Universidad Autónoma de Madrid

Controllability of transport network systems by the regular linear systems approach

Yassine EL GANTOUH

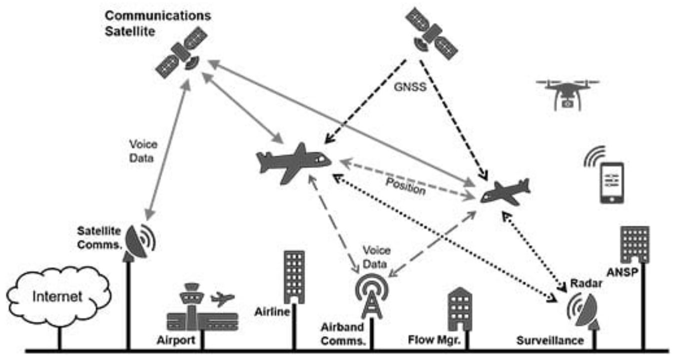
Universidad Autónoma de Madrid - Dycon project

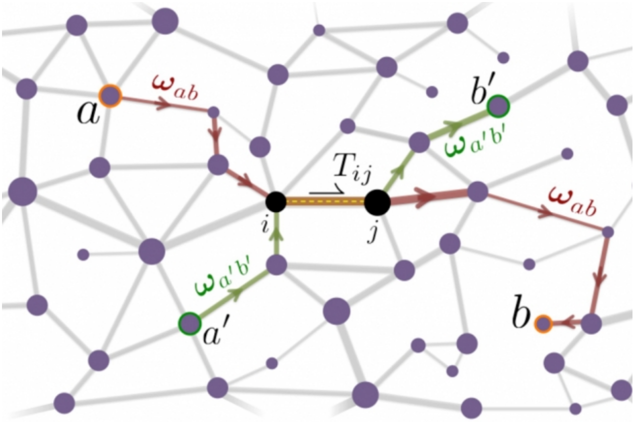
Advanced Seminar of the Chair of Dynamical Systems
Univesity of Passau
09 May 2022

Outline

Introduction: Transport networks

Transport networks
Introduction





- Gallons of gas flowing through pipelines.
- Phone calls transmitted in a communication system.
- Vehicular traffic.
- Biological networks.
- Planes flying in certain areas of airspace.

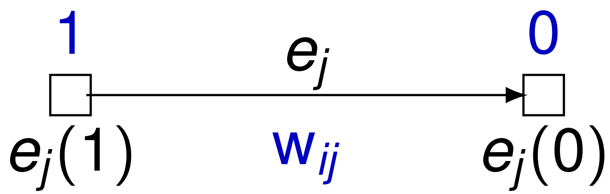
1. Microscopic Models:
   - Considers the evolution in time of the process happening in the vertices of the graph.
   - The dynamical system can hence be described by an ODE.

2. Macroscopic Models:
   - Consider the continuous evolution of the aggregate quantities of flows.
   - The evolution in time is then governed by a system of transport equations.

Transport networks

Metric Graphs

$$G = \{(v_1, \cdots, v_n, \cdots), (e_1, \cdots, e_m, \cdots)\}$$



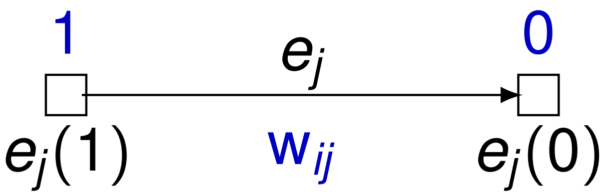
(*) $$\sum_{j=1}^{\infty} w_{ij} = 1, \quad \forall i = 1, 2, \ldots, n, \ldots.$$

Transport networks

Metric Graphs

Graphs Matrices:(*outgoing incidence matrix* I^{out}, *incoming incidence matrix* I^{in}, *weighted transposed adjacency matrix* $\mathbb{A}$)

$$G = \{(v_1, \cdots, v_n, \cdots), (e_1, \cdots, e_m, \cdots)\}$$



$$(\mathrm{I}^{out})_{ij} := \imath_{ij}^{out} = \begin{cases} 1, & \text{if } v_i \bullet\!\xrightarrow{e_j} \\ 0, & \text{if not,} \end{cases}, \qquad (\mathrm{I}^{in})_{ij} := \imath_{ij}^{in} = \begin{cases} 1, & \text{if } \xrightarrow{e_j}\!\bullet\, v_i \\ 0, & \text{if not,} \end{cases},$$

$$(\mathbb{A})_{il} := \left(\mathrm{I}^{in}(\mathrm{I}_{w}^{out})^{\top}\right)_{il} = \begin{cases} w_{lj}, & \text{if } \exists\, e_j \quad v_l \bullet\!\xrightarrow{e_j}\!\bullet\, v_i \\ 0, & \text{if not.} \end{cases},$$

Transport networks

Vertex Delay Problems

- along every edge e_j:

$$\frac{\partial}{\partial t} z_j(t,x) = c_j(x) \frac{\partial}{\partial x} z_j(t,x) + q_j(x).z_j(t,x), \; x \in (0,1), \; t \geq 0,$$

- initial conditions:

$$z_j(0,x) = g_j(x), \;\; z_j(\theta,x) = \varphi_j(\theta,x), \quad \theta \in [-r,0], \; x \in (0,1),$$

- boundary conditions in every v_i:

$$\imath_{ij}^{out} c_j(1) z_j(t,1) = \mathrm{w}_{ij} \sum_{k=1}^{\infty} \imath_{ik}^{in} [c_k(0) z_k(t,0) + L_k(z_k(t+.,.))] + \mathrm{b}_{il} u_l(t)$$

for $l = 1, \ldots, N$.

$$(*) \; \& \; (\mathrm{BC}) \Longrightarrow \sum \text{incoming flow} = \sum \text{outgoing flow}$$

Abstract formulation

- $X = L^p([0,1],\ell^1)$, $Y := L^p([-r,0],X)$, $\partial X = \ell^1$, $U = \mathbb{C}^N, p \in [1,+\infty)$.
- $$\begin{aligned} A_m f &= c(\cdot)\partial_x f + q(\cdot) f, \\ D(A_m) &= \{f \in (W^{1,p}([0,1],\ell^1) : c(1)f(1) \in \mathrm{Range}(\mathrm{I}_w^{out})^\top\}. \end{aligned}$$
- $\mathbb{G}f := \mathrm{I}^{out} c(1) f(1), \qquad \mathbb{M}f := \mathrm{I}^{in} c(0) f(0), \quad f \in W^{1,p}([0,1],\ell^1).$
- $(\mathbb{L}\varphi)_{ik} = \imath_{ik}^{in} \int_0^1 \int_{-r}^0 d[\eta_k(\theta)] c_k(x) \varphi_k(\theta, x) dx$, $\varphi \in W^{1,p}([-r,0],X)$.
- $K = (\mathrm{b}_{il})_{\mathbb{N}\times N}$.

