot\Delta_j S_3\|_{L^2}+\|K_j^2\|_{L^2})\\
&\quad+(\|u_j\|_{L^2}+\varepsilon_\ell\|\nabla r_j\|_{L^2})(\|\dot\Delta_j S_4\|_{L^2}+\|K_j^3\|_{L^2})+C_2\|(w_j,r_j,u_j)\|_{L^2}^2\|\mathrm{div}\,v\|_{L^\infty}\\
&\quad+\varepsilon_\ell\Big(h_3\|\mathrm{div}\,u_j\|_{L^2}^2+h_4\|\mathcal{A}_{\mu,\lambda}\,u_j\|_{L^2}\|\nabla r_j\|_{L^2}+\eta\|u_j\|_{L^2}\|\nabla r_j\|_{L^2}+h_6\|\nabla w_j\|_{L^2}\|\nabla r_j\|_{L^2}\Big)\\
&\quad+\varepsilon_\ell(\|\mathrm{div}\,u_j\|_{L^2}\|v\cdot\nabla r_j\|_{L^2}+\|\nabla r_j\|_{L^2}\|v\cdot\nabla u_j\|_{L^2}).
\end{aligned}
\tag{4.36}
$$

At this stage, we are ready to estimate the right-hand side of (4.36). Using (4.31) and $C_1\le 1\le C_2$, we have

$$
\begin{aligned}
&C_2\|w_j\|_{L^2}(\|\dot\Delta_j S_2\|_{L^2}+\|K_j^1\|_{L^2})+C_2\|r_j\|_{L^2}(\|\dot\Delta_j S_3\|_{L^2}+\|K_j^2\|_{L^2})+\|u_j\|_{L^2}(\|\dot\Delta_j S_4\|_{L^2}+\|K_j^3\|_{L^2})\\
&\le C_2\|(w_j,r_j,u_j)\|_{L^2}\,\|(\dot\Delta_j S_2,\ \dot\Delta_j S_3,\ \dot\Delta_j S_4,\ K_j^1,\ K_j^2,\ K_j^3)\|_{L^2}\\
&\le\frac{2C_2}{C_1}\,\mathcal{L}_j\,\|(\dot\Delta_j S_2,\ \dot\Delta_j S_3,\ \dot\Delta_j S_4,\ K_j^1,\ K_j^2,\ K_j^3)\|_{L^2}.
\end{aligned}
$$

Since $\varepsilon_\ell\le\frac{5}{24}$ and $j\le 0$, we have

$$
\begin{aligned}
&C_2\varepsilon_\ell\|\mathrm{div}\,u_j\|_{L^2}(\|\dot\Delta_j S_3\|_{L^2}+\|K_j^2\|_{L^2})+\varepsilon_\ell\|\nabla r_j\|_{L^2}(\|\dot\Delta_j S_4\|_{L^2}+\|K_j^3\|_{L^2})\\
&\le C_2\varepsilon_\ell\frac{12}{5}\|(u_j,r_j)\|_{L^2}(\dot\Delta_j S_3,\ \dot\Delta_j S_4,\ K_j^2,\ K_j^3)\|_{L^2}\le\frac{C_2}{C_1}\,\mathcal{L}_j\,\|(\dot\Delta_j S_3,\ \dot\Delta_j S_4,\ K_j^2,\ K_j^3)\|_{L^2}.
\end{aligned}
$$

Obviously, using again (4.31), we obtain

$$
C_2\|(w_j,r_j,u_j)\|_{L^2}^2\|\mathrm{div}\,v\|_{L^\infty}\le\frac{4C_2}{C_1^2}\mathcal{L}_j^2\,\|\mathrm{div}\,v\|_{L^\infty}.
$$

Owing to the fact that $j\le 0$, we use the following inequalities

$$
\|\nabla u_j\|_{L^2}\le\frac{12}{5}\|u_j\|_{L^2},\quad\|\nabla r_j\|_{L^2}\le\frac{12}{5}\|r_j\|_{L^2},\quad\|\nabla w_j\|_{L^2}\le\frac{12}{5}\|w_j\|_{L^2},
\tag{4.37}
$$

and Young's inequality to write

$$
\varepsilon_\ell h_3\|\mathrm{div}\,u_j\|_{L^2}^2\le\varepsilon_\ell h_3(\frac{12}{5})^2\|u_j\|_{L^2}^2\le 6\,h_3\varepsilon_\ell\|u_j\|_{L^2}^2,
$$

$$\varepsilon_\ell \eta \|u_j\|_{L^2}\|\nabla r_j\|_{L^2} \le \frac{8\eta^2}{h_5}\varepsilon_\ell\|u_j\|_{L^2}^2 + \frac{h_5}{8}\varepsilon_\ell\|\nabla r_j\|_{L^2}^2,$$

$$\begin{aligned}\varepsilon_\ell h_6\|\nabla w_j\|_{L^2}\|\nabla r_j\|_{L^2} &\le \frac{8h_6^2}{h_5}\varepsilon_\ell\|\nabla w_j\|_{L^2}^2 + \frac{h_5}{8}\varepsilon_\ell\|\nabla r_j\|_{L^2}^2\\ &\le \frac{48h_6^2}{h_5}\varepsilon_\ell\|w_j\|_{L^2}^2 + \frac{h_5}{8}\varepsilon_\ell\|\nabla r_j\|_{L^2}^2.\end{aligned}$$

Using once more (4.31) and $\varepsilon_\ell \le \frac{5}{24}$, we arrive at

$$\varepsilon_\ell\|\mathrm{div}\, u_j\|_{L^2}\|v\cdot\nabla r_j\|_{L^2} \le \varepsilon_\ell(\frac{12}{5})^2\|u_j\|_{L^2}\|r_j\|_{L^2}\|v\|_{L^\infty} \le \frac{5}{C_1^2}\,\mathcal{L}_j^2\,\|v\|_{L^\infty},$$

$$\varepsilon_\ell\|\nabla r_j\|_{L^2}\|v\cdot\nabla u_j\|_{L^2} \le \varepsilon_\ell(\frac{12}{5})^2\|u_j\|_{L^2}\|r_j\|_{L^2}\|v\|_{L^\infty} \le \frac{5}{C_1^2}\,\mathcal{L}_j^2\,\|v\|_{L^\infty}.$$

Similar arguments lead to the following estimate:

$$\|\mathcal{A}_{\mu,\lambda}\,u_j\|_{L^2} \le \frac{12}{5}\mu\|\nabla u_j\|_{L^2} + \frac{12}{5}(\mu+\lambda)\|\mathrm{div}\, u_j\|_{L^2},$$

then, using the fact that $0 \le \max\{\mu, \mu+\lambda\} \le \nu$, we readily have

$$\|\mathcal{A}_{\mu,\lambda}\,u_j\|_{L^2}^2 \le 12\,\nu\,(\mu\|\nabla u_j\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div}\, u_j\|_{L^2}^2),$$

and thus

$$\begin{aligned}\varepsilon_\ell h_4\|\mathcal{A}_{\mu,\lambda}\,u_j\|_{L^2}\|\nabla r_j\|_{L^2} &\le \varepsilon_\ell(\frac{8}{h_5}h_4^2\|\mathcal{A}_{\mu,\lambda}\,u_j\|_{L^2}^2 + \frac{h_5}{8}\|\nabla r_j\|_{L^2}^2)\\ &\le \frac{96h_4^2\nu}{h_5}\varepsilon_\ell\big(\mu\|\nabla u_j\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div}\, u_j\|_{L^2}^2\big) + \frac{h_5}{8}\varepsilon_\ell\|r_j\|_{L^2}^2.\end{aligned}$$

Inserting the above estimates into (4.36) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + (\frac{h_2h_6}{c_1} - \frac{48h_6^2}{h_5}\nu\varepsilon_\ell)\frac{\|w_j\|_{L^2}^2}{\nu} + \frac{h_5}{2}\varepsilon_l\|\nabla r_j\|_{L^2}^2 + (\eta - 6h_3\varepsilon_\ell - \frac{8\eta^2}{h_5}\varepsilon_\ell)\|u_j\|_{L^2}^2\\ &\quad+(h_4 - \frac{96\,h_4^2\nu}{h_5}\varepsilon_\ell)\,(\mu\|\nabla u_j\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div}\, u_j\|_{L^2}^2)\\ &\quad\le\frac{3C_2}{C_1}\,\mathcal{L}_j\,\|(\dot\Delta_j S_2,\ \dot\Delta_j S_3,\ \dot\Delta_j S_4,\ K_j^1,\ K_j^2,\ K_j^3)\|_{L^2} + \frac{14C_2}{C_1^2}\mathcal{L}_j^2\,\big(\|\mathrm{div}\, v\|_{L^\infty} + \|v\|_{L^\infty}\big).\end{aligned} \tag{4.38}$$

Let us choose $\varepsilon_\ell > 0$ such that

$$\begin{aligned}
\varepsilon_\ell &\le \min\{\frac{5h_5}{24h_3}, \frac{5}{24}\},\\
\frac{h_2h_6}{2h_1} &\le \frac{h_2h_6}{h_1} - \frac{48h_6^2}{h_5}\nu\varepsilon_\ell,\\
\frac{\eta}{2} &\le \eta - 6h_3\varepsilon_\ell - \frac{8\eta^2}{h_5}\varepsilon_\ell,\\
\frac{h_4}{2} &\le h_4 - \frac{96h_4^2}{h_5}\nu\varepsilon_\ell
\end{aligned}$$

for example one may take

$$\varepsilon_\ell = \frac{1}{192}\min\{\frac{h_5}{h_3},\ 1,\ \frac{h_2h_5}{h_1h_6\,\nu},\ \frac{h_5\eta}{h_3h_5+\eta^2},\ \frac{h_5}{h_4\,\nu}\}.$$

Under the consideration of such ε_ℓ, we further define

$$\kappa := \frac{1}{4}\min\{\frac{h_2h_6}{h_1\,\nu},\ h_5\,\varepsilon_\ell,\ \eta\}\quad (\ge \min\{\frac{h_2h_6}{2h_1\,\nu},\ \frac{h_5\varepsilon_\ell}{2}(\frac{5}{6})^2,\ \frac{\eta}{2}\}). \tag{4.39}$$

Combining the facts that $\frac{5}{6}\,2^j\|r_j\|_{L^2} \le \|\nabla r_j\|_{L^2}$ and for all $j \le 0$, $2^{2j}\|(w_j,u_j)\|_{L^2}^2 \le \|(w_j,u_j)\|_{L^2}^2$ and using (4.31), one is able to rewrite (4.38) as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2 + \frac{\kappa}{4C_2^2}\,2^{2j}\,\mathcal{L}_j^2\\
&\le \frac{3C_2}{C_1}\,\mathcal{L}_j\,\|(\dot\Delta_j S_2,\ \dot\Delta_j S_3,\ \dot\Delta_j S_4,\ K_j^1,\ K_j^2,\ K_j^3)\|_{L^2} + \frac{14C_2}{C_1^2}\mathcal{L}_j^2\,\big(\|\mathrm{div}\,v\|_{L^\infty} + \|v\|_{L^\infty}\big).
\end{aligned} \tag{4.40}$$

Owing to Proposition A.2.4, we know that for $s_1 \in (-\frac{d}{2}, \frac{d}{2}-1]$ the product is continuous from

$$B^{\frac{d}{2}} \times B^{s_1} \quad \text{to} \quad B^{s_1} \quad \text{and from} \quad B^{\frac{d}{2}-1} \times B^{s_1+1} \quad \text{to} \quad B^{s_1} \tag{4.41}$$

Then, using the commutator estimate (A.4), the product law (4.41) and the fact that $s_1 \le s_2 - 1$, we obtain

$$\begin{aligned}
\|K_j^1\|_{L^2} &\le C2^{-js_1}q_j\Big(\|\nabla v\|_{B^{\frac{d}{2}}}\|\nabla w\|_{B^{s_1-1}} + \|H_1\,\mathrm{div}\,u\|_{B^{s_1}} + \frac{1}{\nu}\|H_2\,w\|_{B^{s_1}}\Big)\\
&\le C2^{-js_1}q_j\Big(\|v\|_{B^{\frac{d}{2}+1}}\|w\|_{B^{s_1}} + \|H_1\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_1+1}} + \|H_2\|_{B^{\frac{d}{2}}}\|\frac{w}{\nu}\|_{B^{s_1}}\Big)\\
\|K_j^2\|_{L^2} &\le C2^{-js_1}q_j\Big(\|\nabla v\|_{B^{\frac{d}{2}}}\|\nabla r\|_{B^{s_1-1}} + \|H_3\,\mathrm{div}\,u\|_{B^{s_1}}\Big)\\
&\le C2^{-js_1}q_j\Big(\|v\|_{B^{\frac{d}{2}+1}}\|r\|_{B^{s_1}} + \|H_3\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_1+1}}\Big).
\end{aligned}$$

The term $\dot\Delta_j(H_5\,\nabla r)$ appearing in K_j^3 deserves some particular attention. Using (4.41) we observe that the product maps

$$B^{\frac{d}{2}+1+s_1-s_2} \times B^{s_2-1} \hookrightarrow B^{s_1}\ (\text{note that } \frac{d}{2}+1+s_1-s_2 \le \frac{d}{2}),$$

and we infer that

$$\begin{aligned}\|\dot\Delta_j(H_5\,\nabla r)\|_{L^2} &\le C2^{-js_1}q_j\Big(\|H_5\,\nabla r^\ell\|_{B^{s_1}}+\|H_5\,\nabla r^h\|_{B^{s_1}}\Big)\\ &\le C2^{-js_1}q_j\Big(\|H_5\|_{B^{\frac d2-1}}\|\nabla r^\ell\|_{B^{s_1+1}}+\|H_5\|_{B^{\frac d2+1+s_1-s_2}}\|\nabla r^h\|_{B^{s_2-1}}\Big)\\ &\le C2^{-js_1}q_j\Big(\|H_5\|_{B^{\frac d2-1}}\|r\|^\ell_{B^{s_1+2}}+\|H_5\|_{B^{\frac d2+1+s_1-s_2}}\|r\|^h_{B^{s_2}}\Big).\end{aligned}\tag{4.42}$$

In conclusion, using once more the commutator estimate (A.4) and (4.41), we obtain

$$\begin{aligned}\|K_j^3\|_{L^2}\le C2^{-js_1}q_j\Big(&\|v\|_{B^{\frac d2+1}}\|u\|_{B^{s_1}}+\|H_4\|_{B^{\frac d2}}\|\mathcal{A}_{\mu,\lambda}u\|_{B^{s_1}}+\|H_6\|_{B^{\frac d2}}\|\nabla w\|_{B^{s_1}}\\ &+(\|H_5\|_{B^{\frac d2-1}}\|r\|^\ell_{B^{s_1+2}}+\|H_5\|_{B^{\frac d2+1+s_1-s_2}}\|r\|^h_{B^{s_2}})\Big).\end{aligned}$$

Define $\mathcal{L}_j^\ell=2^{s_1}\,\mathcal{L}_j$. Recalling assumption (4.27) and using that $\nu<1$, by interpolation we have

$$\begin{aligned}&2^{js_1}\|(K_j^1,K_j^1,K_j^3)\|_{L^2}\\ &\le Cq_j\Big(\|\nabla v\|_{B^{\frac d2}}\|(w,r,u)\|_{B^{s_1}}+c_0(\|\frac{w}{\nu}\|_{B^{s_1}}+\|w\|_{B^{s_1+1}}+\|u\|_{B^{s_1+1}}\\ &\qquad\qquad+\|\mathcal{A}_{\mu,\lambda}\,u\|_{B^{s_1}}+\|r\|^\ell_{B^{s_1+2}}+\|r\|^h_{B^{s_2}})\Big)\\ &\le CK\|\nabla v\|_{B^{\frac d2}}\mathcal{L}_j^\ell+C\|\nabla v\|_{B^{\frac d2}}\Big(q_j\|(w,r,u)\|_{B^{s_1}}-K\mathcal{L}_j^\ell\Big)\\ &\qquad+Cc_0\,q_j\Big(\|\frac{w}{\nu}\|_{B^{s_1}}+\|(w,u)\|_{B^{s_1+1}}+\|\mathcal{A}_{\mu,\lambda}\,u\|_{B^{s_1}}+\|r\|^\ell_{B^{s_1+2}}+\|r\|^h_{B^{s_2}}\Big)\end{aligned}$$

where K is a positive large constant to be fixed later. Applying Lemma A.1.1 from the Appendix, we get

$$\begin{aligned}\mathcal{L}_j^\ell(t)+\kappa\,2^{2j}\int_0^t\mathcal{L}_j^\ell\le\exp\Big(\frac{CKC_2^2}{C_1^2}V(t)\Big)\Big\{&\mathcal{L}_j^\ell(0)+\int_0^t q_j\,\Big(\|(S_2,S_3,S_4)\|_{B^{s_1}}\\ &+\|\nabla v\|_{B^{\frac d2}}\Big(q_j\,\|(w,r,u)\|_{B^{s_1}}-K\mathcal{L}_j^\ell\Big)\Big)+\int_0^t c_0(\|\frac{w}{\nu}\|^\ell_{B^{s_1}}+\|\frac{w}{\nu}\|^h_{B^{s_2-1}}+\|u\|_{B^{s_1+1}})\\ &+\int_0^t c_0(\|\mathcal{A}_{\mu,\lambda}\,u\|_{B^{s_1}}+\|r\|^\ell_{B^{s_1+2}}+\|r\|^h_{B^{s_2}}\Big\}\end{aligned}\tag{4.43}$$

Note that $\mathcal{L}_j^\ell(t)$ may be replaced by $\sup_{[0,t]}\mathcal{L}_j^\ell$ in the left-hand side of the above inequality. Thanks to (4.31) and after summation on $j\le 0$, we conclude that

$$\begin{aligned}&\|(w,r,u)\|^\ell_{\widetilde L^\infty_t(B^{s_1})}+\kappa\,\|(w,r,u)\|^\ell_{L^1_t(B^{s_1+2})}\\ &\le\exp\Big(\frac{CKC_2^2}{C_1^2}V(t)\Big)\Big\{\|(w_0,r_0,u_0)\|^\ell_{B^{s_1}}+\int_0^t\|\nabla v\|_{B^{\frac d2}}\Big(\|(w,r,u)\|_{B^{s_1}}-K\sum_{j\le 0}\mathcal{L}_j^\ell\Big)\\ &\quad+\int_0^t\Big(\|(S_2,S_3,S_4)\|_{B^{s_1}}+c_0(\|\frac{w}{\nu}\|_{B^{s_1}}+\|(w,u)\|_{B^{s_1+1}}\\ &\qquad\qquad+\|\mathcal{A}_{\mu,\lambda}\,u\|_{B^{s_1}}+\|r\|^\ell_{B^{s_1+2}}+\|r\|^h_{B^{s_2}})\Big)\Big\}\end{aligned}\tag{4.44}$$

which is the first part of the low frequencies a priori estimates.

4.4.2 High frequencies analysis

Throughout this part, we suppose that $j > 0$. First of all, recall the definitions of C_1, C_2 from (4.32) and notice that the assumptions (4.26), (4.27) ensure that

$$\frac{1}{2}h_i \leq h_i + H_i \leq \frac{3}{2}h_i \qquad i = 1, \cdots, 6, \tag{4.45}$$

$$\frac{C_1}{3} \leq \frac{h_6}{3h_1} \leq \frac{h_6 + H_6}{h_1 + H_1} \leq \frac{3h_6}{h_1} \leq 3C_2, \qquad \frac{C_1}{3} \leq \frac{h_5}{3h_3} \leq \frac{h_5 + H_5}{h_3 + H_3} \leq \frac{3h_5}{h_3} \leq 3C_2, \tag{4.46}$$

and

$$\|\nabla\Big(\frac{h_6 + H_6}{h_1 + H_1}\Big)\|_{L^\infty_{t,x}} + \|\nabla\Big(\frac{h_5 + H_5}{h_3 + H_3}\Big)\|_{L^\infty_{t,x}} \leq 6c_0\frac{h_1 + h_6}{h_1^2} + 6c_0\frac{h_5 + h_3}{h_3^2} \leq C_3c_0, \tag{4.47}$$

$$\begin{aligned}\|\partial_t\Big(\frac{h_6 + H_6}{h_1 + H_1}\Big)(t)\|_{L^\infty_x} + \|\partial_t\Big(\frac{h_5 + H_5}{h_5 + H_3}\Big)(t)\|_{L^\infty_x} &\leq C_3\|(\partial_t H_1, \partial_t H_3, \partial_t H_5, \partial_t H_6)(t)\|_{L^\infty_x} \\ &\leq C_3 H(t)\end{aligned} \tag{4.48}$$

with $C_3 := 6\dfrac{h_1 + h_6}{h_1^2} + 6\dfrac{h_5 + h_3}{h_3^2}$.

Using spectral localization properties, we have

$$2^{-2j}\|\nabla r_j\|_{L^2} \leq \frac{12}{5}2^{-j}\|r_j\|_{L^2} \leq \frac{24}{5}\|r_j\|_{L^2},$$

thus we easily obtain that as soon as $\varepsilon_h \leq \min\{\dfrac{5h_5}{144h_3}, \dfrac{5}{48}\}$, (4.30) is satisfied in the following way

$$\frac{C_1^2}{9}\|(w_j, r_j, u_j)\|_{L^2}^2 \leq \mathcal{L}_j^2(t) \leq 4C_2^2\|(w_j, r_j, u_j)(t)\|_{L^2}^2. \tag{4.49}$$

Indeed, one has

$$\begin{aligned}\mathcal{L}_j^2 &= \int_{\mathbb{R}^d}\Big(\frac{h_6 + H_6}{h_1 + H_1}w_j^2 + \frac{h_5 + H_5}{h_3 + H_3}r_j^2 + |u_j|^2 + 2\varepsilon_h 2^{-2j}u_j \cdot \nabla r_j\Big) \\ &\geq \frac{h_6}{3h_1}\|w_j\|_{L^2}^2 + \frac{h_5}{3h_3}\|r_j\|_{L^2}^2 + \|u_j\|^2 - \frac{24}{5}\varepsilon_h(\|u_j\|_{L^2}^2 + \|r_j\|_{L^2}^2) \\ &\geq \min\{\frac{h_6}{6h_1}, \frac{h_5}{6h_3}, \frac{1}{6}\}\|(w_j, r_j, u_j)\|_{L^2} \geq \frac{C_1^2}{9}\|(w_j, r_j, u_j)\|_{L^2}^2\end{aligned}$$

and

$$\mathcal{L}_j^2 \leq \frac{3h_6}{h_1}\|w_j\|_{L^2}^2 + \frac{3h_5}{h_3}\|r_j\|_{L^2}^2 + \|u_j\|^2 + \frac{24}{5}\varepsilon_l(\|u_j\|_{L^2}^2 + \|r_j\|_{L^2}^2) \leq 4C_2^2\|(w_j, r_j, u_j)\|_{L^2}^2.$$

Slightly different from (4.33), after applying the operator $\dot{\Delta}_j$ to the three equations of (4.25), page 12, we write

$$\begin{cases} \partial_t w_j + v\cdot\nabla w_j + (h_1+H_1)\operatorname{div} u_j + (h_2+H_2)\dfrac{w_j}{\nu} = \dot\Delta_j S_2 + T_j^1, \\ \partial_t r_j + v\cdot\nabla r_j + (h_3+H_3)\operatorname{div} u_j = \dot\Delta_j S_3 + T_j^2, \\ \partial_t u_j + v\cdot\nabla u_j - (h_4+H_4)\mathcal{A}_{\mu,\lambda}u_j + \eta\, u_j + (h_5+H_5)\nabla r_j + (h_6+H_6)\nabla w_j = \dot\Delta_j S_4 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +T_j^3, \end{cases} \tag{4.50}$$

where T_j^1, T_j^2, T_j^3 are commutators defined by

$$\begin{cases} T_j^1 = [v,\dot\Delta_j]\nabla w + [H_1,\dot\Delta_j]\operatorname{div} u + \dfrac{1}{\nu}[H_2,\dot\Delta_j]w, \\ T_j^2 = [v,\dot\Delta_j]\nabla r + [H_3,\dot\Delta_j]\operatorname{div} u, \\ T_j^3 = [v,\dot\Delta_j]\nabla u - [H_4,\dot\Delta_j]\mathcal{A}_{\mu,\lambda}u + [H_5,\dot\Delta_j]\nabla r + [H_6,\dot\Delta_j]\nabla w. \end{cases}$$

Now, multiplying the first equation of (4.50) by $\dfrac{h_6+H_6}{h_1+H_1}w_j$ and the second equation by $\dfrac{h_5+H_5}{h_3+H_3}r_j$, we get from (4.45)-(4.48) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\frac{h_6+H_6}{h_1+H_1}w_j^2 + \frac{h_2h_6}{6h_1}\frac{\|w_j\|_{L^2}^2}{\nu} + \int_{\mathbb{R}^d}(h_6+H_6)w_j\operatorname{div} u_j \\ &\le \frac{1}{2}\int_{\mathbb{R}^d}\Big\{w_j^2\,\partial_t\Big(\frac{h_6+H_6}{h_1+H_1}\Big) + \frac{h_6+H_6}{h_1+H_1}w_j^2\operatorname{div} v + w_j^2\, v\cdot\nabla\Big(\frac{h_6+H_6}{h_1+H_1}\Big) + 2(\dot\Delta_j S_2 + T_j^1)\frac{h_6+H_6}{c_1+H_1}w_j\Big\} \\ &\quad\le \|w_j\|_{L^2}^2\Big\{\Big\|\partial_t\Big(\frac{h_6+H_6}{h_1+H_1}\Big)\Big\|_{L^\infty} + \Big\|\frac{h_6+H_6}{h_1+H_1}\Big\|_{L^\infty}\|\operatorname{div} v\|_{L^\infty} + \Big\|\nabla\Big(\frac{h_6+H_6}{h_1+H_1}\Big)\Big\|_{L^\infty}\|v\|_{L^\infty}\Big\} \\ &\qquad\qquad + \|w_j\|_{L^2}\Big\|\frac{h_6+H_6}{h_1+H_1}\Big\|_{L^\infty}(\|\dot\Delta_j S_2\|_{L^2} + \|T_j^1\|_{L^2}) \\ &\le \|w_j\|_{L^2}^2\Big(C_3H(t) + 3C_2\|\operatorname{div} v\|_{L^\infty} + C_3c_0\|v\|_{L^\infty}\Big) + 3C_2\|w_j\|_{L^2}(\|\dot\Delta_j S_2\|_{L^2} + \|T_j^1\|_{L^2}) \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\frac{h_5+H_5}{h_3+H_3}r_j^2 + \int_{\mathbb{R}^d}(h_5+H_5)r_j\operatorname{div} u_j \\ &= \frac{1}{2}\int_{\mathbb{R}^d}\Big\{r_j^2\,\partial_t\Big(\frac{h_5+H_5}{h_3+H_3}\Big) + \frac{h_5+H_5}{h_3+H_3}r_j^2\operatorname{div} v + r_j^2\, v\cdot\nabla\Big(\frac{h_5+H_5}{h_3+H_3}\Big) + 2(\dot\Delta_j S_3 + T_j^2)\frac{h_5+H_5}{h_3+H_3}r_j\Big\} \\ &\quad\le \|r_j\|_{L^2}^2\Big\{\Big\|\partial_t\Big(\frac{h_5+H_5}{h_3+H_3}\Big)\Big\|_{L^\infty} + \Big\|\frac{h_3+H_5}{h_3+H_3}\Big\|_{L^\infty}\|\operatorname{div} v\|_{L^\infty} + \Big\|\nabla\Big(\frac{h_5+H_5}{h_3+H_3}\Big)\Big\|_{L^\infty}\|v\|_{L^\infty}\Big\} \\ &\qquad\qquad + \|r_j\|_{L^2}\Big\|\frac{h_5+H_5}{h_3+H_3}\Big\|_{L^\infty}(\|\dot\Delta_j S_3\|_{L^2} + \|T_j^2\|_{L^2}) \\ &\le \|r_j\|_{L^2}^2\Big(C_3H(t) + 3C_2\|\operatorname{div} v\|_{L^\infty} + C_3c_0\|v\|_{L^\infty}\Big) + 3C_2\|r_j\|_{L^2}(\|\dot\Delta_j S_3\|_{L^2} + \|T_j^2\|_{L^2}). \end{aligned} \tag{4.52}$$

