On the controllability of the water-waves equation in bounded domains

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Summary

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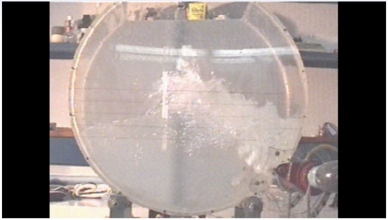
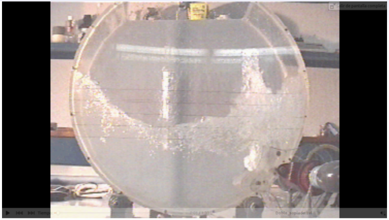
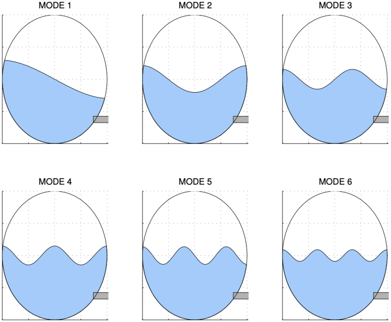
A copper conversion process in mining

- Copper converters carry out the smelting and conversion process of copper concentrate.
- The injection of air jets into the molten mixture through hoses on the walls is essential. The interaction between the air and the mixture produces the chemical reactions necessary for the conversion process.
- The air jets cause excessive splashing and agitation of the fluid. This splashing damages the internal walls, shortening the life of the converter.





M. Rosales, A. Valencia, and R. Fuentes. A methodology for controlling slopping in copper converters by using lateral and bottom gas injection. International Journal of Chemical Reactor Engineering, 7(1), 2009.

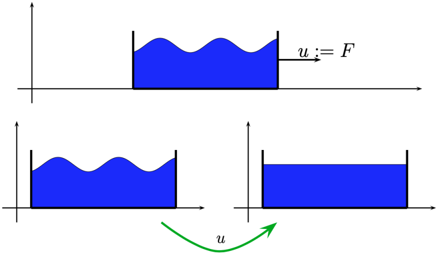


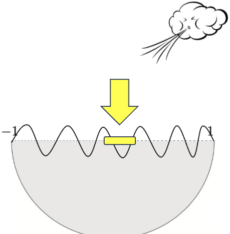
E. Godoy, A. Osses, J. Ortega, and A. Valencia. Modeling and control of surface gravity waves in a model of a copper converter. Applied Mathematical Modelling, 32(9):1696-1710, 2008.

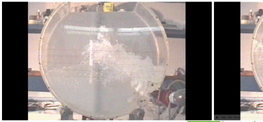
Note that the surface has its own dynamics based on physical laws.

The types of control we can exercise

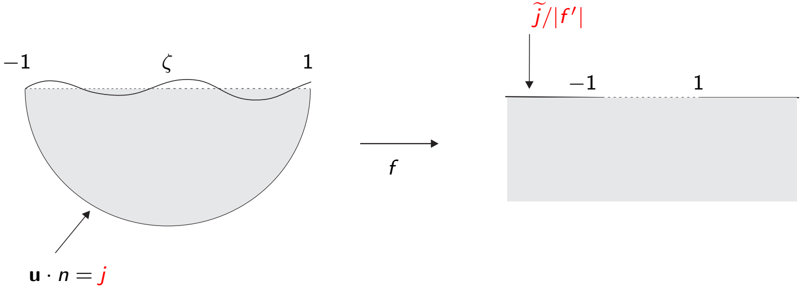
(Suggested by P. Rouchon)







What is our strategy?



We want to compute and control the oscillations frequencies of the free surface, in a bounded container. We put ourselves in the frame of an incompressible, non viscous fluid (water), in contact with solid walls.

