Lipschitz stability in inverse problems for semi-discrete parabolic operators

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The goal in the semi-discrete setting

A brief presentation of the inverse problem

The discrete source inverse problem, $h = \frac{1}{N+1}$

$$g = (g_{ij}) \to \begin{cases} \frac{d}{dt} y_{ij} - \Delta_h y_{ij} & = & g_{ij}(t) & t \in (0,T),\ i,j \in [[1,N]] \\ y_{0,j} = y_{N+1,j} & = & 0 & t \in (0,T),\ j \in [[1,N]] \\ y_{i,0} = y_{i,N+1} & = & 0 & t \in (0,T),\ i \in [[1,N]] \\ y(0)_{ij} & = & y^0_{ij} & i,j \in [[1,N]] \end{cases}$$

$$\Delta_h y_{i,j} := \frac{y_{i+1,j} + y_{i-1,j} + y_{i,j+1} + y_{i,j-1} - 4y_{i,j}}{h^2}$$

We want to recover g from measurements of y

Let $\vartheta \in (0,T)$

$$\int_0^T \|g\|^2_{\mathcal{W}} dt \le C \left(\|y(\vartheta)\|^2_{\mathcal{W}} + \int_0^T (\|y_t\|^2_\omega + \|y\|^2_\omega) dt \right) + o(h)$$

where $\omega \subset \mathcal{W} := [[1,N]] \times [[1,N]]$

A brief presentation of the inverse problem

The discrete source inverse problem, $h = \frac{1}{N+1}$

$$f = (f_{ij}) \rightarrow \begin{cases} \frac{d}{dt} y_{ij} - \Delta_h \, y_{ij} + q_{ij} y_{ij} & = & R_{ij}(t) f_{ij} & t \in (0,T), \; i,j \in [[1,N]] \\ y_{0,j} = y_{N+1,j} & = & 0 & t \in (0,T), \; j \in [[1,N]] \\ y_{i,0} = y_{i,N+1} & = & 0 & t \in (0,T), \; i \in [[1,N]] \\ y(0)_{ij} & = & y^0_{ij} & i,j \in [[1,N]] \end{cases}$$

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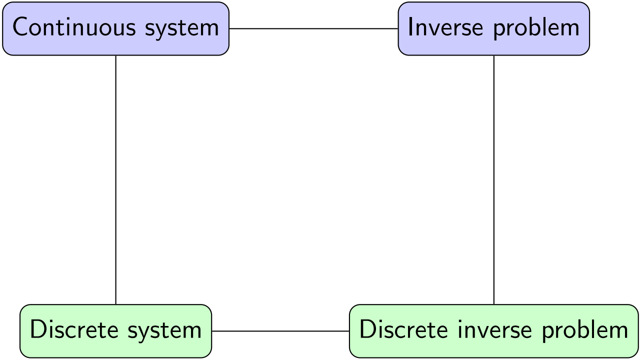
Scheme



Continuous system
Inverse problem
Discrete system
Discrete inverse problem

The results in the continuous case

The source inverse problem

$$g \rightarrow \begin{cases} y_t - \mathcal{A}\, y &= g(t,x) & (0,T)\times\Omega \\ y &= 0 & (0,T)\times\partial\Omega \\ y(0,x) &= y^0(x) & \Omega \end{cases}$$

$$\mathcal{A}y = \sum_{i,j=1}^{N} \partial_i(a_{i,j}(t,x)\partial_j y(t,x)) - \sum_{i=1}^{N} b_i(t,x)\partial_i y(t,x) - c(t,x)y(t,x)$$

$$|\partial_t g(t,x)| \leq C|g(T/2,x)|, \quad [0,T]\times\overline{\Omega} \tag{1}$$

Lipschitz stability in inverse parabolic problems by the Carleman estimate

Oleg Yu Imanuvilov and Masahiro Yamamoto (1998)

The source inverse problem

$$g \to \begin{cases} y_t - \mathcal{A}\,y = g(t,x) & (0,T)\times\Omega \\ y = 0 & (0,T)\times\partial\Omega \\ y(0,x) = y^0(x) & \Omega \end{cases}$$

$$|\partial_t g(t,x)| \le C|g(T/2,x)|, \quad [0,T]\times\overline{\Omega} \tag{1}$$

Theorem 3.2 (Imanuvilov and Yamamoto (1998))

