Approximations of the best constants

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April 2025, Bilbao

What is a best constant?

In general it is related with a minimization procedure for a Rayleigh quotient

$$C_{A,B,X} = \inf_{u \in X} \frac{a(u,u)}{b(u,u)}, \quad C_{A,B,X} = \inf_{u \in X} \frac{\|u\|_A}{\|u\|_B},$$

Questions:

- Is the minimum attained? is the class of minimizers $\mathcal{M} \subset X$ non empty?

$$C_{A,B,X} = \min_{u \in X} \frac{\|u\|_A}{\|u\|_B},$$

- How big is $\mathcal{M} \subset X$? Is $\mathcal{M}$ invariant under some elementary transforms? $u \in \mathcal{M} \Rightarrow u(\lambda(x - x_0)) \in \mathcal{M}$?
- Does $\mathcal{M}$ has only one element? or $\mathcal{M} = \mathrm{span}\{\psi_1\}$?
- Regularity of the minimizers
- Can we compute exactly $C_{A,B,X}$?
- Can we approximate $C_{A,B,X}$ by some $C^h_{A_h,B_h,X_h}$?

Examples: One-dimensional Laplace eigenvalue problem

Let $\Omega = (0, \pi)$ and

$$\lambda_1(\Omega) = \inf_{u \in C_c^1(\Omega), u \not\equiv 0} \frac{\int_\Omega |u'|^2 dx}{\int_\Omega u^2 dx}.$$

- $\lambda_1(\Omega) = 1$ but there is no compactly supported $C^1((0, \pi))$ function to archive the min
- The minimum is attained in $H_0^1((0,1))$, $\lambda_1(\Omega) = 1$, and $\mathcal{M} = \operatorname{span}\{\sin(x)\}$ solution of the elliptic problem

$$\begin{cases} -u''(x) = \lambda u(x), \ x \in (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases}$$

- You can take $H_0^1 \setminus \operatorname{span}\{\sin(x)\}$ and minimize on the new space... you obtain an $\lambda_2(\Omega)$ which is again simple, etc....
- In general $u^k(x) = \sin(kx) \in C^\infty(\Omega), H^2(\Omega) \cap H_0^1(\Omega)$

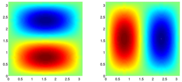
2D examples

When $\Omega = (0,\pi) \times (0,\pi)$ you begin to see the first difference

$$\begin{cases} -\Delta u(x,y) = \lambda u(x,y), \ (x,y) \in (0,\pi) \times (0,\pi), \\ u(x,y) = 0, (x,y) \in \partial\Omega. \end{cases}$$

$$\lambda_{m,n} = m^2 + n^2, u_{m,n}(x,y) = \sin(mx)\sin(ny)$$

- We have multiplicity $5 = 1^2 + 2^2 = 2^2 + 1^2$



- For L-shape domains- no explicit solutions, no explicit eigenvalues, problems with H^2-regularity (Grisvard)
- For sectors, annulus, Sphere and Spherical Shell, Ellipse and Elliptical Annulus, Equilateral Triangle there are explicit eigenvalues/eigenfunctions

(Grebenkov, Nguyen, Geometrical Structure of Laplacian Eigenfunctions, Siam Review 2013)

Other exemples, Sobolev's inequality

$$S_1(p,N) = \inf_{u\in \dot W^{1,p}(\mathbb{R}^N)} \frac{\|\nabla u\|_{L^p(\mathbb{R}^N)}}{\|u\|_{L^{p^*}(\mathbb{R}^N)}} \tag{1}$$

- $1 < p < N$, $\inf = \min$,
- $S_1(p,N)$ and $\mathcal{M}$ are explicit

$$U_{a,b,x_0}(x) = \frac{a}{(1+b|x-x_0|^{\frac{p}{p-1}})^{\frac{N-p}{p}}},\ a\in\mathbb{R}\setminus\{0\},\ b>0,\ x_0\in\mathbb{R}^N$$

$$-\mathrm{div}(|\nabla U|^{p-2}\nabla U) = S^p\|U\|^{p-p^*}_{L^{p^*}(\mathbb{R}^N)}|U|^{p^*-2}U,\ \text{in } \mathbb{R}^N.$$