Multiplying the velocity equation by u_j and observing that

$$\begin{aligned} -\int_{\mathbb{R}^d}(h_4+H_4)\mathcal{A}_{\mu,\lambda}u_j\cdot u_j &= \mu\int_{\mathbb{R}^d}\nabla u_j : \nabla\big((h_4+H_4)u_j\big) + (\mu+\lambda)\int_{\mathbb{R}^d}\operatorname{div} u_j\ \operatorname{div}\big((h_4+H_4)u_j\big) \\ &\ge \frac{h_4}{2}\mu\|\nabla u_j\|_{L^2}^2 + \frac{h_4}{2}(\mu+\lambda)\|\operatorname{div} u_j\|_{L^2}^2 \\ &\quad - (\mu\|\nabla u_j\|_{L^2} + (\mu+\lambda)\|\operatorname{div} u_j\|_{L^2})\|u_j\|_{L^2}\|\nabla H_4\|_{L^\infty}, \end{aligned}$$

we get that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_j\|_{L^2}^2+\frac{h_4}{2}\mu\|\nabla u_j\|_{L^2}^2+\frac{h_4}{2}(\mu+\lambda)\|\operatorname{div}u_j\|_{L^2}^2+\eta\,\|u_j\|_{L^2}^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-\int_{\mathbb{R}^d}(h_6+H_6)w_j\operatorname{div}u_j+(h_5+H_5)r_j\operatorname{div}u_j\\
&\le\int_{\mathbb{R}^d}\Big\{|u_j|^2\operatorname{div}v+(\dot\Delta_jS_4+T_j^3)u_j+w_j\,u_j\cdot\nabla H_6+r_j\,u_j\cdot\nabla H_5\Big\}\\
&\quad+(\mu\|\nabla u_j\|_{L^2}+(\mu+\lambda)\|\operatorname{div}u_j\|_{L^2})\|u_j\|_{L^2}\|\nabla H_4\|_{L^\infty}\\
&\le\|u_j\|_{L^2}^2\|\operatorname{div}v\|_{L^\infty}+\|u_j\|_{L^2}(\|\dot\Delta_jS^4\|_{L^2}+\|T_j^3\|_{L^2}+\|w_j\|_{L^2}\|\nabla H_6\|_{L^\infty}+\|r_j\|_{L^2}\|\nabla H_5\|_{L^\infty})\\
&\quad+(\mu\|\nabla u_j\|_{L^2}+(\mu+\lambda)\|\operatorname{div}u_j\|_{L^2})\|u_j\|_{L^2}\|\nabla H_4\|_{L^\infty}\\
&\le\|u_j\|_{L^2}^2\|\operatorname{div}v\|_{L^\infty}+c_0\|u_j\|_{L^2}\Big(\|w_j\|_{L^2}+\|r_j\|_{L^2}+\mu\|\nabla u_j\|_{L^2}+(\mu+\lambda)\|\operatorname{div}u_j\|_{L^2}\Big)\\
&\quad+\|u_j\|_{L^2}(\|\dot\Delta_jS^4\|_{L^2}+\|T_j^3\|_{L^2}).
\end{aligned}
$$

Summing up the resulting inequalities together, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\Big(\frac{h_6+H_6}{h_1+H_1}w_j^2+\frac{h_5+H_5}{h_3+H_3}r_j^2+|u_j|^2\Big)+\frac{h_2h_6}{6h_1}\frac{\|w_j\|_{L^2}^2}{\nu}+\eta\,\|u_j\|_{L^2}^2\\
&\quad+\frac{h_4}{2}\mu\|\nabla u_j\|_{L^2}^2+\frac{h_4}{2}(\mu+\lambda)\|\operatorname{div}u_j\|_{L^2}^2\\
&\le\|(w_j,r_j,u_j)\|_{L^2}^2\Big(C_3H(t)+3C_2\|\operatorname{div}v\|_{L^\infty}+C_3h_0\|v\|_{L^\infty}\Big)+3C_2\|w_j\|_{L^2}(\|\dot\Delta_jS_2\|_{L^2}+\|T_j^1\|_{L^2})\\
&\quad+3C_2\|r_j\|_{L^2}(\|\dot\Delta_jS_3\|_{L^2}+\|T_j^2\|_{L^2})+\|u_j\|_{L^2}(\|\dot\Delta_jS_4\|_{L^2}+\|T_j^3\|_{L^2})\\
&\quad+c_0\|u_j\|_{L^2}\Big(\|w_j\|_{L^2}+\|r_j\|_{L^2}+\mu\|\nabla u_j\|_{L^2}+(\mu+\lambda)\|\operatorname{div}u_j\|_{L^2}\Big).
\end{aligned}\tag{4.53}
$$

Observe that the left-hand side of (4.53) does not encode any decay properties for r. To recover such information we must differentiate in time the second part of the Lyapunov functional. Note that ∇r_j verifies

$$
\partial_t\nabla r_j+\nabla\left(v\cdot\nabla r_j\right)+\nabla\left((h_3+H_3)\operatorname{div}u_j\right)=\nabla(\dot\Delta_jS_3+T_j^2).\tag{4.54}
$$

Hence, multiplying this equation by u_j, and the equation of u_j from (4.50) by ∇r_j and summing up the results we end up with

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(u_j\cdot\nabla r_j)+h_5\|\nabla r_j\|_{L^2}^2\\
&=\int_{\mathbb{R}^d}\Big\{v\cdot\nabla r_j\operatorname{div}u_j+(h_3+H_3)\,|\operatorname{div}u_j|^2-(v\cdot\nabla u_j)\cdot\nabla r_j+(h_4+H_4)\mathcal{A}_{\mu,\lambda}u_j\cdot\nabla r_j\\
&\quad-\eta\,u_j\cdot\nabla r_j-(h_6+H_6)\nabla w_j\cdot\nabla r_j-(\dot\Delta_jS_3+T_j^2)\operatorname{div}u_j+(\dot\Delta_jS_4+T_j^3)\cdot\nabla r_j-H_5|\nabla r_j|^2\Big\}\\
&\le2\|v\|_{L^\infty}\|\nabla r_j\|_{L^2}\|\operatorname{div}u_j\|_{L^2}+(h_3+\|H_3\|_{L^\infty})\|\operatorname{div}u_j\|_{L^2}^2+(h_4+\|H_4\|_{L^\infty})\|\mathcal{A}_{\mu,\lambda}u_j\|_{L^2}\|\nabla r_j\|_{L^2}\\
&\quad+\eta\,\|u_j\|_{L^2}\|\nabla r_j\|_{L^2}+(h_6+\|H_6\|_{L^\infty})\|\nabla w_j\|_{L^2}\|\nabla r_j\|_{L^2}+(\|\dot\Delta_jS_3\|_{L^2}+\|T_j^2\|_{L^2})\|\operatorname{div}u_j\|_{L^2}\\
&\quad+(\|\dot\Delta_jS_4\|_{L^2}+\|T_j^3\|_{L^2})\|\nabla r_j\|_{L^2}+\frac{h_5}{2}\|\nabla r_j\|_{L^2}^2.
\end{aligned}\tag{4.55}
$$

Using the spectral localization of r_j, u_j and $\varepsilon_h \le \frac{5}{48}$, we have

$$\begin{aligned}\varepsilon_h 2^{-2j} 2\|v\|_{L^\infty}\|\nabla r_j\|_{L^2}\|\mathrm{div}\, u_j\|_{L^2} &\le 2\varepsilon_h(\frac{12}{5})^2\|v\|_{L^\infty}\|r_j\|_{L^2}\|u_j\|_{L^2}\\ &\le \|v\|_{L^\infty}(\|r_j\|_{L^2}^2+\|u_j\|_{L^2}^2).\end{aligned}$$

Using Young's inequality and (4.45), we have

$$\varepsilon_h 2^{-2j}(h_3+\|H_3\|_{L^\infty})\|\mathrm{div}\, u_j\|_{L^2}^2 \le \varepsilon_h(\frac{12}{5})^2\frac{3h_3}{2}\|u_j\|_{L^2}^2 \le 9h_3\varepsilon_h\|u_j\|_{L^2}^2,$$

$$\begin{aligned}\varepsilon_h 2^{-2j}(h_6+\|H_6\|_{L^\infty})\|\nabla w_j\|_{L^2}\|\nabla r_j\|_{L^2} &\le \varepsilon_h 2^{-2j}\frac{3h_6}{2}\|\nabla w_j\|_{L^2}\|\nabla r_j\|_{L^2}\\ &\le \varepsilon_h 2^{-2j}(\frac{h_5}{16}\|\nabla r_j\|_{L^2}+\frac{16}{h_5}(\frac{3h_6}{2})^2\|\nabla w_j\|_{L^2}^2)\\ &\le \varepsilon_h 2^{-2j}\frac{h_5}{16}\|\nabla r_j\|_{L^2}+256\frac{h_6^2}{h_5}\varepsilon_h\|w_j\|_{L^2}^2.\end{aligned}$$

Moreover, taking into account that $\max\{\mu,\mu+\lambda\}\le\nu$, we obtain that

$$\begin{aligned}&\varepsilon_h 2^{-2j}(h_4+\|H_4\|_{L^\infty})\|\mathcal{A}_{\mu,\lambda}u_j\|_{L^2}\|\nabla r_j\|_{L^2}\\ &\le\varepsilon_h 2^{-j}\frac{3h_4}{2}\frac{12}{5}(\mu\|\nabla u_j\|_{L^2}+(\mu+\lambda)\|\mathrm{div}\, u_j\|_{L^2})\|\nabla r_j\|_{L^2}\\ &\le\varepsilon_h 2^{-2j}\frac{h_5}{16}\|\nabla r_j\|_{L^2}^2+\varepsilon_h\frac{16}{h_5}(\frac{3h_4}{2}\frac{12}{5})^2(\mu\|\nabla u_j\|_{L^2}+(\mu+\lambda)\|\mathrm{div}\, u_j\|_{L^2})^2\\ &\le 2^{-2j}\frac{h_5}{16}\varepsilon_h\|\nabla r_j\|_{L^2}^2+512\frac{h_4^2}{c_5}\nu\varepsilon_h(\mu\|\nabla u_j\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}\, u_j\|_{L^2}^2).\end{aligned}$$

As $j\ge -1$, we use the following rough inequalities:

$$\begin{aligned}\varepsilon_h 2^{-2j}\eta\,\|u_j\|_{L^2}\|\nabla r_j\|_{L^2} &\le \varepsilon_h 2^{-2j}(\frac{16\eta^2}{h_5}\|u_j\|_{L^2}^2+\frac{h_5}{16}\|\nabla r_j\|_{L^2}^2)\\ &\le 2^{-2j}\frac{h_5}{16}\varepsilon_h\|\nabla r_j\|_{L^2}^2+\frac{64\eta^2}{h_5}\varepsilon_h\|u_j\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}\varepsilon_h 2^{-2j}(\|\dot\Delta_j S_3\|_{L^2}+\|T_j^2)\|_{L^2})\|\mathrm{div}\, u_j\|_{L^2} &\le \varepsilon_h 2^{-2j}\frac{12}{5}2^j(\|\dot\Delta_j S_3\|_{L^2}+\|T_j^2)\|_{L^2})\|u_j\|_{L^2}\\ &\le \frac{24}{5}\varepsilon_h\|u_j\|_{L^2}(\|\dot\Delta_j S_3\|_{L^2}+\|T_j^2)\|_{L^2}),\end{aligned}$$

$$\begin{aligned}\varepsilon_h 2^{-2j}(\|\dot\Delta_j S_4\|_{L^2}+\|T_j^3)\|_{L^2})\|\nabla r_j\|_{L^2} &\le \varepsilon_h 2^{-2j}\frac{12}{5}2^j(\|\dot\Delta_j S_4\|_{L^2}+\|T_j^3)\|_{L^2})\|r_j\|_{L^2}\\ &\le \frac{24}{5}\varepsilon_h\|r_j\|_{L^2}(\|\dot\Delta_j S_4\|_{L^2}+\|T_j^3)\|_{L^2}).\end{aligned}$$

Inserting the above estimates into (4.55), one has

$$
\begin{aligned}
&\varepsilon_h 2^{-2j}\Big(\frac{d}{dt}\int_{\mathbb{R}^d}(u_j\cdot\nabla r_j)+\frac{h_5}{4}\|\nabla r_j\|_{L^2}^2\Big)\\
&\le\|v\|_{L^\infty}(\|r_j\|_{L^2}^2+\|u_j\|_{L^2}^2)+256\frac{h_6^2}{h_5}\varepsilon_h\|w_j\|_{L^2}^2+(9h_3\varepsilon_h+\frac{64\eta^2}{h_5}\varepsilon_h)\|u_j\|_{L^2}^2\\
&\quad+\frac{24}{5}\varepsilon_h\|u_j\|_{L^2}(\|\dot\Delta_j S_3\|_{L^2}+\|T_j^2)\|_{L^2})+\frac{24}{5}\varepsilon_h\|r_j\|_{L^2}(\|\dot\Delta_j S_4\|_{L^2}+\|T_j^3)\|_{L^2})\\
&\quad+512\frac{h_4^2}{h_5}\nu\varepsilon_h(\mu\|\nabla u_j\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}\,u_j\|_{L^2}^2).
\end{aligned}
\tag{4.56}
$$

Remembering the definition of $\mathcal{L}_j$, summing up (4.53) and (4.56) and keeping in mind that $\varepsilon_h\le\frac{5}{48}$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2+(\frac{h_2h_6}{6h_1}-256\frac{h_6^2}{h_5}\nu\varepsilon_h)\frac{\|w_j\|_{L^2}^2}{\nu}+2^{-2j}\frac{h_5}{4}\varepsilon_h\|\nabla r_j\|_{L^2}^2+(\eta-9h_3\varepsilon_h-\frac{64\eta^2}{h_5}\varepsilon_h)\|u_j\|_{L^2}^2\\
&\quad+(\frac{h_4}{2}-512\frac{h_4^2}{h_5}\nu\varepsilon_h)\big(\mu\|\nabla u_j\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}\,u_j\|_{L^2}^2\big)\\
&\le\|(w_j,r_j,u_j)\|_{L^2}^2\Big(C_3H(t)+3C_2\|\mathrm{div}\,v\|_{L^\infty}+(C_3c_0+1)\|v\|_{L^\infty}\Big)\\
&\quad+c_0\|u_j\|_{L^2}\Big(\|w_j\|_{L^2}+\|r_j\|_{L^2}+\mu\|\nabla u_j\|_{L^2}+(\mu+\lambda)\|\mathrm{div}\,u_j\|_{L^2}\Big)\\
&\quad+3C_2\|(w_j,r_j,u_j)\|_{L^2}\|\dot\Delta_jS_2,\dot\Delta_jS_3,\dot\Delta_jS_4,\ T_j^1,T_j^2,\ T_j^3)\|_{L^2}.
\end{aligned}
\tag{4.57}
$$

Let us choose $\varepsilon_h>0$ such that

$$
\begin{aligned}
\varepsilon_h&\le\min\{\frac{5h_5}{144h_3},\frac{5}{48}\},\\
\frac{h_2h_6}{12h_1}&\le\frac{h_2h_6}{6c_1}-256\frac{h_6^2}{h_5}\nu\varepsilon_h,\\
\frac{\eta}{2}&\le\eta-9h_3\varepsilon_h-\frac{64\eta^2}{h_5}\varepsilon_h,\\
\frac{h_4}{4}&\le\frac{h_4}{2}-512\frac{h_4^2}{c_5}\nu\varepsilon_h,
\end{aligned}
$$

for example one may take

$$
\varepsilon_h=\frac{1}{3072}\min\Big\{\frac{h_5}{h_3},\ 1,\ \frac{h_2h_5}{h_1h_6\,\nu},\ \frac{h_5\eta}{h_3h_5+\eta^2},\ \frac{h_5}{h_4\,\nu}\Big\}.
$$

Compared to ε_ℓ that has been defined in the previous section, we see that $\frac{1}{16}\varepsilon_\ell\le\varepsilon_h\le\varepsilon_\ell$. Thus, one has $2^{-2j}\frac{h_5}{4}\varepsilon_h\|\nabla r_j\|_{L^2}^2\ge\frac{h_5}{256}\varepsilon_\ell\|r_j\|_{L^2}^2$. We now update the definition of κ in (4.39) by

$$
\kappa:=\frac{1}{256}\min\{\frac{h_2h_6}{h_1\,\nu},\ h_5\,\varepsilon_\ell,\ \eta\}\ (\ge\min\{\frac{h_2h_6}{12h_1\,\nu},\ \frac{h_5}{256}\varepsilon_\ell,\ \frac{\eta}{2}\}).
\tag{4.58}
$$

Then, using (4.49) we are able to rewrite (4.57) as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathcal{L}_j^2+\frac{\kappa}{4C_2^2}\mathcal{L}_j^2 \leq & \frac{9}{C_1^2}\mathcal{L}_j^2\Big(C_3H(t)+3C_2\|\mathrm{div}\, v\|_{L^\infty}+(C_3c_0+1)\|v\|_{L^\infty}\Big)\\ &+\frac{3h_0}{C_1}\mathcal{L}_j\left(\|w_j\|_{L^2}+\|u_j\|_{L^2}+\mu\|\nabla u_j\|_{L^2}+(\mu+\lambda)\|\mathrm{div}\, u_j\|_{L^2}\right)\\ &+\frac{3C_2}{C_1}\mathcal{L}_j\,\|\dot{\Delta}_jS_2,\dot{\Delta}_jS_3,\dot{\Delta}_jS_4,\ T_j^1,T_j^2,\ T_j^3)\|_{L^2}. \end{aligned}\tag{4.59}$$

Estimating T_j^1,T_j^2,T_j^3 requires some particular attention. For example, the term $[H_5,\dot{\Delta}_j]\nabla r$ in T_j^3, has to be rewritten

$$[H_5,\dot{\Delta}_j]\nabla r=[H_5,\dot{\Delta}_j]\nabla r_j^\ell+[H_5,\dot{\Delta}_j]\nabla r^h$$

and by Proposition A.2.3, we have

$$2^{js_2}\,\|[H_5,\dot{\Delta}_j]\,\nabla r^h\|_{L^2}\leq Cq_j\|\nabla H_5\|_{B^{\frac{d}{2}}}\|\nabla r^h\|_{B^{s_2-1}}\leq Cq_j\|\nabla H_5\|_{B^{\frac{d}{2}}}\|r\|^h_{B^{s_2}},$$

$$\begin{aligned}2^{js_2}\,\|[H_5,\dot{\Delta}_j]\,\nabla r^\ell\|_{L^2}&\leq 2^{(\frac{d}{2}+1)j}\,\|[H_5,\dot{\Delta}_j]\nabla r^\ell\|_{L^2}\\ &\leq Cq_j\|\nabla H_5\|_{B^{\frac{d}{2}}}\|\nabla r^\ell\|_{B^{\frac{d}{2}}}\leq Cq_j\|\nabla H_5\|_{B^{\frac{d}{2}}}\|r\|^\ell_{B^{s_1+2}}.\end{aligned}$$

Above, the conditions that

$$s_1+2,s_2\leq\frac{d}{2}+1$$

are crucial. The other terms can be estimated using similar argument, we obtain

$$\begin{aligned}\|T_j^1\|_{L^2}\leq C2^{-js_2}q_j\Big(&\|\nabla v\|_{B^{\frac{d}{2}}}\|\nabla w\|_{B^{s_2-1}}+\|\nabla H_1\|_{B^{\frac{d}{2}}}\|\mathrm{div}\, u\|_{B^{s_2-1}}\\ &+\|\nabla H_2\|_{B^{\frac{d}{2}}}(\|\frac{w}{\nu}\|^\ell_{B^{s_1+1}}+\|\frac{w}{\nu}\|^h_{B^{s_2-1}})\Big),\\ \|T_j^2\|_{L^2}\leq C2^{-js_2}q_j\Big(&\|\nabla v\|_{B^{\frac{d}{2}}}\|\nabla r\|_{B^{s_2-1}}+\|\nabla H_3\|_{B^{\frac{d}{2}}}\|\mathrm{div}\, u\|_{B^{s_2-1}}\Big)\end{aligned}$$

and

$$\begin{aligned}\|T_j^3\|_{L^2}\leq & C2^{-js_2}q_j\Big(\|\nabla v\|_{B^{\frac{d}{2}}}\|\nabla u\|_{B^{s_2-1}}+\|\nabla H_4\|_{B^{\frac{d}{2}}}\|\mathcal{A}_{\mu,\lambda}u\|_{B^{s_2-1}}\\ &+\|\nabla H_5\|_{B^{\frac{d}{2}}}(\|r\|^\ell_{B^{s_1+2}}+\|r\|^h_{B^{s_2}})+\|\nabla H_6\|_{B^{\frac{d}{2}}}\|\nabla w\|_{B^{s_2-1}}\Big).\end{aligned}$$

Thanks to smallness assumption (4.27), we gather that

$$\begin{aligned}&2^{js_2}\|(T_j^1,T_j^2,T_j^3)\|_{L^2}\\ &\leq Cq_j\|\nabla v\|_{B^{\frac{d}{2}}}\|(w,r,u)\|_{B^{s_2}}+Cc_0\,q_j\Big(\|(w,u)\|_{B^{s_2}}+\|(\frac{1}{\nu}w,\ \mathcal{A}_{\mu,\lambda}\,u)\|_{B^{s_2-1}}\\ &\qquad\qquad+\|r\|^\ell_{B^{s_1+2}}+\|r\|^h_{B^{s_2}}\Big)\\ &\leq CK2^{s_2j}\mathcal{L}_j\|\nabla v\|_{B^{\frac{d}{2}}}+C\|\nabla v\|_{B^{\frac{d}{2}}}\Big(q_j\|(w,r,u)\|_{B^{s_2}}-K2^{s_2j}\mathcal{L}_j\Big)\\ &+Cc_0\,q_j\Big(\|(w,u)\|_{B^{s_2}}+\|\mathcal{A}_{\mu,\lambda}\,u\|_{B^{s_2-1}}+(\|\frac{w}{\nu}\|^\ell_{B^{s_1+1}}+\|\frac{w}{\nu}\|^h_{B^{s_2-1}})+\|r\|^\ell_{B^{s_1+2}}+\|r\|^h_{B^{s_2}}\Big).\end{aligned}$$

Defining $\mathcal{L}_j^h = 2^{js_2}\mathcal{L}_j$ and using Gronwall's Lemma implies that

$$\begin{aligned}
&\mathcal{L}_j^h(t) + \frac{\kappa}{4C_2^2}\int_0^t \mathcal{L}_j^h \\
&\quad \leq \exp\Big(\frac{CKC_2^2}{C_1^2}\big(C_3(H(t)+V(t))+V(t)\big)\Big) \\
&\qquad \Big\{\mathcal{L}_j^h(0) + \int_0^t q_j\Big(\|(S_2,S_3,S_4)\|_{B^{s_2}} + \|\nabla v\|_{B^{\frac{d}{2}}}\Big(q_j\,\|(w,r,u)\|_{B^{s_2}} - K\mathcal{L}_j^h\Big)\Big) \\
&\qquad + \int_0^t c_0\big(\|(w,u)\|_{B^{s_2}} + \|\mathcal{A}_{\mu,\lambda}\,u\|_{B^{s_2-1}} + (\|\frac{w}{\nu}\|^{\ell}_{B^{s_1+1}} + \|\frac{w}{\nu}\|^{h}_{B^{s_2-1}}) + (\|r\|^{\ell}_{B^{s_1+2}} + \|r\|^{h}_{B^{s_2}})\big)\Big\}.
\end{aligned}$$

Note that $\mathcal{L}_j^h(t)$ may be replaced by $\sup_{[0,t]}\mathcal{L}_j^h(t)$ in the left-hand side of the above inequality. Thanks to (4.49), we conclude after summation over $j \geq -1$, that

$$\begin{aligned}
&\|(w,r,u)\|^h_{\widetilde{L}^\infty_t(B^{s_2})} + \kappa\,\|(w,r,u)\|^h_{L^1_t(B^{s_2})} \\
&\quad \leq \exp\Big(\frac{CKC_2^2}{C_1^2}\big(C_3(H(t)+V(t))+V(t)\big)\Big) \\
&\qquad \Big\{\|(w_0,r_0,u_0)\|^h_{B^{s_2}} + \int_0^t \Big(\|(S_2,S_3,S_4)\|_{B^{s_2}} + \|\nabla v\|_{B^{\frac{d}{2}}}\Big(\|(w,r,u)\|_{B^{s_2}} - K\sum_{j\geq -1}\mathcal{L}_j^h\Big)\Big) \\
&\qquad + c_0\int_0^t \Big(\|(w,u)\|_{B^{s_2}} + \|\mathcal{A}_{\mu,\lambda}\,u\|_{B^{s_2-1}} + (\|\frac{w}{\nu}\|^{\ell}_{B^{s_1+1}} + \|\frac{w}{\nu}\|^{h}_{B^{s_2-1}})\Big) \\
&\qquad\qquad + c_0\int_0^t (\|r\|^{\ell}_{B^{s_1+2}} + \|r\|^{h}_{B^{s_2}})\Big\}. \qquad (4.60)
\end{aligned}$$

We are now going to show that inequalities (4.44) and (4.60) entails a decay for w and u. In fact, one finds from (4.31) and (4.49) that

$$\begin{aligned}
&\int_0^t \|\nabla v\|_{B^{\frac{d}{2}}}\Big(\|(w,r,u)\|_{B^{s_1}} - K\sum_{j\leq 0}\mathcal{L}_j^{\ell}\Big) + \int_0^t \|\nabla v\|_{B^{\frac{d}{2}}}\Big(\|(w,r,u)\|_{B^{s_2}} - K\sum_{j\geq -1}\mathcal{L}_j^h\Big) \\
&\leq \int_0^t \|\nabla v\|_{B^{\frac{d}{2}}}\Big(\|(w,r,u)\|_{B^{s_1}\cap B^{s_2}} - \frac{C_1K}{3}\Big(\sum_{j\leq 0}2^{js_1}\|(w_j,r_j,u_j)\|_{L^2} + \sum_{j\geq -1}2^{js_2}\|(w_j,r_j,u_j)\|_{L^2}\Big) \\
&\leq \int_0^t \|\nabla v\|_{B^{\frac{d}{2}}}\Big(\|(w,r,u)\|_{B^{s_1}\cap B^{s_2}} - \frac{C_1K}{6}\Big(\|(w,r,u)\|_{B^{s_1}} + \|(w,r,u)\|_{B^{s_2}}\Big) \\
&\leq 0
\end{aligned}$$

when we choose $K = \dfrac{12}{C_1}$.