Let g satisfy (1). If y is the solution of the parabolic problem, then we have

$$\|g\|_{L^2(\Omega)} \le C\left(\|y(T/2,\cdot)\|_{H^2(\Omega)} + \|e^{s\alpha}y_t\|_{L^2(Q_\omega)} + \|e^{s\alpha}y\|_{L^2(Q_\omega)}\right)$$

where $Q_\omega = (0,T)\times\omega$

Lipschitz stability in inverse parabolic problems by the Carleman estimate

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The source inverse problem

$$f \to \begin{cases} y_t - \mathcal{A}\, y = R(t,x) f(x) & (0,T) \times \Omega \\ y = 0 & (0,T) \times \partial\Omega \\ y(0,x) = y^0(x) & \Omega \end{cases}$$

$$|R(T/2, x)| \geq \alpha > 0, \quad \overline{\Omega} \tag{2}$$

Theorem 3.2 (Imanuvilov and Yamamoto (1998))

Let $R \in C^{1,0}(\mathbb{R})$ satisfy (2). If y is the solution of the parabolic problem, then we have

$$\|f\|_{L^2(\Omega)} \leq C \left(\|y(T/2, \cdot)\|_{H^2(\Omega)} + \|e^{s\alpha} y_t\|_{L^2(Q_\omega)} + \|e^{s\alpha} y\|_{L^2(Q_\omega)} \right)$$

where $Q_\omega = (0,T) \times \omega$

The strategy for the proof in the continuous setting

- To develop a Carleman estimate for the parabolic system, in the fashion

$$E(y) \preceq \int_Q e^{2s\alpha}|g|^2 + Obs(y)$$

- To improve the Carleman inequality

$$s\int_\Omega e^{2s\alpha(T/2)}|y(T/2)|^2 + E(y) \preceq \int_Q e^{2s\alpha}|g|^2 + Obs(y)$$

- Under the assumption for g and in the case of $\partial_t \mathcal{A} \neq 0$, it is needed

$$\int_Q s\theta(t)e^{2s\alpha}|g|^2 \preceq \sqrt{s}\int_\Omega e^{2s\alpha(T/2)}|g(T/2)|^2$$

- Finally, by following the method introduced in Bukhgeim and Klibanov, the proof is completed

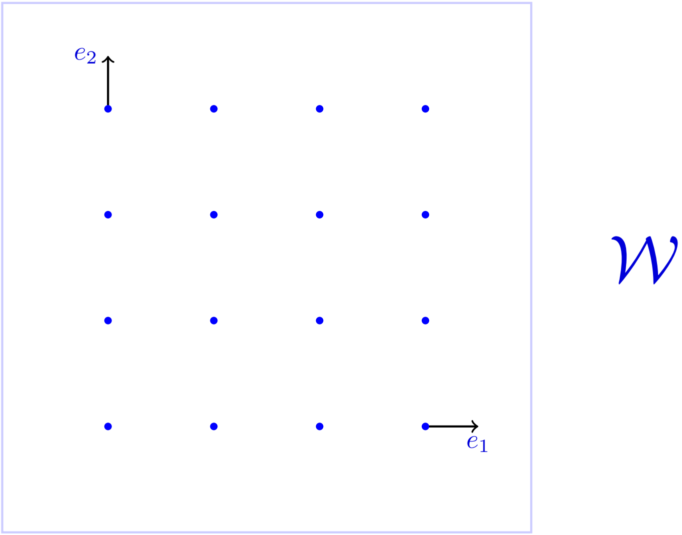
Carleman, $\alpha(t,x)=\theta(t)\varphi(x)$

For $\theta(t)=\frac{1}{t(T-t)}$, and $\varphi(x)=e^{\lambda\psi(x)}-e^{2\lambda\|\psi\|_\infty}$