If $\mathbf{u}$ is the velocity of the fluid (Euler):

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + f_{ext}.$$

These system is typically formulated in terms of the velocity potential function φ such that $\mathbf{u} = \nabla\varphi$:

$$\Delta\varphi = 0$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 = -\frac{1}{\rho}p + gy,$$

Together with impermeability boundary conditions at the solid walls

$$\frac{\partial\varphi}{\partial n} = 0.$$

Since dynamics is usually understood on the surface, the system is complemented with a dynamic boundary condition (the mass conservation) on $y = \zeta$

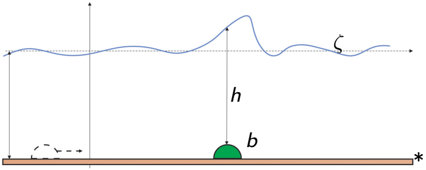
$$\zeta_t = \frac{\partial\varphi}{\partial n}.$$

Some practical problems and general concepts

- The design of an "optimal" bottom generating specific waves (Zuazua '15).

$$\zeta_t + (hV)_x = b_t$$

$$V_t + \zeta_x + \epsilon V V_x = -\epsilon/2 b_{ttx}$$



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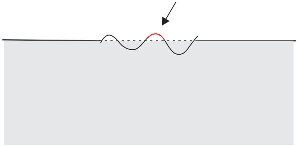

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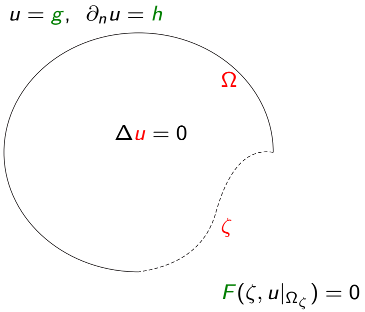
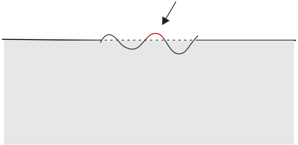
- Finally, in a free boundary problem, in addition to having a function u as an unknown, we also have an interface (boundary) that is unknown (free) and is part of the problem.



Some practical problems and general concepts

- The design of an "optimal" bottom generating specific waves (Zuazua '15).
- The design of surfing facilities (wavegarden.com).
- An inverse problem about the detection of a specific bathymetry from measurements on the surface.
- Finally, in a free boundary problem, in addition to having a function u as an unknown, we also have an interface (boundary) that is unknown (free) and is part of the problem.
- A controllability problem consists of using a parameter of the equation to bring its solution to a desired state.

$$\phi_{tt} + \mathcal{A}\phi = v\chi_\omega$$

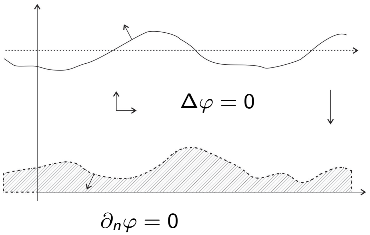


Given ϕ_0, ϕ_1, $T > 0$, to find v s.t $\phi(T) = \overline{\phi}$

Exact
To zero: $\phi(T) = 0$
Approximated: $\|\phi(T) - \overline{\phi}\| < \epsilon$

Summary

As a free boundary problem, we have a system inside the domain, complemented with impermeability boundary condition on the solid walls and *conservation laws* in the free surface:



$$\begin{cases} \Delta\varphi = 0, & \Omega\times(0,T), \\ \zeta_t = \partial_n\varphi, & y=\zeta, \\ \varphi_t + \frac{1}{2}|\nabla\varphi|^2 + \zeta = 0, & y=\zeta, \\ \partial_n\varphi = 0, & y = b. \end{cases}$$

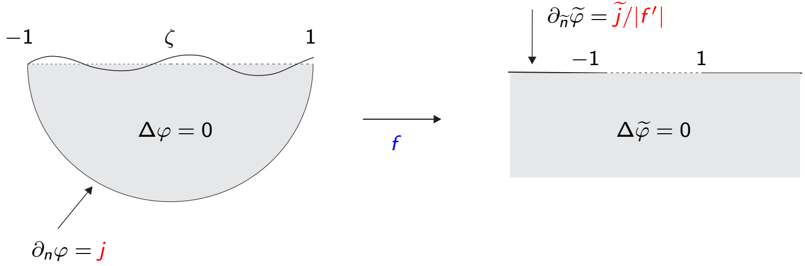
The problem is settled (decoupled), for $\phi = \varphi|_{y=\zeta}$, as