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- $K = (\mathrm{b}_{il})_{\mathbb{N}\times N}$.

$$\Sigma_{TN}\begin{cases} \dot z(t) = A_m z(t), & t > 0, \\ z(0) = g,\ z_0 = \varphi & \\ \mathbb{G}z(t) = \mathbb{M}z(t) + \mathbb{L}z_t + Ku(t), & t > 0, \end{cases}$$

First of all, let us recall some basic facts about this kind of systems and their controllability properties:

1. System Σ_{TN} is well-posed if for every $g \in X$, $\varphi \in Y$, and $u \in L^p([0, +\infty); U)$ there exists a unique solution $z \in C([-r, +\infty); X)$ that depends continuously on the initial data g, φ and the control u.
2. System Σ_{TN} is said to be exactly controllable if it can be steered from any initial data to any target by choosing the control as a function of time in an appropriate way.
3. System Σ_{TN} is said to be approximate controllable if it can be steered from any initial data to a state arbitrarily close to a target by choosing a suitable control.

Transport networks

Finite Networks

- M. Kramar and E. Sikolya (2005)
- T. Matrai and E. Sikolya (2007)
- F.M. Hante, G. Leugering and T.I. Seidman (2009)
- K.J. Engel, M. Kramar Fijavž, R. Nagel, E. Sikolya (2008)
- F. Bayazit B. Dorn, and A. Rhandi. (2012)
- K.J. Engel, M. Kramar, B. KLöss, R. Nagel and E. Sikolya. (2010)
- S. Boulite, H. Bouslous, M. El Azzouzi and L. Maniar (2013)
- J. Banasiak and P. Namayanja (2014)
- M Kramar FijavÅ¿, D Mugnolo, S Nicaise (2021)

Infinite Networks

- B. Dorn et al. (2008)
- D. Kunszenti-Kovács (2009)
- P. Namayanja (2018)

Feedback theory of infinite dimensional linear systems

Feedback theory of infinite dimensional linear systems

- Banach spaces $X, \partial X, U, Z$ with $Z \subset X$.
- $A_m : Z \to X$ be a closed (differential) linear operator.
- $G, M : Z \to \partial X$ and $K \in \mathcal{L}(U, \partial X)$.

$$(\Sigma) \begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \quad z(0) = x, \\ Gz(t) = Mz(t) + Ku(t), & t > 0. \end{cases}$$

G. Greiner (1987), D. Salamon, O.J. Staffans (2005), A. Chen and K. Morris (2003), H. Zwart et al. (2010), S. Hadd et al (2015) .

- Banach spaces $X, \partial X, U, Z$ with $Z \subset X$.
- $A_m : Z \to X$ be a closed (differential) linear operator.
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$$

To (Σ) we associate the following operator

$$
\mathfrak{A} = A_m, \qquad D(\mathfrak{A}) = \{x \in Z : Gx = Mx\}.
$$

- Well-posedness of (Σ).
- Characterizing the boundary approximate controllability of (Σ).

$$
\text{(BIOS)} \begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \\ z(0) = x, & \\ Gz(t) = v(t), & t > 0, \\ y(t) = Mz(t) + Ku(t), & t > 0. \end{cases}
$$

with the feedback law "$u = y$".

$$\text{(BIOS)} \begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \\ z(0) = x, & \\ Gz(t) = v(t), & t > 0, \\ y(t) = Mz(t), & t > 0. \end{cases}$$

Definition

The system (BIOS) is called well-posed if, the operator $A := (A_m)_{|\ker G}$ generates a C_0-semigroup $\mathbb{T}$ on X and for some (hence for all) $\alpha > 0$, there is a constant $c_\alpha > 0$ such that the following inequality holds for all solutions of (BIOS):

$$\|z(\alpha)\|_X^p + \|y\|_{L^p([0,\alpha];\partial X)}^p \leq c_\alpha \left(\|x\|_X^p + \|v\|_{L^p([0,\alpha];\partial X)}^p\right).$$

Consider the linear systems

$$
\text{(OS)} \begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \\ z(0) = x, & \\ Gz(t) = 0, & t > 0, \\ y(t) = Mz(t), & t > 0. \end{cases}
\qquad
\text{(CS)} \begin{cases} \dot{z}(t) = A_m z(t), & t > 0, \\ z(0) = x, & \\ Gz(t) = v(t), & t > 0. \end{cases}
$$

Main Assumptions:

H1. $A := (A_m)_{|D(A)}$ with $D(A) = \ker G$ generates a strongly continuous semigroup $\mathbb{T} := (T(t))_{t\geq 0}$ on X.

H2. $\mathrm{Range}\,(G) = \partial X$.

Consider the observed linear system

$$
\text{(OS)} \begin{cases} \dot{z}(t) = Az(t), \quad z(0) = x, \quad t > 0, \\ y(t) = Mz(t), \qquad\qquad\qquad\quad t \geq 0, \end{cases}
$$

Consider the observed linear system

$$\text{(OS)} \begin{cases} \dot{z}(t) = Az(t), & z(0) = x, \quad t > 0, \\ y(t) = Mz(t), & t > 0, \end{cases}$$

We say that the system (OS) is well-posed if the output function $y(\cdot)$ can be extended to a function $y \in L^p_{loc}([0, +\infty), \partial X)$ such that

$$\|y(\cdot; x)\|_{L^p([0,\alpha],\partial X)} \leq \gamma \|x\|_X \qquad (x \in X)$$

for any $\alpha > 0$ and some constant $\gamma := \gamma(\alpha) > 0$.