Thus, we conclude from summing up (4.44) and (4.60) that

$$
\begin{aligned}
&\|(w,r,u)\|^{\ell}_{\widetilde{L}^{\infty}_t(B^{s_1})} + \|(w,r,u)\|^{h}_{\widetilde{L}^{\infty}_t(B^{s_2})} + \kappa\Big(\|(w,r,u)\|^{\ell}_{L^1_t(B^{s_1+2})} + \|(w,r,u)\|^{h}_{L^1_t(B^{s_2})}\Big)\\
&\quad\le \exp\Big(\frac{CC_2^2}{C_1^2}\big(C_3H(t)+\max\{C_3,1\}V(t)\big)\Big)\\
&\qquad\Big\{\|(w_0,r_0,u_0)\|^{\ell}_{B^{s_1}} + \|(w_0,r_0,u_0)\|^{h}_{B^{s_2}} + \int_0^t \|(S_2,S_3,S_4)\|_{B^{s_1}\cap B^{s_2}}\\
&\qquad + c_0\int_0^t\Big((\|\frac{w}{\nu}\|_{B^{s_1}\cap B^{s_2-1}} + \|\mathcal{A}_{\mu,\lambda}\,u\|_{B^{s_1}\cap B^{s_2-1}} + \|(w,u)\|_{B^{s_1+1}\cap B^{s_2}}\Big)\\
&\qquad\qquad + c_0\int_0^t \|r\|^{\ell}_{B^{s_1+2}} + \|r\|^{h}_{B^{s_2}}\Big\} \qquad (4.61)
\end{aligned}
$$

which is the first part of the high frequencies a priori estimates.

4.4.3 The damping effects and estimation for time derivatives

We will now recover uniform estimates concerning the decay and time derivatives of our solutions. We have the following lemma.

Lemma 4.4.1. *Under the hypotheses of Proposition 4.3.1, we have*

$$
\begin{aligned}
&\int_0^t(\|(\partial_t w,\frac{w}{\nu})\|^{\ell}_{B^{s_1}} + \|(\partial_t w,\frac{w}{\nu})\|^{h}_{B^{s_2-1}}) + \int_0^t(\|(\partial_t u,u,\eta u,\partial_t r)\|^{\ell}_{B^{s_1+1}} + \|(\partial_t u,\eta u,\partial_t r)\|^{h}_{B^{s_2-1}})\\
&\qquad + \int_0^t(\|\mu\Delta u,(\mu+\lambda)\nabla\mathrm{div}\,u\|^{\ell}_{B^{s_1+1}} + \|\mu\Delta u,(\mu+\lambda)\nabla\mathrm{div}\,u\|^{h}_{B^{s_2-1}})\\
&\le C\Big(\|(w_0,r_0,u_0)\|_{B^{s_1}\cap B^{s_2}} + \int_0^t\|(v,\nabla v)\|_{B^{\frac{d}{2}}}\|(w,r,u)\|_{B^{s_1}\cap B^{s_2}} + \int_0^t\|(S_2,S_3,S_4)\|_{B^{s_1}\cap B^{s_2}}\\
&\qquad + \int_0^t(\|r\|^{\ell}_{B^{s_1+2}} + \|(w,r,u)\|^{h}_{B^{s_2}})\Big). \qquad (4.62)
\end{aligned}
$$

Proof. **Damping effects**

First, we investigate the damping effects in the equation of w and u. Let us look at the equation of w in System (4.25), which reads

$$
\partial_t w + v\cdot\nabla w + \frac{h_2}{\nu}\,w = -\frac{1}{\nu}H_2\,w - (h_1+H_1)\,\mathrm{div}\,u + S_2.
$$

We infer from Proposition 4.7.2 (take $r_1 = s_1, r_2 = s_2 - 1$) that

$$
\begin{aligned}
\|w(t)\|^{\ell}_{B^{s_1}} + \frac{h_2}{\nu}\int_0^t\|w\|^{\ell}_{B^{s_1}} &\le \|w_0\|^{\ell}_{B^{s_1}} + C\int_0^t\|v\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_1+1}}\\
&\quad + \int_0^t\Big(\|\frac{1}{\nu}H_2\,w\|^{\ell}_{B^{s_1}} + \|(h_1+H_1)\,\mathrm{div}\,u\|^{\ell}_{B^{s_1}} + \|S_2\|^{\ell}_{B^{s_1}}\Big)
\end{aligned}
$$

and

$$\|w(t)\|^h_{B^{s_2-1}} + \frac{h_2}{\nu}\int_0^t \|w\|^h_{B^{s_2-1}} \le \|w_0\|^h_{B^{s_2-1}} + C\int_0^t \|\nabla v\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_2-1}} + \int_0^t \Big(\|\frac{1}{\nu}H_2\, w\|^h_{B^{s_2-1}} + \|(h_1+H_1)\,\mathrm{div}\, u\|^h_{B^{s_2-1}} + \|S_2\|^h_{B^{s_2-1}}\Big).$$

By assumption (4.27) and $s_1 \le s_2 - 1$, one has

$$\frac{1}{\nu}\|H_2\, w\|^\ell_{B^{s_1}} \le \frac{C}{\nu}\|H_2\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_1}} \le \frac{Cc_0}{\nu}(\|w\|^\ell_{B^{s_1}} + \|w\|^h_{B^{s_2-1}}),$$

$$\|(h_1+H_1)\mathrm{div}\, u\|^\ell_{B^{s_1}} \le C(h_1 + \|H_1\|_{B^{\frac{d}{2}}})\|u\|_{B^{s_1+1}},$$

$$\frac{1}{\nu}\|H_2\, w\|^h_{B^{s_2-1}} \le \frac{C}{\nu}\|H_2\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_2-1}} \le \frac{Cc_0}{\nu}(\|w\|^\ell_{B^{s_1}} + \|w\|^h_{B^{s_2-1}}),$$

$$\|(h_1+H_1)\mathrm{div}\, u\|^h_{B^{s_2-1}} \le C(h_1 + \|H_1\|_{B^{\frac{d}{2}}})\|u\|_{B^{s_2}}.$$

Assuming that $c_0 \le \dfrac{h_2}{2C}$, we conclude that

$$\begin{aligned}\|w(t)\|^\ell_{B^{s_1}} + \|w(t)\|^h_{B^{s_2-1}} + \frac{h_2}{2\nu}\int_0^t (\|w\|^\ell_{B^{s_1}} + \|w\|^h_{B^{s_2-1}}) \\ \le \|w_0\|^\ell_{B^{s_1}} + \|w_0\|^h_{B^{s_2-1}} + C\int_0^t (\|v\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_1+1}} + \|\nabla v\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_2-1}}) \\ + C\int_0^t (\|u\|_{B^{s_1+1}} + \|u\|_{B^{s_2}}) + C\int_0^t (\|S_2\|^\ell_{B^{s_1}} + \|S_2\|^h_{B^{s_2-1}}).\end{aligned} \tag{4.63}$$

The estimate of the time derivative comes readily from the equation of w, we have

$$\begin{aligned}\int_0^t (\|\partial_t w\|^\ell_{B^{s_1}} + \|\partial_t w\|^h_{B^{s_2-1}}) \\ \le \int_0^t \Big(\|v\cdot\nabla w\|^\ell_{B^{s_1}} + \|(h_2+H_2)\frac{w}{\nu}\|^\ell_{B^{s_1}} + \|(h_1+H_1)\mathrm{div}\, u\|^\ell_{B^{s_1}} + \|S_2\|^\ell_{B^{s_1}} \\ + \|v\cdot\nabla w\|^h_{B^{s_2-1}} + \|(h_2+H_2)\frac{w}{\nu}\|^h_{B^{s_2-1}} + \|(h_1+H_1)\mathrm{div}\, u\|^h_{B^{s_2-1}} + \|S_2\|^h_{B^{s_2-1}}\Big) \\ \le C\int_0^t \Big(\|v\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_1+1}\cap B^{s_2}} + (\|\frac{w}{\nu}\|^\ell_{B^{s_1}} + \|\frac{w}{\nu}\|^h_{B^{s_2-1}}) + \|u\|_{B^{s_1+1}\cap B^{s_2}} + (\|S_2\|^\ell_{B^{s_1}} + \|S_2\|^h_{B^{s_2-1}})\Big).\end{aligned}$$

This combined with (4.63) implies that

$$\begin{aligned}\|w(t)\|^{\ell}_{B^{s_1}}+\|w(t)\|^{h}_{B^{s_2-1}}+\frac{h_2}{4\nu}\int_0^t(\|w\|^{\ell}_{B^{s_1}}+\|w\|^{h}_{B^{s_2-1}})+\frac{h_2}{4C}\int_0^t(\|\partial_t w\|^{\ell}_{B^{s_1}}+\|\partial_t w\|^{h}_{B^{s_2-1}})\\ \leq \|w_0\|^{\ell}_{B^{s_1}}+\|w_0\|^{h}_{B^{s_2-1}}+C\int_0^t(\|v\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_1}\cap B^{s_2}}+\|\nabla v\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_2-1}})\\ +C\int_0^t\Big(\|u\|_{B^{s_1+1}\cap B^{s_2}}+(\|S_2\|^{\ell}_{B^{s_1}}+\|S_2\|^{h}_{B^{s_2-1}})\Big),\end{aligned}$$

which further gives

$$\begin{aligned}\frac{h_2}{4\nu}\int_0^t(\|w\|^{\ell}_{B^{s_1}}+\|w\|^{h}_{B^{s_2-1}})+\frac{h_2}{4C}\int_0^t(\|\partial_t w\|^{\ell}_{B^{s_1}}+\|\partial_t w\|^{h}_{B^{s_2-1}})\\ \leq \|w_0\|^{\ell}_{B^{s_1}}+\|w_0\|^{h}_{B^{s_2}}+C\int_0^t\|(v,\nabla v)\|_{B^{\frac{d}{2}}}\|w\|_{B^{s_1}\cap B^{s_2}}\\ +C\int_0^t\Big(\|u\|_{B^{s_1+1}\cap B^{s_2}}+(\|S_2\|^{\ell}_{B^{s_1}}+\|S_2\|^{h}_{B^{s_2-1}})\Big).\end{aligned}\tag{4.64}$$

Next, we consider the equation of u in System (4.25) and we write that

$$\begin{aligned}\partial_t u+v\cdot\nabla u-h_4\mu\Delta u-h_4(\lambda+\mu)\nabla\operatorname{div}u+\eta u\\ =H_4\mu\Delta u+H_4(\lambda+\mu)\nabla\operatorname{div}u-(h_5+H_5)\nabla r-(h_6+H_6)\nabla w+S_4.\end{aligned}$$

Applying Proposition 4.7.3 (take $r_1=s_1+1, r_2=s_2-1$) to the above equation gives

$$\begin{aligned}\|u(t)\|^{\ell}_{B^{s_1+1}}+\|u(t)\|^{h}_{B^{s_2-1}}+\mu\int_0^t(\|\nabla u\|^{\ell}_{B^{s_1+2}}+\|\nabla u\|^{h}_{B^{s_2}})\\ +(\lambda+\mu)\int_0^t(\|\operatorname{div}u\|^{\ell}_{B^{s_1+2}}+\|\operatorname{div}u\|^{h}_{B^{s_2}})+\eta\int_0^t(\|u\|^{\ell}_{B^{s_1+1}}+\|u\|^{h}_{B^{s_2-1}})\\ \lesssim \|u_0\|^{\ell}_{B^{s_1+1}}+\|u_0\|^{h}_{B^{s_2-1}}+\int_0^t(\|v\|_{B^{\frac{d}{2}+1}}\|u\|_{B^{s_1+1}\cap B^{s_2-1}}+\|v\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_1+2}\cap B^{s_2}})\\ +\int_0^t\Big(\mu\|H_4\Delta u\|^{\ell}_{B^{s_1+1}}+(\lambda+\mu)\|H_4\nabla\operatorname{div}u\|^{\ell}_{B^{s_1+1}}+\|(h_5+H_5)\nabla r\|^{\ell}_{B^{s_1+1}}+\|(h_6+H_6)\nabla w\|^{\ell}_{B^{s_1+1}}\Big)\\ +\int_0^t\Big(\mu\|H_4\Delta u\|^{h}_{B^{s_2-1}}+(\lambda+\mu)\|H_4\nabla\operatorname{div}u\|^{h}_{B^{s_2-1}}+\|(h_5+H_5)\nabla r\|^{h}_{B^{s_2-1}}+\|(h_6+H_6)\nabla w\|^{h}_{B^{s_2-1}}\Big)\\ +\int_0^t\|S_4\|^{\ell}_{B^{s_1+1}}+\|S_4\|^{h}_{B^{s_2-1}}.\end{aligned}$$

Then, we use the low and high frequencies decomposition, product law (A.7) and that $s_2-1\leq s_1+1$ to obtain

$$\begin{aligned}\mu\|H_4\Delta u\|^{\ell}_{B^{s_1+1}}&\leq \mu(\|H_4\Delta u^{\ell}\|^{\ell}_{B^{s_1+1}}+\|H_4\Delta u^{h}\|^{\ell}_{B^{s_2-1}})\\ &\leq C\mu\|H_4\|_{B^{\frac{d}{2}}}(\|\nabla u\|^{\ell}_{B^{s_1+2}}+\|\nabla u\|^{h}_{B^{s_2}}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(\lambda+\mu)\|H_4\nabla\mathrm{div}\,u\|^{\ell}_{B^{s_1+1}} &\le C(\lambda+\mu)\|H_4\|_{B^{\frac{d}{2}}}(\|\mathrm{div}\,u\|^{\ell}_{B^{s_1+2}}+\|\mathrm{div}\,u\|^{h}_{B^{s_2}}),\\ \|(h_5+H_5)\nabla r\|^{\ell}_{B^{s_1+1}} &\le C(h_5+\|H_5\|_{B^{\frac{d}{2}}})(\|r\|^{\ell}_{B^{s_1+2}}+\|r\|^{h}_{B^{s_2}}),\\ \|(h_6+H_6)\nabla w\|^{\ell}_{B^{s_1+1}} &\le C(h_6+\|H_6\|_{B^{\frac{d}{2}}})(\|w\|^{\ell}_{B^{s_1+2}}+\|w\|^{h}_{B^{s_2}}).\end{aligned}$$

For the terms pertaining to the high frequencies part, using product law (A.7) and low and high frequencies decomposition, and the fact that $s_2-1\le s_1+1$, we have

$$\begin{aligned}\mu\|H_4\Delta u\|^{h}_{B^{s_2-1}} &\le C\mu(\|H_4\Delta u^{\ell}\|^{h}_{B^{s_1+1}}+\|H_4\Delta u^{h}\|^{h}_{B^{s_2-1}})\\ &\le C\mu\|H_4\|_{B^{\frac{d}{2}}}(\|\nabla u\|^{\ell}_{B^{s_1+2}}+\|\nabla u\|^{h}_{B^{s_2}})\end{aligned}$$

and similarly

$$\begin{aligned}(\lambda+\mu)\|H_4\nabla\mathrm{div}\,u\|^{h}_{B^{s_1+1}} &\le C(\lambda+\mu)\|H_4\|_{B^{\frac{d}{2}}}(\|\mathrm{div}\,u\|^{\ell}_{B^{s_1+2}}+\|\mathrm{div}\,u\|^{h}_{B^{s_2}}),\\ \|(h_5+H_5)\nabla r\|^{h}_{B^{s_2-1}} &\le C(h_5+\|H_5\|_{B^{\frac{d}{2}}})(\|r\|^{\ell}_{B^{s_1+2}}+\|r\|^{h}_{B^{s_2}}),\\ \|(h_6+H_6)\nabla w\|^{h}_{B^{s_2-1}} &\le C(h_6+\|H_6\|_{B^{\frac{d}{2}}})\left(\|w\|^{\ell}_{B^{s_1+2}}+\|w\|^{h}_{B^{s_2}}\right).\end{aligned}$$

Choosing c_0 small enough, we see that

$$\begin{aligned}&\|u(t)\|^{\ell}_{B^{s_1+1}}+\|u(t)\|^{h}_{B^{s_2-1}}+\mu\int_0^t(\|\nabla u\|^{\ell}_{B^{s_1+2}}+\|\nabla u\|^{h}_{B^{s_2}})\\ &\qquad+(\lambda+\mu)\int_0^t(\|\mathrm{div}\,u\|^{\ell}_{B^{s_1+2}}+\|\mathrm{div}\,u\|^{h}_{B^{s_2}})+\eta\int_0^t(\|u\|^{\ell}_{B^{s_1+1}}+\|u\|^{h}_{B^{s_2-1}})\\ &\le C(\|u_0\|^{\ell}_{B^{s_1+1}}+\|u_0\|^{h}_{B^{s_2-1}})+C\int_0^t(\|\nabla v\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_1+1}\cap B^{s_2-1}}+\|v\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_2}})\\ &\qquad+C\int_0^t(\|S_4\|^{\ell}_{B^{s_1+1}}+\|S_4\|^{h}_{B^{s_2-1}})+C\int_0^t(\|(w,r)\|^{\ell}_{B^{s_1+2}}+\|(w,r)\|^{h}_{B^{s_2}}).\end{aligned}$$

Time derivatives estimates

The estimates of the time derivative $\partial_t u$ will be obtained through the equation of u. Notice that

$$\begin{aligned}\|v\cdot\nabla u\|^{\ell}_{B^{s_1+1}} &\le \|v\cdot\nabla u^{\ell}\|^{\ell}_{B^{s_1+1}}+\|v\cdot\nabla u^{h}\|^{\ell}_{B^{s_2-1}}\\ &\le C\|v\|_{B^{\frac{d}{2}}}(\|u\|^{\ell}_{B^{s_1+1}}+\|u\|^{h}_{B^{s_2}})\le C\|v\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_1+1}\cap B^{s_2}}\end{aligned}$$

and

$$\|v\cdot\nabla u\|^{h}_{B^{s_2-1}}\le C\|v\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_2}}.$$

Then one immediately has

$$\int_0^t (\|(\partial_t u, \eta u)\|^\ell_{B^{s_1+1}} + \|(\partial_t u, \eta u)\|^h_{B^{s_2-1}}) + \int_0^t (\|\mu\Delta u, (\mu+\lambda)\nabla \mathrm{div}\, u\|^\ell_{B^{s_1+1}} + \int_0^t \|\mu\Delta u, (\mu+\lambda)\nabla \mathrm{div}\, u\|^h_{B^{s_2-1}})$$
$$\leq C(\|u_0\|^\ell_{B^{s_1+1}} + \|u_0\|^h_{B^{s_2-1}}) + C\int_0^t (\|\nabla v\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_1+1}\cap B^{s_2-1}} + \|v\|_{B^{\frac{d}{2}}}\|u\|_{B^{s_1+1}\cap B^{s_2}})$$
$$+ C\int_0^t (\|S_4\|^\ell_{B^{s_1+1}} + \|S_4\|^h_{B^{s_2-1}}) + C\int_0^t (\|(w,r)\|^\ell_{B^{s_1+2}} + \|(w,r)\|^h_{B^{s_2}}). \quad (4.65)$$

At last, we show the estimate for the time derivative of r. Recall that r satisfies

$$\partial_t r = -v\cdot\nabla r + (h_3 + H_3)\mathrm{div}\, u + S_3,$$

thus, using that $s_2 - 1 \leq s_1 + 1$, we have

$$\int_0^t \|\partial_t r\|^h_{B^{s_2-1}} \leq \int_0^t \Big(\|v\cdot\nabla r\|^h_{B^{s_2-1}} + \|(h_3+H_3)\mathrm{div}\, u\|^h_{B^{s_2-1}} + \|S_3\|^h_{B^{s_2-1}}\Big)$$
$$\leq C\int_0^t \Big(\|v\|_{B^{\frac{d}{2}}}\|r\|_{B^{s_2}} + (h_3 + \|H_3\|_{B^{\frac{d}{2}}})(\|u\|^\ell_{B^{s_1+2}} + \|u\|^h_{B^{s_2}})\Big) + \int_0^t \|S_3\|^h_{B^{s_2-1}},$$

and

$$\int_0^t \|\partial_t r\|^\ell_{B^{s_1+1}} \leq \int_0^t \Big(\|v\cdot\nabla r\|^\ell_{B^{s_1+1}} + \|(h_3+H_3)\mathrm{div}\, u\|^\ell_{B^{s_1+1}} + \|S_3\|^\ell_{B^{s_1+1}}\Big)$$
$$\leq C\int_0^t \Big(\|v\|_{B^{\frac{d}{2}}}(\|r\|^\ell_{B^{s_1+2}} + \|r\|^h_{B^{s_2}}) + (h_3 + \|H_3\|_{B^{\frac{d}{2}}})(\|u\|^\ell_{B^{s_1+2}} + \|u\|^h_{B^{s_2}})\Big)$$
$$+ \int_0^t \|S_3\|^\ell_{B^{s_1+1}}.$$

Combining (4.64), (4.65) and the above two inequalities and keeping in mind that $\eta \geq 1$, we conclude that (4.62) is satisfied. □

Completion of the proof of Proposition 4.3.1

First, we multiply (4.62) by a small constant (far less than κ) and add it to (4.61). Then applying Gronwall's lemma and taking c_0 small enough, we are able to obtain (4.28). The proof of Proposition 4.3.1 is completed. ■

4.5 Proof of Theorem 4.3.1

Here we expose the main arguments one has to use to obtain the existence of a unique global-in-time solution for System (4.23). We follow the scheme explained in details in [36]. As we mentioned in the Remark 4.3.1, we are not able to consider the case $\eta \to \infty$, so, for simplicity, from now on we assume $\eta = 1$. Moreover, in this section we fix the value of the regularity indexes $s_1 = \frac{d}{2} - 1$ and $s_2 = \frac{d}{2} + 1$, it is the best setting in which we can derive our global existence result. Note that the couple (s_1, s_2) satisfies the conditions in Theorem 4.3.1.

4.5.1 Existence scheme

Here we expose a classical iterative method to build a solution.

Iterative existence scheme

We consider the sequence $(Z^n)_{n\in\mathbb{N}} = (y^n, w^n, r^n, u^n)_{n\in\mathbb{N}}$ with smoothed out initial data

$$(y_0^n, w_0^n, r_0^n, u_0^n) = (\dot{S}_n y_0, \dot{S}_n w_0, \dot{S}_n r_0, \dot{S}_n u_0)$$

where $\dot{S}_n f = \sum_{j\leq n} \dot{\Delta}_j f_j$ and define the first term of the sequence $Z^0 = (0,0,0,0)$. Then, assuming that Z^n is smooth and globally well defined, we choose Z^{n+1} as the solution of the following linear system (the existence and uniqueness of solutions for an analogous system can be found in e.g. [33])

$$\begin{cases} \partial_t y^{n+1} + u^n \cdot \nabla y^{n+1} = 0, \\ \partial_t w^{n+1} + u^n \cdot \nabla w^{n+1} + \left(\bar{F}_1 + G_1^n\right) \operatorname{div} u^{n+1} + \left(\bar{F}_2 + G_2^n\right) \dfrac{w^{n+1}}{\nu} = 0, \\ \partial_t r^{n+1} + u^n \cdot \nabla r^{n+1} + \left(\bar{F}_3 + G_3^n\right) \operatorname{div} u^{n+1} = F_4^n \dfrac{(w^n)^2}{\nu}, \\ \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} + u^{n+1} + (\bar{F}_0 + G_0^n)\nabla r^{n+1} + (\gamma_+ - \gamma_-)\,(\bar{F}_0 + G_0^n)\nabla w^{n+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = (\bar{F}_0 + G_0^n)\mathcal{A}_{\mu,\lambda} u^{n+1} \\ (y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1})_{t=0} = (y_0^{n+1}, w_0^{n+1}, r_0^{n+1}, u_0^{n+1}) \end{cases} \tag{4.66}$$

where $G_i^n \triangleq G_i(w^n, r^n, u^n)$ for $i \in \overline{0,3}$. In the next part, we prove uniform estimates for Z^n in $E^{\frac{d}{2}-1,\frac{d}{2}+1}$.