$$\begin{cases} \Delta\varphi = 0, & \Omega\times(0,T), \\ \varphi|_{y=\zeta} = \phi, & \\ \partial_n\varphi|_{y=b} = 0, & \end{cases} \iff \begin{cases} \zeta_t = \partial_n\varphi \\ \phi_t + \zeta + \frac{1}{2}|\nabla_x\phi|^2 + \frac{(\partial_n\varphi + \nabla_x\zeta\cdot\nabla_x\phi)^2}{2(1+|\nabla_x\zeta|^2)} = 0, \end{cases}$$

which after linearization becomes

$$\phi_{tt} + \partial_n\varphi = 0, \quad y = \zeta.$$

- This linearization makes sense in the shallow water regime: $\zeta = \epsilon\eta$ and $\varphi = C + \epsilon\overline{\varphi}$.
- Since the domain is "almost flat" we can solve, explicitly, the problem for $\partial_n\varphi = \phi_y$.



$$(1) \qquad \begin{cases} \Delta\varphi = 0, & \mathbb{R}^2_-, \\ \varphi_{tt} + |f'|\varphi_y = 0, & y' = 0, |x| < 1, \\ \partial_n\varphi = 0, & |x| > 1. \end{cases}$$

By taking the Fourier transform in x:

$$\hat{\varphi}_{yy} - k^2\varphi = 0 \implies \hat{\varphi} = \hat{\varphi}(0)e^{|k|y} \implies \hat{\varphi}_y|_{y=0} = \hat{\varphi}(0)\frac{1}{i}sgn(k)ik$$

$$\implies \hat{\varphi}_y|_{y=0} = \hat{\varphi}_x(0)\widehat{\frac{\sqrt{2}}{\sqrt{\pi}x}} \implies \varphi_y(0) = \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{\phi_x(t,\xi,0)}{x-\xi}d\xi.$$

A mathematical framework

Summarizing, the explicit solution for the Laplace equation $\Delta\varphi = 0$ in the (lower) half-plane is

$$\varphi_y = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\phi_x(t,\xi,0)}{x-\xi} d\xi =: -H(\phi_x),$$

or equivalently

$$\varphi_x(t,x,y)|_{y=0} = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\partial_n\varphi(t,\xi,0)}{x-\xi} d\xi = H(\partial_n\varphi) \qquad \text{(airfoil equation)}.$$

Since we are interested in bounded domains and the effects of walls, the conservation laws $\phi_{tt} + \varphi_y = 0$, $x \in (-1,1)$ and $y = 0$, can be written as

$$\phi_{tt} + \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} P.V. \int_{-1}^{1} \sqrt{1-\xi^2} \frac{\phi_x(\xi)}{x-\xi} d\xi = 0.$$

Given $w = \sqrt{1-x^2}$ let

$$L^2_w(-1,1) = \left\{ v : \int_{-1}^{1} w v^2 < \infty \right\}, \qquad L^2_{w^{-1}}(-1,1) = \left\{ v : \int_{-1}^{1} w^{-1} v^2 < \infty \right\},$$

$$H^{1/2}_{w^{-1}}(-1,1) = \left\{ \psi \in L^2_{w^{-1}} : \|\psi\|^2_{H^{1/2}_{w^{-1}}} = \|\psi\|^2_{w^{-1}} + (\mathcal{A}\psi, \psi)_{L^2} < \infty \right\},$$

and then

$$H^{1/2}_{w^{-1}} \subset L^2_{w^{-1}} \subset L^2 \subset L^2_w.$$

Lemma

Given $\phi \in H^{1/2}_{w^{-1}}$, it holds

$$\partial_n \varphi(x) = \frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^{1*} \frac{\sqrt{1-\xi^2}\phi_x(\xi)}{x-\xi} d\xi = \partial_x \left(\frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1*} \frac{\phi(\xi)}{\sqrt{1-\xi^2}(x-\xi)} d\xi \right) \equiv \mathcal{A}.$$