- Infinitely many minimizers, an $N+2$ manifold
- $\mathcal{M}\subset \dot W^{2,p}(\mathbb{R}^N)\cap L^{p^*}(\mathbb{R}^N)$ but $\mathcal{M}\not\subset L^p(\mathbb{R}^N)$
- In the case of a bounded domain $S_1(p,N,\Omega)=S_1(p,N,\mathbb{R}^N)$ but inf in $W^{1,p}_0(\Omega)$ is not attained

Hardy's inequality

$$S_2(N,\Omega) = \inf_{u\in H^1(\Omega)} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|\frac{u}{|x|}\|^2_{L^2(\Omega)}},\ 0\in\Omega \tag{2}$$

- $S_2(N,\Omega) = S_2(\mathbb{R}^N) = \frac{(N-2)^2}{4}$, $N\neq 2$
- S_2 is not attained in $H^1(\Omega)$. Any possible minimizer should be of the form

$$u(r) = r^{-\frac{N-2}{2}}(a_1 + a_2\log(r)) \notin H^1_{loc}(\Omega)$$

but it belongs to a bigger space $\mathcal{H}(\Omega)$, the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_H = \int_\Omega |\nabla u|^2 - S_2(N,\Omega)\frac{u^2}{|x|^2}, \tag{3}$$

- $\mathcal{H}$ is isometric with the space $W_0^{1,2}(|x|^{-(N-2)}dx,\Omega)$

$$u = Tv = |x|^{-(N-2)/2}v.$$

$$S_2(N,\Omega)=\inf_{u\in H^1(\Omega)}\frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}, \quad (4)$$

$$S_3(\Omega)=\inf_{u\in C_c^\infty(\Omega)}\frac{\|\nabla u\|^2_{L^2(\Omega)}-S_2\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}. \quad (5)$$

- S_3 corresponds to the first eigenvalue of a singular operator in a weighted Sobolev space

$$u=Tv=|x|^{-(N-2)/2}v$$

$$\tilde{L}v=-|x|^{N-2}\nabla\cdot(|x|^{-(N-2)}\nabla v)=-\Delta v+(N-2)\frac{x}{|x|^2}\cdot\nabla v.$$

$$D(\tilde{L})=\{v\in W_0^{1,2}(|x|^{-(N-2)/2}dx,\Omega),\ \tilde{L}\in L^2(|x|^{-(N-2)/2}dx,\Omega)\}.$$

I. Peral, F. Soria, Elliptic and Parabolic Equations involving the Hardy-Leray Potential, 2021

Main references

- I. Babuška, J.E. Osborn, Eigenvalue problems. In Handbook of Numerical Analysis, Vol. II, North-Holland, Amsterdam, pp. 641-787 (1991).
- D. Boffi, Finite element approximation of eigenvalue problems, Acta Numerica, pp. 1-120 (2010).
- Some excelent web notes/slides by Jaap van der Vegt, Daniele Boffi, etc...

New one found by E.Z.

- E. Ernst and P. Le Tallec (2004), "Numerical Approximation of Poincaré and Friedrichs Constants."

Discrete constants and numerical approximations

Main references

- I. Babuška, J.E. Osborn, Eigenvalue problems. In Handbook of Numerical Analysis, Vol. II, North-Holland, Amsterdam, pp. 641-787 (1991).
- D. Boffi, Finite element approximation of eigenvalue problems, Acta Numerica, pp. 1-120 (2010).
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- E. Ernst and P. Le Tallec (2004), "Numerical Approximation of Poincaré and Friedrichs Constants."

via ChatGPT "This paper is often cited for its comprehensive numerical methods to compute these constants using finite element approximations."