We then select.

Definition

The operator $C := M_{|D(A)}$ is called an admissible observation operator for A if for some (hence all) $\alpha > 0$, there exists a constant $\gamma := \gamma(\alpha) > 0$ such that

$$\int_0^\alpha \|CT(t)x\|_{\partial X}^p dt \leq \gamma^p \|x\|^p \qquad (x \in D(A)).$$

We also say that (C, A) is admissible.

One can associate with $C \in \mathcal{L}(D(A), \partial X)$ the following operator

$$C_\Lambda x := \lim_{\mu \to \infty} C\mu R(\mu, A)x$$

with $D(C_\Lambda) := \{x \in X : \text{the above limit exists}\}$ called the Yosida extension of C for A. The introduced operator makes possible to give a simple pointwise representation of the extended output function y in terms of the observation operator C.

Weiss (Isr. J. Math, 1989)

For an admissible observation operator C we have $T(t)x \in D(C_\Lambda)$ for all $x \in X$ a.e. $t \geq 0$ and

$$y(t; x) = C_\Lambda T(t)x, \qquad \text{a.e. } t \geq 0 \ \ (x \in X).$$

Feedback theory of infinite dimensional linear systems

Consider the boundary controlled equation

$$(\mathrm{CS})\begin{cases}\dot{z}(t)=A_m z(t), \quad z(0)=x, \quad t>0,\\ Gz(t)=v(t), \qquad\qquad\qquad t\geq 0,\end{cases}$$

For any $\mu\in\rho(A)$, we have

$$D(A_m)=D(A)\oplus \ker(\mu-A_m)$$

and the following inverse

$$D_\mu := \left(G_{|\ker(\mu-A_m)}\right)^{-1}\in\mathcal{L}(\partial X,X)$$

exists. Now we define the control operator

$$B:=(\mu-A_{-1})D_\mu\in\mathcal{L}(\partial X,X_{-1}), \qquad \mu\in\rho(A).$$

- G. Greiner, Perturbing the boundary conditions of a generator, Houston J. Math., 13 (1987), 213–229.

The system (CS) is reformulated as the following distributed one

$$\dot{z}(t) = A_{-1}z(t) + Bv(t), \quad z(0) = x, \quad t \geq 0.$$

The system (CS) is reformulated as the following distributed one

$$\dot{z}(t) = A_{-1}z(t) + Bv(t), \quad z(0) = x, \quad t \geq 0.$$

An integral solution of the above equation is given by

$$z(t) = T(t)x + \Phi_t v \in X_{-1},$$

for any $t \geq 0$, $X \in X$ and $v \in L^p([0, +\infty), \partial X)$, where

$$\Phi_t v := \int_0^t T_{-1}(t-s)Bv(s)ds.$$

Definition

We say that the operator B is an admissible control operator for A, if for some $\tau > 0$, we have $\Phi_\tau v \in X$ for any $v \in L^p([0, +\infty), \partial X)$. In this case, we also say that (A, B) is admissible.

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Weiss (SICON, 1989)

If (A, B) is admissible, then for any $t \geq 0$,

$$\Phi_t \in \mathcal{L}(L^p([0, +\infty), \partial X), X).$$

Moreover, the solution of (CS) satisfies $z(t) \in C([0, +\infty); X)$, for any $t \geq 0$, $x \in X$ and $v \in L^p([0, +\infty), \partial X)$.

In the same spirit as for the above two reformulations, we transform the system (BIOS) to the following distributed input-output system:

$$\text{(DS)} \begin{cases} \dot{z}(t) = A_{-1}w(t) + Bv(t), & z(0) = x, \quad t > 0, \\ y(t) = Cz(t), & t \geq 0, \end{cases}$$

where we recall

$$C := M_{|D(A)}, \qquad B := (\mu - A_{-1})D_\mu \in \mathcal{L}(\partial X, X_{-1}), \qquad \mu \in \rho(A).$$

We then select the following definition.

Definition

We say that the system (DS) (or the triple (A, B, C)) is well-posed if, (A, B) is admissible and the output function y of the system is extended to a function $y \in L^p_{loc}([0, +\infty), \partial X)$ such that

$$\|y(\cdot; x, v)\|_{L^p([0,\alpha],\partial X)} \leq \kappa \left(\|x\|_X + \|v\|_{L^p([0,\alpha],\partial X)}\right)$$

for any $x \in X$, any $\alpha > 0$ and any $v \in L^p([0, \alpha], \partial X)$, where $\kappa := \kappa(\alpha) > 0$ is a constant.

Feedback theory of infinite dimensional linear systems

Input-output system

In order to shed more light on the well-posedness of the triple (A, B, C), we define the following dense spaces

$$W^{1,p}_{0,\alpha}(\partial X) := \left\{ v \in W^{1,p}([0,\alpha]; \partial X) : v(0) = 0 \right\}, \qquad \alpha > 0.$$

Now assume that (A, B) is admissible and (without loss of generality) $0 \in \rho(A)$. An integration by parts shows that

$$\Phi_t v = D_0 v(t) - R(0, A)\Phi_t \dot{v} \in Z$$

for any $t \in [0, \alpha]$, and $v \in W^{1,p}_{0,\alpha}(\partial X)$. This allows us to introduce the following map

$$(\mathbb{F}v)(t) := M\Phi_t v, \qquad v \in W^{1,p}_{0,\alpha}(\partial X),\ a.e.\ t \in [0, \alpha].$$

For any $\alpha > 0$ and $(x, v) \in D(A) \times W^{1,p}_{0,\alpha}(\partial X)$, the output function of the system (DS) is given by

$$y(t; x, v) = CT(t)x + (\mathbb{F}v)(t), \qquad t \in [0, \alpha].$$

Proposition

The triple (A, B, C) is well-posed if and only if the following assertions hold:

1. (A, B) and (C, A) are admissible.
2. for $\alpha > 0$, there exists a constant $\kappa := \kappa(\alpha) > 0$ such that

$$\|\mathbb{F}v\|_{L^p([0,\alpha];\partial X)} \leq \kappa \|v\|_{L^p([0,\alpha];\partial X)}, \qquad \forall\, v \in W^{1,p}_{0,\alpha}(\partial X).$$

In this case, the operator $\mathbb{F}$ is extendable to $\mathcal{L}(L^p([0,\alpha];\partial X))$ and the extended output function y of the system (DS) satisfies

$$y(t; x, v) = C_\Lambda T(t)x + (\mathbb{F}v)(t)$$

for any $x \in X$, $v \in L^p_{loc}([0, +\infty), \partial X)$ and a.e. $t > 0$. The operator $\mathbb{F}$ is called the extended input-output control operator.