First step: uniform estimates

We shall use the following classical inductive argument:
We claim that there exists constants c and N, such that if we assume that $\|Z_0\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}} < c$ then for all $n \in \mathbb{N}$, we have

$$\|Z^n\|_{E^{\frac{d}{2}-1,\frac{d}{2}+1}} \leq Nc. \tag{4.67}$$

This is obviously true for $n = 0$, let's assume that it is true for some fixed $n \in \mathbb{N}$ and prove it for $n+1$. First, looking at the equation of y^{n+1} and applying proposition 4.7.2 with $a = 0$ and $f = 0$, we get

$$\left\|y^{n+1}(\tau)\right\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}} + \int_0^t \left\|\partial_t y^{n+1}\right\|_{B^{\frac{d}{2}}} d\tau \leq \|y_0\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}} \exp\left(C\int_0^T \|u^n(\tau)\|_{B^{\frac{d}{2}+1}} d\tau\right)$$

Then, using (4.67), we get

$$\left\|y^{n+1}\right\|_{L_T^\infty(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})} + \int_0^t \left\|\partial_t y^{n+1}\right\|_{B^{\frac{d}{2}}} \leq ce^{Nc}. \tag{4.68}$$

Then, to recover some estimates for $(w^{n+1}, r^{n+1}, u^{n+1})$, we need to apply Proposition 4.3.1 to (4.66) without the equation of y^{n+1}, to do so, we have to show that for all $i \in \overline{0,3}$, we have

$$\|G_i^n\|_{L_T^\infty(L^\infty)} \leq CNc. \tag{4.69}$$

To that matter, expressing ρ as a function of r, w, y and using (4.67), we get

$$\begin{aligned}\|\rho^n\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})} &\leq C(\|r^n\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})} + \|w^n\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})} + \|y^n\|_{\widetilde{L}^\infty_T B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})}) \\ &\leq CNc. \end{aligned} \tag{4.70}$$

Next, as the G_i are smooth functions vanishing at origin, we can apply the composition Proposition A.2.6 and, for all $i \in \overline{1,3}$, obtain

$$\|G_i\,(r^n, w^n, y^n)\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})} \leq C\,\|(r^n, w^n, y^n)\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})} \leq CNc. \tag{4.71}$$

This combined with the inductive hypothesis (4.67) tells us that (4.69) is satisfied.

Therefore, applying Propositions 4.3.1 to (4.66) and adding the resulting inequality to (4.68), we obtain

$$\begin{aligned}&\|(y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1})\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}} + \int_0^t \|(w^{n+1}, r^{n+1}, u^{n+1})\|_{B^{\frac{d}{2}+1}} \\ &+ \int_0^t \|\frac{w^{n+1}}{\nu}\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}} + \int_0^t \|u^{n+1}\|_{B^{\frac{d}{2}}} \quad + \int_0^t \|(\partial_t y^{n+1}, \partial_t w^{n+1}, \partial_t r^{n+1}, \partial_t u^{n+1})\|_{B^{\frac{d}{2}}} \\ &\leq Ce^{CNc} + \exp(CV^n(t))\Big(\|(w_0, r_0, u_0)\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}} + \int_0^t \|F_4^n \frac{(w^n)^2}{\nu}\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}}\Big),\end{aligned}$$

where $V^n(t) = \int_0^t (\sum_{i=0}^3 \|\partial_t G_i^n\|_{B^{\frac{d}{2}}} + \|v^n\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}})$. Then, in order to obtain the desired estimate, we need to control the right hand side terms using an inductive argument. Defining $G_4^n := F_4 - \bar{F}_4$, thanks to (4.71) and (4.67), we have

$$\begin{aligned}\int_0^t \|(F_4^n)\frac{(w^n)^2}{\nu}\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1}} &\leq C(\bar{F}_4 + \|G_4^n\|_{L^\infty_T(B^{\frac{d}{2}})})\|w^n\|_{L^\infty_T(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})}\|\frac{w^n}{\nu}\|_{L^1_T(B^{\frac{d}{2}})} \\ &\leq C(1 + \|G_4^n\|_{L^\infty_T(B^{\frac{d}{2}})})\|Z^n\|^2_{E^{\frac{d}{2}-1,\frac{d}{2}+1}} \\ &\leq CN^2c^2.\end{aligned}$$

Concerning V^n, it is clear that $\int_0^t \|v^n\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}} \leq Cc$. Thus we are left with controlling the terms with time-derivative. For all $i \in \overline{0,3}$, we have

$$\partial_t G_i^n = \frac{\partial G_i^n}{\partial y}\partial_t y^n + \frac{\partial G_i^n}{\partial r}\partial_t r^n + \frac{\partial G_i^n}{\partial w}\partial_t w^n. \tag{4.72}$$

Using that the G_i^n are smooth functions and (4.67), we obtain

$$\int_0^t \sum_{i=1}^3 \|\partial_t G_i^n\|_{B^{\frac{d}{2}}} \leq CNc.$$

Gathering all those estimates, we obtain

$$\|Z^{n+1}\|_{E^{\frac{d}{2}-1,\frac{d}{2}+1}} + \int_0^t \|\frac{w^{n+1}}{\nu}\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}} + \int_0^t \|\eta u^{n+1}\|_{B^{\frac{d}{2}}} \leq Ce^{CNc}(c + CN^2c^2).$$

Thus, choosing c small enough and a suitable N, the inductive hypothesis is fulfilled for $n+1$, and thus for all $n \in \mathbb{N}$.

Second step: Existence of a solution

Here we show that the sequence $(Z^n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^d)$ to a solution Z of (4.23) which has the desired regularity properties. The following lemma will imply that $(Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a suitable space and also the uniqueness of the solution.

Lemma 4.5.1. *Let $U = (y_1, w_1, r_1, u_1)$ and $V = (y_2, w_2, r_2, u_2)$ be two solutions of (4.23) having, respectively, U_0 and V_0 as initial data and such that $U, V \in E_T^{\frac{d}{2}-1,\frac{d}{2}+1}$. We set $\tilde{V} = U - V$, it satisfies*

$$\|\tilde{V}\|_{E_T^{\frac{d}{2}-1,\frac{d}{2}}} \leq C(\|\tilde{V}_0\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}} + R(T)\|\tilde{V}\|_{E_T^{\frac{d}{2}-1,\frac{d}{2}}}) \tag{4.73}$$

with

$$R(T) = \|V\|_{L^\infty_T(B^{\frac{d}{2}+1}\cap B^{\frac{d}{2}-1})} + \frac{1}{\nu}(\|w_1\|_{L^1_T(B^{\frac{d}{2}})} + \|w_1\|^2_{L^1_T(B^{\frac{d}{2}})} + \|V\|_{L^\infty_T(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}})}\|(w_1, w_2)\|_{L^1_T(B^{\frac{d}{2}})}).$$

Proof. Let first consider the case $d \geq 3$. Observe that $\tilde{V}$ is a solution of

$$\begin{cases} \partial_t\tilde{y} + u_1\cdot\nabla\tilde{y} = \tilde{u}\nabla y_2, \\ \partial_t\tilde{w} + u_1\cdot\nabla\tilde{w} + \big(\bar{F}_1 + G_1(w_1, r_1, y_1)\big)\mathrm{div}\,\tilde{u} + \bar{F}_2\tilde{w} = \mathcal{R}_1 + R_1(U) - R_1(V), \\ \partial_t\tilde{r} + u_1\cdot\nabla\tilde{r} + \big(\bar{F}_3 + G_3(w_1, r_1, y_1)\big)\mathrm{div}\,\tilde{u} = \mathcal{R}_2 + R_2(U) - R_2(V), \\ \partial_t\tilde{u} + u_1\cdot\nabla\tilde{u} - (\bar{F}_0 + G_{0,1})\mathcal{A}_{\mu,\lambda}\tilde{u} + \eta\tilde{u} + (\bar{F}_0 + G_{0,1})\nabla\tilde{r} + (\gamma_+ - \gamma_-)\,(\bar{F}_0 + G_{0,1})\nabla\tilde{w} = \mathcal{R}_3 \end{cases} \tag{4.74}$$

where

$$\begin{aligned} \mathcal{R}_1 &= -\tilde{u}\nabla w_2 - (G_1(w_1, r_1, y_1) - G_1(w_2, r_2, y_2))\mathrm{div}\,u_2, \\ \mathcal{R}_2 &= \tilde{u}\nabla r_2 - (G_3(w_1, r_1, y_1) - G_3(w_2, r_2, y_2))\mathrm{div}\,u_2, \\ \mathcal{R}_3 &= -\tilde{u}\nabla u_2 - (G_{0,1} - G_{0,2})\,(-\mathcal{A}_{\mu,\lambda}u_2 + \nabla r_2 + (\gamma_+ - \gamma-)\nabla w_2)\,, \\ R_1(U) &= -G_2(w_1, r_1, y_1)\frac{w_1}{\nu} \quad \text{and} \quad R_2(U) = -F_4(w_1, r_1, y_1)\frac{w_1^2}{\nu}. \end{aligned}$$

From similar arguments as for system (4.66), we can apply Proposition 4.3.1 to the three last equations of (4.74) and Proposition 4.7.2 for the equation of $\tilde{y}$ in the case $r_1 = \dfrac{d}{2} - 1$ and $r_2 = \dfrac{d}{2}$, we obtain

$$\begin{aligned} \|\tilde{V}\|_{E_T^{\frac{d}{2}-1,\frac{d}{2}}} \leq\ & \exp\Big(C\int_0^t V\Big)\Big(\|\tilde{V}_0\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}} \\ & + \int_0^t \|(\tilde{u}\nabla y_2, \mathcal{R}_1, R_1(U) - R_1(V), \mathcal{R}_2, R_2(U) - R_2(V), \mathcal{R}_3)\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}}\Big) \end{aligned}$$

where $V(t) = \sum_{i=0}^{3} \|\partial_t G_i(w_1, r_1, y_1)\|_{B^{\frac{d}{2}}} + \|u_1\|_{B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}}$. Again, using from the smoothness of the G_i^n and (4.72), it is clear that there exists a C such that

$$\int_0^t V \leq C.$$

Concerning the source terms, since

$$R_1(U) - R_1(V) = -G_2(w_1, r_1, y_1)\frac{w_1}{\nu} + G_2(w_2, r_2, y_2)\frac{w_1}{\nu} - G_2(w_2, r_2, y_2)\frac{w_1}{\nu} + G_2(w_2, r_2, y_2)\frac{w_2}{\nu}$$

and similarly for R_2, using composition Proposition A.2.6, for $i = 1, 2$, we obtain

$$\begin{aligned}\|R_i(U) - R_i(V)\|_{B^{\frac{d}{2}}} \lesssim\ & \frac{1}{\nu}\|w_1\|_{B^{\frac{d}{2}}}\|\tilde{V}\|_{B^{\frac{d}{2}}} + \frac{1}{\nu}\|\tilde{w}\|_{B^{\frac{d}{2}}}\|V\|_{B^{\frac{d}{2}}} \\ & + \frac{1}{\nu}\|w_1\|^2_{B^{\frac{d}{2}}}\|\tilde{V}\|_{B^{\frac{d}{2}}} + \frac{1}{\nu}\|\tilde{w}\|_{B^{\frac{d}{2}}}\|(w_1, w_2)\|_{B^{\frac{d}{2}}}\|V\|_{B^{\frac{d}{2}}}.\end{aligned}$$

Similarly

$$\begin{aligned}\|R_i(U) - R_i(V)\|_{B^{\frac{d}{2}-1}} \lesssim\ & \frac{1}{\nu}\|w_1\|_{B^{\frac{d}{2}}}\|\tilde{V}\|_{B^{\frac{d}{2}-1}} + \frac{1}{\nu}\|\tilde{w}\|_{B^{\frac{d}{2}}}\|V\|_{B^{\frac{d}{2}-1}} \\ & + \frac{1}{\nu}\|w_1\|^2_{B^{\frac{d}{2}}}\|\tilde{V}\|_{B^{\frac{d}{2}-1}} + \frac{1}{\nu}\|\tilde{w}\|_{B^{\frac{d}{2}}}\|(w_1, w_2)\|_{B^{\frac{d}{2}}}\|V\|_{B^{\frac{d}{2}-1}}.\end{aligned}$$

Using composition Proposition A.2.6, Corollary 2.66 from [3] and product law we obtain

$$\int_0^T \|(\dot{\Delta}_j\mathcal{R}_1, \dot{\Delta}_j\mathcal{R}_2, \dot{\Delta}_j\mathcal{R}_3)\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}} \lesssim \|\tilde{V}\|_{L^\infty_T(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}})}\|V\|_{E^{\frac{d}{2}-1,\frac{d}{2}+1}}.$$

Gathering those estimates, we obtain

$$\|\tilde{V}\|_{E_T^{\frac{d}{2}-1,\frac{d}{2}}} \leq C(\|\tilde{V}_0\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}} + R(T)\|\tilde{V}\|_{E_T^{\frac{d}{2}-1,\frac{d}{2}}}),$$

which is the desired estimate in the case $d \geq 3$. The above proof fails for $d = 2$ as some right-hand side terms have to be estimated in Besov spaces with a regularity index equal to zero (such as e.g. $G_2(w_2, r_2, y_2) - G_2(w_1, r_1, y_1)$). To overcome this difficulty one must adapt the proof to Chemin-Lerner spaces defined in [3, 24, 25] with third index $r = \infty$ and to estimate the difference of solutions with logarithmic interpolation inequality. For more details you may refer to [3] p.445-447. □

Applying Lemma 4.5.1 with $U = Z^n$ and $V = Z^{n+1}$ and using that thanks to the uniform bounds (4.67) the right-hand side of (4.73) can be absorbed by its left hand side, we infer that $(Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $E^{\frac{d}{2}-1,\frac{d}{2}+1}$ and therefore there exists a Z such that $(Z^n)_{n\in\mathbb{N}}$ converges strongly toward Z in $E^{\frac{d}{2}-1,\frac{d}{2}+1}$.

We are now left with proving that Z is a solution of (4.23) and indeed satisfies the stated regularity properties. The proof of such results are quite classical, we would like to omit details here, and advice the reader to the lecture [36] and the paper [33] about the study of Navier-Stokes equation.

4.5.2 Uniqueness

As a direct consequence of Lemma 4.5.1, the following result implies the uniqueness of our solution.

Lemma 4.5.2. *Let U and V be two solutions of (4.23) with the same initial data such that $U, V \in E_T^{\frac{d}{2}-1,\frac{d}{2}+1}$. There exists a constant $K > 0$ such that if*

$$\|V\|_{L_T^\infty(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}+1})} \leq K \tag{4.75}$$

then $U \equiv V$.

Proof. Let $\tilde{V} = U - V$, since $U_0 = V_0$, lemma 4.5.1 implies that

$$\|\tilde{V}\|_{E_T^{\frac{d}{2}-1,\frac{d}{2}}} \leq CR(T)\|\tilde{V}\|_{E_T^{\frac{d}{2}-1,\frac{d}{2}}}$$

with

$$R(T) = \|V\|_{L_T^\infty(B^{\frac{d}{2}+1}\cap B^{\frac{d}{2}-1})} + \frac{1}{\nu}(\|w_1\|_{L_T^1(B^{\frac{d}{2}})} + \|w_1\|^2_{L_T^1(B^{\frac{d}{2}})} + \|V\|_{L_T^\infty(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}})}\|(w_1,w_2)\|_{L_T^1(B^{\frac{d}{2}})}).$$

As we have

$$\limsup_{T\to 0_+} R(T) \leq C\|V\|_{L_T^\infty(B^{\frac{d}{2}+1}\cap B^{\frac{d}{2}-1})},$$

choosing K such that $CK < 1$ we deduce that $\|\tilde{V}\|_{E_T^{\frac{d}{2}-1,\frac{d}{2}}} = 0$ for a $T > 0$ small enough. Therefore, $U = V$ on $[0, T]$. Then, a classical bootstrap argument allows to show that it is also true for $T = +\infty$. □

4.6 Relaxation limit

In this section we assume that $d \geq 3$ as they are some limitations for $d = 2$, due to negative regularity indexes.

4.6.1 Recovering a solution for the Kapila system

Here we establish the strong convergence locally-in-space of system (*BN*) to system (*K*), which proves Theorem 4.2.2. To do that we assume that we have a solution $(\alpha_+^\nu, \alpha_-^\nu, \rho_+^\nu, \rho_-^\nu, u^\nu)$ of (*BN*) satisfying the properties described in Theorem 4.6.1 which are similar to the properties obtained in the a priori estimates used to prove the existence Theorem 4.2.1.

First of all, we will derive the equation of α_+ of the limit System (*K*) by following an idea of Bresch and Hillairet in [13]. In System (*K*), the second equation reads:

$$\alpha_+(\partial_t\rho_+ + u\cdot\nabla\rho_+ + \rho_+\operatorname{div} u) + \rho_+(\partial_t\alpha_+ + u\cdot\nabla\alpha_+) = 0.$$

Multiplying this equation by $P_+'(\rho_+)$ and using that $\rho_+P_+'(\rho_+) = \gamma_+P_+(\rho_+)$, we get

$$\alpha_+(\partial_t P_+ + u\cdot\nabla P_+ + \gamma_+P_+\operatorname{div} u) + \gamma_+P_+(\partial_t\alpha_+ + u\cdot\nabla\alpha_+) = 0, \tag{4.76}$$

and similarly for $\alpha_-\rho_-$. Then multiplying the resulting equations respectively by α_- and α_+ we obtain

$$\alpha_+\alpha_-\big(\partial_t(P_+ - P_-) + u\cdot\nabla(P_+ - P-) + (\gamma_+P_+ - \gamma_-P_-)\mathrm{div}\, u\big) \\ + (\alpha_-\gamma_+P_+ + \alpha_+\gamma_-P_-)(\partial_t\alpha_+ + u\cdot\nabla\alpha_+) = 0. \quad (4.77)$$

Then, since for system (4.2) there holds $P_+ = P_-$, we have

$$\partial_t\alpha_+ + u\cdot\nabla\alpha_+ = -\frac{(\gamma_+ - \gamma_-)\alpha_+\alpha_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}\,\mathrm{div}\, u.$$

Taking this equation into (4.76), one finds that System (K)-(4.7) is equivalent to the following system

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ \partial_t\alpha_+ + u\cdot\nabla\alpha_+ = -\dfrac{(\gamma_+ - \gamma_-)\alpha_+\alpha_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}\,\mathrm{div}\, u, \\ \partial_t P + u\cdot\nabla P = -\dfrac{\gamma_+\gamma_- P}{\gamma_+\alpha_- + \gamma_-\alpha_+}\mathrm{div}\, u, \\ \rho(\partial_t u + u\cdot\nabla u) + \nabla P + \rho u = 0, \\ \rho = \alpha_+\rho_+ + \alpha_-\rho_-, \\ P = P_+(\rho_+) = P_-(\rho_-), \\ (\alpha_+, \alpha_-, P_+, P_-, u)|_{t=0} = (\alpha_{+0}, \alpha_{-0}, P_0, P_0, u_0). \end{cases} \quad (4.78)$$

For simplicity, we use $P^\nu_\pm$ to represent $P_\pm(\rho^\nu_\pm)$ respectively. Since the solution $(\alpha^\nu_+, \alpha^\nu_-, \rho^\nu_+, \rho^\nu_+, u^\nu)$ is regular enough, the equations for $P^\nu_\pm$ can be obtained rigorously as in Section 3, we have

$$\partial_t P^\nu_\pm + u^\nu\cdot\nabla P^\nu_\pm + \gamma_\pm P^\nu_\pm\,\mathrm{div}\, u^\nu = \mp\frac{\gamma_\pm\alpha^\nu_\mp P^\nu_\pm}{\nu}(P^\nu_+ - P^\nu_-)$$

and thus

$$\frac{\alpha^\nu_+\alpha^\nu_-}{\nu}\left(P^\nu_+ - P^\nu_-\right) = \frac{-\alpha^\nu_+\alpha^\nu_-}{\gamma_+\alpha^\nu_- P^\nu_+ + \gamma_-\alpha^\nu_+ P^\nu_-}\Big(\partial_t(P^\nu_+ - P^\nu_-) + u^\nu\cdot\nabla(P^\nu_+ - P^\nu_-) + (\gamma_+P^\nu_+ - \gamma_-P^\nu_-)\mathrm{div}\, u^\nu\Big).$$

Substituting this equation into the equations of α^ν_+ and $P^\nu_\pm$, we have

$$\begin{cases} \alpha^\nu_+ + \alpha^\nu_- = 1, \\ \partial_t\alpha^\nu_+ + u^\nu\cdot\nabla\alpha^\nu_+ = \Gamma_1\Big(\partial_t(P^\nu_+ - P^\nu_-) + u^\nu\cdot\nabla(P^\nu_+ - P^\nu_-)\Big) - \dfrac{\alpha^\nu_+\alpha^\nu_-(\gamma_+P^\nu_+ - \gamma_-P^\nu_-)}{\gamma_+\alpha^\nu_- P^\nu_+ + \gamma_-\alpha^\nu_+ P^\nu_-}\mathrm{div}\, u^\nu, \\ \partial_t P^\nu_+ + u^\nu\cdot\nabla P^\nu_+ = \Gamma_2\Big(\partial_t(P^\nu_+ - P^\nu_-) + u^\nu\cdot\nabla(P^\nu_+ - P^\nu_-)\Big) - \dfrac{\gamma_+\gamma_- P^\nu_+ P^\nu_-}{\gamma_+\alpha^\nu_- P^\nu_+ + \gamma_-\alpha^\nu_+ P^\nu_-}\,\mathrm{div}\, u^\nu, \\ \partial_t P^\nu_- + u^\nu\cdot\nabla P^\nu_- = (\Gamma_2 - 1)\Big(\partial_t(P^\nu_+ - P^\nu_-) + u^\nu\cdot\nabla(P^\nu_+ - P^\nu_-)\Big) - \dfrac{\gamma_+\gamma_- P^\nu_+ P^\nu_-}{\gamma_+\alpha^\nu_- P^\nu_+ + \gamma_-\alpha^\nu_+ P^\nu_-}\,\mathrm{div}\, u^\nu, \\ \rho^\nu(\partial_t u^\nu + u^\nu\cdot\nabla u^\nu) + \nabla P^\nu + \rho^\nu u^\nu = \mathcal{A}_{\mu,\lambda}u^\nu, \\ \rho^\nu = \alpha^\nu_+\rho^\nu_+ + \alpha^\nu_-\rho^\nu_-, \\ P^\nu = \alpha^\nu_+ P^\nu_+ + \alpha^\nu_- P^\nu_-, \\ (\alpha^\nu_+, \alpha^\nu_-, P^\nu_+, P^\nu_-, u^\nu)|_{t=0} = (\alpha^\nu_{+0}, \alpha^\nu_{-0}, P^\nu_{+0}, P^\nu_{-0}, u^\nu_0) \end{cases} \quad (4.79)$$

where

$$\Gamma_1 := -\frac{\alpha_+^\nu \alpha_-^\nu}{\gamma_+ \alpha_-^\nu P_+^\nu + \gamma_- \alpha_+^\nu P_-^\nu}, \quad \Gamma_2 := \frac{\gamma_+ \alpha_-^\nu P_+^\nu}{\gamma_+ \alpha_-^\nu P_+^\nu + \gamma_- \alpha_+^\nu P_-^\nu}.$$

Here and after, Γ_i are some regular functions of variables $(\alpha_+^\nu, \alpha_-^\nu, P_+^\nu, P_-^\nu)$ and $\bar{\Gamma}_i := \Gamma_i(\bar{\alpha}_+,\ \bar{\alpha}_-,\ \bar{P})$. Thus with the composition Proposition A.2.6 for multivariate functions, we have

$$\|\Gamma_i - \bar{\Gamma}_i\|_{L^\infty(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} \leq C\|(\alpha_\pm^\nu - \bar{\alpha}_\pm, P_\pm^\nu - \bar{P})\|_{L^\infty(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} \leq CM_1. \tag{4.80}$$

From the bounds of the existence theorem, we see that $\frac{1}{\nu}(P_+^\nu - P_-^\nu)$ is uniformly bounded in $L^1(B^{\frac{d}{2}})$. Therefore, $\partial_t(P_+^\nu - P_-^\nu)$ converges to zero when ν goes to zero in the sense of distributions, and the product law $B^{\frac{d}{2}-1} \times B^{\frac{d}{2}} \hookrightarrow B^{\frac{d}{2}-1}$ yields

$$\|u^\nu \cdot \nabla(P_+^\nu - P_-^\nu)\|_{L^1(B^{\frac{d}{2}-1})} \leq C\|u^\nu\|_{L^\infty(B^{\frac{d}{2}})} \|P_+^\nu - P_-^\nu\|_{L^1(B^{\frac{d}{2}})} \to 0 \quad \text{as} \quad \nu \to 0.$$

In particular, this implies that the first terms in the right-hand sides of equations of $\alpha_+^\nu, P_+^\nu, P_-^\nu$ converge to zero respectively in the sense of distributions, since it is easy to find that $\Gamma_1, \Gamma_2 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$.

At this stage, with the uniform bounds of the existence theorems in hand, one may perform the classical weak compactness method to show that there exists a function $(\alpha_+^0, \alpha_-^0, P_+^0, P_-^0, u^0)$ such that

$$(\alpha_+^0 - \bar{\alpha}_+, \alpha_-^0 - \bar{\alpha}_-, P_+^0 - \bar{P}_+, P_-^0 - \bar{P}_-, u^0) \in L^\infty(\mathbb{R}^+; B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})$$

$$\text{and} \ (\alpha_+^\nu, \alpha_-^\nu, P_+^\nu, P_-^\nu, u^\nu) \to (\alpha_+, \alpha_-, P_+^0, P_-^0, u^0) \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d) \quad \text{as} \quad \nu \to 0.$$

Moreover, with this strong convergence one can further show that $(\alpha_+^0, \alpha_-^0, P_+^0, P_-^0, u^0)$ is a solution to the Cauchy problem (4.78). To prove the uniqueness of the solution one has to perform similar computations as in the proof of Theorem 4.2.1 with the help of Proposition 4.6.1. Therefore we can conclude Theorem 4.2.2.

However, this weak compactness method does not allow us to obtain an explicit rate of convergence, we shall work in this direction in the following.