Back to the 1D Laplace eigenvalue problem

$$\begin{cases} -u''(x) = \lambda u(x), \ x \in (0,\pi), \\ u(0) = u(\pi) = 0. \end{cases}$$

- Weak formulation: Find $\lambda \in R$, $u, v \in V = H_0^1((0,\pi))$ and $u \in V$, such that

$$\int_0^\pi u'(x)v'(x)dx = \lambda \int_0^\pi u(x)v(x)dx, \ \forall \ v \in V$$

- Galerkin approximation: Consider $V_h = \mathrm{span}\{\phi_1, \dots, \phi_N\} \subset V$, $h = 1/N$. Find $\lambda_h \in \mathbb{R}$, $u_h \in V_h$ such that

$$\int_0^\pi u_h'(x)v'(x)dx = \lambda_h \int_0^\pi u_h(x)v(x)dx, \ \forall \ v \in V_h$$

- $\lambda_1 = 1$, $\lambda_{1h} = \frac{6}{h^2}\frac{1-\cos(h)}{2+\cos(h)}$, $u^1(x) = \sin(x)$, $u_h^1(jh) = \sin(jh)$, $j = 0, \dots, N$,

Comparison of the discrete and continuous eigenvalues

- $\lambda_1 = 1 \leq \lambda_{1h} = \frac{6}{h^2}\frac{1-\cos(h)}{2+\cos(h)}$ as a consequence of the minimization argument

$$\inf_{V} \frac{\int_0^\pi u'^2}{\int_0^\pi u^2} \leq \min_{V_h} \frac{\int_0^\pi u'^2}{\int_0^\pi u^2}$$

  Warning: always the discrete value is greater than the continuous one since $V_h \subset V$

- Taylor's expansion

$$\lambda_{1h} = \lambda_1 + \frac{h^2}{12} + O(h^4), \text{ as } h \to 0,$$

- A similar estimate for eigenfunctions

$$\|u^1 - u_h^1\|_V = O(h)$$

- What you can say in the cases when the explicit values are not available ?

General case

• Always the approximation space V_h is taken as a subspace of the continuous one to have the same weak formulation

$$\lambda_1 = \min_V \frac{\|u\|_A^2}{\|u\|_B^2} \leq \min_{V_h} \frac{\|u\|_A^2}{\|u\|_B^2} = \lambda_{1h}$$

• Question: How close is λ_{1h} to λ_1?
• Answer: in general for P_1, piecewise linear finite elements, almost all the references say (Raviart-Thomas Lemma 6.4-2, Brenner-Scott, etc...)

$$\lambda_1 \leq \lambda_{1h} \leq \frac{\|\Pi_{V_h} u\|_A^2}{\|\Pi_{V_h} u\|_B^2} = \frac{\|u\|_A^2 + \|u - \Pi_{V_h} u\|_A^2}{\|\Pi_{V_h} u\|_B^2} \leq \lambda_1 + O(h^2)$$

by using $\|u - \Pi_{V_h} u\|_A \lesssim O(h)$, $\|u\|_B^2 = 1$, $\|\Pi_{V_h} u\|_B^2 \geq 1 - 2(u - \Pi_{V_h} u, u)$
Warning: The triangle inequality looses an h
• Question: Can we prove that in fact

$$\lambda_1 < \lambda_1 + Ah^2 \leq \lambda_{1h} \leq \lambda + Bh^2?$$

Answer: YES but you have to find the right reference

- I. Babuška and J. Osborn. Eigenvalue problems. In *Handbook of numerical analysis, Vol. II*, volume II of *Handb. Numer. Anal.*, pages 641–787. North-Holland, Amsterdam, 1991.
  Section 8, p. 700
- F. Chatelin. *Spectral approximation of linear operators*, volume 65 of *Classics in Applied Mathematics*. SIAM 2011. Reprint of the 1983 original
  Prop. 6.30, p. 315

For general operators and general approximation spaces V_h of the Hilbert space V

$$\lambda_{1h}(V_h) - \lambda_1(V) \simeq d_V(u_1, V_h)^2,$$

where u_1 is the first eigenvalue of the continuous problem
Problem: how you really compute

$$d_V(u_1, V_h)^2 = \min_{v \in V_h} \int_\Omega |\nabla u_1 - \nabla v_h|^2 dx?$$

How you compute a distance?

