Partially dissipative hyperbolic systems, global existence in L^p spaces and relaxation limit[1]

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[1] Results obtained in collaboration with Raphaël Danchin

1. Introduction
   - General presentation
   - Beauchard and Zuazua's method

2. Our approach
   - New ideas
   - Main results

3. Extensions
   - Damped Baer-Nunziato system
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Introduction

We consider n-component quasilinear hyperbolic systems of the form:

$$\frac{\partial V}{\partial t} + \sum_{j=1}^{d} A^j(V)\frac{\partial V}{\partial x_j} = \frac{Q(V)}{\varepsilon}, \quad \text{where} \quad V : \begin{cases} \mathbb{R}_+ \times \mathbb{R}^d \to & \mathcal{O}_V \subset \mathbb{R}^n \\ (t \quad , \quad x) \to & V = V(t,x) \end{cases} \tag{1}$$

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More precisely we are interested in global strong and smooth solution being small perturbation of a constant state $\bar{V}\in\mathbb{R}^n$ such that $Q(\bar{V})=0$.

$\rightarrow$ There is a surprisingly strong connection between this question and problems related to control theory and Villani's hypocoercivity theory.

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- This can be written as:

$$Q(V) = \begin{pmatrix} 0_{\mathbb{R}^{n_1}} \\ q(V) \end{pmatrix} \text{ where } q(V) \in \mathbb{R}^{n_2}, n_1, n_2 \in \mathbb{N} \text{ and } n_1 + n_2 = n. \quad (2)$$

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- In the linear case, the system reads

$$\partial_t V + \sum_{j=1}^{d} A^j \partial_{x_j} V = \begin{pmatrix} 0 & 0 \\ 0 & -L \end{pmatrix} V \quad (3)$$

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And since $\mathcal{L}^2 \sim \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2$, we can obtain time-decay estimates.

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- $\rightarrow$ Ensures that **the damping is strong enough** to prevent small solutions (perturbations of constant states) from developing singularities in finite time.

- Under this condition, they are able to prove the following decay estimates:

$$\|V^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda t}\|V_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)},$$
$$\|V^\ell(t)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \leq Ct^{-\frac{d}{2}}\|V_0\|_{L^1(\mathbb{R}^d,\mathbb{R}^n)} \tag{5}$$

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- Here we develop a more explicit method partially based on Beauchard and Zuazua's paper (ARMA 2011).

Beauchard and Zuazua's method

Another point of view

Their paper (ARMA 2011) is based on the following classical proposition.

Proposition

Let A and L be two real matrices $n \times n$. The followings properties are equivalent:

1. *the couple (A, L) satisfies the (SK) condition;*
2. *the couple (A, L) satisfies the Kalman rank condition: the rank of $\begin{pmatrix} L & AL & \dots & A^{n-1}L \end{pmatrix}$ is n;*

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$\rightarrow$ link between the stability of these systems, control theory and Villani's hypocoercivity theory.

Inspired by this result, Beauchard and Zuazua designed the following Lyapunov functional:

$$\mathcal{L}^2 \triangleq \|V\|_{L^2}^2 + \int_{\mathbb{R}^d} \min(\rho, \rho^{-1})\mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(LA_\omega^{k-1} \widehat{V} \cdot LA_\omega^k \widehat{V} \big)$$

for $n-1$ positives parameters $\varepsilon_1, \cdots \varepsilon_{n-1}$.

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Differentiating in time $\mathcal{L}$ and adjusting the coefficients ε_k, we obtain:

$$\frac{d}{dt}\mathcal{L}^2 + \mathcal{H} \leq 0 \quad \text{where} \quad \mathcal{H} \triangleq \frac{1}{2}\int_{\mathbb{R}^d} \sum_{k=0}^{n-1} \varepsilon_k \min(1, \rho^2)|LA_\omega^k \widehat{V}(\xi)|^2 \, d\xi.$$

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- If $\varepsilon_1, \cdots, \varepsilon_{n-1}$ are chosen small enough, then $\mathcal{L} \sim \|V\|_{L^2}^2$
- And, if the (SK) condition is satisfied then

$$\mathcal{H} \geq \kappa \min(1, |\xi|^2)\mathcal{L},$$

which brings us to

$$\frac{d}{dt}\mathcal{L} + \kappa \min(1, |\xi|^2)\mathcal{L} \leq 0$$

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- More explicit method than Kawashima's $\rightarrow$ very useful to study the high frequencies of non-linear systems.
- Their method covers some cases where condition (SK) is not verified.
- However, it does not tell the full story about the low frequencies, which turns out to be essential to study the relaxation limit issue.