Feedback theory of infinite dimensional linear systems

Input-output system

In order to give a complete representation of the extended output function y, we need the following subclass of well-posed linear systems.

Weiss (Trans. Am. Math. Soc., 1994)

A well-posed triple (A, B, C) with extended input-output control operator $\mathbb{F}$ is called regular (with feedthrough zero) if for any $v \in \partial X$, the following limit

$$\lim_{\tau \longmapsto 0} \frac{1}{\tau} \int_0^\tau (\mathbb{F}(\chi_{\mathbb{R}_+} \cdot v))(\sigma) d\sigma = 0,$$

exists in ∂X.

Weiss(Trans. Am. Math. Soc., 1994)

Let (A, B, C) be a regular triple with extended input-output control operator $\mathbb{F}$. Then $\Phi_t v \in D(C_\Lambda)$ and $(\mathbb{F}v)(t) = C_\Lambda \Phi_t v$ for any $v \in L^p([0, +\infty), \partial X)$ and a.e. $t \geq 0$. In particular, we have the state trajectory and the output function of the system (DS) satisfy

$$z(t; x, v) \in D(C_\Lambda), \qquad y(t; x, v) = C_\Lambda z(t; x, v),$$

for any $x \in X$, $v \in L^p_{loc}([0, +\infty), \partial X)$ and a.e. $t \geq 0$.

- G. Weiss, "Regular linear systems with feedback," *Math. Control Signals Systems* , vol. 7, pp. 23–57, 1994.
- O.J. Staffans, "Well-posed Linear Systems," Cambridge Univ. Press, 2005.

Hadd, Manzo and Rhandi (DCDS, 2015)

Assume that (A, B, C) is a regular triple with $1 \in \rho(\mathbb{F})$. Then the operator $\mathfrak{A}$ coincides with the following one

$$\mathfrak{A} = A_m \qquad D(\mathfrak{A}) = \{x \in Z : Gx = Mx\},$$

and hence generates a strongly continuous semigroup $\mathfrak{T} := (\mathfrak{T}(t))_{t\geq 0}$ on X. Furthermore, we have $\mathfrak{T}(s)x \in D(C_\Lambda)$ for any $x \in X$ and a.e. $s \geq 0$, and

$$\mathfrak{T}(t)x = T(t)x + \int_0^t T_{-1}(t-s)BC_\Lambda\mathfrak{T}(s)x\, ds, \qquad t \geq 0, \ (\text{on } X).$$

In addition, (Σ) is is equivalent to the following well-posed open-loop system

$$(\mathrm{CP}) \begin{cases} \dot{z}(t) = \mathfrak{A}_{-1} z(t) + BKu(t), \quad t \geq 0, \\ z(0) = x, \end{cases}$$

and therefore has a unique mild solution $z : [0, +\infty) \to X$ satisfying

$$D(C_\Lambda) \ni z(t) = \mathfrak{T}(t)x + \int_0^t \mathfrak{T}_{-1}(t-s)BKu(s)\, ds,$$

for a.e. $t \geq 0$ and for any $x \in X$ and any $u \in L^p_{loc}([0, +\infty), U)$.

Given a time $t > 0$, we shall be concerned with the final state

$$z(t; x, u) = \mathfrak{T}(t)x + \Phi_t^{\mathfrak{A}} K u,$$

and the following space of reachable states from $x = 0$ in time t

$$\mathcal{R}_t := \{ z(t; 0, u) \; : u \in L^p([0, t]; U) \} .$$

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and the following space of reachable states from $x = 0$ in time t

$$\mathcal{R}_t := \{z(t; 0, u) \ : u \in L^p([0, t]; U)\} .$$

Then the notion of approximate controllability in infinite time is defined as follows.

Definition

Assume that (A, B, C) is a regular triple with $1 \in \rho(\mathbb{F})$. We say that the open-loop system (CP) is approximately controllable if, the reachable set in finite time $\underset{t>0}{\cup} \mathcal{R}_t$ is dense in X.

El gantouh, Hadd and Rhandi (EECT, 2021)

Assume that (A, B, C) is a regular triple with $1 \in \rho(\mathbb{F})$. Then the open-loop system (CP) is approximately controllable if and only if, for $1 \in \rho(MD_\mu)$ and any $\varphi \in X'$ we have

$$\left\langle (I_X - D_\mu M)^{-1} D_\mu K u, \varphi \right\rangle = 0, \qquad \forall\, u \in U$$

implies that $\varphi = 0$. Here X' denotes the dual space of X.

Corollary

Assume that (A, B, C) is a regular triple such that $1 \in \rho(\mathbb{F})$ and $1 \in \rho(MD_\mu)$. Then the system Σ is boundary approximately controllable if and only if

$$\mathrm{Cl}\left(\mathrm{Range}\, D_\mu (I_{\partial X} - MD_\mu)^{-1} K\right) = X.$$

Controllability of Transport network systems

(1)- Well-posedness of Σ_{TN}

Abstract formulation

- $X = L^p([0,1],\ell^1)$, $Y := L^p([-r,0],X)$, $\partial X = \ell^1$, $U = \mathbb{C}^N, p \in [1,+\infty)$.
- $$\begin{aligned} A_m f &= c(\cdot)\partial_x f + q(\cdot) f, \\ D(A_m) &= \{f \in (W^{1,p}([0,1],\ell^1) : c(1)f(1) \in \text{Range}(I_w^{out})^\top\}. \end{aligned}$$
- $\mathbb{G}f := \mathrm{I}^{out}c(1)f(1), \qquad \mathbb{M}f := \mathrm{I}^{in}c(0)f(0), \quad f \in W^{1,p}([0,1],\ell^1)$.
- $(\mathbb{L}\varphi)_{ik} = \imath_{ik}^{in}\int_0^1 \int_{-r}^0 d[\eta_k(\theta)]c_k(x)\varphi_k(\theta,x)dx$, $\varphi \in W^{1,p}([-r,0],X)$
- $K = (\mathrm{b}_{il})_{\mathbb{N}\times N}$.