In general all the estimates are looking to the "worst case scenario"
• for all $u \in D(L) = \{u \in H_0^1(\Omega), \Delta u \in L^2(\Omega)\} ? = ? H^2(\Omega)$,

$$d_V(u, V_h) = \inf_{v \in V_h} \|u - v_h\|_V \leq h\|u\|_{D(L)}$$

but for the first eigenvalue you maybe know/have more regularity, is not any function in the domain of an operator is the first eigenvalue
• Warning: when you don't have it you have to use the "wcs" :(
• Why Taylor's expansion can still work in more general domains?

$$\lambda_{1h}(V_h) - \lambda_1(V) \simeq d_V(u_1, V_h)^2 \simeq h^2?$$

In fact we have to prove a lower bound

$$d_V(u_1, V_h) \gtrsim h$$

Easy to understand in 1D - use the convexity

- $|u_1''| = \lambda_1 |u_1| \geq c > 0$ on $(1/3, 2/3)$

Lemma

For any $-\infty < a < b < \infty$ *and* $u \in C^2((a,b))$

$$\int_a^b |u'(r) - A|^2 dr \geq \frac{(b-a)^3}{12} \inf_{r \in [a,b]} |u''(r)|^2, \ \forall \ A \in \mathbb{R}.$$

Then

$$\min_{v_h \in P_1^h} \int_{1/3}^{2/3} |u_1' - v_h'|^2 \geq \frac{c^2 h^2}{12}$$

- The same argument works if we know apriori a minimizer: Bessel functions for Hardy's inequality in balls of $\mathbb{R}^N$, or the Talentians for the Sobolev ineq, etc....
- Similar lower bounds can be obtained in higher dimensions: you only need the function to not be flat in one direction, i.e. one eigenvalue of the the Hessian should not vanish

What we did?

Theorem (Sobolev constants ver. 1)

Let us consider $\Omega \in \mathbb{R}^N$, $N \geq 2$, *and* V_h *the space of linear finite elements space on* Ω. *Then the corresponding discrete minimizers,* S_{1h} *satisfy*

$$h^{\gamma(p,N)\max\{2,p\}} \lesssim S_{1h}(p,N) - S_1(p,N) \lesssim h^{\gamma(p,N)\min\{2,p\}},$$

where

$$\gamma(p,N) = \frac{N-p}{N-p+p(p-1)},$$

• When $p = 2$ the Sobolev constant is approximated as follows:

$$S_{1h}(p,N) - S_1(p,N) \simeq h^{\frac{2(N-2)}{N}}.$$

• $N = 3$, $p = 2$ this convergence rate, $h^{2/3}$, is better than the upper bound $h^{1/3}$ obtained in [P. F. Antonietti and A. Pratelli. Finite element approximation of the Sobolev constant. Numer. Math. 2011.]

Theorem (Hardy)

Let us consider $\Omega \in \mathbb{R}^N$, $N \geq 3$, and V_h the space of linear finite elements space on Ω. Then the corresponding discrete minimizers satisfy

$$S_{2h} - S_2 \simeq \frac{1}{|\log h|^2}, \quad S_2(N,\Omega) = \inf_{u \in H^1(\Omega)} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}$$

$$S_{3h} - S_3 \simeq \frac{1}{|\log h|}, \quad S_3(\Omega) = \inf_{u \in C_c^\infty(\Omega)} \frac{\|\nabla u\|^2_{L^2(\Omega)} - S_2\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

• L.I. et. all, J. Convex Analysis 2024 for the 1D case,
• L.I., E. Zuazua 2025, $N \geq 2$
• For the lower bound: Hardy inequality with logarithmic reminders
• For the upper bound: take a pseudo-minimizer $u(x) = |x|^{-\frac{N-2}{2}}$ regularize it and approximate the regularization with P_h^1 functions
• Second estimate should be easier: approximate singular functions $u_1 = |x|^{-\frac{N-2}{2}} J_0(|x|)$ with piecewise functions: $d_{\mathcal{H}}(u_1, V_h) \simeq |\log h|^{-1/2}$

Hardy - The Lower bound

Key Inequality

$$\int_\Omega |\nabla v_h|^2 dx - S_2 \int_\Omega v_h^2 |x|^{-2} dx \gtrsim \int_\Omega |\nabla v_h|^2 |\log|x||^{-2} dx$$