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- $\rightarrow$ More accurate information on the qualitative properties of the constructed solutions and a larger class of initial data under control.

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- $\rightarrow$ More accurate information on the qualitative properties of the constructed solutions and a larger class of initial data under control.
- To have a better understanding of where our improvement fits in, we need to keep in mind the following embeddings:

$$H^s(s > \frac{d}{2} + 1) \hookrightarrow B^{\frac{d}{2}+1}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{p}}_{p,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1} (p > 2) \hookrightarrow \mathcal{C}^1_b.$$

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- Since we work with homogeneous spaces, we can use their scaling properties and study the system with $\varepsilon = 1$. Then we recover the exact ε-dependency of the norm by scaling back.
- This allows to obtain optimal uniform estimates and therefore new results concerning the relaxation limit issue such as explicit convergence rates.

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$$f = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f \quad \text{and} \quad \text{supp}(\widehat{\dot{\Delta}_j f}) \subset \{\xi \in \mathbb{R}^d \text{ t.q. } \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}.$$

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- Low-frequency norms: $\|f\|^\ell_{\dot{\mathbb{B}}^{s'}_{p,1}} \triangleq \sum_{j\leq J_0} 2^{js'}\|\dot{\Delta}_j f\|_{L^p}$.

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A spectral analysis of the matrix

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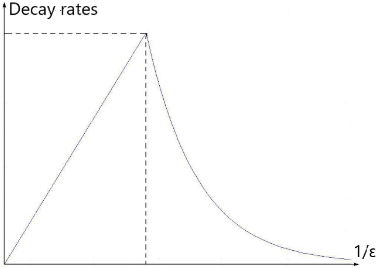

- This phenomena makes things difficult when trying to study the relaxation limit of concrete systems.

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- Let us now transcribe this spectral analysis into concrete properties.

Low frequencies in a simple case

Back to the damped p-system:

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→ This allows to prove a well-posedness result with the low frequencies of the solution being bounded in L^p spaces, with $2 \leq p \leq 4$.

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In the general case, the system can be rewritten as follows:

$$\begin{cases} \partial_t Z_1 + \displaystyle\sum_{k=1}^{d} \Big(A^k_{1,1}(V)\partial_k Z_1 + A^k_{1,2}(V)\partial_k Z_2 \Big) = 0, \\ \partial_t Z_2 + \displaystyle\sum_{k=1}^{d} \Big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2 \Big) + \dfrac{L_2 Z_2}{\varepsilon} = 0. \end{cases}$$

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We define the damped mode:

$$W \triangleq Z_2 + \varepsilon \sum_{k=1}^{d} L_2^{-1}(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2) = -{L_2}^{-1}\partial_t Z_2.$$

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$$\begin{cases} \partial_t W + \frac{L_2 W}{\varepsilon} = g \\ \partial_t Z_1 - \varepsilon \sum_{k=1}^{d} \sum_{\ell=1}^{d} \bar{A}_{1,2}^k L_2^{-1} \bar{A}_{2,1}^\ell \partial_k \partial_\ell Z_1 = f \end{cases} \tag{7}$$

where f and g are controllable in the low-frequency regime.

General case

To study the equation of Z_1, we have the following property

Lemma

Assume that $\forall\ k \in \{1, \cdots, d\},\ \bar{A}^k_{1,1} = 0$. *The following assertions are equivalent:*

- *the system satisfy the (SK) condition at* $\bar{V}$*;*
- *the operator* $\mathcal{A} := \sum_{k=1}^d \sum_{\ell=1}^d \bar{A}^k_{1,2} L_2^{-1} \bar{A}^\ell_{2,1} \partial_k \partial_\ell$ *is strongly elliptic.*

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$\rightarrow$ We may study the equations of W and Z_1 separately, the former as a damped equation and the latter as a heat equation.