Finally, the water-waves problem in this case can be written as

$$(2) \qquad \begin{cases} \phi_{tt} + \mathcal{A}\phi = 0, & (t,x) \in (0,\infty) \times (-1,1), \\ \phi(0,x) = \phi_0(x), & x \in (-1,1), \\ \phi_t(0,x) = \phi_1(x), & x \in (-1,1). \end{cases}$$

Theorem

Given $u \in L^2_w$, there exists a unique weak solution $\phi \in H^{1/2}_{w^{-1}}$ for the problem

$$\mathcal{A}\phi = u.$$

Theorem (existence)

Let $T > 0$, $[\phi_0, \phi_1] \in H^{1/2}_{w^{-1}} \times L^2$ and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz. Then, system

$$\begin{cases} \phi_{tt} + \mathcal{A}\phi = f(\phi), & (t,x) \in (0,\infty) \times (-1,1), \\ \phi(0,x) = \phi_0(x), & x \in (-1,1), \\ \phi_t(0,x) = \phi_1(x), & x \in (-1,1), \end{cases}$$

has a unique solution

$$[\phi, \phi_t] \in C([0,T]; H^{1/2}_{w^{-1}} \times L^2).$$

Some spectral properties

$$\lambda\phi = \mathcal{A}\phi.$$

By using the (orthogonal base of) Tchebyshev polynomials

$$\phi(\xi) = \sum_{n=0}^{\infty} a_n T_n(\xi) \quad \Longrightarrow \quad \mathcal{A}\phi = \sum_{n=1}^{\infty} n a_n \frac{T_n(x')}{\sqrt{1-x'^2}}.$$

Moreover, if $\psi = \sum b_n T_n$,

$$(\mathcal{A}\phi, \psi)_{L^2} = \int_{-1}^{1} \left(\sum_{n=1}^{\infty} n a_n \frac{T_n(x)}{\sqrt{1-x^2}} \right) \left(\sum_{m=0}^{\infty} b_m T_m(x) \right) = \frac{\pi}{2} \sum_{n=1}^{\infty} n a_n b_n.$$

Therefore,a weak version of the equation is:

$$\mathcal{A}\phi = u \quad \overset{\text{weak}}{\Longleftrightarrow} \quad (\mathcal{A}\phi, \psi)_{L^2} = (u, \psi)_{L^2} \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} n a_n b_n = \sum_{n=1}^{\infty} u_n b_n.$$

$(\mathcal{A}\phi, \psi)_{L^2}$ es:

- Continuity: $|(\mathcal{A}\phi, \psi)_{L^2}| \leq \frac{2}{\pi} \|\phi\|_{H^{1/2}_{w-1}} \|\psi\|_{H^{1/2}_{w-1}}$
- Coerciveness: $(\mathcal{A}\phi, \phi)_{L^2} \geq C\|\widetilde{\phi}\|^2_{H^{1/2}_{w-1}}$

Theorem

There exists a Hilbert base $\{e_n\}_{n\geq 1}$ of L^2 and $\{\lambda_n\}_{n\geq 1}$ of real numbers $\lambda_n > 0 \ \forall n$, $\lambda_n \to +\infty$ such that

$$e_n \in H^{\frac{1}{2}}_{w-1}, \qquad \mathcal{A}e_n = \lambda_n e_n.$$

Indeed: given $v \in L^2_w$, let $\mathcal{A}\phi = v$ and $\mathcal{T} : L^2_w \to L^2_w$ given by $\mathcal{T}v = \phi$. Since $H^{1/2}_{w-1} \overset{c}{\hookrightarrow} L^2_{w-1} \hookrightarrow L^2$ (operator $\mathcal{T}$ is compact) and self-adjoint, and

$$(\mathcal{T}v, v)_{L^2} = (\phi, \mathcal{A}\phi)_{L^2} \geq 0, \quad \forall v \in L^2_w.$$

From the Spectral decomposition theorem L^2 admits a Hilbert base of eigenvalues of $\mathcal{T}$, such that $\mathcal{T}e_n = \mu_n e_n$, $e_n \in H^{1/2}_{w-1}$.