- $S_{2h} = \frac{\int_\Omega |\nabla v_h|^2 dx}{\int_\Omega v_h^2 |x|^{-2} dx}$
- $S_{2h} - S_2 = \frac{\int_\Omega |\nabla v_h|^2 dx - S_2 \int_\Omega v_h^2 |x|^{-2} dx}{\int_\Omega v_h^2 |x|^{-2} dx} \gtrsim \frac{\int_\Omega |\nabla v_h|^2 |\log|x||^{-2} dx}{\int_\Omega v_h^2 |x|^{-2} dx}.$

$$\begin{aligned}
\int_\Omega |\nabla v_h|^2 |\log|x||^{-2} dx &= \sum_{T \in \mathcal{T}_h} \int_T |\nabla v_h|^2 |\log|x||^{-2} dx \\
&\gtrsim \frac{1}{|\log h|^2} \sum_{T \in \mathcal{T}_h} \int_T |\nabla v_h|^2 dx \\
&= \frac{1}{|\log h|^2} \int_\Omega |\nabla v_h|^2 = \frac{S_{2h}}{|\log h|^2} \int_\Omega v_h^2 |x|^{-2} dx.
\end{aligned}$$

Hardy - The upper bound

- $u(x) = |x|^{-N/2+1} \notin H^1(\Omega)$
- regularize u: $u_\varepsilon = u\eta_\varepsilon$

$$\eta_\varepsilon(x) = \begin{cases} 0, & |x| < \varepsilon^2, \\ \xi\left(\frac{\log(|x|/\varepsilon^2)}{\log(1/\varepsilon)}\right), & |x| \in (\varepsilon^2, \varepsilon), \\ 1, & |x| > \varepsilon, \end{cases}$$

- $A_\varepsilon = \int_\Omega |\nabla u_\varepsilon|^2 dx - S_2 \int_\Omega \frac{u_\varepsilon^2 dx}{|x|^2} \lesssim |\log \varepsilon|^{-1}$
- $B_\varepsilon = \int_\Omega \frac{u_\varepsilon^2 dx}{|x|^2} \simeq |\log \varepsilon|$
- $$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\int_\Omega u_\varepsilon^2 |x|^{-2}} \simeq |\log \varepsilon|^{-2}$$
- $\int_B |\nabla(u_\varepsilon - \Pi_h u_\varepsilon)|^2 dx \leq h^2 \|u_\varepsilon\|^2_{H^2(B)} \lesssim \frac{h^2}{\varepsilon^{4-2\mu}}$

$$S_{1h} - S_1 \lesssim \frac{1}{|\log \varepsilon|^2} + \frac{h^2}{\varepsilon^{4-2\mu}} \simeq \frac{1}{|\log h|^2}$$

Few details in the Sobolev case

- Take $\Omega = B_1$, and approximate it with a polygonal domain $B_h \subset B_1$
- Take $\mathcal{T}_h$ a triangularization of B_h
- Consider the space $\tilde{V}_h$ of piecewise linear functions on each triangle of $\mathcal{T}_h$
- Consider $V_h \subset H^1(B_h) \cap C(\overline{B}_h)$ to be the space of functions in $\tilde{V}_h$ extended by zero in $B \setminus B_h$.

$$S_h := \min_{w_h \in V_h} \frac{\|Dw_h\|_{L^2(\mathbb{R}^N)}}{\|w_h\|_{L^{2^*}(\mathbb{R}^N)}}$$

Goal: $N = 3$, $p = 2$:

$$S_h - S \simeq h^{2/3}$$

Warning: The nonhilbertian case $p \neq 2$ is more difficult

The Sobolev deficit

$$\delta(u) = \frac{\|Du\|_{L^p(\mathbb{R}^N)}}{\|u\|_{L^{p*}(\mathbb{R}^N)}} - S_{p,N}, \ \forall\ u \in \dot{W}^{1,p}(\mathbb{R}^N).$$