Main results

Recalling setting and assumptions

We look at multi-dimensional first order n-component systems in $\mathbb{R}^d$:

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{d} A^k(V)\frac{\partial V}{\partial x_k} + \frac{LV}{\varepsilon} = 0,$$

such that:

- The maps A^k are symmetric valued $\rightarrow$ hyperbolicity of the system.
- $L + {}^T L$ is nonnegative $\rightarrow$ only *partial* dissipation occurs.
- An Hörmander's hypoellipticity-like condition is satisfied: the condition (SK): $\ker L \cap \{\text{eigenvectors of } A\} = \{0\}$

Main example of application:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \partial_t(\rho v) + \mathrm{div}\,(\rho v \otimes v) + \nabla P + \dfrac{1}{\varepsilon}\rho v = 0, \end{cases} \tag{8}$$

Well-posedness result

Theorem (Danchin, C-B '22)

Let $d \geq 1$, $p \in [2,4]$ and $\varepsilon > 0$. There exists $k_p \in \mathbb{Z}$ and $c_0 = c_0(p) > 0$ such that for all $J_\varepsilon \triangleq \lfloor -\log_2 \varepsilon \rfloor + k_p$, if we assume

$$\|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

then the system admits a unique solution Z satisfying

$$X_{p,\varepsilon}(t) \lesssim \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \quad \text{for all } t \geq 0, \text{ and where}$$

$$X_{p,\varepsilon}(t) \triangleq \varepsilon \|Z\|^{h}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|Z\|^{h}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \varepsilon^{-\frac{1}{2}} \|Z_2\|_{L^2_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}$$
$$+ \|Z\|^{\ell}_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon \|Z_1\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \|W\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}.$$

Decay estimates

Theorem (Danchin, C-B '22)

Assuming additionally that $Z_0 \in \dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}$ *for* $\sigma_1 \in \left]-\frac{d}{2}, \frac{d}{2}\right]$ *then there exists* $C > 0$ *such that*

$$\|Z(t)\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} \leq C \|Z_0\|_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}}, \quad \forall t \geq 0.$$

Moreover, if $\sigma_1 > 1 - d/2$,

$$\langle t \rangle \triangleq \sqrt{1+t^2}, \quad \alpha_1 \triangleq \frac{\sigma_1 + \frac{d}{2} - 1}{2} \quad \text{and} \quad C_0 \triangleq \|Z_0\|^{\ell}_{\dot{\mathbb{B}}^{-\sigma_1}_{2,\infty}} + \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}},$$

then Z *satisfies the following decay estimates:*

$$\sup_{t\geq 0} \left\| \langle t \rangle^{\frac{\sigma+\sigma_1}{2}} Z(t) \right\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \leq CC_0 \quad \text{if} \quad -\sigma_1 < \sigma \leq d/2 - 1,$$

$$\sup_{t\geq 0} \left\| \langle t \rangle^{\frac{\sigma+\sigma_1}{2}+\frac{1}{2}} Z_2(t) \right\|^{\ell}_{\dot{\mathbb{B}}^{\sigma}_{2,1}} \leq CC_0 \quad \text{if} \quad -\sigma_1 < \sigma \leq d/2 - 2,$$

$$\text{and} \quad \sup_{t\geq 0} \left\| \langle t \rangle^{2\alpha_1} Z(t) \right\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \leq CC_0.$$

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Introducing the slow-time variable $\tau = \varepsilon t$ and using the diffusive change of variable:

$$(\tilde{\rho}_\varepsilon, \tilde{J}_\varepsilon)(\tau, x) = (\rho, \frac{\rho u}{\varepsilon})(t, x). \tag{9}$$