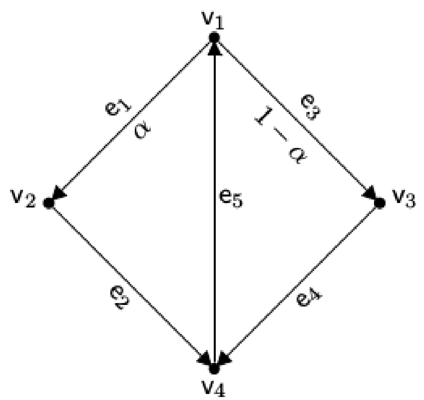
$$\Sigma_{TN}\begin{cases} \dot z(t) = A_m z(t), & t>0, \\ z(0) = g, \; z_0 = \varphi \\ \mathbb{G}z(t) = \mathbb{M}z(t) + \mathbb{L}z_t + Ku(t), & t>0, \end{cases}$$

Observation

Note that for any $g \in D(A_m)$, we have

$$\frac{c_{j_1}(1)g_{j_1}(1)}{w_{ij_1}} = \frac{c_{j_2}(1)g_{j_2}(1)}{w_{ij_2}} = \cdots = \frac{c_{j_l}(1)g_{j_l}(1)}{w_{ij_l}},$$

for $e_{j_1}, \cdots, e_{j_l}$ incidents edges having tails at the vertex v_i. This can be seen as a constraint on the solution at edges having the same tail ($z(t,1) \in \text{Range}(\mathrm{I}_w^{out})^\top$, $t \geq 0$). It is designed to takes into account that only a certain subset of mass distributions can be obtained.



- The mass distributions on the edges e_1 and e_3 will always satisfy the ratio $\frac{\alpha}{1-\alpha}$.

Controllability of Transport Network Systems

Well-posedness

As we work with delay system then is more convenient to introduce the new state space

$$\mathscr{X} := X \times Y \quad \text{with norm} \quad \left\| \binom{g}{\varphi} \right\| := \|g\|_X + \|\varphi\|_Y.$$

Furthermore, we introduce the spaces

$$\mathscr{Z} := D(A_m) \times D(Q_m), \quad \mathscr{U} := \ell^1 \times X,$$

and the following matrix operator

$$\mathscr{A}_m := \begin{pmatrix} A_m & 0 \\ 0 & Q_m \end{pmatrix} \qquad \mathscr{A}_m : \mathscr{Z} \to \mathscr{X},$$

$$\mathscr{G} := \begin{pmatrix} \mathbb{G} & 0 \\ 0 & \delta_0 \end{pmatrix}, \qquad \mathscr{M} := \begin{pmatrix} \mathbb{M} & \mathbb{L} \\ I_X & 0 \end{pmatrix} \qquad \mathscr{G}, \mathscr{M} : \mathscr{Z} \to \mathscr{U},$$

$$\mathfrak{B} := \binom{K}{0} \qquad \mathfrak{B} : U \to \mathscr{U}.$$

Now by selecting the new state

$$w(t) = \begin{pmatrix} z(t) \\ z_t \end{pmatrix}, \qquad t \geq 0,$$

the delay boundary control problem Σ_{TN} is reformulated in $\mathscr{X}$ as the following free delay perturbed boundary value control system

$$(VDP)\begin{cases} \dot{w}(t) & = & \mathscr{A}_m w(t), & t > 0, \\ w(0) & = & w^0, & \\ \mathscr{G} w(t) & = & \mathscr{M} w(t) + \mathfrak{B} u(t), & t > 0, \end{cases}$$

To (VDP) we associate the following operator

$$
\begin{aligned}
D(\mathfrak{A}) &:= \left\{ \binom{g}{\varphi} \in D(\mathscr{A}_m) : \mathscr{G}\binom{g}{\varphi} = \mathscr{M}\binom{g}{\varphi} \right\} \\
&= \left\{ \binom{g}{\varphi} \in D(\mathscr{A}_m) : \mathbb{G}g = \mathbb{M}g + \mathbb{L}\varphi,\ g = \varphi(0) \right\},
\end{aligned}
$$

To (*VDP*) we associate the following operator

$$D(\mathfrak{A}) = \left\{ \binom{g}{\varphi} \in D(\mathscr{A}_m) : \mathbb{G}g = \mathbb{M}g + \mathbb{L}\varphi,\ g = \varphi(0) \right\},$$

In order to apply the results of the previous sections, let assume:

Main assumptions

(A1) $c_j(\cdot), q_j(\cdot) \in L^\infty([0,1])$ such that $c_j(x) \geq \gamma_1, q_j(x) \leq \gamma_2$ for $j \in \mathbb{N}$, $x \in [0,1]$ and some constants $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}$.

(A2) Each vertex has only finitely many outgoing edges.

(A3) The weights w_{ij} satisfies $(*)$.

(A4) For any $i \in \mathbb{N}$, $\eta_i : [-r, 0] \to \mathbb{R}$ is a function of bounded variations such that $|\eta_i|([-\varepsilon, 0]) \to 0$ as $\varepsilon \to 0$, and for any $\theta \in [-r, 0]$, $\eta(\theta) = \mathrm{diag}(\eta_i(\theta))_{i\in\mathbb{N}} \in \ell^1$.

In view of **(A1)**, we define

$$\tau_j(x_1, x_2) := \int_{x_1}^{x_2} \frac{dx}{c_j(x)},\ \xi_j(x_1, x_2) := \int_{x_1}^{x_2} \frac{q_j(x)}{c_j(x)} dx,$$

for every $j \in \mathbb{N}$ and $x_1, x_2 \in [0, 1]$.