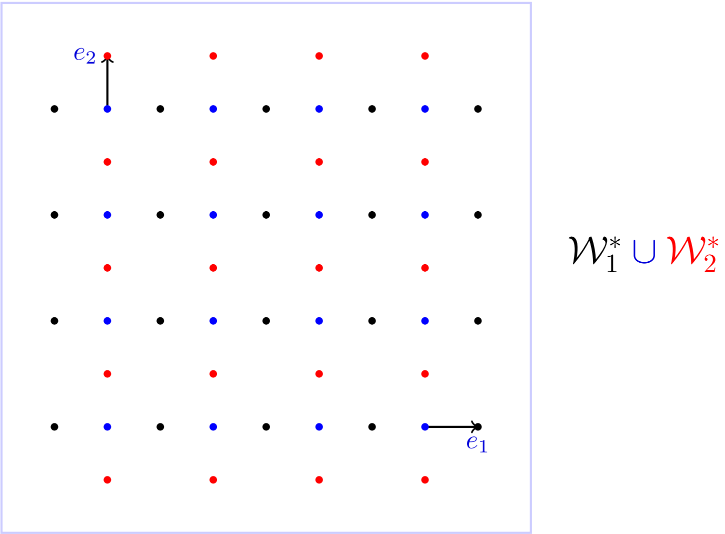
Carleman for $p=0,1$

$$\int_Q (s\theta)^{p-1}\left(|y_t|^2+\sum_{i,j}|\partial^2_{i,j}y|^2\right)e^{2s\theta\varphi}+\int_Q\left((s\theta)^{p+1}|\nabla y|^2+(s\theta)^{3+p}|y|^2\right)e^{2s\theta\varphi}$$
$$\le C\left(\int_Q e^{2s\theta\varphi}(s\theta)^p|g|^2+\int_{(0,T)\times\omega}(s\theta)^{p+3}e^{2s\theta\varphi}|y|^2\right)$$

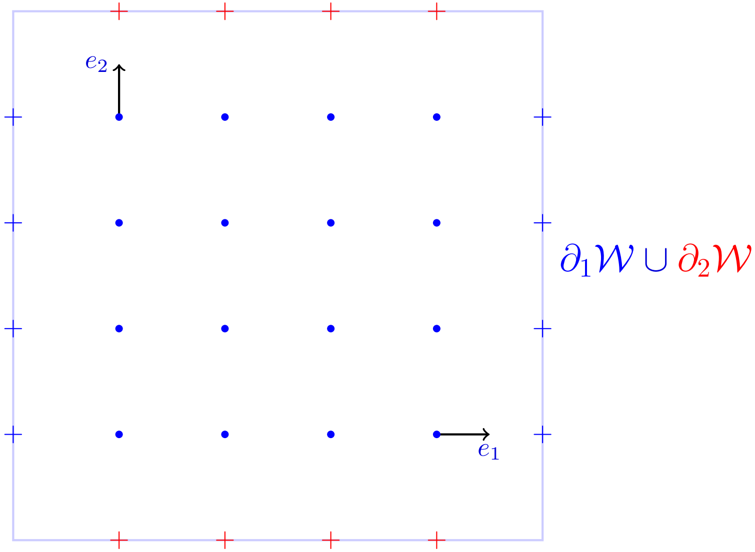
Some preliminaries in the discrete setting



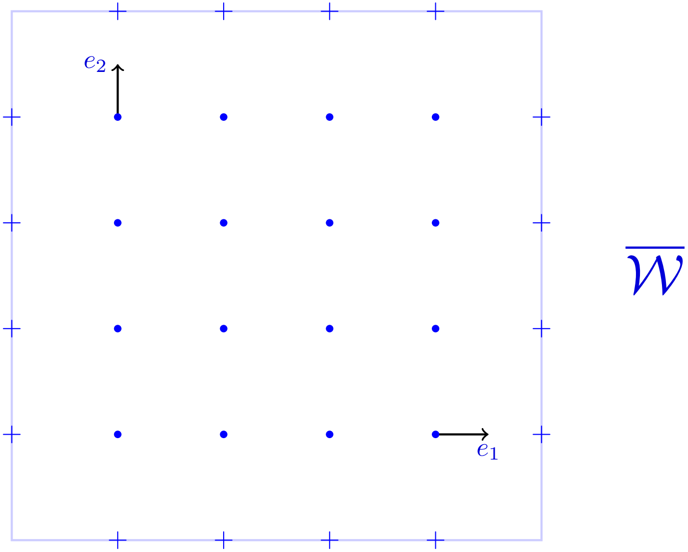
e_2
e_1
$\mathcal{W}$



e_2
e_1
$\mathcal{W}_1^* \cup \mathcal{W}_2^*$



e_2
e_1
$\partial_1 \mathcal{W} \cup \partial_2 \mathcal{W}$



e_2
e_1
$\overline{\mathcal{W}}$

Spatial Discrete Configuration

- Let us consider $n \geq 1$, $N \in \mathbb{N}$, and $h = \frac{1}{N+1}$
- We define the Cartesian grid of $(0,1)^n$ as:

$$\mathcal{W} := (0,1)^n \cap h\mathbb{Z}^n \tag{3}$$

- $C(\mathcal{W})$: set of functions defined on $\mathcal{W}$.