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Then, formally, we have

$$-\tilde{J}_\varepsilon - \nabla P(\widetilde{\rho}_\varepsilon) \overset{*}{\rightharpoonup} 0 \quad \text{et donc} \quad \partial_\tau \tilde{\rho}_\varepsilon - \Delta P(\tilde{\rho}_\varepsilon) \overset{*}{\rightharpoonup} 0.$$

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And therefore, we expect $\widetilde{\rho}_\varepsilon$ to converge to $\mathcal{N}$, the solution of the porous media equation:

$$\begin{cases} \partial_\tau \mathcal{N} - \Delta P(\mathcal{N}) = 0, \\ \mathcal{N}_{\tau=0} = \rho_0 \end{cases} \tag{11}$$

Relaxation result

Theorem (Danchin, C-B '21)

Let $d \geq 1$, $p \in [2,4]$ and $\varepsilon > 0$. Let $\bar{\rho}$ be a strictly positive constant and $(\rho - \bar{\rho}, v)$ be the solution obtained with the previous theorem.

Let the positive function $\mathcal{N}_0$ such that $\mathcal{N}_0 - \bar{\rho}$ is small enough in $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$, and let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$ be the unique solution associated to the Cauchy problem:

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \end{cases}$$

If we assume that

$$\|\widetilde{\rho}_0^\varepsilon - \mathcal{N}_0\|_{\mathbb{B}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^\infty(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^1(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \widetilde{v}^\varepsilon \right\|_{L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

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$$c \triangleq \frac{2}{\gamma - 1}\sqrt{\frac{\partial P}{\partial \rho}} = \frac{(4\gamma A)^{\frac{1}{2}}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}. \tag{12}$$

Then, setting $\check{\gamma} = \dfrac{\gamma - 1}{2}$ and $\widetilde{c} = c - \bar{c}$, the Euler system reads

$$\begin{cases} \partial_t \widetilde{c} + v \cdot \nabla \widetilde{c} + \check{\gamma}(\widetilde{c} + \bar{c})\mathrm{div}\, v = 0, \\ \partial_t v + v \cdot \nabla v + \check{\gamma}(\widetilde{c} + \bar{c})\nabla \widetilde{c} + \dfrac{1}{\varepsilon} v = 0. \end{cases} \tag{13}$$

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This system is symmetric and satisfies the (SK) condition.
We use the diffusive change of variable:

$$(\widetilde{c}^\varepsilon, \tilde{v}^\varepsilon)(\tau, x) = (c, \frac{v}{\varepsilon})(t, x).$$

Thus the pair $(\widetilde{c}^\varepsilon, \tilde{v}^\varepsilon)$ satisfies

$$\begin{cases} \partial_t \tilde{c}^\varepsilon + \tilde{v}^\varepsilon \cdot \nabla \tilde{c}^\varepsilon + \tilde{\gamma}\tilde{c}^\varepsilon \mathrm{div}\, \tilde{v}^\varepsilon) = 0, \\ \varepsilon^2 (\partial_t \tilde{v}^\varepsilon + \tilde{v}^\varepsilon \cdot \nabla \tilde{v}^\varepsilon) + \tilde{\gamma}\widetilde{c}^\varepsilon \nabla \widetilde{c}^\varepsilon + \tilde{v}^\varepsilon = 0. \end{cases} \tag{14}$$

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Lemma

Let $(\widetilde{c}, v)$ be the solution from the existence theorem. Then, for all $t \geq 0$, $\widetilde{\rho}^{\varepsilon}$ satisfies

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With this in hand we can estimate the difference of the solutions of:

$$\partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \quad \text{and} \quad \partial_t \tilde{\rho}^{\varepsilon} + \operatorname{div}(\widetilde{\rho}^{\varepsilon}\,\tilde{v}^{\varepsilon}) = 0.$$

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Using the damped mode $\tilde{W}^\varepsilon = \widetilde{v}^\varepsilon + \dfrac{\nabla P(\widetilde{\rho}^\varepsilon)}{\rho}$, the equation of $\widetilde{\rho}^\varepsilon$ can be rewritten:

$$\partial_t \widetilde{\rho}^\varepsilon - \Delta P(\widetilde{\rho}^\varepsilon) = S^\varepsilon \quad \text{with} \quad S^\varepsilon = -\mathrm{div}(\widetilde{\rho}^\varepsilon \widetilde{W}^\varepsilon).$$