4.6.2 Convergence rate

4.6.2.1 Presentation of the problem and strategy

To tackle this problem, let us first define the difference of two solutions by

$$(\delta\alpha_+,\ \delta\alpha_-, \delta\rho_+,\ \delta\rho_-,\ \delta u) := (\alpha_+^\nu - \alpha_+, \alpha_-^\nu - \alpha_-,\ \rho_+^\nu - \rho_+,\ \rho_-^\nu - \rho_-,\ u^\nu - u).$$

We have to admit that it seems hard to obtain decay rate for the terms containing $\partial_t(P_+^\nu - P_-^\nu)$ in the system (4.79). To avoid this problem we will replace the equation of P_+^ν by the equation of $Q_+^\nu := P_+^\nu - \Gamma_2(P_+^\nu - P_-^\nu)$. Moreover as in the equation of $\delta\alpha_+$ there would be a linear higher-order term of δu (i.e. $\operatorname{div} \delta u$) which would be hard to handle, we will replace the equation of $\delta\alpha_+$by the equation of $Y_+^\nu - Y_+$. This last modification is similar to the one we did to prove the global well-posedness for system (BN), it does not seem suitable to use the equations of $\alpha_\pm$ as

we cannot recover L^1-in-time integrability on this unknown.
More precisely, we consider the following differences

$$\delta Y_+ := \frac{\alpha_+^\nu \rho_+^\nu}{\alpha_+^\nu \rho_+^\nu + \alpha_-^\nu \rho_+^\nu} - \frac{\alpha_+ \rho_+}{\alpha_+ \rho_+ + \alpha_- \rho_-}, \quad \delta Q_+ := P_+^\nu - \Gamma_2 (P_+^\nu - P_-^\nu) - P_+, \tag{4.81}$$

so that we have the obvious relationships

$$\begin{cases} \delta\alpha_+ = \delta Y_+ \dfrac{\rho^\nu \rho}{\rho_+^\nu \rho_-} + \dfrac{\alpha_+ \alpha_-^\nu \rho_-}{\rho_+^\nu \rho_-} \delta\rho_+ + \dfrac{\alpha_+ \alpha_-^\nu \rho_+}{\rho_+^\nu \rho_-} \delta\rho_-, \\ \delta P_+ := P_+^\nu - P_+ = \delta Q_+ + \Gamma_2 (P_+^\nu - P_-^\nu), \\ \delta P_- := P_-^\nu - P_- = \delta Q_+ + (\Gamma_2^\nu - 1)(P_+^\nu - P_-^\nu), \\ \delta P := P^\nu - P = \delta Q_+ + (\Gamma_2 - \alpha_-^\nu)(P_+^\nu - P_-^\nu). \end{cases} \tag{4.82}$$

Notice that we also have

$$\begin{cases} \delta\alpha_+ + \delta\alpha_- = 0, \\ \delta\rho_\pm = (\frac{1}{A_\pm} P_\pm^\nu)^{\frac{1}{\gamma_\pm}} - (\frac{1}{A_\pm} P_\pm)^{\frac{1}{\gamma_\pm}}, \\ \delta\rho = (\rho_+^\nu - \rho_-^\nu)\delta\alpha_+ + \alpha_+ \delta\rho_+ + \alpha_- \delta\rho_-. \end{cases} \tag{4.83}$$

From System (4.78) and System (4.79), we obtain the following system for $(\delta Y_+, \delta Q_+, \delta u)$

$$\begin{cases} \partial_t \delta Y_+ + u^\nu \cdot \nabla \delta Y_+ = \delta S_1, \\ \partial_t \delta Q_+ + u^\nu \cdot \nabla \delta Q_+ + \big(\bar\Gamma_3 + (\Gamma_3 - \bar\Gamma_3)\big) \operatorname{div} \delta u = \delta S_2, \\ \partial_t \delta u + u^\nu \cdot \nabla \delta u + \delta u + \big(\dfrac{1}{\bar\rho} + (\dfrac{1}{\rho^\nu} - \dfrac{1}{\bar\rho})\big) \nabla \delta Q_+ = \delta S_3, \\ (\delta Y_+, \delta Q_+, \delta u)|_{t=0} = (0,0,0) \end{cases} \tag{4.84}$$

where $\Gamma_3 := \dfrac{\gamma_+ \gamma_- P_+^\nu P_-^\nu}{\gamma_+ \alpha_-^\nu P_+^\nu + \gamma_- \alpha_+^\nu P_-^\nu}$, $\Gamma_4 := \dfrac{\gamma_+ P_+^\nu - \gamma_- P_-^\nu}{\gamma_+ \alpha_-^\nu P_+^\nu + \gamma_- \alpha_+^\nu P_-^\nu}$ and

$$\begin{cases} \delta S_1 = -\delta u \cdot \nabla Y_+, \\ \delta S_2 = -\delta u \cdot \nabla P_+ - (P_+^\nu - P_-^\nu)(\partial_t \Gamma_2 + u^\nu \cdot \nabla \Gamma_2) - \dfrac{\gamma_+ \gamma_-}{\gamma_+ \alpha_- + \gamma_- \alpha_+} (\delta Q_+ + \Gamma_4 \, \delta\alpha_+) \operatorname{div} u \\ \delta S_3 = -\delta u \cdot \nabla u + \dfrac{1}{\rho^\nu} \mathcal{A}_{\mu,\lambda} \, u^\nu + \dfrac{\nabla P_+}{\rho^\nu \rho} \, \delta\rho - \dfrac{1}{\rho^\nu} \nabla \Big((\Gamma_2 - \alpha_-^\nu)(P_+^\nu - P_-^\nu) \Big). \end{cases}$$

At this moment, one may find that the advantages of our choice of $(\delta Y_+, \delta Q_+)$ is that all the source terms can be represented by the quantities depending on the differences $(\delta Y_+, \delta Q_+, \delta u)$ or on $P_+^\nu - P_-^\nu$ which will be proven to have a convergence rate of at least $\sqrt{\nu}$ in $L_T^\infty(B^s)$ and $L_T^1(B^s)$ for a suitable s. The necessity of L_T^∞ decay comes from the fact that some factors of $P_+^\nu - P_-^\nu$ will only belong to $L_T^1(B^s)$. Moreover, the first equation in System (4.84) is a transport equation, while the coupling between the second and the third equations is covered by Proposition 4.3.1 or, more precisely, by eliminating all the "w" factors in Proposition 4.3.1, one can obtain the following proposition:

Proposition 4.6.1. *Given functions E_1, E_2, E_3, E_4 and positive constants e_1, e_2 such that*

$$E_1, \cdots, E_4,\ v \in \mathcal{C}^1(\mathbb{R}_+ \times \mathcal{S}(\mathbb{R}^d)), \qquad \|E_i\|_{L^\infty(0,T;\mathbb{R}^d)} \leq \frac{1}{2}\, e_i, \qquad i = 1, 2.$$

Let $-\frac{d}{2} < s_1 \leq \frac{d}{2} - 1$ and $s_1 \leq s_2 - 1 \leq s_1 + 1$. Suppose that (q, u) is a solution of the following linear System (4.85) on time interval $[0, T)$

$$\begin{cases} \partial_t q + v \cdot \nabla q + \big(e_1 + E_1\big)\mathrm{div}\, u = E_3, \\ \partial_t u + v \cdot \nabla u + u + \big(e_2 + E_2\big)\nabla q = E_4, \\ (q, u)|_{t=0} = (q_0, u_0). \end{cases} \tag{4.85}$$

There exists a positive constant e_0 depends only on e_1, e_2 and dimension d such that if

$$\|(E_1, E_2)\|_{L^\infty_t(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} \leq e_0,$$

then the following estimate holds on $[0, T)$

$$\begin{aligned} &\|(q,u)\|^\ell_{L^\infty_t(B^{s_1})} + \|(q,u)\|^h_{L^\infty_t(B^{s_2})} + \|q\|^\ell_{L^1_t(B^{s_1+2})} + \|u\|^\ell_{L^1_t(B^{s_1+1})} + \|(q,u)\|^h_{L^1_t(B^{s_2})} \\ \leq &\exp(CE(t))\Big(\|(q_0,u_0)\|^\ell_{B^{s_1}} + \|(q_0,u_0)\|^h_{B^{s_2}} + \|(E_3,E_4)\|_{L^1_t(B^{s_1} \cap B^{s_2})}\Big), \end{aligned}$$

where $E(t) = \displaystyle\int_0^t \big(\|(\partial_t E_1, \partial_t E_2)\|_{B^{\frac{d}{2}}} + \|v\|_{B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}}\big)\,.$

We are now in the position of stating our convergence result.

Theorem 4.6.1. *Let $d \geq 3$. Let $\nu \in (0, 1]$ and assume that (4.5) are (4.8) are satisfied. Given any $T \in (0, \infty]$, suppose that $(\alpha^\nu_+, \alpha^\nu_-, P^\nu_+, P^\nu_-, u^\nu)$ (resp. $(\alpha_+, \alpha_-, P_+, P_-, u)$) is the solution to the Cauchy problem (4.79) (resp. (4.78)) that satisfies*

$$\|(\delta Y_{+0}, \delta Q_{+0}, P^\nu_{\pm 0} - P_{\pm 0}, u^\nu_0 - u_0)\|_{B^{\frac{d}{2}-\frac{3}{2}} \cap B^{\frac{d}{2}-\frac{1}{2}}} \leq C\sqrt{\nu}, \tag{4.86}$$

where δY_{+0}, δQ_{+0} are the initial data of the differences δY_+ and δQ_+ defined in (4.81),

$$\begin{cases} (\alpha^\nu_+ - \bar\alpha_+, \alpha^\nu_- - \bar\alpha_-, \rho^\nu_+ - \bar\rho_+, \rho^\nu_- - \bar\rho_-, u^\nu) \in \mathcal{C}([0,T); B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}), \\ (\alpha_+ - \bar\alpha_+, \alpha_- - \bar\alpha_-, \rho_+ - \bar\rho_+, \rho_- - \bar\rho_-, u) \in \mathcal{C}([0,T); B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1}), \end{cases} \tag{4.87}$$

and that there exist positive constants M_0, M_1 independent of ν such that

$$\|(\alpha^\nu_\pm - \bar\alpha_\pm, \rho^\nu_\pm - \bar\rho)\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} + \|(\alpha_\pm - \bar\alpha_\pm, \rho_\pm - \bar\rho)\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} \leq M_0, \tag{4.88}$$

$$\begin{aligned} &\|\frac{1}{\nu}(P^\nu_+ - P^\nu_-)\|_{\widetilde{L}^1_T(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}})} + \|\ \mathcal{A}_{\mu,\lambda}\, u^\nu\|_{\widetilde{L}^1_T(B^{\frac{d}{2}})} + \|P^\nu_+ - P^\nu_-\|_{\widetilde{L}^1_T(B^{\frac{d}{2}+1})} \\ &+\|(\partial_t \alpha^\nu_\pm,\ \partial_t \rho^\nu_\pm,\ \nabla u^\nu,\ \nabla u,)\|_{\widetilde{L}^1_T(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}})} + \|(u^\nu,\ u)\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}-1} \cap B^{\frac{d}{2}+1})} \leq M_1. \end{aligned} \tag{4.89}$$

Then there exists a constant C such that we have the following estimate for all $t \in [0,T)$,

$$\|(\delta Y_+, \delta Q_+, u^\nu - u)\|_{L^\infty_T(B^{\frac{d}{2}-\frac{3}{2}} \cap B^{\frac{d}{2}-\frac{1}{2}})} + \|\delta Q_+\|^\ell_{L^1_T(B^{\frac{d}{2}+\frac{1}{2}})} + \|\delta Q_+\|^h_{L^1_T(B^{\frac{d}{2}-\frac{1}{2}})} + \|\delta u\|_{L^1_T(B^{\frac{d}{2}-\frac{1}{2}})} \leq \sqrt{\nu}\, \exp(CM_1^4).$$

Above, the constants M_0, M_1 and C depend only on $\bar{\alpha}_\pm$, $\bar{\rho}_\pm$, $\bar{P}$ and the dimension d.

Remark 4.6.1. *We shall suppose that M_0 is small (say $M_0 \ll 1$ without loss of generality), however M_1 need not to be small.*

Remark 4.6.2. *The assumption (4.86) can be lowered to $\mathcal{O}(\nu^\gamma)$ with $\gamma > 0$ but then we would end up with a convergence rate equal to ν^β with $\beta = \min\{\frac{1}{2}, \gamma\}$.*

All the properties assumed in the above theorem are satisfied by the solution coming from Theorem 4.2.1 thanks to the properties established in Theorem 4.3.1 and therefore this result immediately implies Theorem 4.2.3. The rest of this section is devoted to the proof of the above theorem.

It is clear that the System (4.84) can be looked as a linear system with a given convection velocity u^ν and coefficients that fall in the range of application of Proposition 4.6.1. Indeed, under condition (4.88) with small enough M_0, the assumptions presented in Proposition 4.6.1 are satisfied. Thus one only needs to find the appropriate regularity indexes in low and high frequencies so that all source terms are bounded and have "enough decay" appropriately. This is the purpose of the following lines.

4.6.2.2 Finding the appropriate spaces

Notice that by interpolation inequality and Young's inequality we have

$$\begin{aligned}
\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}} &\leq \|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}}} \|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}-1}} \\
&\leq \sqrt{\nu}\, \|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}}} \|\Delta u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}-1}} \\
&\leq \sqrt{\nu}\, \big(\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|_{B^{\frac{d}{2}}} + \|u^\nu\|_{B^{\frac{d}{2}+1}}\big).
\end{aligned}$$

Then the uniform bounds (4.89) imply that

$$\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|_{L^1(B^{\frac{d}{2}-\frac{1}{2}})} \leq M_{11}\sqrt{\nu}, \tag{4.90}$$

and

$$\begin{aligned}
\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|_{B^{\frac{d}{2}-\frac{3}{2}}} &\leq \|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}-1}} \|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}-2}} \\
&\leq \sqrt{\nu}\, \|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}-1}} \|\Delta u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}-2}} \\
&\leq \sqrt{\nu}\, \big(\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|_{B^{\frac{d}{2}-1}} + \|u^\nu\|_{B^{\frac{d}{2}}}\big).
\end{aligned}$$

Then the uniform bounds (4.89) imply that

$$\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|_{L^1(B^{\frac{d}{2}-\frac{3}{2}})} \leq M_{11}\sqrt{\nu}. \tag{4.91}$$

Remark 4.6.3. *In fact, for any $\theta \in (0,1)$ we have*

$$\begin{aligned}\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|_{L^1(B^{\frac{d}{2}-\theta})} &\leq \|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{\theta}_{L^1(B^{\frac{d}{2}-1})}\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{1-\theta}_{L^1(B^{\frac{d}{2}})}\\ &\leq \Big(\nu\,\|\Delta\, u^\nu\|_{L^1(B^{\frac{d}{2}-1})}\Big)^{\theta}\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{1-\theta}_{L^1(B^{\frac{d}{2}})}\\ &\leq \nu^\theta\|u^\nu\|^{\theta}_{L^1(B^{\frac{d}{2}+1})}\|\mathcal{A}_{\mu,\lambda}\, u^\nu\|^{1-\theta}_{L^1(B^{\frac{d}{2}})} \leq C(\theta)M_1\,\nu^\theta.\end{aligned}$$

This means that we could get a better decay rate for this term if we change the space where we estimate the difference of solutions, like ν^θ with $\theta \in [\frac{1}{2},1)$. However, some of the source terms containing $\alpha_\pm$ will have a decay rate bounded by $\sqrt{\nu}$ only in $B^{\frac{d}{2}-\frac{1}{2}}$ and $B^{\frac{d}{2}-\frac{3}{2}}$ so it is not relevant to use this idea until we find a way to improve the decay rate for the terms containing volume fractions.

Now, from the bounds (4.89), we know that $\|P^\nu_+ - P^\nu_-\|_{L^1(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}})}$ converges to zero at the rate ν, but the convergence rates in the L^∞-in-time based spaces are not clear. We have to emphasis that this kind of convergence rates are very important, since when handling the source term $(P^\nu_+ - P^\nu_-)\partial_t\Gamma_2$ lying in δS_2, one could only get $\partial_t\Gamma_2 \in L^1(B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}})$ from the bounds (4.89), therefore obtaining convergence rate in L^∞-in-time for $P^\nu_+ - P^\nu_-$ is necessary.
In fact we have such a result in Chemin-Lerner type spaces.

Proposition 4.6.2. *Let (P^ν_+, P^ν_-) satisfying the condition of Theorem 4.6.1, we have*

$$\|P^\nu_+ - P^\nu_-\|_{\widetilde{L}^\infty(B^{\frac{d}{2}-\frac{3}{2}}\cap B^{\frac{d}{2}-\frac{1}{2}})} \leq C\sqrt{\nu}. \tag{4.92}$$

Proof. It is easy to check that

$$\begin{aligned}&\|\gamma_+\alpha^\nu_- P^\nu_+ + \gamma_-\alpha^\nu_+ P^\nu_- - (\gamma_+\bar{\alpha}_-\bar{P} + \gamma_-\bar{\alpha}_+\bar{P})\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}\\ &\leq C\|(\alpha^\nu_+ - \bar{\alpha}_+, \alpha^\nu_- - \bar{\alpha}_-, P^\nu_+ - \bar{P}, P^\nu_- - \bar{P})\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}\\ &\leq CM_0\end{aligned}$$

and therefore it is small enough. So we can apply Proposition 4.7.1 to (4.6.1) with $s \in [\frac{d}{2} - \frac{3}{2},\ \frac{d}{2} - \frac{1}{2}]$, one has

$$\begin{aligned}\|P^\nu_+ - P^\nu_-\|_{\widetilde{L}^\infty(B^s)} \leq& C\|P^\nu_{+0} - P^\nu_{-0}\|_{B^s}\\ &+ C\sqrt{\nu}\Big(\|u^\nu\nabla(P^\nu_+ - P^\nu_-)\|_{\widetilde{L}^2(B^s)} + \|(\gamma_+P^\nu_+ - \gamma_-P^\nu_-)\operatorname{div} u^\nu\|_{\widetilde{L}^2(B^s)}\Big).\end{aligned} \tag{4.93}$$

Notice that thanks to (4.86) and the fact that $P_{+0} = P_{-0}$, we have $\|P^\nu_{+0} - P^\nu_{-0}\|_{B^s} \leq C\sqrt{\nu}$.
Now, recalling bounds (4.88) and (4.89), using the product law

$$B^{\frac{d}{2}} \times B^{\frac{d}{2}-\frac{3}{2}} \hookrightarrow B^{\frac{d}{2}-\frac{3}{2}} \tag{4.94}$$

and interpolation inequalities we have

$$\begin{aligned}\|u^\nu\cdot\nabla(P_+^\nu-P_-^\nu)\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac{3}{2}})}&\le C\,\|u^\nu\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}\|\nabla(P_+^\nu-P_-^\nu)\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac{3}{2}})}\\&\le C\,\|u^\nu\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}\|P_+^\nu-P_-^\nu\|^{\frac12}_{\widetilde{L}^1(B^{\frac{d}{2}})}\|P_+^\nu-P_-^\nu\|^{\frac12}_{\widetilde{L}^\infty(B^{\frac{d}{2}-1})}\\&\le CM_0^{\frac12}M_1^{\frac32}\end{aligned}$$

and

$$\begin{aligned}&\|(\gamma_+P_+^\nu-\gamma_-P_-^\nu)\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac{3}{2}})}\\&\quad\le\gamma_+\|(P_+^\nu-\bar P)\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac{3}{2}})}+\gamma_-\|(P_-^\nu-\bar P)\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac{3}{2}})}\\&\qquad\qquad+(\gamma_++\gamma_-)\bar P\,\|\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac{3}{2}})}\\&\quad\le C\Big(\gamma_+\|(P_-^\nu-\bar P)\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}+\gamma_-\|(P_+^\nu-\bar P)\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}+(\gamma_++\gamma_-)\bar P\Big)\,\|\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac{3}{2}})}\\&\quad\le C(\gamma_++\gamma_-)(M_0+\bar P)\,\|u^\nu\|^{\frac12}_{\widetilde{L}^1(B^{\frac{d}{2}})}\|u^\nu\|^{\frac12}_{\widetilde{L}^\infty(B^{\frac{d}{2}-1})}\\&\quad\le C(\gamma_++\gamma_-)(M_0+\bar P)\,M_1.\end{aligned}$$

Once again, using the following product law

$$B^{\frac{d}{2}}\times B^{\frac{d}{2}-\frac12}\hookrightarrow B^{\frac{d}{2}-\frac12}\tag{4.95}$$

and interpolation inequalities we have

$$\begin{aligned}\|u^\nu\cdot\nabla(P_+^\nu-P_-^\nu)\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac12})}&\le C\,\|u^\nu\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}\|\nabla(P_+^\nu-P_-^\nu)\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac12})}\\&\le C\,\|u^\nu\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}\|P_+^\nu-P_-^\nu\|^{\frac12}_{\widetilde{L}^1(B^{\frac{d}{2}+1})}\|P_+^\nu-P_-^\nu\|^{\frac12}_{\widetilde{L}^\infty(B^{\frac{d}{2}})}\\&\le CM_0^{\frac12}M_1^{\frac32}\end{aligned}$$

and

$$\begin{aligned}&\|(\gamma_+P_+^\nu-\gamma_-P_-^\nu)\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac12})}\\&\quad\le\gamma_+\|(P_+^\nu-\bar P)\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac12})}+\gamma_-\|(P_-^\nu-\bar P)\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac12})}\\&\qquad\qquad+(\gamma_++\gamma_-)\bar P\,\|\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac12})}\\&\quad\le C\Big(\gamma_+\|(P_-^\nu-\bar P)\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}+\gamma_-\|(P_+^\nu-\bar P)\|_{\widetilde{L}^\infty(B^{\frac{d}{2}})}+(\gamma_++\gamma_-)\bar P\Big)\,\|\operatorname{div}u^\nu\|_{\widetilde{L}^2(B^{\frac{d}{2}-\frac12})}\\&\quad\le C(\gamma_++\gamma_-)(M_0+\bar P)\,\|u^\nu\|^{\frac12}_{\widetilde{L}^1(B^{\frac{d}{2}+1})}\|u^\nu\|^{\frac12}_{\widetilde{L}^\infty(B^{\frac{d}{2}})}\\&\quad\le C(\gamma_++\gamma_-)(M_0+\bar P)\,M_1.\end{aligned}$$

Plugging the last four inequalities into (4.93), one gets the desired inequality. □

In conclusion, from this preliminary analysis it seems that the indexes $s_1=\frac{d}{2}-\frac32$ and

$s_2 = \frac{d}{2} - \frac{1}{2}$ are good choices when applying Proposition 4.6.1 to the System (4.84).

4.6.2.3 Derivation of the convergence rate

Applying Proposition 4.6.1 to the equations of δQ_+ and δu in (4.84) with $s_1 = \frac{d}{2} - \frac{3}{2}$ and $s_2 = \frac{d}{2} - \frac{1}{2}$, Proposition 4.7.2 twice to the equation of δY_+, one time with s_1 and the other time with s_2, and summing the resulting estimates together, we obtain

$$\begin{aligned}
&\|(\delta Y_+, \delta Q_+, \delta u)\|_{\widetilde{L}^\infty_t(B^{\frac{d}{2}-\frac{3}{2}} \cap B^{\frac{d}{2}-\frac{1}{2}})} + \|(\delta Q_+, \delta u)\|^h_{L^1_t(B^{\frac{d}{2}-\frac{1}{2}})} + \|\delta Q_+\|^\ell_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}})} + \|\delta u\|^\ell_{L^1_t(B^{\frac{d}{2}-\frac{1}{2}})} \\
&\leq \exp(CE(t)) \left(\|(\delta Y_{+0}, \delta Q_{+0}, \delta u_0)\|_{B^{\frac{d}{2}-\frac{3}{2}} \cap B^{\frac{d}{2}-\frac{1}{2}}} + \|(\delta S_1, \delta S_2, \delta S_3)\|_{L^1_t(B^{\frac{d}{2}-\frac{3}{2}} \cap B^{\frac{d}{2}-\frac{1}{2}})} \right), \quad (4.96)
\end{aligned}$$

where $E(t) = \displaystyle\int_0^t \left(\|(\partial_t(\Gamma_3 - \bar{\Gamma}_3), \partial_t(\frac{1}{\rho^\nu} - \frac{1}{\bar{\rho}}))\|_{B^{\frac{d}{2}}} + \|u^\nu\|_{B^{\frac{d}{2}} \cap B^{\frac{d}{2}+1}} \right)$.

From the bounds (4.88) and (4.89), $E(t)$ is clearly uniformly bounded and from the hypothesis (4.86), we have $\|(\delta Y_{+0}, \delta Q_{+0}, \delta u_0)\|_{B^{\frac{d}{2}-\frac{3}{2}} \cap B^{\frac{d}{2}-\frac{1}{2}}} \leq C\sqrt{\nu}$. Let us now check that the source terms lying in δS_1, δS_2 and δS_3 in the spaces $L^1_t(B^{\frac{d}{2}-\frac{3}{2}} \cap^{\frac{d}{2}-\frac{1}{2}})$. To do so, we have to split the analysis depending on the frequency regime.

Step 1: Low regularity estimates

Here we use repeatedly the product law Proposition A.2.4. Concerning δS_1, we have

$$\begin{aligned}
\|\delta u \cdot \nabla Y_+\|_{B^{\frac{d}{2}-\frac{3}{2}}} &\leq C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|\nabla(Y_+ - \bar{Y}_+)\|_{B^{\frac{d}{2}-1}} \\
&\leq C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|Y_+ - \bar{Y}_+\|_{B^{\frac{d}{2}}} \leq CM_0\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}. \quad (4.97)
\end{aligned}$$

For δS_2 and δS_3, we have

$$\begin{aligned}
\|\delta u \cdot \nabla P_+\|_{B^{\frac{d}{2}-\frac{3}{2}}} &\leq C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|\nabla(P_+ - \bar{P}_+)\|_{B^{\frac{d}{2}-1}} \\
&\leq C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|P_+ - \bar{P}\|_{B^{\frac{d}{2}}} \leq CM_0\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}, \quad (4.98)
\end{aligned}$$

$$\|\delta u \cdot \nabla u\|_{B^{\frac{d}{2}-\frac{3}{2}}} \leq C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|\nabla u\|_{B^{\frac{d}{2}-1}} \leq C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|u\|_{B^{\frac{d}{2}}}, \quad (4.99)$$

and

$$\begin{aligned}
\|(P^\nu_+ - P^\nu_-)(\partial_t\Gamma_2 + u^\nu \cdot \nabla\Gamma_2)\|_{B^{\frac{d}{2}-\frac{3}{2}}} \leq& C\|P^\nu_+ - P^\nu_-\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|\partial_t\Gamma_2\|_{B^{\frac{d}{2}-1}} \quad (4.100) \\
&+ C\|P^\nu_+ - P^\nu_-\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|u^\nu \cdot \nabla\Gamma_2\|_{B^{\frac{d}{2}-1}}.
\end{aligned}$$

Thanks to the estimate (4.92) and bounds (4.89), by product law (4.95) the previous can be bounded as follows

$$\begin{aligned}
&\|(P^\nu_+ - P^\nu_-)(\partial_t\Gamma_2 + u^\nu \cdot \nabla\Gamma_2)\|_{B^{\frac{d}{2}-\frac{3}{2}}} \\
&\leq C\sqrt{\nu}\, M_1 \|\partial_t\Gamma_2\|_{B^{\frac{d}{2}-1}} + \|P^\nu_+ - P^\nu_-\|_{B^{\frac{d}{2}-\frac{1}{2}}} \|\Gamma_2 - \bar{\Gamma}_2\|_{B^{\frac{d}{2}}} \|u^\nu\|_{B^{\frac{d}{2}}}. \quad (4.101)
\end{aligned}$$

Before estimating the last term in δS_2, we need the following lemma.