- For $\mathcal{W}$, with the translation operators, we can construct two dual sets given by

$$\mathcal{W}_i^* := \left\{ x - \frac{h}{2} e_i \middle| x \in \mathcal{W} \right\} \bigcup \left\{ x + \frac{h}{2} e_i \middle| x \in \mathcal{W} \right\}$$

- For these meshes, we define their boundary in direction e_i by

$$\partial_i \mathcal{W} := (\mathcal{W}_i^*)_i^* \setminus \mathcal{W}.$$

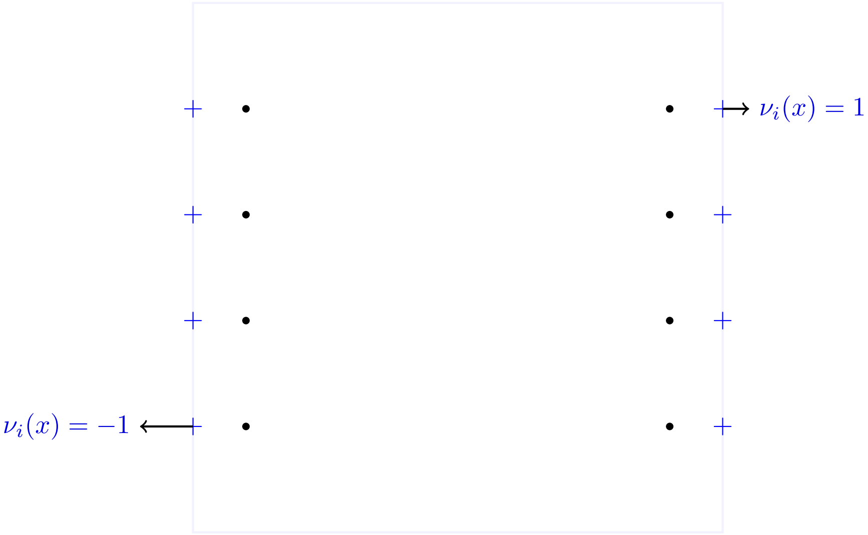
Moreover, we define

$$\partial \mathcal{W} := \bigcup_{k=1}^{n} \partial_k \mathcal{W}, \quad \text{and} \quad \overline{\mathcal{W}} := \bigcup_{k=1}^{n} (\mathcal{W}_k^*)_k^*. \tag{4}$$

- For $u \in C(\overline{\mathcal{W}})$, the translation operators are defined as $\boldsymbol{\tau}_{\pm i}u(x) = u(x \pm \frac{h}{2}e_i),$ in $\mathcal{W}_i^*$.
- We note that $\boldsymbol{\tau}_{\pm i} : C(\overline{\mathcal{W}}) \longrightarrow C(\mathcal{W}_i^*).$
- Moreover, for $u \in C(\mathcal{W}_i^*)$ we have that $\boldsymbol{\tau}_{\pm i}u \in C(\mathcal{W})$.
- We define the averaging operator and the difference operator for $u \in C(\overline{\mathcal{W}})$ as

$$A_i u = \frac{\boldsymbol{\tau}_{+i}u + \boldsymbol{\tau}_{-i}u}{2}, \quad D_i u = \frac{\boldsymbol{\tau}_{+i}u - \boldsymbol{\tau}_{-i}u}{h}, \quad \text{in } \mathcal{W}_i^*$$

respectively.