Lemma

Let the assumptions **(A1)**-**(A3)** be satisfied. Then, the operator $(\mathscr{A}_m, D(\mathscr{A}_m))$ is closed.

Lemma

Let the assumption **(A4)** *be satisfied and assume that* $\sup_{i\in\mathbb{N}} \|c_i\|_\infty < \infty$. *If we denote by* d_λ *for* $\lambda \in \mathbb{C}$ *the Dirichlet maps associated with* Q_m *and* δ_0, *then* $d_\lambda : X \to Y$ *is given by*

$$(d_\lambda g)(\theta, x) = e^{\lambda\theta} g(x), \quad \theta \in [-r, 0],\ x \in [0, 1].$$

Moreover, if we denotes $\beta := (-Q_{-1})d_0 : X \to Y_{-1,Q}$, *then the triple* $(Q, \beta, \mathbb{L}_{|D(Q)})$ *is a regular triplet with control*

$$(\Phi_t^\beta v)(\theta, x) = \begin{cases} v(t+\theta, x), & -t \le \theta \le 0, \\ 0, & -r \le \theta \le -t, \end{cases}$$

for any $t \ge 0$ *and* $x \in [0, 1]$, *and* $v \in L^p(\mathbb{R}^+, \ell^1)$.

El gantouh, Hadd and Rhandi (2022)

Let the assumptions **(A1)**-**(A3)** be satisfied. The Dirichlet operator associated with A_m and $\mathbb{G}$ is given by

$$D_\mu v = \mathrm{diag}(e^{\xi_j(\cdot,1)-\mu\tau_j(\cdot,1)})_{j\in\mathbb{N}} c^{-1}(1)(\mathrm{I}_{\mathrm{w}}^{out})^\top v, \qquad v \in \ell^1.$$

Moreover, we define

$$B := (\mu - A_{-1})D_\mu : \ell^1 \to X_{-1,A}, \quad \mu \in \rho(A)(= \mathbb{C}).$$

Then (A, B) is admissible with control maps $\Phi_t^A : L^p(\mathbb{R}^+, \ell^1) \to X$ are given by, for any $j \in \ell^1$,

$$(\Phi_t^A v)_j(x) = \sum_{i=1}^\infty \mathrm{w}_{ij} e^{\xi_j(x,1)} c_j^{-1}(1) v_i(t - \tau_j(x, 1))\chi_{[\tau_j(x,1),+\infty)}(t).$$

Furthermore, if $\sup\{\|c_i\|_\infty : i \in \mathbb{N}\} < \infty$ and $\inf_{j\in\mathbb{N}} \tau_j(0, 1) > 0$, then the triple $(A, B, \mathbb{M}_{|D(A)})$ is regular on X, ℓ^1, ℓ^1.

Thus, we have the following generation result.

El gantouh, Hadd and Rhandi (2022)

Let the assumptions **(A1)** to **(A4)** be satisfied and assume that $\sup_{i\in\mathbb{N}} \|c_i\|_\infty < \infty$ and $\inf_{j\in\mathbb{N}} \tau_j(0,1) > 0$. Then the operator $\mathfrak{A}$ generates a strongly continuous semigroup $\mathfrak{T} := (\mathfrak{T}(t))_{t\geq 0}$ on $\mathscr{X}$. Moreover, we have $\mu \in \rho(\mathfrak{A})$ if and only if $1 \in \rho(\mathbb{A}_\mu)$, where we set

$$\mathbb{A}_\mu = \mathbb{B}\left[c(0)\,\mathrm{diag}\,(e^{\xi_j(0,1)-\mu\tau_j(0,1)})_{j\in\mathbb{N}} + \Xi(\mu)\right] c(1)^{-1}(\mathcal{I}_\mathrm{w}^-)^\top.$$

with $\Xi(\mu) := \mathrm{diag}\,\left(\int_0^1\int_{-r}^0 d[\eta_j](\theta)e^{\mu\theta}c_j(x)e^{\xi_j(x,1)-\mu\tau_j(x,1)}dx\right)_{j\in\mathbb{N}}$. In this case,

$$R(\mu,\mathfrak{A}) = \begin{pmatrix} \Gamma(\mu)R(\mu,A) & \Omega(\mu)R(\mu,Q) \\ d_\mu\Gamma(\mu)R(\mu,A) & R(\mu,Q)+d_\mu\Omega(\mu)R(\mu,Q) \end{pmatrix}$$

where $\Gamma(\mu) := I_X + D_\mu(I_{\ell^1} - \mathbb{A}_\mu)^{-1}[\mathbb{M} + \mathbb{L}d_\mu]$ and $\Omega(\mu) = D_\mu(I_{\ell^1} - \mathbb{A}_\mu)^{-1}\mathbb{L}$.

El gantouh, Hadd and Rhandi (2022)

Let the assumptions **(A1)-(A4)** be satisfied and assume that $\sup_{i\in\mathbb{N}} \|c_i\|_\infty < \infty$ and $\inf_{j\in\mathbb{N}} \tau_j(0,1) > 0$. Then the system (VDP) has a unique strong solution $w(\cdot)$ satisfying

$$w(t) \in D((\mathbb{M}_{|D(A)})_\Lambda) \times D((\mathbb{L}_{|D(Q)})_\Lambda), \qquad \text{for a.e. } t \geq 0,$$

$$w(t; w^0, u) = \mathfrak{T}(t)w^0 + \int_0^t \mathfrak{T}_{-1}(t-s)\widetilde{\mathcal{B}}u(s)\,ds,$$

for all $t \geq 0$, $w^0 := \left(\begin{smallmatrix} g \\ \varphi \end{smallmatrix}\right) \in \mathscr{X}$ and $u \in L^p_+(\mathbb{R}_+, U)$, where the operator $\widetilde{\mathcal{B}}$ is given by

$$\widetilde{\mathcal{B}} = \begin{pmatrix} (\mu - A_{-1})\mathbb{D}_\mu K \\ (\mu - Q_{-1})\xi_\mu \end{pmatrix} = \begin{pmatrix} BK \\ \beta \end{pmatrix} \in \mathcal{L}(U, \mathscr{X}_{-1,\mathfrak{A}}).$$