Lemma 4.6.1. *Let $d \geq 3$. For s such that $\frac{d}{2} - \frac{3}{2} < s \leq \frac{d}{2} - \frac{1}{2}$, we have*

$$\|(\delta\alpha_+,\ \delta\alpha_-,\ \delta\rho_+,\ \delta\rho_-,\ \delta\rho, \delta P)\|_{B^s} \leq C(M_0+1)\Big(\|(\delta Y_+, \delta Q_+)\|_{B^s} + \|P_+^\nu - P_-^\nu\|_{B^s}\Big). \tag{4.102}$$

Proof. Thanks to the relations in (4.82), by product law (4.95) we have

$$\begin{aligned}\|(\delta P_+,\ \delta P_-)\|_{B^s} &\leq C\|\delta Q_+\|_{B^s} + C\big(\|\Gamma_2 - \bar{\Gamma}_2\|_{L^\infty(B^{\frac{d}{2}})}\|P_+^\nu - P_-^\nu\|_{B^s} + \bar{\Gamma}_2\,\|P_+^\nu - P_-^\nu\|_{B^s}\big)\\ &\leq C\|\delta Q_+\|_{B^s} + C(M_0+1)\|P_+^\nu - P_-^\nu\|_{B^s}.\end{aligned} \tag{4.103}$$

Using relations in (4.83) and a decomposition argument similar to (4.103) yields

$$\|\delta\alpha_+\|_{B^s} \leq C(M_0+1)\,\|(\delta Y_+, \delta\rho_+, \delta\rho_-)\|_{B^s}. \tag{4.104}$$

Define $f(x) = (\frac{1}{A_+}x)^{\frac{1}{\gamma_+}}$. We are now going to control $\delta\rho_+$ by δP_+ but the composition Proposition A.2.6 cannot be applied readily since $f'(0) \neq 0$. Still, we can rewrite $\delta\rho_+$ as

$$\begin{aligned}\delta\rho_+ &= \Big(f(x+\bar{P}) - f'(\bar{P})\,x\Big)\Big|_{P_+-\bar{P}}^{P_+^\nu-\bar{P}} + \Big(f'(\bar{P})\,x\Big)\Big|_{P_+-\bar{P}}^{P_+^\nu-\bar{P}}\\ &= \Big(f(x+\bar{P}) - f'(\bar{P})\,x\Big)\Big|_{P_+-\bar{P}}^{P_+^\nu-\bar{P}} + f'(\bar{P})\,\delta P_+,\end{aligned}$$

then, since $s > \frac{d}{2} - \frac{3}{2} \geq 0$ as $d \geq 3$, we can apply the composition Proposition A.2.6 to the first quantity of right-hand side and obtain

$$\begin{aligned}\|\delta\rho_+\|_{B^s} &\leq f'(\bar{P})\|\delta P_+\|_{B^s} + C\|(P_+^\nu - \bar{P},\ P_+ - \bar{P})\|_{B^s}\|\delta P_+\|_{B^s}\\ &\leq C(M_0+1)\|\delta P_+\|_{B^s}.\end{aligned}$$

Proceeding similarly with $\delta\rho_-$, and using combining the resulting estimates with (4.82), we conclude that (4.102) is satisfied. □

Now, using (4.94), (4.95) and the composition proposition, we get

$$\begin{aligned}&\|\frac{\gamma_+\gamma_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}(\delta Q_+ + \Gamma_4\delta\alpha_+)\,\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{3}{2}}}\\ \leq&\Big(\|\frac{\gamma_+\gamma_-}{\gamma_+\alpha_- + \gamma_-\alpha_+} - \frac{\gamma_+\gamma_-}{\gamma_+\bar{\alpha}_- + \gamma_-\bar{\alpha}_+}\|_{B^{\frac{d}{2}}} + 1\Big)\Big(\|\delta Q_+\,\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{3}{2}}} + \|\Gamma_4\,\delta\alpha_+\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{3}{2}}}\Big)\\ \leq&C(M_0+1)\Big(\|\delta Q_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\,\|u\|_{B^{\frac{d}{2}}} + \|\Gamma_4 - \bar{\Gamma}_4\|_{B^{\frac{d}{2}-1}}\,\|\delta\alpha_+\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{1}{2}}} + \bar{\Gamma}_4\|\delta\alpha_+\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{3}{2}}}\Big)\\ \leq&C(M_0+1)\Big(\|\delta Q_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\,\|u\|_{B^{\frac{d}{2}}} + M_0\,\|\delta\alpha_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\|u\|_{B^{\frac{d}{2}+1}} + \bar{\Gamma}_4\|\delta\alpha_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\|u\|_{B^{\frac{d}{2}}}\Big)\\ \leq&C(M_0+1)^2\Big(\|\delta Q_+\|_{B^{\frac{d}{2}-\frac{1}{2}}} + \|\delta\alpha_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\Big)\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}.\end{aligned}$$

Combining this with (4.102), we obtain

$$\begin{aligned}&\|\frac{\gamma_+\gamma_-}{\gamma_+\alpha_- + \gamma_-\alpha_+}(\delta Q_+ + \Gamma_4\delta\alpha_+)\,\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{3}{2}}}\\ \leq&C(M_0+1)^3\Big(\|(\delta Y_+, \delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}} + \|P_+^\nu - P_-^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}}\Big)\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}.\end{aligned} \tag{4.105}$$

Concerning the terms in δS_3, using a product law and (4.91), we have

$$\begin{aligned}\|\frac{1}{\rho^\nu}\mathcal{A}_{\mu,\lambda}\,u^\nu\|_{B^{\frac{d}{2}-\frac{3}{2}}} &\leq C\|\frac{1}{\rho^\nu}-\frac{1}{\bar\rho}\|_{B^{\frac{d}{2}-1}}\,\|\mathcal{A}_{\mu,\lambda}\,u^\nu\|_{B^{\frac{d}{2}-\frac{3}{2}}}+C\,\|\mathcal{A}_{\mu,\lambda}\,u^\nu\|_{B^{\frac{d}{2}-\frac{3}{2}}}\\ &\leq C\,\nu\,(M_0+1)\,\|u^\nu\|_{B^{\frac{d}{2}+\frac{1}{2}}}\leq C\,\nu\,(M_0+1)\,(\|u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}}}+\|u^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}+1}}),\end{aligned}\tag{4.106}$$

and by (4.102),

$$\begin{aligned}\|\frac{\nabla P_+}{\rho^\nu\rho}\delta\rho\|_{B^{\frac{d}{2}-\frac{3}{2}}} \leq& C\|\nabla(P_+-\bar P)\|_{B^{\frac{d}{2}-1}}\|\frac{1}{\rho^\nu\rho}\delta\rho\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ \leq& C\|P_+-\bar P\|_{B^{\frac{d}{2}}}(\|\frac{1}{\rho^\nu\rho}-\frac{1}{\bar\rho^2}\|_{B^{\frac{d}{2}}}+\frac{1}{\bar\rho^2})\|\delta\rho\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ \leq& CM_0(M_0+1)\|\delta\rho\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ \leq& CM_0(M_0+1)\Big(\|(\delta Y_+,\delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}}+\|P^\nu_+-P^\nu_-\|_{B^{\frac{d}{2}-\frac{1}{2}}}\Big).\end{aligned}\tag{4.107}$$

We estimate the last term of δS_3 in the following way

$$\begin{aligned}&\|\frac{1}{\rho^\nu}\nabla\Big((\Gamma_2-\alpha^\nu_-)(P^\nu_+-P^\nu_-)\Big)\|_{B^{\frac{d}{2}-\frac{3}{2}}}\\ &\leq C\Big(\|\frac{1}{\rho^\nu}-\frac{1}{\bar\rho}\|_{B^{\frac{d}{2}}}+\frac{1}{\bar\rho}\Big)\,\|(\Gamma_2-\alpha^\nu_-)\,(P^\nu_+-P^\nu_-)\|_{\frac{d}{2}-\frac{1}{2}}\\ &\leq C(M_0+1)\Big(\|\Gamma_2-\bar\Gamma_2\|_{B^{\frac{d}{2}}}+\|\alpha^\nu_--\bar\alpha_-\|_{B^{\frac{d}{2}}}+\bar\Gamma_2+\bar\alpha^\nu_-\Big)\|P^\nu_+-P^\nu_-\|_{\frac{d}{2}-\frac{1}{2}}\\ &\leq C(M_0+1)^2\|P^\nu_+-P^\nu_-\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}}.\end{aligned}\tag{4.108}$$

Summing up (4.97)-(4.101) and (4.105)-(4.108) together, integrating over $[0,t]$ and assuming without loss of generality that $M_0\ll 1$, we conclude that

$$\begin{aligned}&\|(\delta S_1,\delta S_2,\delta S_3)\|_{L^1_t(B^{\frac{d}{2}-\frac{3}{2}})}\\ \leq& CM_0\int_0^t\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+C\int_0^t\big(\|\delta u\|_{B^{\frac{d}{2}-\frac{3}{2}}}+\|(\delta Y_+,\delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}}\big)\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}\\ &+CM_1^2\int_0^t\sqrt{\nu}\|\partial_t\Gamma_2\|_{B^{\frac{d}{2}-1}}+\nu\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}+\|P^\nu_+-P^\nu_-\|_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}}\Big)\\ \leq& CM_0\int_0^t\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+C\int_0^t\big(\|\delta u\|_{B^{\frac{d}{2}-\frac{3}{2}}}+\|(\delta Y_+,\delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}}\big)\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}\\ &+CM_1^3\sqrt{\nu},\end{aligned}\tag{4.109}$$

where we used that $\|\partial_t\Gamma_2\|_{L^1_T(B^{\frac{d}{2}-1})}\leq M_1$ which can be directly obtained from writing the equation verified by $\partial_t\Gamma_2$.

Step 2: High regularity estimates

We will now show how to control the same sources terms in $L^1_T(B^{\frac{d}{2}-\frac{1}{2}})$. Concerning δS_1, we

have

$$\begin{aligned}\|\delta u\cdot\nabla Y_+\|_{B^{\frac{d}{2}-\frac{1}{2}}} &\le C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}\,\|\nabla(Y_+-\bar{Y}_+)\|_{B^{\frac{d}{2}}}\\ &\le C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}\,\|Y_+-\bar{Y}_+\|_{B^{\frac{d}{2}+1}}\le CM_0\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}.\end{aligned}\tag{4.110}$$

Similarly as before, we have

$$\|\delta u\cdot\nabla P_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\le CM_0\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}},\tag{4.111}$$

$$\|\delta u\cdot u\|_{B^{\frac{d}{2}-\frac{1}{2}}}\le C\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}\|u\|_{B^{\frac{d}{2}}},\tag{4.112}$$

and

$$\begin{aligned}\|(P_+^\nu-P_-^\nu)(\partial_t\Gamma_2+u^\nu\cdot\nabla\Gamma_2)\|_{B^{\frac{d}{2}-\frac{1}{2}}} \le& C\|P_+^\nu-P_-^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}}\,\|\partial_t\Gamma_2\|_{B^{\frac{d}{2}}}\\ &+C\|P_+^\nu-P_-^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}}\|u^\nu\cdot\nabla\Gamma_2\|_{B^{\frac{d}{2}}}.\end{aligned}\tag{4.113}$$

Thanks to the estimate (4.92) and bounds (4.89), by a product law the previous term can be bounded as follows

$$\begin{aligned}&\|(P_+^\nu-P_-^\nu)(\partial_t\Gamma_2+u^\nu\cdot\nabla\Gamma_2)\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ &\quad\le C\sqrt{\nu}\,M_1\,\|\partial_t\Gamma_2\|_{B^{\frac{d}{2}}}+\|P_+^\nu-P_-^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}}\|\Gamma_2-\bar{\Gamma}_2\|_{B^{\frac{d}{2}}}\|u^\nu\|_{B^{\frac{d}{2}}}.\end{aligned}\tag{4.114}$$

For the last term of δS_2, we have

$$\begin{aligned}&\|\frac{\gamma_+\gamma_-}{\gamma_+\alpha_-+\gamma_-\alpha_+}(\delta Q_++\Gamma_4\delta\alpha_+)\,\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ &\le\Big(\|\frac{\gamma_+\gamma_-}{\gamma_+\alpha_-+\gamma_-\alpha_+}-\frac{\gamma_+\gamma_-}{\gamma_+\bar{\alpha}_-+\gamma_-\bar{\alpha}_+}\|_{B^{\frac{d}{2}}}+1\Big)\Big(\|\delta Q_+\,\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+\|\Gamma_4\,\delta\alpha_+\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{1}{2}}}\Big)\\ &\le C(M_0+1)\Big(\|\delta Q_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\,\|\mathrm{div}\,u\|_{B^{\frac{d}{2}}}+\|\Gamma_4-\bar{\Gamma}_4\|_{B^{\frac{d}{2}}}\,\|\delta\alpha_+\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+\bar{\Gamma}_4\|\delta\alpha_+\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{1}{2}}}\Big)\\ &\le C(M_0+1)\Big(\|\delta Q_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\,\|\mathrm{div}\,u\|_{B^{\frac{d}{2}}}+M_0\,\|\delta\alpha_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\|\mathrm{div}\,u\|_{B^{\frac{d}{2}}}+\bar{\Gamma}_4\|\delta\alpha_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\|\mathrm{div}\,u\|_{B^{\frac{d}{2}}}\Big)\\ &\le C(M_0+1)^2\Big(\|\delta Q_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}+\|\delta\alpha_+\|_{B^{\frac{d}{2}-\frac{1}{2}}}\Big)\|u^\nu\|_{B^{\frac{d}{2}+1}}.\end{aligned}$$

Combining this with (4.102) gives that

$$\begin{aligned}&\|\frac{\gamma_+\gamma_-}{\gamma_+\alpha_-+\gamma_-\alpha_+}(\delta Q_++\Gamma_4\delta\alpha_+)\,\mathrm{div}\,u\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ &\le C(M_0+1)^3\Big(\|(\delta Y_+,\delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}}+\|P_+^\nu-P_-^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}}\Big)\|u^\nu\|_{B^{\frac{d}{2}+1}}.\end{aligned}\tag{4.115}$$

Now, with regard to the terms relating to δS_3, using a product law and (4.90) yields

$$\begin{aligned}\|\frac{1}{\rho^\nu}\mathcal{A}_{\mu,\lambda}\,u^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}} &\le C\|\frac{1}{\rho^\nu}-\frac{1}{\bar{\rho}}\|_{B^{\frac{d}{2}-1}}\,\|\mathcal{A}_{\mu,\lambda}\,u^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}}+C\,\|\mathcal{A}_{\mu,\lambda}\,u^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ &\le C\,\nu\,(M_0+1)\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}},\end{aligned}\tag{4.116}$$

and by (4.102),

$$\begin{aligned}\|\frac{\nabla P_+}{\rho^\nu\rho}\delta\rho\|_{B^{\frac{d}{2}-\frac{1}{2}}} \leq& C\|\nabla(P_+-\bar{P})\|_{B^{\frac{d}{2}}}\|\frac{1}{\rho^\nu\rho}\delta\rho\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ \leq& C\|P_+-\bar{P}\|_{B^{\frac{d}{2}+1}}(\|\frac{1}{\rho^\nu\rho}-\frac{1}{\bar{\rho}^2}\|_{B^{\frac{d}{2}}}+\frac{1}{\bar{\rho}^2})\|\delta\rho\|_{B^{\frac{d}{2}}}\\ \leq& CM_0(M_0+1)\|\delta\rho\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ \leq& CM_0(M_0+1)\Big(\|(\delta Y_+,\delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}}+\|P_+^\nu-P_-^\nu\|_{B^{\frac{d}{2}-\frac{1}{2}}}\Big).\end{aligned} \tag{4.117}$$

The last term in δS_3 is estimated in the following way

$$\begin{aligned}&\|\frac{1}{\rho^\nu}\nabla\Big((\Gamma_2-\alpha_-^\nu)(P_+^\nu-P_-^\nu)\Big)\|_{B^{\frac{d}{2}-\frac{1}{2}}}\\ &\leq C\Big(\|\frac{1}{\rho^\nu}-\frac{1}{\bar{\rho}}\|_{B^{\frac{d}{2}}}+\frac{1}{\bar{\rho}}\Big)\|(\Gamma_2-\alpha_-^\nu)\,(P_+^\nu-P_-^\nu)\|_{B^{\frac{d}{2}+\frac{1}{2}}}\\ &\leq C(M_0+1)^2\|P_+^\nu-P_-^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+\frac{1}{2}}}.\end{aligned} \tag{4.118}$$

Using the interpolation inequality

$$\begin{aligned}\|P_+^\nu-P_-^\nu\|_{B^{\frac{d}{2}+\frac{1}{2}}} \leq& \|P_+^\nu-P_-^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}}}\|P_+^\nu-P_-^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}+1}}\\ \leq& M_1\|P_+^\nu-P_-^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}}}\end{aligned}$$

we have the desired estimate for this term.

Then, summing up (4.110)-(4.114) and (4.115)-(4.118) together, integrating over $[0,t]$, we conclude that

$$\begin{aligned}&\|(\delta S_1,\delta S_2,\delta S_3)\|_{L^1_t(B^{\frac{d}{2}-\frac{1}{2}})}\\ &\leq CM_0\int_0^t\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+C\int_0^t\big(\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+\|(\delta Y_+,\delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}}\big)\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}\\ &\quad+CM_1^2\int_0^t\sqrt{\nu}\|\partial_t\Gamma_2\|_{B^{\frac{d}{2}}}+\nu\|u^\nu\|_{B^{\frac{d}{2}+1}}+\|P_+^\nu-P_-^\nu\|^{\frac{1}{2}}_{B^{\frac{d}{2}-1}\cap B^{\frac{d}{2}}}\Big)\\ &\leq CM_0\int_0^t\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+C\int_0^t\big(\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+\|(\delta Y_+,\delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}}\big)\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}\\ &\quad+CM_1^3M_0\sqrt{\nu},\end{aligned} \tag{4.119}$$

where we used again that $\|\partial_t\Gamma_2\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}\leq M_1$.

Step 3: Conclusion

Gathering (4.109) and (4.119), we obtain

$$\begin{aligned}\|(\delta S_1,\delta S_2,\delta S_3)\|_{L^1_t(B^{\frac{d}{2}-\frac{3}{2}}\cap B^{\frac{d}{2}-\frac{1}{2}})} \leq\ & CM_0\int_0^t\|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}}+C\int_0^t\|\delta u\|_{B^{\frac{d}{2}-\frac{3}{2}}\cap B^{\frac{d}{2}-\frac{1}{2}}}\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}\\ &+C\int_0^t\|(\delta Y_+,\delta Q_+)\|_{B^{\frac{d}{2}-\frac{1}{2}}}\|u^\nu\|_{B^{\frac{d}{2}}\cap B^{\frac{d}{2}+1}}\\ &+CM_1^3\sqrt{\nu}.\end{aligned} \tag{4.120}$$

Therefore, using (4.120) and (4.86) in (4.96) and Gronwall's lemma, we have

$$\|(\delta Y_+, \delta Q_+, \delta u)(t)\|_{B^{\frac{d}{2}-\frac{3}{2}}} + \|(\delta Y_+, \delta Q_+, \delta u)(t)\|_{B^{\frac{d}{2}-\frac{1}{2}}} + \int_0^t \|\delta Q_+\|^{\ell}_{B^{\frac{d}{2}+\frac{1}{2}}} + \int_0^t \|\delta Q_+\|^{h}_{B^{\frac{d}{2}-\frac{1}{2}}} + \int_0^t \|\delta u\|_{B^{\frac{d}{2}-\frac{1}{2}}} \leq \sqrt{\nu}\, \exp(CM_1^4)$$

which concludes the proof of Theorem 4.6.1. ■

4.7 Some basic linear problems

In this section we prove some estimates for the damped transport equations, a purely damped equation and the Lamé system that we used in this chapter.

Proposition 4.7.1. *Let $s \in (-\frac{d}{2}, \frac{d}{2}]$, $H_2 \in \mathcal{C}([0,T);\mathcal{S})$ and h_2 a positive constant. Let w be a solution of the following damped equation with variable coefficient on $[0,T)$,*

$$\begin{cases} \partial_t w + (h_2 + H_2)\dfrac{w}{\nu} = f, \\ w(t,x)|_{t=0} = w_0. \end{cases} \tag{D}$$

There exists a positive constant $\bar{h}_2$ such that if

$$\|H_2\|_{\widetilde{L}^\infty_T(B^{\frac{d}{2}})} \leq \bar{h}_2 < \frac{h_2}{2},$$

then the following estimate holds on $[0,T)$,

$$\frac{1}{\sqrt{\nu}}\|w\|_{\widetilde{L}^\infty_t(B^s)} + \frac{1}{2\sqrt{h_2}}\|\partial_t w\|_{\widetilde{L}^2_t(B^s)} \leq \frac{1}{\sqrt{\nu}}\|w_0\|_{B^s} + \max(\frac{1}{2\sqrt{h_2}}, 4h_2)\,\|f\|_{\widetilde{L}^2_t(B^s)}.$$

Proof. We first rewrite the equation into the form

$$\big(\frac{1}{h_2+H_2} - \frac{1}{h_2} + \frac{1}{h_2}\big)\,\partial_t w + \frac{w}{\nu} = \frac{f}{h_2+H_2}.$$

Applying the operator $\dot{\Delta}_j$ to it, we get

$$\frac{1}{h_2}\partial_t w_j + \frac{w_j}{\nu} = -\dot{\Delta}_j\big((\frac{1}{h_2+H_2} - \frac{1}{h_2})\partial_t w)\big) + \dot{\Delta}_j(\frac{f}{h_2+H_2}).$$

Taking the L^2 inner product with $\partial_t w_j$ leads to

$$\frac{1}{2\nu}\frac{d}{dt}\|w_j\|^2_{L^2} + \frac{1}{h_2}\|\partial_t w_j\|^2 \leq \|\dot{\Delta}_j((\frac{1}{h_2+H_2} - \frac{1}{h_2})\partial_t w)\|_{L^2}\,\|\partial_t w_j\|_{L^2} + \|\dot{\Delta}_j(\frac{f}{h_2+H_2})\|_{L^2}\,\|\partial_t w_j\|_{L^2}.$$

Using Young's inequality, we get

$$\frac{1}{2\nu}\frac{d}{dt}\|w_j\|^2_{L^2} + \frac{1}{h_2}\|\partial_t w_j\|^2_{L^2} \leq 4h_2\|\dot{\Delta}_j((\frac{1}{h_2+H_2} - \frac{1}{h_2})\partial_t w)\|^2_{L^2} + 4h_2\|\dot{\Delta}_j(\frac{f}{h_2+H_2})\|^2_{L^2} + \frac{1}{2h_2}\|\partial_t w_j\|^2_{L^2}.$$

Integrating on the time interval $[0,t]$ for any $t\in[0,T)$ and using Young's inequality again, we readily obtain

$$\frac{1}{\sqrt{\nu}}\|w_j(t)\|_{L^2}+\frac{1}{\sqrt{h_2}}\|\partial_t w_j\|_{L^2_t(L^2)}\leq\frac{1}{\sqrt{\nu}}\|w_0\|_{L^2}+4\sqrt{h_2}\,\|\dot\Delta_j\big((\frac{1}{h_2+H_2}-\frac{1}{h_2})\partial_t w\big)\|_{L^2_t(L^2)}$$
$$+4\sqrt{h_2}\|\dot\Delta_j(\frac{f}{h_2+H_2})\|_{L^2_t(L^2)}.$$

Multiplying both sides by 2^{js} and summing up for $j\in\mathbb{Z}$, we get

$$\frac{1}{\sqrt{\nu}}\|w\|_{\widetilde{L}^\infty_t(B^s)}+\frac{1}{\sqrt{h_2}}\|\partial_t w\|_{\widetilde{L}^2_t(B^s)}\leq\frac{1}{\sqrt{\nu}}\|w_0\|_{B^s}+4\sqrt{h_2}\,\|(\frac{1}{h_2+H_2}-\frac{1}{h_2})\partial_t w\|_{\widetilde{L}^2_t(B^s)}$$
$$+4\sqrt{h_2}\,\|\frac{f}{h_2+H_2}\|_{\widetilde{L}^2_t(B^s)}.$$

The product law $B^s\times B^{\frac{d}{2}}\hookrightarrow B^s$ and composition lemma entail that we have

$$4\sqrt{h_2}\,\|(\frac{1}{h_2+H_2}-\frac{1}{h_2})\partial_t w\|_{\widetilde{L}^2_t(B^s)}\leq 4\sqrt{h_2}\,C\|H_2\|_{\widetilde{L}^\infty_t(B^{\frac{d}{2}})}\|\partial_t w\|_{\widetilde{L}^2_t(B^s)}\leq\frac{1}{2\sqrt{h_2}}\,\|\partial_t w\|_{\widetilde{L}^2_t(B^s)}$$

whenever $\bar h_2\leq\dfrac{1}{8Ch_2}$. Handling the other right-hand side term in a similar manner completes the proof of the proposition. □

Recall that the constant coefficient damped transport equation reads:

$$\begin{cases}\partial_t w+v\cdot\nabla w+a\,w=f,\\ w(t,x)|_{t=0}=w_0.\end{cases}\tag{4.121}$$

We are going to prove the following proposition.