$\nu_i(x) = 1$
$\nu_i(x) = -1$

- We define the outward normal for $x \in \partial_i \mathcal{W}$ as

$$\nu_i(x) := \begin{cases} 1 & \tau_{-i}(x) \in \mathcal{W}_i^* \text{ and } \tau_{+i}(x) \notin \mathcal{W}_i^*, \\ -1 & \tau_{-i}(x) \notin \mathcal{W}_i^* \text{ and } \tau_{+i}(x) \in \mathcal{W}_i^*, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

- For each $x \in \partial_i \mathcal{W}$, we also introduce the trace operator for $u \in C(\mathcal{W}_i^*)$ as

$$t_r^i(u)(x) := \begin{cases} \tau_{-i} u(x) & \nu_i(x) = 1, \\ \tau_{+i} u(x) & \nu_i(x) = -1, \\ 0 & \nu_i(x) = 0. \end{cases} \tag{6}$$

- We have the following integral by parts for the difference and average operators

$$\int_{\mathcal{W}} u D_i(v) = -\int_{\mathcal{W}_i^*} v D_i u + \int_{\partial_i \mathcal{W}} u t_r^i(v) \nu_i,$$

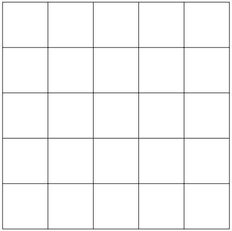
and

$$\int_{\mathcal{W}} u A_i(v) = \int_{\mathcal{W}_i^*} v A_i u - \frac{h}{2} \int_{\partial_i \mathcal{W}} u t_r^i(v),$$

Difficulties in the semi-discrete setting

Unique Continuation Property for Discrete Problems, in $2D$

In $\mathbb{R}^2$ for the discrete case, if $\Delta_h u \equiv 0$ on $\mathcal{W}$ and $u = \partial_n u = 0$ on a part of the boundary, then $u \neq 0$ on $\mathcal{W}$.



$$\Delta_h u_{i,j} := \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2} = 0$$

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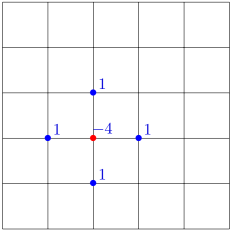

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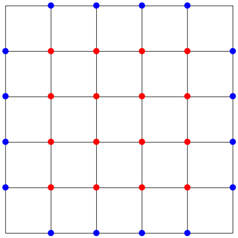

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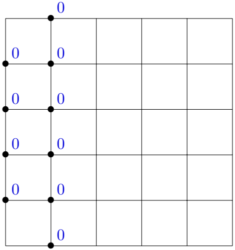

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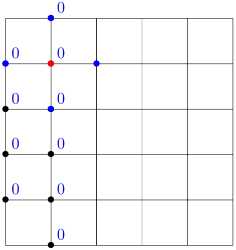

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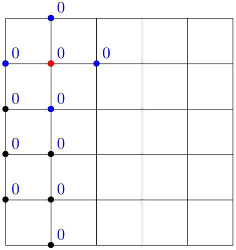

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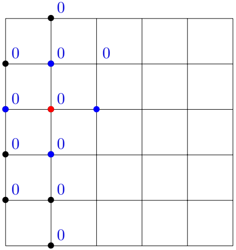

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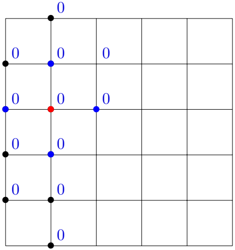

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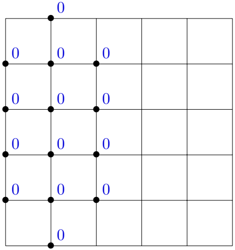

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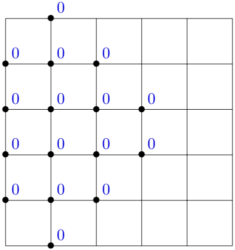

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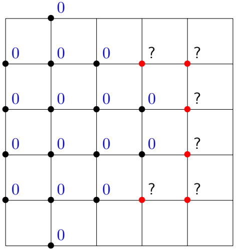

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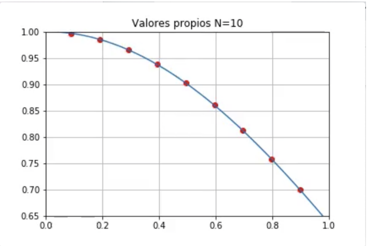
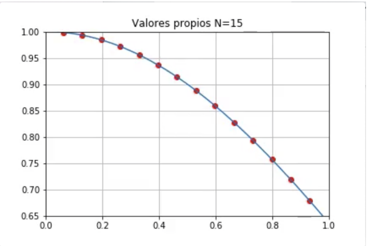
Comments

Several authors justify the loss of the unique continuation property because the discrete spectrum does not converge to the continuum.