Observability

$$\phi_{tt} + \mathcal{A}\phi = 0.$$

Let

$$\phi = \sum_{n=1}^{\infty} \left(A_n \cos(\theta_n t) + B_n \sin(\theta_n t)\right) e_n, \qquad \theta_n = \sqrt{\lambda_n},$$

where

$$\phi_0 = \sum_{n=1}^{\infty} A_n e_n, \quad \phi_1 = \sum_{n=1}^{\infty} \theta_n B_n e_n.$$

Then

$$A_n = \int_{-1}^{1} \phi_0 e_n, \quad B_n = \frac{1}{\theta_n} \int_{-1}^{1} \phi_1 e_n.$$

Finally

$$\begin{aligned} \|\phi\|^2_{L^2(L^2)} &= \int_0^T \int_{-1}^{1} \phi^2 \\ &\geq \int_0^T \sum_n \left(A_n^2 \cos^2(\theta_n t) + B_n^2 \sin^2(\theta_n t) + 2A_n B_n \sin(\theta_n t) \cos(\theta_n t)\right) \\ &\geq CT \sum_n \left[{A_n}^2 + {B_n}^2\right]. \end{aligned}$$

On the unique continuation and Approximate control

Lemma

Let $\mathcal{I} \subset (-1,1)$ an open set. If

$$\phi = \mathcal{A}\phi = 0 \text{ in } \mathcal{I},$$

then $\phi \equiv 0$ in $(-1,1)$.

Proposition

Let $T > 0$, $[\phi_0, \phi_1] \in H^{1/2}_{w^{-1}} \times L^2$, $f : \mathbb{R} \to \mathbb{R}$ such that $\exists c > 0 : |f(x)| \leq c|x|$. If $\phi = 0$ in an open $M \subset (0,T) \times (-1,1)$, where $\phi \in C(0,T; H^{1/2}_{w^{-1}})$ is a solution of

$$\begin{cases} \phi_{tt} + \mathcal{A}\phi = f(\phi), & (t,x) \in (0,T) \times (-1,1), \\ \phi(0,x) = \phi_0(x), & x \in (-1,1), \\ \phi_t(0,x) = \phi_1(x), & x \in (-1,1), \end{cases} \tag{3}$$

then

$$\phi \equiv 0, \quad \text{in } (0,T) \times (-1,1).$$

Theorem

Let $T > 0$ and $[\phi_0, \phi_1] \in H^{1/2}_{w^{-1}} \times L^2$. System

$$(4) \qquad \begin{cases} \phi_{tt} + \mathcal{A}\phi = v\mathbf{1}_{\mathcal{I}}, & (t,x) \in (0,T) \times (-1,1), \\ \phi(0,x) = \phi_0(x), & x \in (-1,1), \\ \phi_t(0,x) = \phi_1(x), & x \in (-1,1). \end{cases}$$

is approximate controllable with a control $v \in L^2(0,T; L^2_w(-1,1))$, in $\mathcal{I} \subset (-1,1)$, i.e., for any $\epsilon > 0$ and $[\phi_0, \phi_1]$, $[g_0, g_1] \in H^{1/2}_{w^{-1}} \times L^2$, there exists a control $v \in L^2(0,T; L^2_w)$ such that the solution of (4) satisfies

$$\|[\phi(T,\cdot), \phi_t(T,\cdot)] - [g_0, g_1]\|_{H^{1/2}_{w^{-1}} \times L^2} \leq \epsilon.$$

- Notice that the (exact) control holds on $(-1, 1)$.
  It would be nice to control only in an open subset of $(-1, 1) \times (0, T)$. So far we obtained approximate control only.
- Same results hold for the general water-waves system (T. Alazard 18')
- Some of this ideas could be implemented for the general water-waves system and the semilinear case.
- It would be interesting to study the inverse problem of source detection, for $\phi_{tt} + \mathcal{A}\phi = h$. One possibility would be through Carleman inequalities for the non local operator H.

References

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