Proposition 4.7.2. *Let $r_1\in(-\frac{d}{2},\frac{d}{2}],r_2\in(-\frac{d}{2},\frac{d}{2}+1]$ and assume that $a\geq 0$. Then there exists a universal constant C such that*

$$\|w(t)\|^h_{B^{r_2}}+a\int_0^t\|w\|^h_{B^{r_2}}\leq\|w_0\|^h_{B^{r_2}}+C\int_0^t\|\nabla v\|_{B^{\frac{d}{2}}}\|w\|_{B^{r_2}}+\int_0^t\|f\|^h_{B^{r_2}}\tag{4.122}$$

and

$$\|w(t)\|^\ell_{B^{r_1}}+a\int_0^t\|w\|^\ell_{B^{r_1}}\leq\|w_0\|^\ell_{B^{r_1}}+C\int_0^t\|v\|_{B^{\frac{d}{2}}}\|w\|_{B^{r_1+1}}+\int_0^t\|f\|^\ell_{B^{r_1}}.\tag{4.123}$$

Proof. Applying $\dot\Delta_j$ to (4.121) yields

$$\partial_t w_j+v\cdot\nabla w_j+a\,w=-[\dot\Delta_j,v]\nabla w+\dot\Delta_j f.$$

Taking the L^2 inner product with w_j and then integrating by parts, one has

$$\frac{1}{2}\frac{d}{dt}\|w_j\|^2_{L^2}+a\,\|w_j\|^2_{L^2}=\frac{1}{2}\int_{\mathbb{R}^d}\operatorname{div}v|w_j|^2-\int_{\mathbb{R}^d}w_j\,[\dot\Delta_j,v]\nabla w+\int_{\mathbb{R}^d}w_j,\dot\Delta_j f$$
$$\leq\|\operatorname{div}v\|_{L^\infty}\|w_j\|^2_{L^2}+\|w_j\|_{L^2}(\|[\dot\Delta_j,v]\nabla w\|_{L^2}+\|\dot\Delta_j f\|_{L^2}).$$

As $-\frac{d}{2} < r_2 \leq \frac{d}{2} + 1$, by Proposition A.2.3, we get

$$\|[\dot{\Delta}_j, v]\nabla w\|_{L^2} \leq C2^{-jr_2} q_j \|\nabla v\|_{B^{\frac{d}{2}}} \|w\|_{B^{r_2}}.$$

Thus by embedding $B^{\frac{d}{2}} \hookrightarrow L^\infty$, we obtain

$$\|w_j(t)\|_{L^2} + a\int_0^t \|w_j\|_{L^2} \leq \|w_j(0)\|_{L^2} + C2^{-jr_2} q_j \int_0^t \|\nabla v\|_{B^{\frac{d}{2}}} \|w\|_{B^{r_2}} + \int_0^t \|\dot{\Delta}_j f\|_{L^2}.$$

Multiplying the factor 2^{jr_2} on both sides and summing up over $j \geq -1$, we get (4.122)
For the low frequencies estimate (4.123), one needs to consider $v \cdot \nabla w$ as a source term, and use the product law $\|v \cdot \nabla w\|_{B^{r_1}} \leq C\|v\|_{B^{\frac{d}{2}}} \|\nabla w\|_{B^{r_1}}$, for $r_1 \in (-\frac{d}{2}, \frac{d}{2}]$. □

Recall that the constant coefficient Lamé system reads:

$$\begin{cases} \partial_t u + v \cdot \nabla u - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \eta u = g, \\ u(t,x)|_{t=0} = u_0. \end{cases} \tag{4.124}$$

We are going to prove the following proposition.

Proposition 4.7.3. *Let $r_1, r_2 \in (-\frac{d}{2}, \frac{d}{2}]$. Assume that $\mu > 0$ and $\mu + \lambda, \eta \geq 0$. Then there exists a universal constant C such that*

$$\begin{aligned} &\|u(t)\|^\ell_{B^{r_1}} + \|u(t)\|^h_{B^{r_2}} + \mu\int_0^t (\|\nabla u\|^\ell_{B^{r_1+1}} + \|\nabla u\|^h_{B^{r_2+1}}) \\ &\qquad + (\lambda+\mu)\int_0^t (\|\operatorname{div} u\|^\ell_{B^{r_1+1}} + \|\operatorname{div} u\|^h_{B^{r_2+1}}) + \eta\int_0^t (\|u\|^\ell_{B^{r_1}} + \|u\|^h_{B^{r_2}}) \\ &\leq C(\|u_0\|^\ell_{B^{r_1}} + \|u_0\|^h_{B^{r_2}}) + C\int_0^t (\|\nabla v\|_{B^{\frac{d}{2}}} \|u\|_{B^{r_1}\cap B^{r_2}} + \|v\|_{B^{\frac{d}{2}}} \|u\|_{B^{r_1+1}\cap B^{r_2+1}}) \\ &\qquad + C\int_0^t (\|g\|^\ell_{B^{r_1}} + \|g\|^h_{B^{r_2}}). \end{aligned}$$

Proof. Applying $\dot{\Delta}_j$ to equation (4.124) yields

$$\partial_t u_j - \mu\Delta u_j - (\lambda+\mu)\nabla \operatorname{div} u_j + \eta u_j = -[\dot{\Delta}_j, v]\nabla u - v \cdot \nabla u_j + \dot{\Delta}_j g. \tag{4.125}$$

Taking the L^2 inner product with u_j, integrating by parts and using that $\lambda + \mu \geq 0$, one gets

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_j\|^2_{L^2} + \mu\|\nabla u_j\|^2_{L^2} + \eta\|u_j\|^2_{L^2} &\leq \frac{1}{2}\int_{\mathbb{R}^d} \operatorname{div} v\, |u_j|^2 - \int_{\mathbb{R}^d} u_j\, [\dot{\Delta}_j, v]\nabla u + \int_{\mathbb{R}^d} u_j\, \dot{\Delta}_j g \\ &\leq \|\operatorname{div} v\|_{L^\infty}\|u_j\|^2_{L^2} + \|u_j\|_{L^2}(\|[\dot{\Delta}_j, v]\nabla u)\|_{L^2} + \|\dot{\Delta}_j g\|_{L^2}). \end{aligned}$$

By Proposition A.2.3, we have

$$\|[\dot{\Delta}_j, v]\nabla u)\|_{L^2} \leq C2^{-r_2 j} q_j \|\nabla v\|_{B^{\frac{d}{2}}} \|u\|_{B^{r_2}}.$$

Using that $\|\nabla u_j\|_{L^2}^2 \geq (\frac{5}{6}2^j\|u_j\|_{L^2})^2 \geq \frac{1}{2}2^{2j}\|u_j\|_{L^2}^2$, we thus get

$$\|u_j(t)\|_{L^2} + \frac{\mu}{2}2^{2j}\int_0^t \|u_j\|_{L^2} + \eta\int_0^t \|u_j\|_{L^2} \\ \leq \|u_j(0)\|_{L^2} + C2^{-r_2 j}q_j\int_0^t \|\nabla v\|_{B^{\frac{d}{2}}}\|u\|_{B^{r_2}} + \int_0^t \|\dot{\Delta}_j g\|_{L^2}\,. \quad (4.126)$$

Multiplying by the factor 2^{jr_2} on both sides, summing up over $j \geq -1$, we further get

$$\|u(t)\|_{B^{r_2}}^h + \frac{\mu}{2}\int_0^t \|u\|_{B^{r_2+2}}^h + \eta\int_0^t \|u\|_{B^{r_2}}^h \\ \leq \|u_0\|_{B^{r_2}}^h + C\int_0^t \|\nabla v\|_{B^{\frac{d}{2}}}\|u\|_{B^{r_2}} + \int_0^t \|g\|_{B^{r_2}}^h\,. \quad (4.127)$$

Similarly, we have

$$\|u(t)\|_{B^{r_1}}^\ell + \frac{\mu}{2}\int_0^t \|u\|_{B^{r_1+2}}^\ell + \eta\int_0^t \|u\|_{B^{r_1}}^\ell \\ \leq \|u_0\|_{B^{r_1}}^\ell + C\int_0^t \|\nabla v\|_{B^{\frac{d}{2}}}\|u\|_{B^{r_1}} + \int_0^t \|g\|_{B^{r_1}}^\ell\,. \quad (4.128)$$

Now, we focus only on the "compressible part" of u that is $\Lambda^{-1}\mathrm{div}\, u$. Applying the 0-th order pseudo-differential operator $\Lambda^{-1}\mathrm{div}$ and $\dot{\Delta}_j$ to (4.124), we find that

$$\partial_t\dot{\Delta}_j(\Lambda^{-1}\mathrm{div}\, u) - (2\mu+\lambda)\Delta(\Lambda^{-1}\mathrm{div}\, u_j) + \eta(\Lambda^{-1}\mathrm{div}\, u_j) = -\dot{\Delta}_j\Lambda^{-1}\mathrm{div}\,(v\cdot\nabla u) + \dot{\Delta}_j\Lambda^{-1}\mathrm{div}\, g.$$

We notice that the above equation is similar to (4.125) and following the way we derived (4.127), we get

$$\|\Lambda^{-1}\mathrm{div}\, u(t)\|_{B^{r_2}}^h + \frac{(2\mu+\lambda)}{2}\int_0^t \|\Lambda^{-1}\mathrm{div}\, u\|_{B^{r_2+2}}^h + \eta\int_0^t \|\Lambda^{-1}\mathrm{div}\, u\|_{B^{r_2}}^h \\ \leq \|u_0\|_{B^{r_2}}^h + \int_0^t (\|v\cdot\nabla u\|_{B^{r_2}} + \|g\|_{B^s}^h)\,.$$

Using product law $\|v\cdot\nabla u\|_{B^{r_2}} \leq C\|v\|_{B^{\frac{d}{2}}}\|\nabla u\|_{B^{r_2}}$ for $r_2 \in (-\frac{d}{2}, \frac{d}{2}]$ and noticing that

$$\|\Lambda^{-1}\mathrm{div}\, u\|_{B^{r_2+2}}^h \geq \frac{5}{12}\|\mathrm{div}\, u\|_{B^{r_2+1}}^h,$$

we further obtain that

$$\frac{(2\mu+\lambda)}{5}\int_0^t \|\mathrm{div}\, u\|_{B^{r_2+1}}^h \leq \|u_0\|_{B^{r_2}}^h + C\int_0^t (\|v\|_{B^{\frac{d}{2}}}\|\nabla u\|_{B^{r_2}} + \|g\|_{B^{r_2}}^h)\,. \quad (4.129)$$

Similarly, we have

$$\frac{(2\mu+\lambda)}{5}\int_0^t \|\mathrm{div}\, u\|_{B^{r_1+1}}^\ell \leq \|u_0\|_{B^{r_1}}^\ell + C\int_0^t (\|v\|_{B^{\frac{d}{2}}}\|\nabla u\|_{B^{r_1}} + \|g\|_{B^{r_1}}^\ell)\,.$$

This, combined with (4.127), (4.128) and (4.129), completes the proof. □

Chapitre 5

Perspectives

5.1 Some broader results and open problems

Generalisation of the results from Chapter 3

An on-going work with Raphaël Danchin concerns the generalisation of the result we obtained in Chapter 3 to more general hyperbolic partially dissipative systems. One has to be careful when including the 0-th order quadratic term $r(Z)$ and the matrix $A^0(V)$ in the computations to obtain global well-posedness. Indeed, when proving uniqueness one has to estimate $r(Z_1) - r(Z_2)$ with the fourth estimate in the product law Lemma A.2.4 which yield some terms in $\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}$ that do not depend on the difference of the solutions $\widetilde{Z}$ and are not under control in low frequencies. This suggests that a more suitable low frequencies framework could be to consider the initial data Z_0^ℓ in $\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p^*}}_{p,1}$. However, in this case, when scaling back, one loses the uniformity of the smallness assumption and thus would not be able to study the limit. One could try to compensate this loss of uniformity by deriving a weaker convergence rate of order $\varepsilon^{1-\frac{d}{p^*}}$.

A more general approach of the damped mode

Here we borrow some arguments from [37] and extend it a bit. We focus on the systems studied in [35] that we already discussed in the introduction:

$$\partial_t V + A(D)V + L(D)V = 0, \quad \text{where} \tag{5.1}$$

- $A(D) = (A_{ij}(D))_{1\leq i,j\leq d}$ for $A_{ij}(D)$ an homogeneous Fourier multiplier of order α,
- $L(D) = (L_{ij}(D))_{1\leq i,j\leq d}$ for $L_{ij}(D)$ an homogeneous Fourier multiplier of order β.

Moreover, we assume that $A(D)$ is skew-symmetric:

$$\mathrm{Re}((A(\xi)\eta)\cdot\eta) = 0 \quad \forall(\xi,\eta)\in\mathbb{R}^d\times\mathbb{C}^n,$$

and that $L(D)$ is partially elliptic :

$$|\xi|^\beta \mathrm{Re}((L(\xi)\eta)\cdot\eta) \geq \kappa|L(\xi)\eta|^2 \quad \forall(\xi,\eta)\in\mathbb{R}^d\times\mathbb{C}^n$$

for κ a nonnegative real number.

Define $R = \mathrm{Im} L(\xi)$ and assume that it is independent of $\xi \in \mathbb{R}^d \setminus \{0\}$, $L(\xi) : R \to R$ and $\mathrm{Ker}\, L(\xi) = R^\perp$. The partial dissipation hypothesis reads:

$$\exists c > 0, \forall \xi \in \mathbb{R}^d, \forall \eta \in R,\ |L(\xi)\eta| \geq c|\xi|^\beta,$$

which, after a suitable change of basis, corresponds to $L(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{L}(\xi) \end{pmatrix}$.

Let $\mathcal{P}$ be the orthogonal projector on R. We define the damped mode W as follows:

$$W \triangleq \mathcal{P}(L(D)V + A(D)V).$$

Using that $\mathcal{P}L(D) = L(D)$ and $\mathcal{P}W = W$, one can obtain

$$\partial_t W + L(D)W = \mathcal{P}A(D)W - \mathcal{P}A(D)(I_d - \mathcal{P})A(D)V$$

Thus, applying the Fourier transform, taking the scalar product with $\widehat{W}$ and using the properties on $L(\xi)$, the skew-symmetry of $A(D)$ and Lemma A.1.1, we get

$$\frac{d}{dt}|\widehat{W}| + c|\xi|^\beta|\widehat{W}| \lesssim |\mathcal{F}(\mathcal{P}A(D)(I_d - \mathcal{P})A(D)V)| \lesssim |\xi|^{2\alpha}|\widehat{V}|$$

Thus, integrating in time yields

$$|\widehat{W}(t,\xi)| + |\xi|^\beta \int_0^t |\widehat{W}| \lesssim |\widehat{W}_0(\xi)| + |\xi|^{2\alpha} \int_0^t |\widehat{V}|.$$

As explained in the introduction, following the computations of Beauchard and Zuazua, we have

$$|\widehat{V}(t,\xi)| + \min(|\xi|^\beta, |\xi|^{2\alpha-\beta}) \int_0^t |\widehat{V}| \lesssim |\widehat{V}_0(\xi)|.$$

- In our partially dissipative setting, we have $\beta < \alpha$, hence, for $|\xi| \leq J_0$, we obtain

$$|\widehat{W}(t,\xi)| + |\xi|^\beta \int_0^t |\widehat{W}| \lesssim |\widehat{W}_0(\xi)| \tag{5.2}$$

- In a partially diffusive setting, $\beta > \alpha$, for $|\xi| \geq J_0$, we obtain

$$|\widehat{W}(t,\xi)| + |\xi|^\beta \int_0^t |\widehat{W}| \lesssim |\widehat{W}_0(\xi)|. \tag{5.3}$$

In both cases we observe a gain of β derivatives for the quantity W. And, as we saw in this section, this allows to recover enhanced properties for the damped component, Z_2 in this chapter.

As we saw in Chapter 3, to avoid some difficult terms in the equations of the damped component, it is possible to consider W as an unknown in the a priori estimates. Consider $\mathcal{P}'$ the projector on $R^\perp$ and apply it to the linear equation (5.1), we have:

$$\partial_t \mathcal{P}'V + \mathcal{P}'A(D)V = 0.$$

Using the definition of W, we may write $V = L^{-1}(D)W - L^{-1}(D)PA(D)V$, hence

$$\partial_t \mathcal{P}'V - \mathcal{P}'A(D)L^{-1}(D)\mathcal{P}A(D)V = -\mathcal{P}'(A(D)L^{-1}(D)W).$$

One fact, that is partially elucidated in the setting of Chapter 3, is that the operator $\mathcal{A} \triangleq \mathcal{P}'A(D)L^{-1}\mathcal{P}A(D)$ of order $2\alpha-\beta$ is actually elliptic under the (SK) condition. And therefore, in low frequencies it is possible to recover time-integrability for the component $\mathcal{P}'V$ directly from its equation i.e. without employing the arguments from Beauchard and Zuazua. Employing these arguments, we have

$$|\mathcal{P}'\widehat{V}(t,\xi)| + |\xi|^{2\alpha-\beta}\int_0^t |\mathcal{P}'\widehat{V}| \leq |\mathcal{P}'\widehat{V}_0(\xi)| + |\xi|^{\alpha-\beta}\int_0^t |\widehat{W}|. \tag{5.4}$$

- In the partially dissipative case, assuming that $2\beta < \alpha$ and adding (5.4) and (5.2), we see that the right-hand side term can be absorbed by the left-hand side if $|\xi| \leq J_0$ for a J_0 small enough.

- In the partially diffusive case, assuming $2\beta > \alpha$, the right-hand side term can also be controlled if $|\xi| \geq J_0$ for a J_0 large enough.

In both cases, we recovered the expected time-integrability for the unknown that does not undergo any effect from $L(D)$ in one of the frequencies regime. And, as this is done in a decoupled way it allows an L^p approach in this specific regime. Concerning the other regime one is still restricted to the energy level and has to construct a Lyapunov functional. In the Chapters 1 and 3, we investigated the specific case $\beta = 0$ and $\alpha = 1$ for quasi-linear systems. For more information concerning the partially diffusive case $\beta = 2$ and $\alpha = 1$, see the work of Hoff [58] and Haspot [56].

Anisotropic behavior

We expect the method developed above to be robust enough to be applied to non-linear systems of any order as well as anisotropic systems.

A first step could be to adapt our method to the system considered by Wan in [98] and Bianchini and Natalini in [87] where they obtained the existence and decay rates for the two-dimensional inviscid Boussinesq equations with a damping term in the velocity equation. In the frequency variable, the linearization of the system reads:

$$\begin{cases} \partial_t \widehat{b} - \dfrac{iN\xi_1}{|\xi|}\widehat{\Omega} = 0 \\[2ex] \partial_t \widehat{\Omega} - \dfrac{iN\xi_1}{|\xi|}\widehat{b} + \alpha\widehat{\Omega} = 0 \end{cases}$$

As explained in [87], the three main difficulties concerning this system are the following ones:

- The corresponding skew-symmetric Fourier multiplier $A(D)$ is of order 0.
- Formally, it doesn't verify the (SK) condition on the submanifold $\xi_1 = 0$.
- The low frequencies behave like $e^{-\frac{\xi_1^2}{|\xi|^2}t}$ while in the partially dissipative setting they behave like the heat flow $e^{-|\xi|^2 t}$.

Still, this system enters in the setting described in the previous subsection. And this new behavior implies a reconsideration of all the regularity setting in low frequency and to use anisotropic

homogeneous Besov spaces as introduced in [3]. While it is not clear if the consideration of a damped mode for this system in low frequencies is appropriate, one should be able to adapt the method developed in this manuscript to study this system.

Global strong small solutions for the non-isentropic Euler-Fourier one dimensional system

One could investigate the potential of the method developed in this manuscript to treat partially diffusive systems. After reformulation, the non-isentropic Euler-Fourier system reads

$$\begin{cases} \partial_t a + \mathrm{div}((1+a)u) = 0 \\ \partial_t u + u \cdot \nabla u + \nabla(a+\theta) = \dfrac{a}{1+a}\nabla a - \theta \dfrac{\nabla a}{1+a}, \\ \partial_t \theta + u \cdot \nabla \theta + \mathrm{div}\, u - \Delta\theta = -\theta \mathrm{div}\, u - \dfrac{a}{1+a}\Delta\theta. \end{cases} \tag{CEF}$$

The main difficulty about this system is that one cannot recover any dispersive effect for the incompressible part of the velocity and therefore it doesn't seem possible to treat the multidimensional case with the method developed above which necessitates L^1-in-time Lipschitz estimates for the velocity.

However, for $d = 1$, we do not have this problem and a study of the linearized system reveals that we can recover a dissipation effect for all the components. The problem we face is that the nonlinear terms cannot be handled in the regularity setting considered in our previous researches, for example, we cannot obtain Lipschitz estimates on the velocity in the low frequencies regime. It is not cleared whether the reformulation of the system we use is not appropriate or the regularity setting is not adapted to the low frequencies analysis. But, following the approach of Beauchard and Zuazua, it is possible to obtain estimates for the high frequencies part by considering different regularity for (a_0, u_0) and θ_0.

High frequencies Strichartz estimates for hyperbolic partially dissipative system

With the approach developed in this manuscript, the imaginary part of the eigenvalues of the system is not taken into account. In other words, we cannot keep track of the possible dispersive properties of the system in high frequencies. Pointing out Strichartz estimates could bring another level of comprehension to those systems.

In our setting, as $\varepsilon \to \infty$ the high frequencies regime covers the whole space of frequencies (according to the definition of the threshold $J_\varepsilon = \lfloor -\log_2 \varepsilon \rfloor$). And therefore Strichartz estimates could help us justifying this limit. However, formally, this limit leads to the compressible Euler system which is only locally well-posed, thus the partially dissipative properties, which are responsible only for the large time behavior, will not be of much help.

Systematic argument

Even if a Lyapunov functional is a great tool to derive well-posedness results as well as decay estimates, one of its drawback is that we cannot obtain directly the whole regularity properties of the damped part of the solutions: this method is always a 2-steps arguments. Finding a more systematic and direct argument would be a great step toward the understanding of partially dissipative and partially viscous systems.

Global strong solution for partially dissipative hyperbolic systems that do not satisfy the (SK) condition

As the arguments borrowed from [4] are mainly implemented to study systems that do not satisfy the (SK) condition, one could investigate such systems with the method developed above. To comprehend a bit more what could be a good condition (weaker than (SK)) to ensure the global existence of partially dissipative hyperbolic systems, it is natural to study systems that do not satisfy it but, still, admit global solutions. In the fourth chapter of this manuscript, we take a step in this direction and obtain a global existence result even though the system does not satisfy the (SK) condition and is not symmetric nor has a positive definite entropy. Mainly, we are able to do that by emphasising a subsystem which satisfies the (SK) condition, and we had to work out weighted norms to compensate the lack of symmetry. Other systems violating the (SK) condition are also under current investigation with the method developed here.

But it is unlikely that finding an optimal long-time existence condition follows from our work here. A more reasonable goal might be to complete the classification, established in [4], of the asymptotic behaviors of the solutions of linear systems that do not satisfy the (SK) condition. This could be done following these steps:

- Tackling this problem relies on the in-depth study of the Kalman variety. The Kalman variety of a linear subspace in a vector space consists of all the endomorphisms that possess an eigenvector in that subspace, it is the exact failure locus of the (SK) condition. It was studied by Ottoviani and Sturmfels [81] and Ottoviani and Shahidi [80]. In these papers, the authors showed that the Kalman variety is an algebraic variety in the matrix space $\mathbb{K}^{n\times n}$ that is irreducible, has codimension $n-d$, of degree $\binom{n}{d-1}$ in $\mathbb{P}^{n^2-1}$ as well as many other properties.

- From their paper, one could deduce a more complete characterization, in terms of the matrices A and B, of the possible sets of degeneracy $\mathcal{D}$ defined in (12).

- Then, characterizing the behaviors of $\mathcal{N}_{*,\varepsilon}(\omega)$ when ω is close to $\mathcal{D}$ should be accessible. And once this is obtained, completing the classification can be done using similar arguments as in [4].

With this complete classification in hand, one would have a perfect understanding of linear systems satisfying the (SK) condition. But one would still have to take into account the non-linearities as sometimes, in the asymptotic analysis, they provide helpful effects for the solution cf. [6].

5.2 Other relaxation problems

Vanishing viscosity and relaxation limit for Baer-Nunziato system to porous media equations

As an extension of Chapter 4, a problem under current investigation with Jin Tan is the study of the relaxation limits $\nu \to 0$ and $\eta \to \infty$ happening at the same time for the damped Baer-Nunziato system. To simplify the problem, one can split the limit problem in two and study the relaxation limit associated with η of the system resulting from the relaxation process

associated with ν. Recall that we justified that as $\nu \to 0$ the solution of (BN) converges (strongly) to the solution of

$$\begin{cases} \alpha_+ + \alpha_- = 1, \\ \partial_t (\alpha_\pm \rho_\pm) + \operatorname{div} (\alpha_\pm \rho_\pm u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \\ P = P_+ (\rho_+) = P_- (\rho_-) . \end{cases} \tag{K}$$

Formally, using similar rescaled variables as (15), one can deduce that any smooth solution of (K) associated to the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ converges, as $\eta \to \infty$, to the solution of the following system:

$$\begin{cases} \beta_+ + \beta_- = 1, \\ \partial_t(\beta_\pm \mathcal{N}_\pm) + \operatorname{div} (\beta_\pm \mathcal{N}_\pm v) = 0, \\ \mathcal{N} v + \nabla Q = 0, \\ \mathcal{N} = \beta_+ \mathcal{N}_+ + \beta_- \mathcal{N}_-, \\ Q = Q_+ (\mathcal{N}_+) = Q_-(\mathcal{N}_-) \end{cases} \tag{2-PME}$$

associated to the initial data $(\beta_{\pm,0}, \mathcal{N}_{\pm,0}, v_0) = (\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$. This limit system can be interpreted as two porous media equations, a Darcy's law and two algebraic closures.

In order to obtain an explicit convergence rate for the relaxation process, as in Chapter 3, we need to recover elliptic properties for the unknown $\mathcal{N}_\pm$. To do that, one can write the equation satisfied by $\mathcal{N}_\pm$ and use Darcy's law to replace the velocity in the equation of $\mathcal{N}_\pm$. Then linearizing around $(\bar{\beta}_\pm, \bar{\mathcal{N}}_\pm)$, the linear term $-c\Delta\mathcal{N}_\pm$, with $c > 0$, will appear. With that in hand, proving the existence of solutions to (2-PME) follows from what we did in Chapter 3 for the porous media equation.

Then, the difficulties to justify the relaxation limit are, on one hand, the fact that the large parameter in front of ρu actually induces that we can only recover very weak decay for the un-directly damped variables ρ_+ and ρ_-. Again, this is linked to the known *overdamping phenomenon* which can be directly deduced from a spectral analysis in our context, to keep it simple, the decay coefficient we obtain behaves like $\min\{\eta, 1/\eta\}$. On the other hand, a method based on rescaling arguments similar to what we did in the previous chapters would necessitate to consider initial data in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$, however the initial system (BN) does not satisfy the condition established in Chapter 2 to work in such a setting, mainly because the main difference between this system and Euler system remains: one cannot recover time-integrability for ρ in any Besov spaces. Nevertheless, in this chapter, we did not fully use the notion of damped mode depicted in Chapter 3. The problematic term that prevents us from being able to work with initial data in $\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1} \cap \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$ lies in the velocity equations, and having a damped mode at hand should actually enable us to "absorb" it.