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The uniform bounds tell us that

$$\|\widetilde{W}^\varepsilon\|_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} = \mathcal{O}(\varepsilon) \quad \text{and} \quad \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^\infty_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} = \mathcal{O}(1).$$

Therefore, using a standard product law and composition lemma, we obtain

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Using that:

- $\|\widetilde{\rho}_0^{\varepsilon} - \mathcal{N}_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} \leq \varepsilon$
- There exists a function H_1 vanishing at $\bar{\rho} = \bar{\mathcal{N}}$ such that

$$P(\widetilde{\rho}^{\varepsilon}) - P(\bar{\rho}) = P'(\bar{\rho})\,(\widetilde{\rho}^{\varepsilon} - \bar{\rho}) + H_1(\widetilde{\rho}^{\varepsilon})\,(\widetilde{\rho}^{\varepsilon} - \bar{\rho}),$$

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$$P(\widetilde{\rho}^{\varepsilon}) - P(\bar{\rho}) = P'(\bar{\rho})\,(\widetilde{\rho}^{\varepsilon} - \bar{\rho}) + H_1(\widetilde{\rho}^{\varepsilon})\,(\widetilde{\rho}^{\varepsilon} - \bar{\rho}),$$

- The condition $P'(\bar{\rho}) > 0$, product law and composition lemma lead to

$$\|\delta D^{\varepsilon}\|_{L^{\infty}_T(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\delta D^{\varepsilon}\|_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \lesssim \varepsilon.$$

Extensions

Multifluid system

In a joint work with C. Burtea and J. Tan, we studied the following damped Baer-Nunziato system:

$$\begin{cases} \partial_t \alpha_\pm + u \cdot \nabla \alpha_\pm = \pm \dfrac{\alpha_+\alpha_-}{2\mu + \lambda}\left(P_+\left(\rho_+\right) - P_-\left(\rho_-\right)\right), \\ \partial_t\left(\alpha_\pm \rho_\pm\right) + \mathrm{div}\left(\alpha_\pm \rho_\pm u\right) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P + \eta \rho u = 0, \\ \rho = \alpha_+\rho_+ + \alpha_-\rho_-, \\ P = \alpha_+ P_+\left(\rho_+\right) + \alpha_- P_-\left(\rho_-\right) \end{cases} \tag{BN}$$

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With similar techniques as the one employed before, we are able to prove the global existence for this system as well as a result concerning the limit $\lambda, \mu \to 0$.

The HPC System

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In a joint work with Q. He and L-Y. Shou, we studied the following hyperbolic-parabolic system:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P(\rho) + \dfrac{1}{\varepsilon}\rho u - \mu\rho\nabla\phi = 0, \\ \partial_t \phi - \Delta\phi - a\rho + b\phi = 0, \qquad x \in \mathbb{R}^d, \quad t > 0, \end{cases} \tag{HPC}$$

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In this case, when $\varepsilon \to 0$, we show that the diffusive-rescaled solution of (HPC) converges strongly to the solution of the Keller-Segel system:

$$\begin{cases} \partial_t\rho - \mathrm{div}\,(\nabla P(\rho) - \mu\rho\nabla\phi) = 0, \\ \rho u = -\nabla P(\rho) + \mu\rho\nabla\phi, \\ -\Delta\phi - a\rho + b\phi = 0, \end{cases} \tag{KS}$$

Other problem under investigation

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- Jin-Xin system
- Anisotropic systems e.g. 2D Boussinesq
- General system of any order (potentially anisotropic)

$$\partial_t V + A(D)V + L(D)V = 0, \quad \text{where}$$

  - $A(D)$ is a skew-symmetric homogeneous Fourier multiplier of order α,
  - $L(D)$ is a partially elliptic homogeneous Fourier multiplier of order β.

Thank you for your attention! 谢谢!!