$$\lambda_{k,h} = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right), \quad \frac{\lambda_{k,h}}{\lambda_k} = \left(\frac{\sin(\pi x_k/2)}{\pi x_k/2}\right)^2, \quad x_k = kh$$

$$|\sqrt{\lambda_{k,h}} - k\pi| \leq Ck^3h^2, \quad \forall k \in \{1, ..., 1/h - 1\}.$$





Example in 2D

Laplacian with homogeneous Dirichlet boundary condition

$$-\Delta_h \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & . \end{pmatrix} = \frac{4}{h^2} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & . \end{pmatrix}$$

$$\Delta_h u_{i,j} := \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$$

The example given by Otared Kavian and presented in

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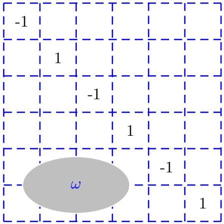
Propagation, observation, and control of waves approximated by finite difference methods.
SIAM Review, 47(2):197–243, 2005.

Non-observability, Example in dimension 2

Laplacian with homogeneous Dirichlet boundary condition

Semi-discrete Heat	$\frac{d}{dt}u - \Delta_h u = \chi_\omega v$
Semi-discrete Wave	$\frac{d^2}{dt^2}u - \Delta_h u = \chi_\omega v$
Full-discrete Heat	$\frac{u^{n+1}-u^n}{\Delta t} - \Delta_h u^{n+1} = \chi_\omega v^n$



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Inverse problem for the semi-discrete parabolic system

Inverse problem for the semi-discrete parabolic system

The source inverse problem

$$g \to \begin{cases} y_t - \mathcal{A}_h\, y &= g(t,x) \quad (0,T)\times\mathcal{W} \\ y &= 0 \quad (0,T)\times\partial\mathcal{W} \end{cases}$$

$$\mathcal{A}_h y := \sum_{i=1}^{d} D_i\left(\gamma_i(t,x) D_i y(t,x)\right) - \sum_{i=1}^{d} b_i(t,x) D_i A_i y(t,y) - c(t,x) y(t,x).$$

$$|\partial_t g(t,x)| \le C|g(T/2,x)|, \quad [0,T]\times\overline{\mathcal{W}} \tag{7}$$

$$\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K} < 0, \quad \theta(t) = \frac{1}{(t+\delta T)(T+\delta T - t)}, \quad t\in[0,T]. \tag{8}$$

Inverse problem for the semi-discrete parabolic system

The source inverse problem

$$g \rightarrow \begin{cases} y_t - \mathcal{A}_h\, y = g(t,x) & (0,T)\times\mathcal{W} \\ y = 0 & (0,T)\times\partial\mathcal{W} \end{cases}$$

Theorem (R.L, J. López and A. Pérez)

Let φ and θ be given by (8). Assume g satisfies (7). Then, there exist positive constants C, C'', $s_0 \geq 1$, $h_0 > 0$, $\varepsilon > 0$, depending on ω, and T, such that, we have the estimate

$$\|g\|_{L^2_h(\mathcal{W})} \leq C\left(\|y(\vartheta,\cdot)\|_{H^2_h(\mathcal{W})} + \|e^{\tau\theta\varphi}\partial_t y\|_{L^2_h(Q_\omega)} + \|e^{\tau\theta\varphi} y\|_{L^2_h(Q_\omega)}\right) + Ce^{-\frac{C''}{h}}\left(\|y(0)\|_{L^2_h(\mathcal{W})} + \|\partial_t y(0)\|_{L^2_h(\mathcal{W})}\right),$$

for all $\tau \geq \tau_0(T+T^2)$, $0 < h \leq h_0$, and exists $0 < \delta \leq 1/2$ depending on h, with $\tau h(\delta T^2)^{-1} \leq \varepsilon$, $y \in \mathcal{C}^1([0,T],\overline{\mathcal{W}})$ and where $Q_\omega := (0,T)\times\omega$.