Therefore the solutions of the vertex delay problem Σ_{TN} are given by

$$
\begin{aligned}
z(t; g, u) &= T(t)g + \Phi_t^A[(\mathbb{M}_{|D(A)})_\Lambda z(\cdot; g, u) + (\mathbb{L}_{|D(Q)})_\Lambda z_t(\cdot; \varphi, u)] + \Phi_t^A K u,\\
z_t(\cdot; \varphi, u) &= S(t)\varphi + \Phi_t^Q z(\cdot; g, u),
\end{aligned}
$$

for all $t \geq 0$, $\binom{g}{\varphi} \in \mathscr{X}$ and $u \in L^p(\mathbb{R}_+, U)$.

Controllability of Transport network systems

(2)- Controllability Criteria

Given a time $\tau > 0$, we shall be concerned with the final state

$$z(\tau; g, u) = T(\tau)g + \Phi_\tau^A[(\mathbb{M}_{|D(A)})_\Lambda z(\cdot; g, u) + (\mathbb{L}_{|D(Q)})_\Lambda z_\tau(\cdot; \varphi, u)] + \Phi_\tau^A K u,$$

$$z_\tau(\cdot; \varphi, u) = S(\tau)\varphi + \Phi_\tau^Q z(\cdot; g, u),$$

$$w(\tau; w^0, u) = \mathfrak{T}(\tau)w^0 + \Phi_\tau^{\mathfrak{A}} k u,$$

and the following spaces of reachable states from $x = 0$ in time τ

$$\mathcal{R}_\tau^X := \{z(\tau, 0, u) \ : u \in L^p([0, \tau]; U)\},$$
$$\mathcal{R}_\tau^Y := \{z(\tau + \cdot, 0, u) \ : u \in L^p([0, \tau]; U)\}$$
$$\mathcal{R}_\tau^{\mathscr{X}} := \{w(\tau, 0, u) \ : u \in L^p([0, \tau]; U)\}.$$

Clearly, we have

$$\mathcal{R}_\tau^X = \mathrm{Range}\, \Phi_\tau^A,$$
$$\mathcal{R}_\tau^Y = \mathrm{Range}\, \Phi_\tau^Q,$$
$$\mathcal{R}_\tau^{\mathscr{X}} = \mathrm{Range}\, \Phi_\tau^{\mathfrak{A}} K.$$

Then the notion of approximate controllability in infinite time is defined as follows.

Definition

Let the assumptions **(A1)**-**(A4)** be satisfied and assume that $\sup_{i\in\mathbb{N}} \|c_i\|_\infty < \infty$ and $\inf_{j\in\mathbb{N}} \tau_j(0,1) > 0$. The system (VDP) (or the pair $(\mathfrak{A}_{-1}, \widetilde{\mathcal{B}})$) is called:

(*i*) boundary X-approximate controllable if $Cl(\underset{\tau>0}{\cup} \mathcal{R}_\tau^X) = X$;

(*ii*) boundary Y-approximate controllable if $Cl(\underset{\tau>0}{\cup} \mathcal{R}_\tau^Y) = Y$;

(*iii*) boundary $\mathscr{X}$-approximate controllable if $Cl(\underset{\tau>0}{\cup} \mathcal{R}_\tau^{\mathscr{X}}) = \mathscr{X}$.

El gantouh, Hadd and Rhandi (2022)

Let the assumptions **(A1)-(A4)** be satisfied and assume that $\sup_{i\in\mathbb{N}}\|c_i\|_\infty < \infty$ and $\inf_{j\in\mathbb{N}}\tau_j(0,1) > 0$. Then the system (VDP) (or Σ_{TN}) is $\mathscr{X}-$approximately controllable if and only if for a large $\lambda > 0$, and for any $g^* \in X'$ and $\varphi^* \in Y'$, we have

$$\langle D_\lambda(I_{\ell^1} - \mathbb{A}_\lambda)^{-1}Ku, g^*\rangle + \left\langle d_\lambda D_\lambda(I_{\ell^1} - \mathbb{A}_\lambda)^{-1}Ku, \varphi^*\right\rangle = 0$$

for any $u \in \mathbb{C}^N$, implies that $g^* = 0$ and $\varphi^* = 0$.

El gantouh, Hadd and Rhandi (2022)

Let the assumptions **(A1)-(A4)** be satisfied and assume that $\sup_{i\in\mathbb{N}} \|c_i\|_\infty < \infty$ and $\inf_{j\in\mathbb{N}} \tau_j(0,1) > 0$. Then we have

(i) The system (VDP) (or Σ_{TN}) is X-approximately controllable if and only if for any $u \in \mathbb{C}^N$ and $g^* \in X'$,

$$\langle D_\lambda (I_{\ell^1} - \mathbb{A}_\lambda)^{-1} K u, g^* \rangle = 0$$

implies that $g^* = 0$.

(ii) The system (VDP) (or Σ_{TN}) is Y-approximately controllable if and only if for any $u \in \mathbb{C}^N$ and $\varphi^* \in Y'$,

$$\left\langle d_\lambda D_\lambda (I_{\ell^1} - \mathbb{A}_\lambda)^{-1} K u, \varphi^* \right\rangle = 0$$

implies that $\varphi^* = 0$.