Bianchi and Egnel [JFA91] (p=2) and Figally and Zgang [Duke22] ($1 < p < N$)

$$\delta(u) \geq c \inf_{v\in\mathcal{M}} \left(\frac{\|Du - Dv\|_{L^p(\mathbb{R}^N)}}{\|Du\|_{L^p(\mathbb{R}^N)}}\right)^{\max\{2,p\}}. \quad (6)$$

Lemma (L.I., E.Z 2024)

There is a constant $C = C(p, N)$ and a positive number ε_0 such that

$$\delta(u) \leq C \inf_{v\in\mathcal{M}} \left(\frac{\|Du - Dv\|_{L^p(\mathbb{R}^N)}}{\|Du\|_{L^p(\mathbb{R}^N)}}\right)^{\min\{2,p\}}. \quad (7)$$

holds for all $u \in W^{1,p}(\mathbb{R}^N)$ satisfying

$$d(u, \mathcal{M}) = \inf_{v\in\mathcal{M}} \|Du - Dv\|_{L^p(\mathbb{R}^N)} \leq \varepsilon_0 \|Du\|_{L^p(\mathbb{R}^N)}.$$

Let $w_h \in V_h$ with $\|Dw_h\|^2_{L^2(\mathbb{R}^N)} = 1$ the function for which S_h above is attained:

$$S_h := \frac{\|Dw_h\|_{L^2(\mathbb{R}^N)}}{\|w_h\|_{L^{2^*}(\mathbb{R}^N)}}$$

Using estimates for the deficit

$$\inf_{U \in \mathcal{M}_1} d_V^2(U, w_h) \lesssim S_h - S \lesssim \inf_{U \in \mathcal{M}_1} d_V^2(U, V_h). \tag{8}$$

Remark: there is an alternative proof for the upper bound using estimates for eigenvalues a la Chatelin for $\mathcal{L} = -U^{2-2^*}\Delta$ on $H = L^2(U^{2^*-2})$

- One still has to prove that

$$h^{2/3} \lesssim \inf_{U \in \mathcal{M}_1} d_V^2(U, w_h), \qquad \inf_{U \in \mathcal{M}_1} d_V^2(U, V_h) \lesssim h^{2/3}$$

Each one is tricky :)

The upper/lower bound

• The simplest case: the upper bound
Take $u = U_{1,\lambda,0}$ and $v_h = I_h(u - u_{|\partial B_h})$ extended with zero outside B_h

$$\int_{\mathbb{R}^N} |DU_\lambda - Dv_h|^2 = \int_{B_h^c} |DU_\lambda|^2 + \int_{B_h} |DU_\lambda - Dv_h|^2 \lesssim \lambda^{-2} + (h\lambda^2)^2$$

and optimize in λ, i,e. $\lambda = h^{-1/3}$.
• The lower bound: take $a_h > 0$, $\lambda_h > 0$ and $x_h \in \mathbb{R}^N$ $(\dim(\mathcal{M}) = N+2)$

$$d_V^2(U_{a_h,\lambda_h,x_h}, w_h) = \inf_{U\in\mathcal{M}} d_V^2(U, w_h) \lesssim S_h - S \lesssim \inf_{U\in\mathcal{M}_1} d_V^2(U, V_h) \lesssim h^{2/3}$$

Few steps: $a_h > 1/2$, $x_h \in B_h$, $(\lambda_h^2 h)_{h>0}$ bounded, otherwise the above upper bound is false. Under these assumption

$$\int_{\mathbb{R}^N} |DU_{a_h,\lambda_h,x_h} - Dv_h|^2 \gtrsim \lambda_h^{-2} + h^2\lambda_h^4 \gtrsim h^{2/3}.$$

Optimal convergence rate

One of the main tools out of any numerical analysis is

$$\inf_{v\in\mathcal{M}}\Big(\frac{\|Du-Dv\|_{L^p(\mathbb{R}^N)}}{\|Du\|_{L^p(\mathbb{R}^N)}}\Big)^{\max\{2,p\}}\lesssim\delta(u)\lesssim\inf_{v\in\mathcal{M}}\Big(\frac{\|Du-Dv\|_{L^p(\mathbb{R}^N)}}{\|Du\|_{L^p(\mathbb{R}^N)}}\Big)^{\min\{2,p\}}$$