Carleman inequality

For

$$\theta(t) = \frac{1}{(t+\delta T)(T+\delta T - t)}, \quad \text{and } \varphi(x) = e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_\infty}$$

Theorem 1.4, from F. Boyer and J. Le Rousseau (2014)

For $\lambda \geq 1$ sufficiently large, there exist C, $s_0 \geq 1$, $h_0 > 0$, $\varepsilon > 0$, depending on ω, ω_0, T, and λ, we have

$$\tau^{-1} \left\| \theta^{-1/2} e^{\tau\theta\varphi} y_t \right\|^2_{L^2_h(Q)} + J(y) \leq C \left(\left\| e^{\tau\theta\varphi} g \right\|^2_{L^2_h(Q)} + \int_{(0,T)\times\omega} \tau^3 \theta^3 e^{2\tau\theta\varphi} |y|^2 dx dt \right) + Ch^{-2} \int_\Omega \left(|y(0,x)|^2 + |y(T,x)|^2 \right) e^{2\tau\theta(0)\varphi} dx,$$

where

$$J(y) := \tau \sum_{i\in[\![1,d]\!]} \left(\left\| \theta^{1/2} e^{\tau\theta\varphi} D_i y \right\|^2_{L^2_h(Q^*)} + \left\| \theta^{1/2} e^{\tau\theta\varphi} A_i D_i y \right\|^2_{L^2_h(Q)} \right) + \tau^3 \left\| \theta^{3/2} e^{\tau\theta\varphi} y \right\|^2_{L^2_h(Q)} . \tag{7}$$

for all $\tau \geq s_0(T+T^2)$, $0 < h \leq h_0$, $0 < \delta \leq 1/2$, $\tau h(\delta T^2)^{-1} \leq \varepsilon$.

Ideas for improve the Carleman

- To consider $\gamma_i(t,x)$ in the Carleman inequality
- To include $\tau^{-1} \sum_{i,j \in [\![1,d]\!]} \int_{Q_{ij}^*} \theta^{-1} \gamma_i \gamma_j e^{2\tau\theta\varphi} |D_{ij}^2 y|^2$
- To improve the Carleman for $p = 1$

New Carleman inequality

Theorem (R.L, J. López and A. Pérez) $p = 0, 1$

$$I_p(y) + J_p(y) \leq C\left(\int_Q e^{2\tau\theta\varphi}(\tau\theta)^p |g|^2 + \int_{(0,T)\times\omega} (\tau\theta)^{p+3} e^{2\tau\theta\varphi}|y|^2\right)$$

$$+ Ch^{-2}\int_{\mathcal{W}} (\tau\theta(0))^p \left(|y(0,x)|^2 + |y(T,x)|^2\right) e^{2\tau\theta(0)\varphi}$$

$$I_p(y) := \int_{\mathcal{W}} \tau^{p+1}|y(T/2)|^2 e^{2\tau\theta(T/2)\varphi} + \int_Q (\tau\theta)^{p-1}|y_t|^2 e^{2\tau\theta\varphi}$$

$$+ \sum_{i,j\in[\![1,d]\!]} \int_{Q^*_{ij}} (\tau\theta)^{p-1}\gamma_i\gamma_j e^{2\tau\theta\varphi}|D_{ij}y|^2$$

$$J_p(y) := \tau^{p+1} \sum_{i\in[\![1,d]\!]} \left(\left\|\theta^{1/2+p/2} e^{\tau\theta\varphi} D_i y\right\|^2_{L^2_h(Q^*)} + \left\|\theta^{1/2+p/2} e^{\tau\theta\varphi} A_i D_i y\right\|^2_{L^2_h(Q)}\right)$$

$$+ \tau^{3+p}\left\|\theta^{3/2+p/2} e^{\tau\theta\varphi} y\right\|^2_{L^2_h(Q)}.$$

Technical Lemmas

Lemma 1

For large $\tau > 0$, there exists a constant $C > 0$ such that for $p = 0, 1$,

$$\int_Q \tau^p \theta^p \left| g\left(\frac{T}{2}, x\right)\right|^2 e^{2\tau\theta\varphi} \leq C\tau^{p-\frac{1}{2}} \int_{\mathcal{W}} \left| g\left(\frac{T}{2}, x\right)\right|^2 e^{2\tau\theta(\frac{T}{2})\varphi}, \qquad \forall \tau \geq \tau_1.$$

Lemma 2

Let y be the solution of the system

$$\partial_t y(t,x) - \mathcal{A}_h y(t,x) = g(t,x), \quad (t,x) \in (0,T) \times \mathcal{W}. \tag{8}$$