Generalizing Giovangigli, Yong et al's work

In the papers [46–49], Giovangigli, Matuszewski and Yong justified locally-in-time the vanishing viscosity and relaxation limit of a large class of symmetrizable systems including a compressible bi-temperature fluid model which turns out to have many similarities with the system we considered in Chapter 4. Assuming that the (SK) condition (or weaker) is satisfied, exhibiting a damped mode to handle the low frequencies and proceeding in a similar fashion as in Chapter

4, it should be possible to study their general system in our setting.

Relaxation limit for the hyperbolic-parabolic chemotaxis (HPC) system

Another on-going project, in collaboration with Shou Ling-Yun and He Qingyou, is the study of the relaxation limit of the hyperbolic-parabolic chemotaxis system to the Keller-Segel system in a similar framework as employed in this manuscript. The HPC system reads:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \dfrac{1}{\varepsilon}\rho u - \mu\rho\nabla\phi = 0, \\ \partial_t \phi - \Delta\phi - A\rho + B\phi = 0, \qquad x \in \mathbb{R}^d, \quad t > 0. \end{cases} \tag{HPC}$$

Introducing the rescaled time variable $\tau = \varepsilon t$ and defining a similar rescaling as for the compressible Euler system:

$$(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)(x,\tau) := (\rho, \frac{u}{\varepsilon}, \phi)(x,t),$$

formally, as $\varepsilon \to 0$ the solution $(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)(x,t)$ converges to the solution of the Keller-Segel system which reads:

$$\begin{cases} \partial_t \rho^* - \operatorname{div}(\nabla P(\rho^*) - \mu\rho^*\nabla\phi^*) = 0, \\ -\Delta\phi^* - A\rho^* + B\phi^* = 0. \end{cases}$$

To justify this relaxation limit, one of the main difficulties lies in the third equation of (HPC), where there is a linear $0-th$ order term depending on the density. To deal with this term, we isolated two damped modes in the low frequencies regime, one similar to the one for the compressible Euler system and one in the equation of ϕ. Again, this allows to diagonalize the system and study it in a decoupled way. In the high frequencies regime, we have to construct a suitable non-linear Lyapunov functional with additional low-order terms to cancel problematic terms arising from the unknown ϕ.

Annexe A

Appendix

A.1 Some useful tools

We often used the following well known result that can be found in [39].

Lemma A.1.1. *Let $p \geq 1$. Let $X : [0,T] \to \mathbb{R}^+$ be a continuous function such that X^p is differentiable. Assume that there exists a constant $B \geq 0$ and a measurable function $A : [0,T] \to \mathbb{R}^+$ such that*

$$\frac{1}{p}\frac{d}{dt}X^p + BX^p \leq AX^{p-1} \quad \textit{a.e. on} \quad [0,T].$$

Then, for all $t \in [0,T]$, we have

$$X(t) + B\int_0^t X \leq X_0 + \int_0^t A.$$

Proof. The case $p = 1$ being obvious, assume that $1 < p < \infty$. Then, we set $X_\varepsilon \triangleq (X^p + \varepsilon^p)^{1/p}$ for $\varepsilon > 0$, and observe that

$$\frac{1}{p}\frac{d}{dt}X_\varepsilon^p + BX_\varepsilon^p \leq AX_\varepsilon^{p-1} + B\varepsilon^p \quad \text{a.e. on} \quad [0,T].$$

Dividing both sides by the positive function X_ε^{p-1}, we get

$$\frac{d}{dt}X_\varepsilon + BX_\varepsilon \leq A + B\varepsilon\left(\frac{\varepsilon}{X_\varepsilon}\right)^{p-1},$$

whence, as $\varepsilon/X_\varepsilon \leq 1$,

$$\frac{d}{dt}X_\varepsilon + BX_\varepsilon \leq A + B\varepsilon.$$

Then, integrating in time and taking the limit as ε tends to 0 yields the desired inequality. □

The next lemma is useful to derive time-decay estimates, its proof can be found in [8].

Lemma A.1.2. *For any $\alpha \geq 0$, $\kappa < 1$ and $t \geq 2$, let $\varphi := \min\{\alpha, \kappa, \alpha + \kappa - 1\}$. Then*

$$\int_0^t \min\{1, (t-\tau)^{-\alpha}\}\min\{1, \tau^{-\kappa}\}d\tau = t^{-\varphi}.$$

A.2 Littlewood-Paley Theory

We present here the Littlewood-Paley theory and some of its properties. Details and corresponding proofs can be found in the Chapter 2 of [3].

A.2.1 Bernstein lemma and dyadic partition of unity

First, we recall the so called Bernstein lemma :

Lemma A.2.1. *Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball. There exists a constant C such that for all nonnegative integer k, for all couple (p,q) $[1,\infty]^2$ with $q \geq p \geq 1$ and for all functions $u \in L^p$, we have*

$$\text{Supp } \hat{u} \subset \lambda\mathcal{B} \implies \|D^k u\|_{L^q} \overset{def}{=} \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},$$

$$\text{Supp } \hat{u} \subset \lambda C \implies C^{-k-1}\lambda^k\|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1}\lambda^k\|u\|_{L^p}.$$

Then, we introduce a dyadic partition of the space.

Proposition A.2.1. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d \,/\, 3/4 \leq |\xi| \leq 8/3\}$. There exists two radial functions $\chi \in \mathcal{D}(B(0,3/4))$ and $\varphi \in \mathcal{D}(\mathcal{C})$ valued in $[0,1]$ such that*

$$\forall \xi \in \mathbb{R}^d\,,\ \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1,$$

$$\forall \xi \in \mathbb{R}^d\backslash\{0\}\,,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1,$$

$$|j-j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j}.) \cap \text{Supp } \varphi(2^{-j}.) = \emptyset,$$

$$j \geq 1 \Rightarrow \text{Supp } \chi \cap \text{Supp } \varphi(2^{-j}.) = \emptyset.$$

The set $\tilde{\mathcal{C}} = \mathcal{B}(0,2/3) + \mathcal{C}$ is an annulus and we have

$$|j-j'| \geq 5 \Rightarrow 2^{j'}\tilde{\mathcal{C}} \cap 2^j\mathcal{C} = \emptyset.$$

Moreover, it holds :

$$\forall \xi \in \mathbb{R}^d,\ \frac{1}{2} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1,$$

$$\forall \xi \in \mathbb{R}^d\backslash\{0\},\ \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1.$$

Definition A.2.1. *We fix two functions χ and φ satisfying the assertions from the previous proposition and we denote $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$ their Fourier inverses. The homogeneous dyadic blocks $\dot{\Delta}_j$ and the low-frequency cut-off operator $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by*

$$\dot{\Delta}_j = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)dy,$$

$$\dot{S}_j = \chi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} \tilde{h}(2^j y)u(x-y)dy.$$

We define $\mathcal{S}'_h$ the space of tempered distributions such that

$$\lim_{j\to-\infty} \|\dot{S}_j\theta\|_{L^\infty} = 0 \quad \textit{for any } \theta \in \mathcal{D}(\mathbb{R}^d).$$

Then, for any $u \in \mathcal{S}'_h$

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \quad \text{and} \quad \dot{S}_j u = \sum_{k\leq j} \dot{\Delta}_j u$$

One can now define the homogeneous Besov spaces:

Definition A.2.2. *Let s be a real number and (p,r) in $[1,\infty]^2$. The homogeneous Besov space $\dot{\mathbb{B}}^s_{p,r}$ is the set of tempered distributions u in $\mathcal{S}'_h$ such that*

$$\|u\|_{\dot{\mathbb{B}}^s_{p,r}} \overset{def}{=} \left(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j u\|^r_{L^p}\right)^{\frac{1}{r}} < \infty.$$

Let us now list some of its most useful properties.

Proposition A.2.2. *Let s be a real number and (p,r) in $[1,\infty]^2$.*

- *The space $\dot{\mathbb{B}}^s_{p,r}$ associated with $\|.\|_{\dot{\mathbb{B}}^s_{p,r}}$ is a normed space. Moreover, if $s < d/p$ or $s = d/p$ and $r = 1$, it's a Banach space.*

- *There exists a constant C, depending only on s, such that for all positive real number λ, we have*
$$C^{-1}\lambda^{s-\frac{d}{p}}\|u\|_{\dot{\mathbb{B}}^s_{p,r}} \leq \|u(\lambda.)\|_{\dot{\mathbb{B}}^s_{p,r}} \leq C\lambda^{s-\frac{d}{p}}\|u\|_{\dot{\mathbb{B}}^s_{p,r}}.$$

- *For $u = u(t,x)$ in $L^r(0,T;\mathbb{B}^s_{p,1})$ we have the following scaling property:*
$$C^{-1}\lambda^{b(s-\frac{d}{p})-\frac{a}{r}}\|u\|_{L^r_T(\dot{\mathbb{B}}^s_{p,1})} \leq \|u(\lambda^a\cdot,\lambda^b\cdot)\|_{L^r_T(\dot{\mathbb{B}}^s_{p,1})} \leq C\lambda^{b(s-\frac{d}{p})-\frac{a}{r}}\|u\|_{L^r_T(\dot{\mathbb{B}}^s_{p,1})}. \tag{A.1}$$

- *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Thus, for all real number s, the space $\dot{\mathbb{B}}^s_{p_1,r_2}$ continuously embeds in $\dot{\mathbb{B}}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

- *Let s_1 and s_2 be two real numbers such that $s_1 < s_2$ and $\theta \in]0,1[$. There exists a constant C such that for all $(p,r) \in [1,\infty]^2$ and for all $u \in \mathcal{S}'_h$, we have*
$$\|u\|_{\dot{\mathbb{B}}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^{\theta}_{\dot{\mathbb{B}}^{s_1}_{p,r}}\|u\|^{1-\theta}_{\dot{\mathbb{B}}^{s_2}_{p,r}},$$
$$\|u\|_{\dot{\mathbb{B}}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_1-s_2}\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\|u\|^{\theta}_{\dot{\mathbb{B}}^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{\dot{\mathbb{B}}^{s_2}_{p,\infty}}.$$

- *Let $\mathcal{C}'$ be an annulus and $(u_j)_{j\in\mathbb{Z}}$ a sequence of functions such that*

$$\text{Supp } \hat{u}_j \subset 2^j\mathcal{C}' \text{ et } \|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\|_{l^r} < \infty.$$

If the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'$ toward u in $\mathcal{S}'_h$, then u is in $\dot{\mathbb{B}}^s_{p,r}$ and

$$\|u\|_{\dot{\mathbb{B}}^s_{p,r}} \leq C_s\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{Z}}\|_{l^r}.$$

- *Let $(s_1, s_2) \in \mathbb{R}^2$ and $1 \leq p_1, p_2, r_1, r_2 \leq \infty$. We assume that (s_1, p_1, r_1) satisfy the following condition*

$$s_1 < \frac{d}{p_1} \text{ ou } s = \frac{d}{p_1} \text{ and } r_1 = 1.$$

The space $\dot{\mathbb{B}}^{s_1}_{p_1,r_1} \cap \dot{\mathbb{B}}^{s_2}_{p_2,r_2}$ with the norm $\|.\|_{\dot{\mathbb{B}}^{s_1}_{p_1,r_1}} + \|.\|_{\dot{\mathbb{B}}^{s_2}_{p_2,r_2}}$ is complete and satisfy the Fatou property : If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $\dot{\mathbb{B}}^{s_1}_{p_1,r_1} \cap \dot{\mathbb{B}}^{s_2}_{p_2,r_2}$, then there exists an element u of $\dot{\mathbb{B}}^{s_1}_{p_1,r_1} \cap \dot{\mathbb{B}}^{s_2}_{p_2,r_2}$ and $u_{\phi(n)}$ a subsequence such that

$$\lim_{n\to\infty} u_{\phi(n)} = u \text{ in } \mathcal{S}' \text{ et } \|u\|_{\dot{\mathbb{B}}^{s_k}_{p_k,r_k}} \leq C \liminf_{n\to\infty} \|u_{\phi(n)}\|_{\dot{\mathbb{B}}^{s_k}_{p_k,r_k}} \text{ pour } k = 1, 2.$$

- *Let $s < 0$ and $1 \leq p, r \leq \infty$. Let u be a distribution in $\mathcal{S}'_h$. Then u belongs to$\dot{\mathbb{B}}^s_{p,r}$ if and only if*

$$(2^{js}\|\dot{S}_j u\|_{L^p})_{j\in\mathbb{Z}} \in l^r.$$

Moreover, for a constant C depending only on d, we have

$$C^{-|s|+1}\|u\|_{\dot{\mathbb{B}}^s_{p,r}} \leq \|(2^{js}\|\dot{S}_j u\|_{L^p})_j\|_{l^r} \leq C(1 + \frac{1}{|s|})\|u\|_{\dot{\mathbb{B}}^s_{p,r}}.$$

- *For all (p, q) in $[1, \infty]$ such that $p \leq q$, the space $\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{q}}_{p,1}$ continuously embeds in L^q. Moreover, if p is finite, then $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ continuously embeds in $\mathcal{C}_0$ the space of continuous function vanishing at infinity. For all $q \in [1, \infty]$, the space L^q continuously embeds in $\dot{\mathbb{B}}^0_{q,\infty}$ and the space $\mathcal{M}$ of the bounded measures $\mathbb{R}^d$ continuously embeds in $\dot{\mathbb{B}}^0_{1,\infty}$.*

A.2.2 Low and high frequencies decomposition

Using from now on the shorthand notation $z_q \triangleq \dot{\Delta}_q z$, for a fixed threshold J_0, we associate to any element z of $\mathcal{S}'_h$, its low and high frequency parts through

$$z^{\ell,J_0} \triangleq \sum_{q\leq J_0} z_q = \dot{S}_{J_0+1}z \quad \text{and} \quad z^{h,J_0} \triangleq \sum_{q\geq J_0} z_q = (\text{Id} - \dot{S}_{J_0})z.$$

We shall constantly use the following Besov semi-norms for low and high frequencies:

$$\|z\|^{\ell,J_0}_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{q\leq J_0} 2^{qs}\|z_q\|_{L^2} \quad \text{and} \quad \|z\|^{h,J_0}_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{q\geq J_0} 2^{qs}\|z_q\|_{L^2},$$

$$\|z\|^{\ell,J_0}_{\dot{\mathbb{B}}^s_{2,\infty}} \triangleq \sup_{q\leq J_0} 2^{qs}\|z_q\|_{L^2} \quad \text{and} \quad \|z\|^{h,J_0}_{\dot{\mathbb{B}}^s_{2,\infty}} \triangleq \sup_{q\geq J_0} 2^{qs}\|z_q\|_{L^2}.$$

Whenever the value of J_0 is clear from the context, the exponent J_0 is omitted in the above notations. Throughout the paper, we shall use repeatedly the following obvious fact: for $r = 1, \infty$ and $s \leq s'$, we have

$$\|z^{\ell,J_0}\|_{\dot{\mathbb{B}}^{s'}_{2,1}} \lesssim \|z\|^{\ell,J_0}_{\dot{\mathbb{B}}^{s'}_{2,1}} \lesssim 2^{J_0(s'-s)}\|z\|^{\ell,J_0}_{\dot{\mathbb{B}}^{s}_{2,1}} \quad \text{and} \quad \|z^{h,J_0}\|_{\dot{\mathbb{B}}^{s}_{2,1}} \lesssim \|z\|^{h,J_0}_{\dot{\mathbb{B}}^{s}_{2,1}} \lesssim 2^{J_0(s-s')}\|z\|^{h,J_0}_{\dot{\mathbb{B}}^{s'}_{2,1}}. \tag{A.2}$$

Thanks to the embedding $\dot{\mathbb{B}}^{s}_{2,1} \hookrightarrow \dot{\mathbb{B}}^{s-d(\frac{1}{2}-\frac{1}{p})}_{p,1}$, we have

$$\|z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+s}_{p,1}} \lesssim \|z\|^{h,J_0}_{\dot{\mathbb{B}}^{\frac{d}{2}+s}_{2,1}} + \|z\|^{\ell,J_0}_{\dot{\mathbb{B}}^{\frac{d}{p}+s}_{p,1}} \quad \text{if} \quad p \geq 2. \tag{A.3}$$

For any Banach space X, index ρ in $[1,\infty]$ and time $T \in [0,\infty]$, we use the notation $\|z\|_{L^\rho_T(X)} \triangleq \big\|\|z(t)\|_X\big\|_{L^\rho(0,T)}$. If $T = +\infty$, then we just write $\|z\|_{L^\rho(X)}$. Finally, in the case where z has n components z_j in X, we keep the notation $\|z\|_X$ for $\sum_{j\in\{1,\cdots,n\}}\|z_j\|_X$.

A.2.3 Paraproduct and properties arising from it

Here we introduce Bony's theory [9] of paradifferential calculus. Let u and v be two tempered distributions in $\mathcal{S}'_h$. We have

$$u = \sum_{j'} \dot{\Delta}_{j'}u \quad \text{and} \quad v = \sum_{j} \dot{\Delta}_{j}v.$$

Thus, at least formally, we have

$$uv = \sum_{j',j} \dot{\Delta}_{j'}u\dot{\Delta}_{j}v.$$

The paradifferential calculus divides this sum into three parts :

- A first part concerns the indexes (j', j) such that the size of Supp $\mathcal{F}(\dot{\Delta}_{j'}u)$is small compared to the size of Supp $\mathcal{F}(\dot{\Delta}_{j}v)$ (i.e., $j' \leq j - N_0$ for a suitable positive integer N_0).
- The second part contains the indices corresponding to those frequencies of u which are large compared with the frequencies of v (i.e. $j' \geq j + N_0$).
- In the last part, we keep the indexes (j, j') for which Supp $\mathcal{F}(\dot{\Delta}_{j'}u)$ and Supp $\mathcal{F}(\dot{\Delta}_{j}v)$ have comparable sizes (i.e., $|j - j'| \leq N_0$)).

This leads to the following definition;

Definition A.2.3. *The homogeneous paraproduct of v by u is defined as follows:*

$$\dot{T}_u v \overset{def}{=} \sum_j \dot{S}_{j-1}u\dot{\Delta}_j v.$$

The homogeneous remainder of u and v is defined by

$$\dot{R}(u,v) = \sum_{|k-j|\leq 1} \dot{\Delta}_k u\dot{\Delta}_j v.$$

At the formal level, we have the so called Bony decomposition :

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v) := \dot{T}_u v + \dot{T}'_v u.$$

The next results state some estimates for the paraproduct and remainder.

Theorem A.2.1. *There exists a constant C such that for all real number s and for all (p,r) in $[1,\infty]^2$, we have, for all $(u,v) \in L^\infty \times \dot{\mathbb{B}}^s_{p,r}$,*

$$\|\dot{T}_u v\|_{\dot{\mathbb{B}}^s_{p,r}} \leq c^{1+|s|}\|u\|_{L^\infty}\|v\|_{\dot{\mathbb{B}}^s_{p,r}}.$$

Moreover, for all $(s,t) \in \mathbb{R}\times]-\infty,0[$ and all $(p,r_1,r_2) \in [1,\infty]^3$, we have, for all $(u,v) \in \dot{\mathbb{B}}^t_{\infty,r_1} \times \dot{\mathbb{B}}^s_{p,r_2}$,

$$\|\dot{T}_u v\|_{\dot{\mathbb{B}}^{s+t}_{p,r}} \leq \frac{C^{1+|s+t|}}{-t}\|u\|_{\dot{\mathbb{B}}^t_{\infty,r_1}}\|v\|_{\dot{\mathbb{B}}^s_{p,r_2}} \quad avec \ \frac{1}{r} \overset{def}{=} min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}.$$

Theorem A.2.2. *There exists a constant C such that the following inequalites are true. Let (s_1,s_2) in $\mathbb{R}^2$ and (p_1,p_2,r_1,r_2) in $[1,\infty]^4$, we assume that*

$$\frac{1}{p} \overset{def}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad and \quad \frac{1}{r} \overset{def}{=} \frac{1}{p_2} + \frac{1}{p_2} \leq 1.$$

If $s_1 + s_2$ is positive, then for all $(u,v) \in \dot{\mathbb{B}}^{s_1}_{p_1,r_1} \times \dot{\mathbb{B}}^{s_2}_{p_2,r_2}$,

$$\|\dot{R}(u,v)\|_{\dot{\mathbb{B}}^{s_1+s_2}_{p,r}} \leq \frac{C^{|s_1+s_2|+1}}{s_1+s_2}\|u\|_{\dot{\mathbb{B}}^{s_1}_{p_1,r_1}}\|v\|_{\dot{\mathbb{B}}^{s_2}_{p_2,r_2}}.$$

For $r = 1$ and$s_1 + s_2 \geq 0$, we have, for all $(u,v) \in \dot{\mathbb{B}}^{s_1}_{p_1,r_1} \times \dot{\mathbb{B}}^{s_2}_{p_2,r_2}$,

$$\|\dot{R}(u,v)\|_{\dot{\mathbb{B}}^{s_1+s_2}_{p,\infty}} \leq C^{|s_1+s_2|+1}\|u\|_{\dot{\mathbb{B}}^{s_1}_{p_1,r_1}}\|v\|_{\dot{\mathbb{B}}^{s_2}_{p_2,r_2}}.$$

To prove commutator estimates one needs the following lemma.

Lemma A.2.2. *Let $1 \leq p,q,r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}\cdot$ Let a be a function with gradient in L^p and b, a function in L^q. There exists a constant C such that*

$$\left\|[\dot{\Delta}_j, a]b\right\|_{L^r} \leq C2^{-j}\|\nabla a\|_{L^q}\|b\|_{L^p} \quad for\ all\ \ j \in \mathbb{Z}.$$

From this lemma and the theorems concerning the paradifferential calculus, one can deduce the following proposition that we shall use repeatedly in the manuscript and refer to it as *commutator estimates.*

Proposition A.2.3. *Let $p \in [1,\infty]$ and define its conjugate exponent p' such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. The following inequalities hold true:*

- *If $-\min\{d/p, d/p'\} < s \leq d/p + 1$, then*

$$2^{qs}\left\|[a,\dot{\Delta}_q]b\right\|_{L^p} \leq Cc_q\|\nabla a\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|b\|_{\dot{\mathbb{B}}^{s-1}_{p,1}} \quad with \quad \sum_{q\in\mathbb{Z}} c_q = 1. \tag{A.4}$$

- *If* $-\min\{d/p, d/p'\} \leq s < d/p+1$, *then*

$$\sup_{q\in\mathbb{Z}} 2^{qs}\|[a,\dot{\Delta}_q]\nabla b\|_{L^p} \leq C\|\nabla a\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|b\|_{\dot{\mathbb{B}}^{s}_{p,\infty}}. \tag{A.5}$$

The next proposition concerns estimates refereed to as *product laws.*

Proposition A.2.4. *Let* (s,p,r) *be in* $]0,\infty[\times[1,\infty]^2$. *Then,* $\dot{\mathbb{B}}^s_{p,r}\cap L^\infty$ *is an algebra and we have*

$$\|ab\|_{\dot{\mathbb{B}}^s_{p,r}} \leq C\big(\|a\|_{L^\infty}\|b\|_{\dot{\mathbb{B}}^s_{p,r}} + \|a\|_{\dot{\mathbb{B}}^s_{p,r}}\|b\|_{L^\infty}\big). \tag{A.6}$$

If, furthermore, $-d/p < s \leq d/p$, *then the following inequality holds:*

$$\|ab\|_{\dot{\mathbb{B}}^s_{p,1}} \leq C\|a\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|b\|_{\dot{\mathbb{B}}^s_{p,1}}. \tag{A.7}$$

Finally, if $-d/p < \sigma_1 \leq d/p$, *then the following inequality holds true:*

$$\|fg\|_{\dot{\mathbb{B}}^{-\sigma_1}_{p,\infty}} \leq C\,\|f\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|g\|_{\dot{\mathbb{B}}^{-\sigma_1}_{p,\infty}}. \tag{A.8}$$

The following proposition concerns the action of smooth function and is refereed to as *composition proposition.*

Proposition A.2.5. *Let* f *be a function in* $\mathcal{C}^\infty(\mathbb{R})$ *such that* $f(0)=0$. *Let* $p\in[1,\infty]$, $(s_1,s_2)\in]0,\infty[^2$ *and* $(p_1,p_2,r_1,r_2)\in[1,\infty]^4$. *We assume that* $s_1 < d/p$ *or that* $s_1 = d/p$ *and* $r_1 = 1$.

Then, for every real-valued function u *in* $\dot{\mathbb{B}}^{s_1}_{p_1,r_1}\cap\dot{\mathbb{B}}^{s_2}_{p_2,r_2}\cap L^\infty$, *the function* $f\circ u$ *belongs to* $\dot{\mathbb{B}}^{s_1}_{p_1,r_1}\cap\dot{\mathbb{B}}^{s_2}_{p_2,r_2}\cap L^\infty$ *and we have*

$$\|f\circ u\|_{\dot{\mathbb{B}}^{s_k}_{p_k,r_k}} \leq C\left(f', \|u\|_{L^\infty}\right)\|u\|_{\dot{\mathbb{B}}^{s_k}_{p_k,r_k}} \quad \text{for } k\in\{1,2\}.$$

As a consequence, if g is a $\mathcal{C}^\infty(\mathbb{R})$ function such that $g'(0)=0$. Then, for all u,v in $\dot{B}^s_{p,1}\cap L^\infty$ with $s>0$ and $(p,r)\in[1,\infty]^2$, we have

$$\|g(v)-g(u)\|_{\dot{B}^s_{p,1}} \leq C\Big(\|v-u\|_{L^\infty}\|(u,v)\|_{\dot{\mathbb{B}}^s_{p,1}} + \|v-u\|_{\dot{\mathbb{B}}^s_{p,1}}\|(u,v)\|_{L^\infty}\Big) \tag{A.9}$$

where C depends on f'', $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$. Finally we present a proposition related to composition operator for multivariate functions that can be found in [89] p.387-388.

Proposition A.2.6. *Let* $m\in\mathbb{N}$ *and* $s>0$. *Let* G *be a function in* $\mathcal{C}^\infty(\mathbb{R}^m)$ *such that* $G(0,..,0)=0$.
Then for every real-valued functions $f_1,..,f_m$ *in* $\mathbb{B}^s_{p,1}\cap L^\infty$, *the function* $G(f_1,..,f_m)$ *belongs to* $\mathbb{B}^s_{p,1}\cap L^\infty$ *and we have*

$$\|G(f_1,..,f_m)\|_{\mathbb{B}^s_{p,1}} \leq \|(f_1,..,f_m)\|_{\mathbb{B}^s_{p,1}}\left(1+C(\|f_1\|_{L^\infty}+...+\|f_m\|_{L^\infty})\right).$$

Remark A.2.1. *More involved estimates are proved along the manuscript to deal with each chapter's specific issues.*

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