Both powers $\min\{2,p\}$, $\max\{2,p\}$ are optimal

Optimal convergence rate

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Both powers $\min\{2,p\}$, $\max\{2,p\}$ are optimal

Key idea: inspired by the numerical approximations of the p-Laplacian equation use the quasi-norms adapted to the solution of the problem

$$|U-v|_{p,2}=\left(\int(|DU|+|D(U-v)|)^{p-2}|D(U-v)|^2dx\right)^{1/2}$$

Thanks to Endre Süli ! (Prague 2024)

J. W. Barrett and W. B. Liu.
Finite element approximation of the p-Laplacian, *Math. Comp.*, 1993.

C. Ebmeyer and W. Liu.
Quasi-norm interpolation error estimates for the piecewise linear finite element approximation of p-Laplacian problems. *Numer. Math.*, 2005.

The final results after almost two years

Theorem

Let $N \geq 2$ and V_h the space of linear finite elements space in the unit ball B. Then the discrete Sobolev minimizer, S_h satisfy

$$S_h(p,N) - S(p,N) \simeq h^{\alpha(p,N)}, \tag{9}$$

where

$$\alpha(p,N) = \frac{2(N-p)}{N+p-2}. \tag{10}$$

Main steps:

- Estimate the deficit in term of the quasi-norms instead of $\|D(u-v)\|_p$
- Adapt the previous proofs at the numerical level (a little more technical as before)

Lemma

(Figalli-Zhang, Duke2022) For any $1 < p < N$ *there is a constant* $c = c(p, N)$ *such that*

$$\delta(u) \geq c \inf_{v \in \mathcal{M}} \frac{\int_{\mathbb{R}^n} (|D(u-v)| + |Dv|)^{p-2} |Du - Dv|^2 \, dx}{\|Du\|^p_{L^p(\mathbb{R}^N)}}, \ \forall \ u \in \dot{W}^{1,p}(\mathbb{R}^N). \tag{11}$$

Lemma

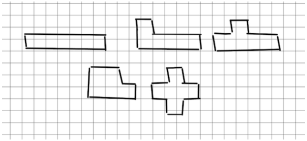
Let $1 < p < N$. *There exists a positive constant* $C(p, N)$ *and a positive number* ε_0 *such that*

$$\delta(u) \leq \frac{C(p,N)}{\|Du\|^p_{L^p(\mathbb{R}^N)}} \begin{cases} \int_{\mathbb{R}^n} (|Du| + |Dv|)^{p-2} |Du - Dv|^2 \, dx, & p \geq 2, \\ \int_{\mathbb{R}^N} (|Dv| + |D(u-v)|)^{p-2} |D(u-v)|^2 dx \\ \qquad + \Big(\int_{\mathbb{R}^N} |Dv|^{p-1} |D(u-v)| \, dx \Big)^2, & 1 < p < 2 \end{cases} \tag{12}$$

holds for all $u \in \dot{W}^{1,p}(\mathbb{R}^N)$ *with* $\delta(u) < S_{p,N}$ *and* $v \in \mathcal{M}$ *such that* $\|D(u-v)\|_{L^p(\mathbb{R}^N)} \leq \varepsilon_0 \|Du\|_{L^p(\mathbb{R}^N)}$.

Questions/Open problems

- Is more difficult to consider discrete or continuous problems? It depends. Some isoperimetric inequalities can be easily analyzed on a square mesh but the corresponding minimization of the first eigenvalue may be tricky. What is the 5 square domain that minimize the first continuous eigenvalue?



- Is very fashionable to work with ML, deep learning model, etc... what is the order of error when approximate with ReLU (The rectified linear unit)? it is only replacing $|x|$ functions in FEM with x^+? what about GELU, SiLU, etc...
- Fractional Sobolev Inequality, Andreea Dima
- There are many "hard analysis" questions that appears when one goes out of the classical Laplace eigenvalue problem as the regularity of the minimizers, how many they are, are they concentrated somewhere?

You need a lot of FUNCTIONAL ANALYSIS!!!

Thanks for your attention!