Then, for $T_0 \in (0,T)$,

$$\int_{\mathcal{W}} |y|^2(t) \leq e^{\tilde{C}(t-T_0)} \left(\int_{\mathcal{W}} |y|^2(T_0) + \int_{T_0}^t \int_{\mathcal{W}} |g|^2 \right). \tag{9}$$

Sketch of the proof

- Following the Bukhgeim-Klibanov method, we obtain the system for $z = \frac{d}{dt}y$.

$$\begin{aligned} &\partial_t z - \mathcal{A}_h z = \mathcal{B}_h y + \partial_t g, && \forall (t,x) \in (0,T) \times \mathcal{W} \\ &z(T/2,x) = \mathcal{C}_h y(T/2,x) + g(T/2,x) && \forall x \in \mathcal{W} \end{aligned}$$

- Using the New Carleman inequality with $p = 0$

$$I_0(z) + J_0(z) \preceq E_0(g_t) + E_0(\mathcal{B}_h y) + Obs_0(z) + Er_0(z) \tag{10}$$

where

$$E_p(u) := \int_Q (\tau\theta)^p e^{\tau\theta\varphi} u^2, \quad Obs_p(u) := \int_{Q_\omega} (\tau\theta)^p e^{\tau\theta\varphi} u^2,$$

$$Er_p(u) := \frac{(\tau\theta(0))^p}{h^2} \int_{\mathcal{W}} (|u(0)|^2 + |u(T)|^2) e^{2\tau\theta(T/2)\varphi}$$

- And for the Carleman inequality with $p = 1$

$$E_0(\mathcal{B}_h y) \preceq I_1(y) + J_1(y) \preceq E_1(g) + Obs_1(y) + Er_1(y) \tag{11}$$

Sketch of the proof

- Combined (10) with (11) we obtain

$$I_0(z) + J_0(z) \preceq E_0(g_t) + E_1(g) + Obs_0(z) + Obs_1(y) + Er_0(z) + Er_1(y)$$

- We observe

$$\tau \int_{\mathcal{W}} e^{2\tau\theta(T/2)\varphi} \left|z(T/2)\right|^2 \leq I_0(z)$$

- And using the condition for z on $t = T/2$, we obtain

$$\begin{array}{rcl} \tau \int_{\mathcal{W}} e^{2\tau\theta(T/2)\varphi} \left|g(T/2)\right|^2 & \preceq & E_0(g_t) + E_1(g) \\ & & +Obs_0(z) + Obs_1(y) \\ & & +\tau \int_{\mathcal{W}} e^{2\tau\theta(T/2)\varphi} \left|\mathcal{C}_h y(T/2)\right|^2 \\ & & +Er_0(z) + Er_1(y) \end{array}$$

Sketch of the proof

- Using Lemma 1 and (7), we obtain

$$E_0(g) + E_1(g) \preceq \sqrt{\tau} \int_{\mathcal{W}} e^{2\tau\theta(T/2)\varphi} \left|g(T/2)\right|^2$$

- Thus, we have

$$\begin{aligned} \tau \int_{\mathcal{W}} e^{2\tau\theta(T/2)\varphi} \left|g(T/2)\right|^2 \quad \preceq \quad & Obs_0(z) + Obs_1(y) \\ & + \tau \int_{\mathcal{W}} e^{2\tau\theta(T/2)\varphi} \left|\mathcal{C}_h y(T/2)\right|^2 \\ & + Er_0(z) + Er_1(y) \end{aligned}$$

- Finally, using Lemma 2 and taking $\delta \to 0$ but $\frac{\tau h}{T^2 \delta} \leq \varepsilon_0$

$$Er_p(u) \preceq \frac{e^{-\frac{C\tau}{\delta}}}{h^3} \int_{\mathcal{W}} \left(|u(0)|^2 + |g(T/2)|^2\right) \preceq e^{-\frac{C}{h}} \int_{\mathcal{W}} \left(|u(0)|^2 + |g(T/2)|^2\right)$$

Comments and path forward

Summary

- We study the case in $\mathbb{R}^N$
- For the semi-discrete problem, we obtain an equivalent version of the UCP plus an error term
- We improve the Carleman inequality for the semi-discrete parabolic problem
- It was possible to control part of the error terms arising in the Carleman inequality

Path forward

- To study a numerical reconstruction algorithm
- To study the full discrete problem (discrete in space and time)
- To study the semi-discrete problem with a stochastic term

Muchas gracias

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