El gantouh, Hadd and Rhandi (2022)

Let the assumptions **(A1)-(A4)** be satisfied and assume that $\sup_{i\in\mathbb{N}} \|c_i\|_\infty < \infty$ and $\inf_{j\in\mathbb{N}} \tau_j(0,1) > 0$. Then, there is $\mu_0 > 0$ such that for any $\mathrm{Re}\lambda > \mu_0$, the vertex delay problem Σ_{TN} is X-approximately controllable if and only if

$$\mathrm{Cl}\left(\mathrm{span}\ \left\{\mathbb{A}_\lambda^k K,\ k = 0, 1, 2, \cdots\right\}\right) = \ell^1.$$

Controllability of Transport Network Systems

A Kalman-type criterion for finite graphs

El gantouh, Hadd and Rhandi (EECT, 2021)

Let the assumptions **(A1)**-**(A4)** be satisfied and assume that $q_j \leq 0$ for all $j = 1, \ldots, m$. Then the free delay transport network system Σ_{TN} is boundary approximately controllable if and only if, for any $\Re e\, \mu > 0$,

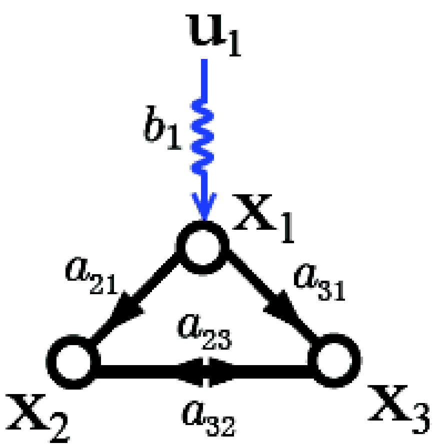
$$\text{Rank}\begin{pmatrix} K & \mathbb{A}_\mu K & \ldots & \mathbb{A}_\mu^{n-1} K \end{pmatrix} = n.$$

Assume that $c_j \equiv c$ and $q_j \equiv 0$ for all $j = 1, \ldots, m$, where c is a positive constant. Then, the the free delay transport network system Σ_{TN} is approximately controllable if and only if

$$\text{Rank}\begin{pmatrix} K & \mathbb{A} K & \ldots & \mathbb{A}^{n-1} K \end{pmatrix} = n.$$

Controllability of Transport Network Systems

Example

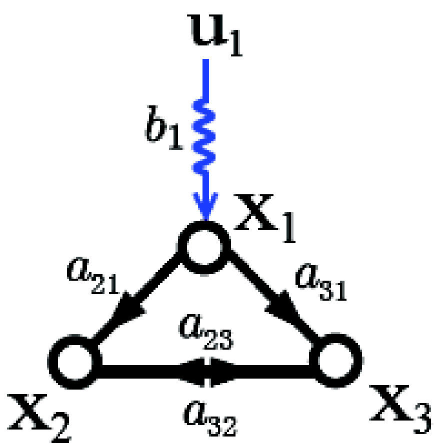


$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}.$$

Controllability of Transport Network Systems

Example



$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\mathbf{C} = \left(B,\ \mathbb{A}B,\ \mathbb{A}^2 B \right) = b_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23}a_{31} \\ 0 & a_{31} & a_{32}a_{21} \end{pmatrix}.$$

Hence,

Rank $\mathbf{C} = 3$ as long as $a_{32}a_{21}^2 \neq a_{23}a_{31}^2$.

Challenges:

- System parameters (link weights), usually unknown especially for biologically networks.
- Rank of the controllability matrix is computationally expensive in large networks (**C** is of dimension $n \times nm$).
- Control of all nodes is practically difficult (if not impossible).

Structural controllability:

- **A structural matrix** $\mathbb{A} \in \mathbb{R}^{n \times n}$ is a matrix having fixed zeros in some locations and arbitrary, independent entries in the remaining locations.
- The two systems $(\mathbb{A}, B)$ and $(\mathbb{A}', B')$ are **structurally equivalent** if there is a one-to-one correspondence between the locations of their fixed zero and nonzero entries.
- A system $(\mathbb{A}, B)$ is called **structurally controllable** if there exists a system structurally equivalent to $(\mathbb{A}, B)$ which is controllable in the usual sense, i.e., fulfilled Kalman's controllability rank condition.

C.T. Lin, "Structural controllability," *IEEE TAC,* vol. 19, pp. 201-208, 1974.

We now introduce the basic technical condition for studying structural controllability.

Definition

An $(n \times s)$ matrix A $(s \geq n)$ is said to be of form (t) for some $1 \leq t \leq n$ if, for some k in the range $s - t < k \leq s$, A contains a zero submatrix of order $(n + s - t - k + 1) \times k$.

We now introduce the basic technical condition for studying structural controllability.

Definition

An $(n \times s)$ matrix A $(s \geq n)$ is said to be of form (t) for some $1 \leq t \leq n$ if, for some k in the range $s - t < k \leq s$, A contains a zero submatrix of order $(n + s - t - k + 1) \times k$.

$$Q_0 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{pmatrix}, \quad Q_1 := \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \\ x & x & x & x & x \\ x & x & x & x & x \end{pmatrix}.$$

Both matrices Q_0 and Q_1 are of form (4), but $k = 5$ for the matrix Q_0 while $k = 4$ for the matrix Q_1.

Controllability of Transport Network Systems

Controllability Criteria

Using the connection between structural controllability and form (t), we obtain.

El gantouh, Hadd and Rhandi (EECT, 2021)

The problem Σ_{TN} is approximately controllable if and only if the matrix

$$\mathfrak{C} := \begin{pmatrix} K & I & 0 & 0 & 0 & \cdots & \cdots & & & 0 \\ 0 & -\mathbb{A} & K & I & 0 & \cdots & \cdots & & & 0 \\ 0 & 0 & 0 & -\mathbb{A} & K & \cdots & \cdots & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & & \cdots & & -\mathbb{A} & K & I & 0 \\ 0 & 0 & 0 & & \cdots & & 0 & 0 & -\mathbb{A} & K \end{pmatrix}.$$

is not of form (n^2).

- R.W. Shields and J. B. Pearson, “Structural controllability of multi-input linear systems,“ *IEEE TAC*, vol. 21, pp. 203–212, 1976.

Conclusion

Conclusion

- Frequency domain characterization for approximate controllability of abstract perturbed boundary control systems.
- Well-posedness and boundary approximate controllability of abstract delay boundary value problems.
- Deducing approximate controllability of transport network system from the structural controllabilty of a finite dimensional system.

(1) Which vertices allow the controllability of the whole network?
(2) What is the minimal controllability time?
(3) What is the optimal control yielding the best assignment of weights?
(4) What if only positive states are allowed?
(5) What if the coefficients are time-dependent?

THANK YOU